1   An informal account of my work

I am primarily interested in questions about billiards in all of their many forms. The study of billiards has been growing steadily over the past 30 years, and it is currently a very active field. I like it because the behavior is global: everything is locally flat, except at isolated vertices, so the system is governed by combinatorics. This leads to beautiful structure, some of which is classical, and much of which is currently being discovered, by me and many others.

In the field of billiards, there are four inter-related areas of study that inform each other; I will explain the relationships between them, and my work on them, in §2:

- *billiards*, a point bouncing around inside a polygon,

- *translation surfaces*, generalizations of the square torus,

- the *sequences* of edges a trajectory hits, and

- *interval exchange transformations*, cutting and reassembling an interval.

I have done work in all four of these areas. Additionally, I have done much of the foundational work to develop a new area of billiards, in which a beam of light refracts through a planar tiling. I named the system *tiling billiards*, led the first two research groups that studied it, and coauthored the first three papers in the area. Now other people are studying it: two mathematicians in France have recently written papers [HR18, R19] giving beautiful proofs of our conjectures, and extending the theory.

In any polygonal billiard table, almost all of the trajectories are aperiodic. Yet I have mostly chosen to focus on *periodic* behavior, because I can draw pictures of the periodic paths, which tend to be beautiful, and also because many simple questions have never been explored. Figure 1 shows several beautiful examples of periodic billiard trajectories on the regular pentagon, which no one had ever seen before we created them in [DL18]. Furthermore, the structure of periodic directions is a generalization of continued fractions, and depends on finitely generated groups.
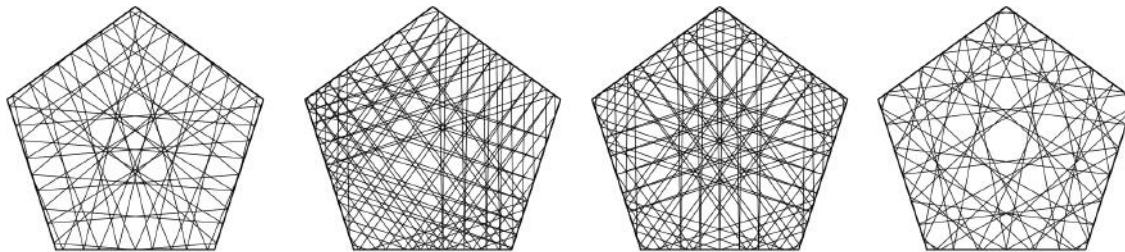

Figure 1: Some periodic billiard paths on the regular pentagon, from [DL18].

Much of my work translates back and forth between the *geometry* of billiard trajectories, and the *combinatorics* of the sequence of edges that such a trajectory hits, studying how each of them determines the other.

2   More details about my work

As described above, to study polygonal billiards, we use tools in four inter-related areas: inner billiards, translation surfaces, sequences, and interval exchange transformations. I'll now describe how they fit together and the work I've done in various areas, and then I'll give more technical details in § 3.

In *billiards*, a point bounces around inside a polygon (red picture in Figure 2). If we label each edge, e.g. A, B, C, D as in the figure, then each trajectory has a corresponding *bounce sequence* of edges that it hits, e.g. $\ldots A, D, B, D, C, \ldots$ In [CCD+18], we showed "how to hear the shape of a billiard table": how to reconstruct the shape of a polygon, from the collection of all possible bounce sequences on it.
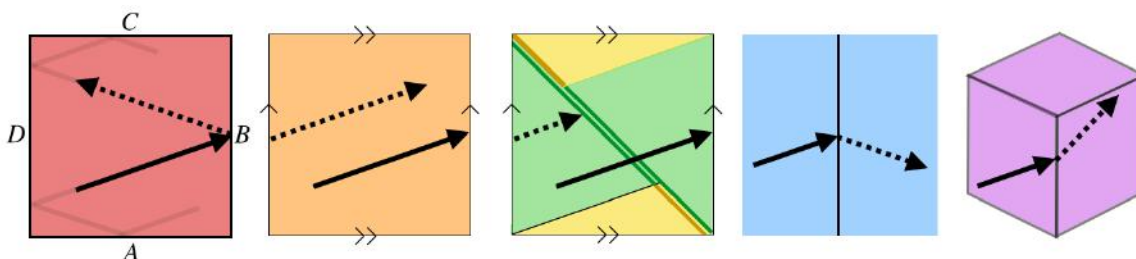

Figure 2: A schematic picture of the relationship between (red) billiards, (orange) translation surfaces, (yellow and green) interval exchange transformations, (blue) tiling billiards, and (purple) geodesics on polyhedra.

In billiards, a trajectory changes direction every time it hits an edge. To make things easier, we can "unfold" the polygon, creating a new copy by reflection each time the point hits an edge, so that the trajectory goes straight. Continuing this process leads to a *translation surface* (orange picture), made from finitely many copies of the original polygon, together with a map to the polygon. The map carries straight lines in the surface to billiard paths. Thus, the study of billiards on the rational polygon reduces to the study of the straight line flow on a translation surface.

As with billiards, we can look at the sequence of edges that a trajectory passes through. In my thesis [D13, D14], I studied the language of such sequences for the double pentagon surface, and other double regular polygons. Later with collaborators, we extended this result to a much larger family of surfaces [DPU18].

In the papers [DFT11, DL18], we classified the directions and behavior of all of the periodic billiard trajectories on the regular pentagon, by using the associated translation surface, the *double pentagon* (see Figure 4).

We can reduce the problem to a one-dimensional system by adding one or more parallel diagonals to the translation surface, transverse to the direction of a given trajectory of interest (yellow and green picture). The trajectory passes through the translation surface exactly as before, but now we only keep track of where it intersects the diagonal(s). This action is equivalent to cutting and reassembling an interval, which we call an *interval exchange transformation* (IET). In the recent pa-

per [CD18], we showed that, while all IETs are measure-preserving, almost every measure-preserving transformation is *not* conjugate to any IET.

In [BDFI18], we showed that the behavior of a *tiling billiards* trajectory (blue picture in Figure 2) on a triangle tilings is equivalent to the orbit of a point in a certain 3-interval orientation-reversing IET. Orientation-reversing IETs have a lot of stable, periodic points, leading to mostly stable, periodic behavior in tiling billiards on triangle tilings. By contrast, almost every trajectory on the triangle-hexagon tiling is unstable, aperiodic and dense, as we showed in [DH18].

One last billiard system is trajectories on polyhedra. In [DDoTY17], we fully characterized all of the vertex-to-vertex paths on the regular tetrahedron and cube (purple picture in Figure 2), using the Stern-Brocot tree of rational numbers. We showed that it is *not* possible to start at a vertex of a cube, travel in a straight line, and return to the same vertex. Some colleagues, inspired by our paper, showed that such a path *is* possible on the dodecahedron by giving an explicit example [AA18], and followed that up with a full characterization of paths on the dodecahedron [AAH18].

I have also done several projects outside of billiards:

- We described the properties of the fractal-like "Thurston set" (Figure 3), which is the points in the complex plane that are Galois conjugates of periodic points of a family of "tent maps" [BDLW18]. Our paper resolved unfinished work from the paper that William Thurston was working on when he died [T14].
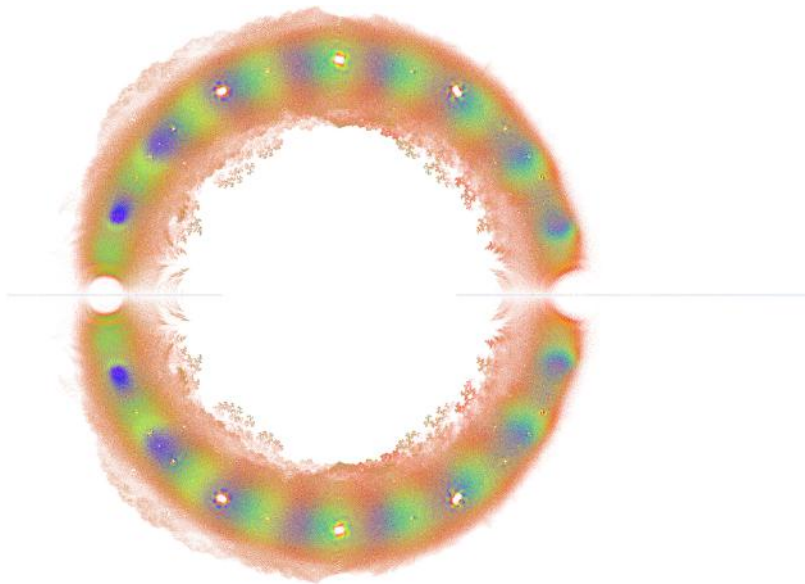

Figure 3: The Thurston set in the complex plane. Each point is the root of a polynomial; points are colored according to its degree.

- With some colleagues at Northwestern, we answered the question, "if you ran a race at an average pace of p minutes per mile, must there have been some mile that you ran in *exactly* p minutes?" in the negative, by showing that it is a novel application of the Horizontal Chord Theorem [BDaD17].

3   Technical details

In this section, I'll give more details about my work described above.

Bounce rigidity. A classic question from a 1960s paper asks, "Can you hear the shape of a drum?" [K66]. We answered this question in the affirmative as it applies to billiards, in our paper "How to hear the shape of a billiard table" [CCD+18]. We gave a constructive method, in which we take the collection of all possible bounce sequences on a table, and use it construct both the cyclic ordering of the edges, and the angles between adjacent edges. We also showed that, to know the exact angles of the table, an infinite amount of information is required; it is not possible to reconstruct the angles from finitely many strings of finitely many sequences. This is a "rigidity" result, in the sense that the collection of bounce sequences determines the shape of the table.

In a companion paper, our colleagues proved that the bounce spectrum uniquely determines the shape of the table: the angles between edges *and* their lengths [DELS18]. Their result is stronger, but it is not constructive.

Cutting sequences on Bouw-Möller surfaces.

The translation surfaces I have studied are *lattice surfaces* (also known as *Veech surfaces*), meaning that their automorphism group forms a lattice, or in other words, that they have a lot of symmetry. Given such a surface, it is natural to ask:

- What happens to a trajectory on the surface, when you apply an automorphism?

I answered this question for the double pentagon surface (see Figure 4) and other double regular polygon surfaces in [D13], for a larger family of surfaces of which those are a subset in [D14], and for an even larger family of surfaces of which *those* are a subset, called *Bouw-Möller surfaces*, in [DPU18]. In all cases, the answer was in terms of what happens to the bi-infinite *cutting sequence* of edges that the trajectory cuts through. Also, understanding the effects of reflections and rotations is easy, so the hard part of the problem was giving the effect of the *shear* automorphism.

Figure 4: The double pentagon surface: a screenshot of my Dance Your Ph.D. video, which explains the above Theorem.

Theorem (Davis [D13]). Let τ be a trajectory on the double pentagon surface, let its associated cutting sequence of symbols be $c(\tau)$, and let the image of τ under the shear be τ'. Then to obtain $c(\tau')$ from $c(\tau)$, keep only the *sandwiched* symbols, those that have the same symbol on both sides.

The main results of [D14, DPU18] are generalizations of the same idea. In [DPU18], we studied the infinite family of Bouw-Möller surfaces. We showed how to take the surface made from m equiangular $2n$-gons, and shear, cut, and paste it to transform it into the surface made from n equiangular $2m$-gons. We used this symmetry to understand the action of the shear on cutting sequences. In particular, we determined how to translate between the geometry of the transformation, and a substitution on the sequences.

Tiling billiards and interval exchange transformations.

Definition. Given a planar tiling, a *tiling billiards trajectory* on the tiling is a straight line within each tile, and refracts across edges between tiles such that the angle of incidence equals the angle of refraction (see Figure 5a).
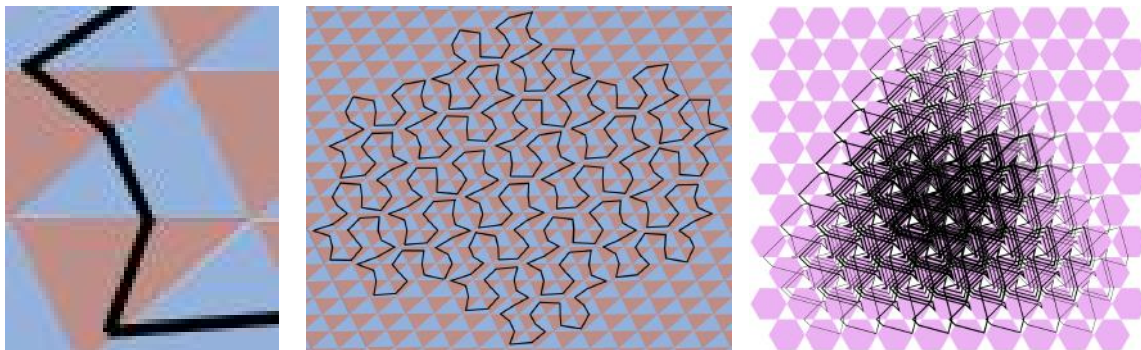

Figure 5: (a) a trajectory on a triangle tiling, which is a closeup of the middle picture (b) a large periodic trajectory on a triangle tiling, approaching the Rauzy fractal (c) a trajectory on the triangle-hexagon tiling

People are excited about tiling billiards because it produces beautiful and intricate patterns, and at the same time it seems more tractable than ordinary billiards in some nice cases. It is also highly connected to existing areas of billiard research:

We can "fold" tiling billiard trajectories, analogous to how we "unfold" billiard trajectories [DDiRS18]. It turns out that, for triangle tilings, all of the triangles fold up to be circumscribed in the same circle, which reduces the system to an *interval exchange transformation* that has particularly nice properties [BDFI18]. We showed that all periodic trajectories are simple closed curves, and that in special cases, a periodic trajectory can form a fractal-like curve (Figure 5b).

Theorem (Baird-Smith, Davis, Fromm, Iyer, [BDFI18]) The behavior of a trajectory on a triangle tiling with angles α, β, γ is described by the action of an orientation-reversing interval exchange transformation with segment lengths α, β, γ.

We also noticed that, for one very specific triangle tiling, there are very long trajectories, and we conjectured that they approach the *Rauzy fractal* under rescaling; the one in Figure 5b is an example. Our colleagues subsequently proved this in [HR18].

The behavior of the triangle-hexagon tiling (Figure 5c) is very different: a general trajectory is unstable and dense.

Theorem (Davis and Hooper, [DH18]) Almost every trajectory on the triangle-hexagon tiling is dense in the entire plane, minus a periodic family of triangle centers.

Figure 5c shows 500 iterations of the trajectory of slope 1, which you can see is filling up the plane (avoiding only small white triangles). Our main tool was an infinite *translation surface* made of rhombi. We essentially showed that the surface acts like a collection of tori, where almost every trajectory is also dense.

Classification of periodic paths on the regular pentagon.

There are two very natural questions to ask about billiards in a polygon:

- Which directions are periodic?

- What is the period in a given periodic direction?

The answers for the square have been well understood for at least a hundred years:

Theorem. The directions of periodic billiard trajectories in the square are those with rational slope p/q. A billiard in that direction is periodic with period $2(p+q)$.

We gave the answers for the double pentagon, which are as elegant as possible: as it is for the square, the period is double the sum of the direction coefficients.

Theorem (Davis and Lelièvre, [DL18]). For a given periodic direction on the double regular pentagon surface, there is a canonical vector in that direction, given by $[a + b\phi, c + d\phi]$, where ϕ is the golden ratio and $a, b, c, d \in \mathbf{N}$. The period of a trajectory in that direction is $2(a + b + c + d)$.

As our main tool, we created a quaternary tree of periodic directions, which acts as a generalization of the Euclidean algorithm. We wrote a `Sage` program to draw a picture of any periodic trajectory; many of them are stunningly beautiful (see Figure 1). We now have many questions about the appearance of the periodic trajectories; see the next section for a partial list. We proved several results about them, including:

Theorem (Davis and Lelièvre, [DL18]). Of the countably infinitely many periodic trajectories on the regular pentagon, 5/6 of them have both rotation and reflection symmetry, and 1/6 of them have only reflection symmetry.

For example, in Figure 1, the first, third and fourth pentagons have rotation symmetry, and the second has only reflection symmetry.

6

4   Future work

Bounce rigidity: finding lengths. We showed how to use the set of all bounce sequences on a billiard table to reconstruct the edge order, and the angles. We would also like a constructive way to use this information to determine the edges' *lengths*.

Periodic billiards on more polygons. Mathematicians understand periodic billiard trajectories on only four shapes of polygons: rectangles, equilateral triangles, 30-60-90 triangles, 45-45-90 triangles, regular hexagons – and now, after my work with Samuel Lelièvre, the regular pentagon. The pentagon was the first "hard" case, because pentagons don't tile the plane. Our next step is to extend our methods to more of the regular polygons, such as the regular heptagon and octagon.

Understanding non-equidistribution. While exploring the pictures of periodic trajectories on the regular pentagon, I noticed that it is possible to find very long periodic trajectories, which can be arbitrarily dense without distributing uniformly in the table (see Figure 6). Curtis McMullen subsequently wrote a paper exploring this phenomenon [M19]; see his recommendation letter for more details. I am currently working on understanding why certain trajectories fail to visit parts of the table completely, as in the picture on the right in Figure 6.
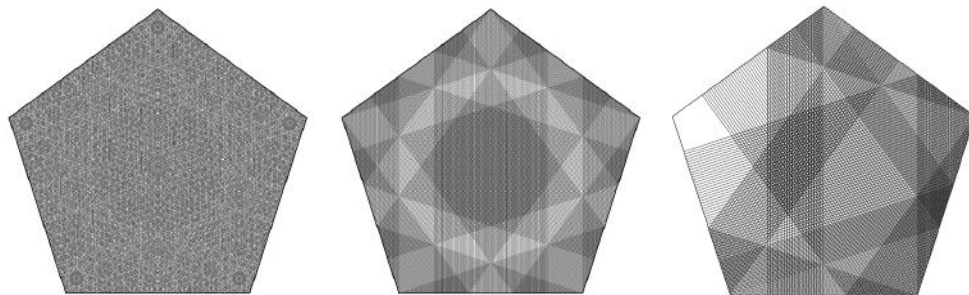

Figure 6: Three very long periodic billiard trajectories on the pentagon, with similar lengths but very different behavior. The one on the left is close to uniformly distributed, the one in the middle is not, and the one on the right misses a corner completely.

Tiling billiards on the aperiodic Penrose tiling. Pat Hooper and I are currently working this. Our preliminary investigations suggest that, surprisingly, almost every trajectory is *periodic*. A long periodic trajectory is in Figure 7a.

Tiling billiards in the wind-tree model. In the past 10 years there has been a lot of interest in the *wind-tree model* of billiards, where there are congruent obstacles ("trees") at every lattice point in a given lattice, and a ball ("wind") bounces around in their complement [HW80, HLT11, De13, DeZ15, ST16, AH17, F18]. With Ferràn

Valdez and Olga Romaskevich, I am working on studying a tiling billiards version of this model. Preliminary work (Figure 7b) suggests that some trajectories are dense, as we previously saw in the triangle-hexagon tiling.
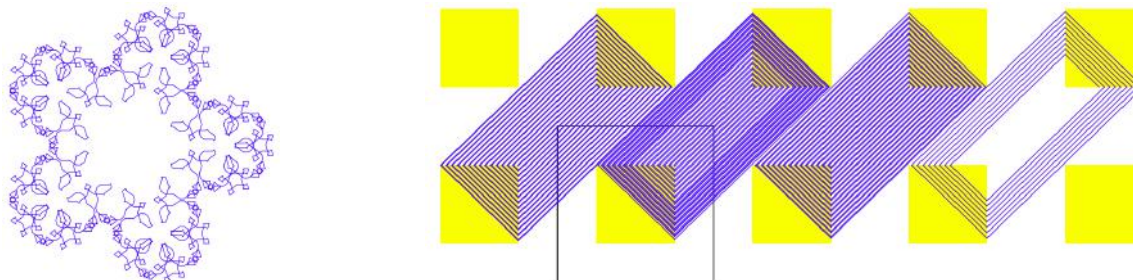

Figure 7: (a) A long periodic trajectory on the Penrose tiling. The tiles themselves are not shown, because they are very small. (b) A trajectory (purple) in the wind-tree model, with square obstacles (yellow), with a fundamental domain outlined.

Undergraduate research on billiards. My undergraduate thesis student Megumi Asada studied billiards on the equilateral triangle and regular hexagon in 2016–2017, and proved some original theorems [A17]. My undergraduate research groups in the summers 2013 and 2016 did excellent foundational work on tiling billiards [DDiRS18, BDFI18]. There are many open problems in many areas of polygonal billiards, flat surfaces, and interval exchange transformations, that are accessible to students. In addition, there are well-developed Sage packages that researchers in the field have written and maintained, which makes it easier to start doing serious computations early in a research project. Indeed, many breakthroughs in the field have developed out of computer experiments.

Undergraduate research on gerrymandering. I have applied to be the director of Summer@ICERM in 2021, to supervise about 20 students working on mathematics problems related to assessing the "fairness" of the new maps that will be proposed after the 2020 Census. The practice of applying geometry and random sampling to the problem of gerrymandering has exploded in the past couple of years, and I see it as a wonderful opportunity to apply our mathematical skills to issues of social justice. In summer 2019, I mentored five high school students at PROMYS at Boston University on a gerrymandering research project; they wrote a very impressive paper showing that, at least for a toy example, "clustering benefits the minority."

References

[A17] Megumi Asada, *Periodic paths on the triangle and hexagon billiard tables*, undergraduate honors thesis (2017).

[AA18] Jayadev Athreya and David Aucilino, *A Trajectory from a Vertex to Itself on the Dodecahedron*, arXiv:1802.00811, preprint (2018).

[AAH18] Jayadev Athreya, David Aucilino, and Pat Hooper, *Platonic solids and high genus covers of lattice surfaces*, arXiv:1811.04131, preprint (2018).

[AH17] Artur Avila and Pascal Hubert, *Recurrence for the wind-tree model*, **Annales de l'Institut Henri Poincaré - Analyse non linéaire**, to appear.

[BDFI18] Paul Baird-Smith, Diana Davis, Elijah Fromm and Sumun Iyer, *Tiling billiards on triangle tilings*, accepted pending revision in **Bulletin of the London Mathematical Society**, arXiv:1809.07876.

[BDLW18] Harrison Bray, Diana Davis, Kathryn Lindsey and Chenxi Wu, *The shape of Thurston's Master Teapot*, arXiv:1902.10805 (submitted 2019).

[BM06] Irene Bouw and Martin Möller: *Teichmüller curves, triangle groups, and Lyapunov exponents*, **Annals of Mathematics**, 2006.

[BDaD17] Keith Burns, Orit Davidovich and Diana Davis, *Average pace and horizontal chords*, **Mathematical Intelligencer**, 39(4), 41–45 (2017).

[CCD+18] Aaron Calderon, Solly Coles, Diana Davis, Justin Lanier, Andre Oliveira, *How to hear the shape of a billiard table*, arXiv:1806.09644 (submitted 2018).

[CD18] Jon Chaika, Diana Davis, *The typical measure preserving transformation is not an interval exchange transformation*, arXiv:1812.10425 (submitted 2018),.

[D13] Diana Davis, *Cutting sequences, regular polygons, and the Veech group*, **Geometriae Dedicata**, Volume 162, Issue 1 (2013).

[D14] Diana Davis, *Cutting sequences on translation surfaces*, **New York Journal of Mathematics**, Volume 20, 2014, 399-429.

[D17] Diana Davis, *Lines in positive genus: An introduction to flat surfaces*, in *Dynamics done with your bare hands*, **EMS Series of Lecture Notes** (2017): arXiv:1507.02571.

[DFT11] Diana Davis, Dmitry Fuchs and Sergei Tabachnikov, *Periodic trajectories in the regular pentagon*, **Moscow Mathematical Journal**, Volume 3, 2011.

[DDiRS18] Diana Davis, Kelsey DiPietro, Jenny Rustad and Alexander St Laurent, *Negative refraction and tiling billiards*, **Advances in Geometry**, 18(2), 133-159 (2018).

[DL18] Diana Davis and Samuel Lelièvre, *Periodic paths on the double pentagon and golden L*, preprint (2018).

[DDoTY17] Diana Davis, Victor Dods, Cynthia Traub and Jed Yang, *Geodesic trajectories on regular polyhedra*, **Discrete Mathematics**, 340(1), 3183–3196 (2017).

[DH18] Diana Davis and W. Patrick Hooper, *Periodicity and ergodicity in the trihexagonal tiling*, **Commentarii Mathematici Helvetici** 93(4), 661–707 (2018).

[DPU18] Diana Davis, Irene Pasquinelli and Corinna Ulcigrai, *Cutting sequences on Bouw-Möller surfaces: an S-adic characterization*, to appear in **Annales scientifiques de l'École normale superieure** (2018).

[De13] Vincent Delecroix, *Divergent trajectories in the periodic wind-tree model.* **Journal of Modern Dynamics** 7(1), 1–29 (2013).

[DeZ15] Vincent Delecroix and Anton Zorich, *Cries and whispers in wind-tree forests.* arXiv:1502.06405 (2015).

[DuELS18] Moon Duchin, Viveka Erlandsson, Chris Leininger, Chandrika Sadanand, *You can hear the shape of a billiard table: Symbolic dynamics and rigidity for flat surfaces*, arXiv:1804.05690 (2018).

[F18] Charles Fougeron, *Diffusion rate of windtree models and Lyapunov exponents*, arXiv:1803.10717 (2018).

[FF07] Dmitry Fuchs and Ekaterina Fuchs, *Closed geodesics on regular polyhedra*, **Moscow Mathematical Journal** 01/2007; 7(2).

[HW80] J. Hardy, J. Weber, *Diffusion in a periodic wind-tree model*, **Journal of Mathematical Physics** 21(7) 1802–1808 (1980).

[HLT11] Pascal Hubert, Samuel Lelièvre, Serge Troubetzkoy, *The Ehrenfest wind-tree model: periodic directions, recurrence, diffusion.* **Journal für die Reine und Angewandte Mathematik** 656, 223–244 (2011).

[HR18] Pascal Hubert, Olga Paris-Romaskevich, *Triangle tiling billiards draw fractals only if aimed at the circumcenter*, preprint (2018), arXiv:1804.00181.

[K66] Mark Kac, *Can one hear the shape of a drum?* American Mathematical Monthly 73, 1–23 (1966).

[LW15] Samuel Lelièvre and Barak Weiss, *Translation surfaces with no convex presentation*, **Geometry and Functional Analysis** 25(6), 1902–1936 (2015).

[ST16] Alba Màlaga Sabogal, Serge Troubetzkoy, *Minimality of the Ehrenfest wind-tree model*, arXiv:1506.00814 (2016).

[M19] Curtis T. McMullen, *Modular symbols for Teichmüller curves*, preprint (2019).

[R19] Olga Paris-Romaskevich, *Trees and flowers on a billiard table*, preprint (2019), arXiv:1907.01178.

[S85a] Caroline Series, *The Geometry of Markoff Numbers*, **The Mathematical Intelligencer**, Vol. 7, No. 3 (1985), pp. 20-29 .

[S85b] Caroline Series, *The modular surface and continued fractions*, **Journal of the London Mathematical Society** (2) 31(1):69–80 (1985).

[S91] Caroline Series, *Ergodic theory, symbolic dynamics, and hyperbolic spaces (Trieste, 1989)*, chapter Geometrical methods of symbolic coding., pages 125–151. Oxford Sci. Publ. Oxford Univ. Press, New York, 1991.

[SSS01] R. A. Shelby, D. R. Smith and S. Schultz, *Experimental Verification of a Negative Index of Refraction*, **Science**, Vol. 292 no. 5514 pp. 77-79 (2001).

[SU10] John Smillie and Corinna Ulcigrai, *Geodesic flow on the Teichmüller disk of the regular octagon, cutting sequences and octagon continued fractions maps*, **Dynamical Numbers**: Interplay between Dynamical Systems and Number Theory, Contemporary Mathematics, vol. 532, American Mathematical Society, Providence, RI, pp. 29–65 (2010).

[SU11] John Smillie and Corinna Ulcigrai, *Beyond Sturmian sequences: coding linear trajectories in the regular octagon*, **Proceedings of the London Mathematical Society** 102(2): 291–340 (2011).

[SPW04] D. Smith, J. Pendry and M. Wiltshire: *Metamaterials and negative refractive index*, **Science**, Vol. 305, pp. 788–792 (2004).

[T14] William P. Thurston, *Entropy in dimension one.* **Frontiers in Complex Dynamics**: In Celebration of John Milnor's 80th Birthday. Princeton University Press, Princeton, NJ (2014), 339–384.