Interval exchange transformations from tiling billiards

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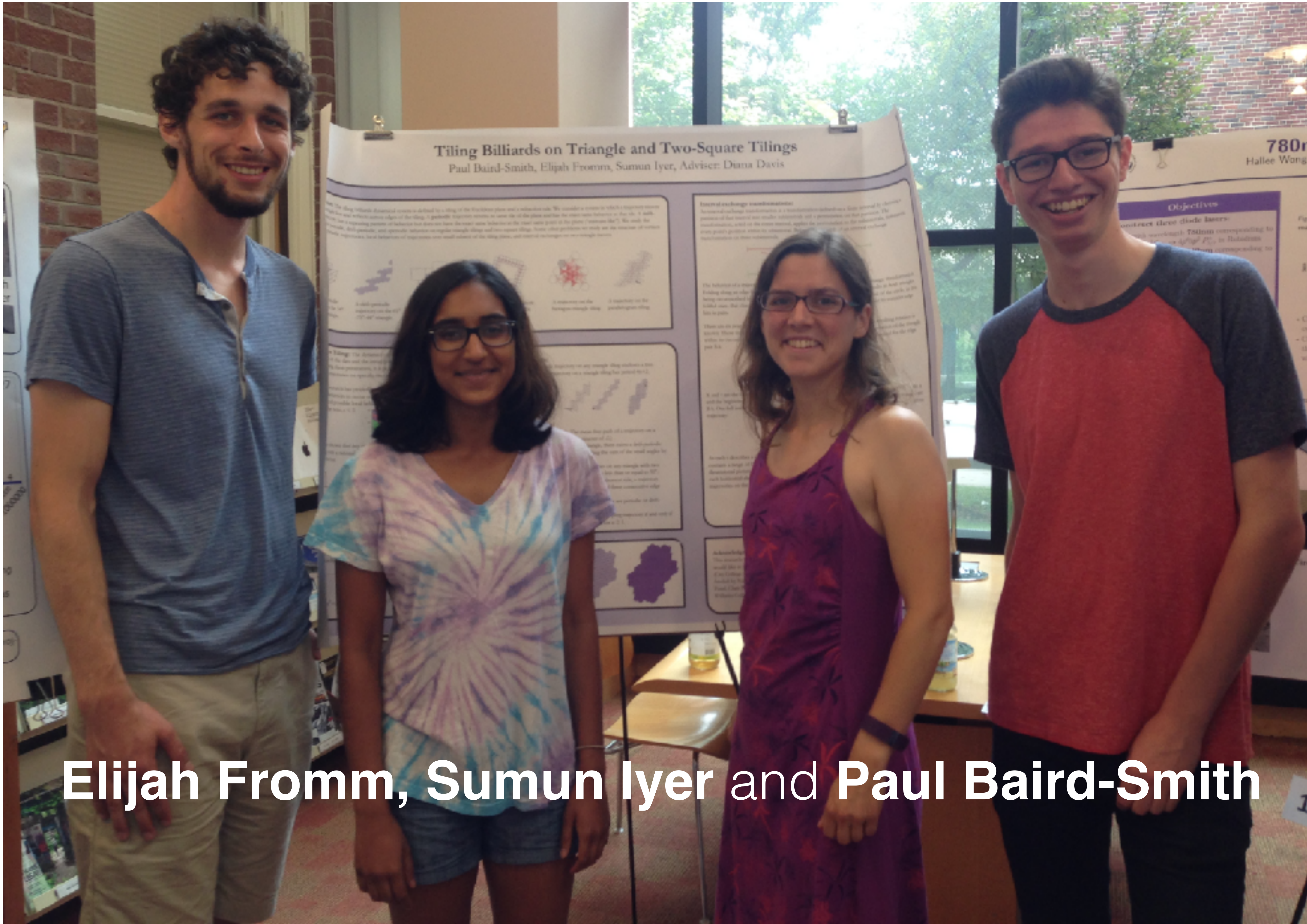
JMM, San Diego

Joint work with:

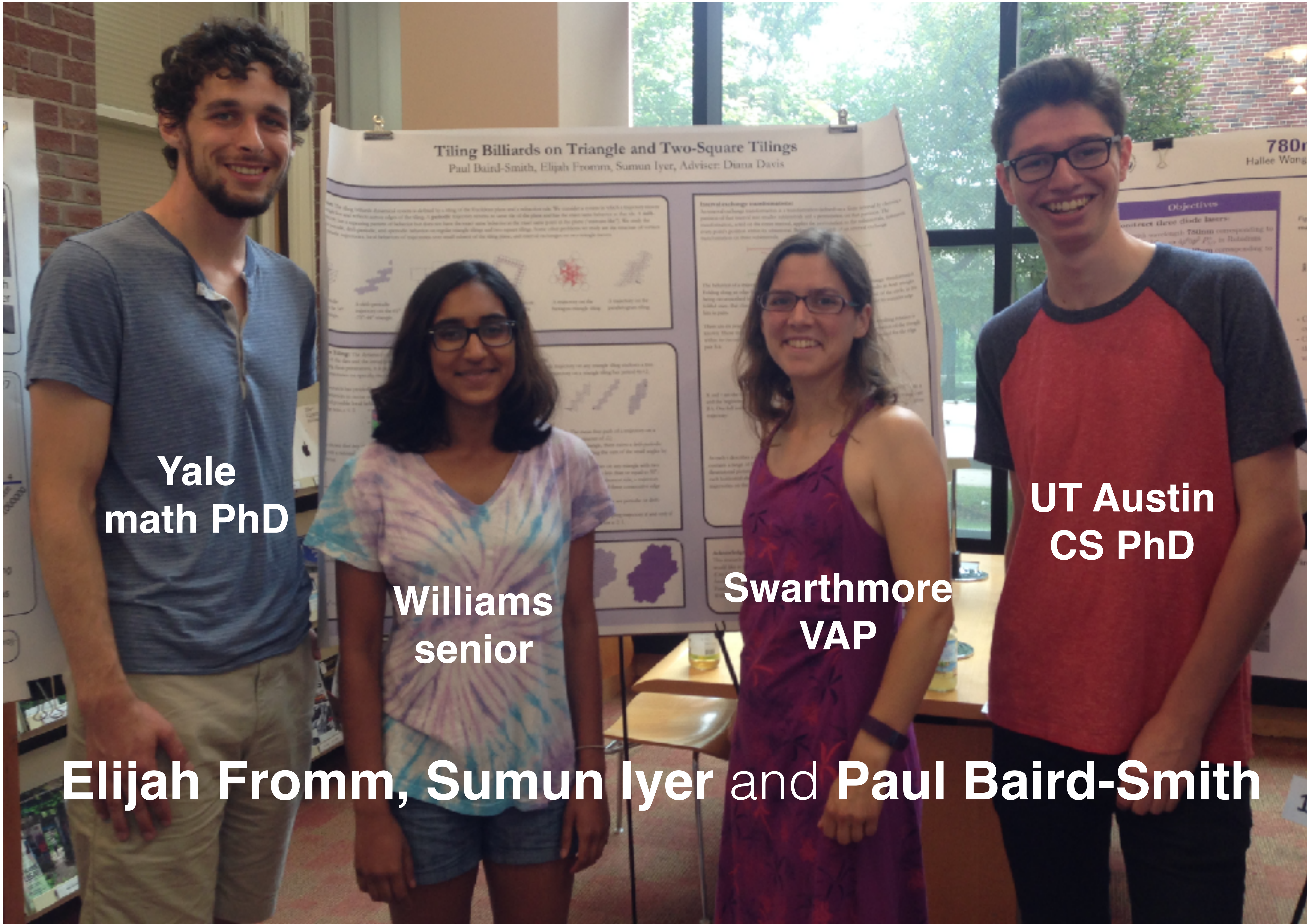
Paul Baird-Smith, Elijah Fromm and Sumun Iyer
SMALL 2016

SMALL 2016 - Billiards group



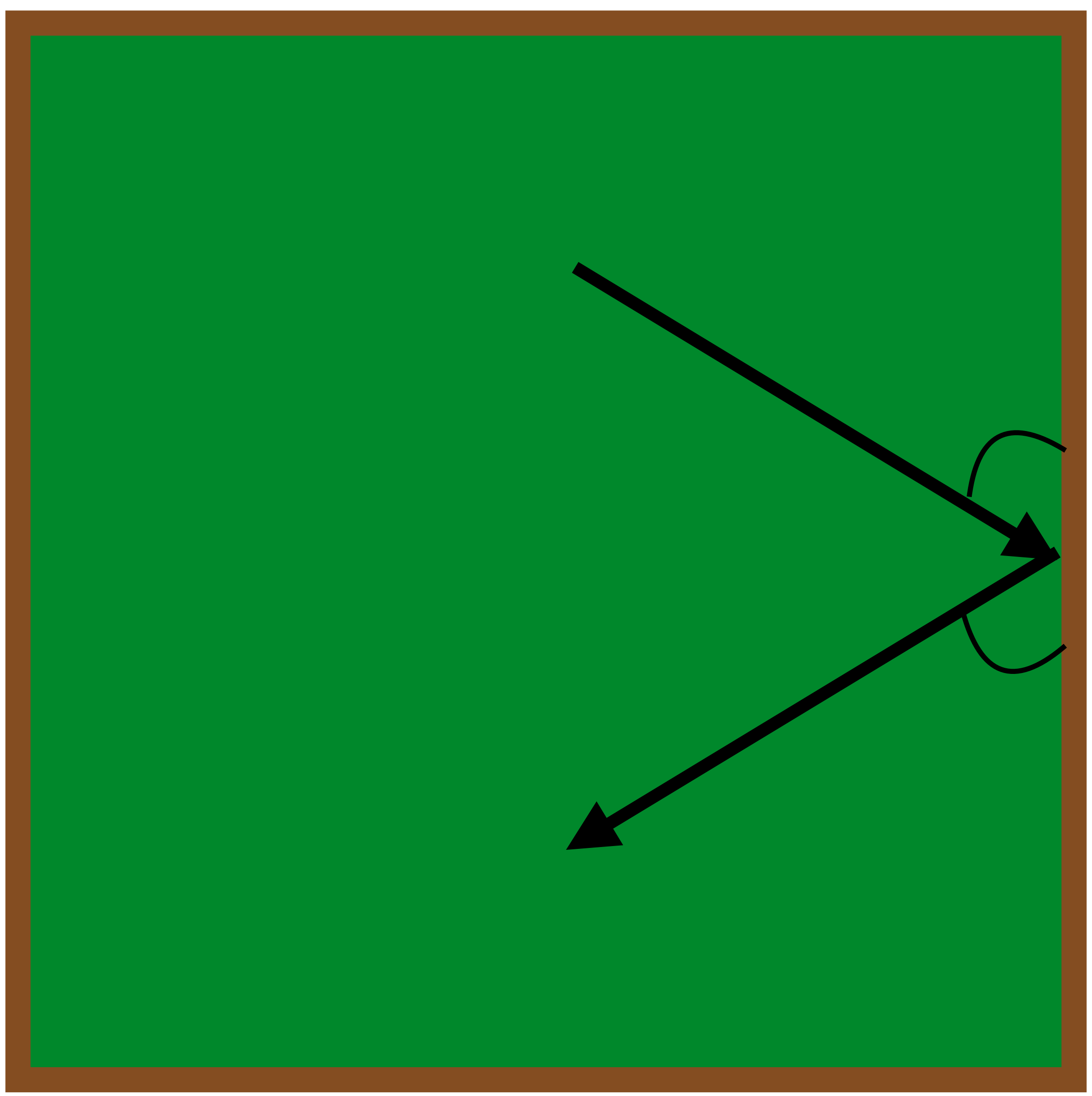
Elijah Fromm, Sumun Iyer and **Paul Baird-Smith**

SMALL 2016 - Billiards group



Standard (Inner) Billiards:

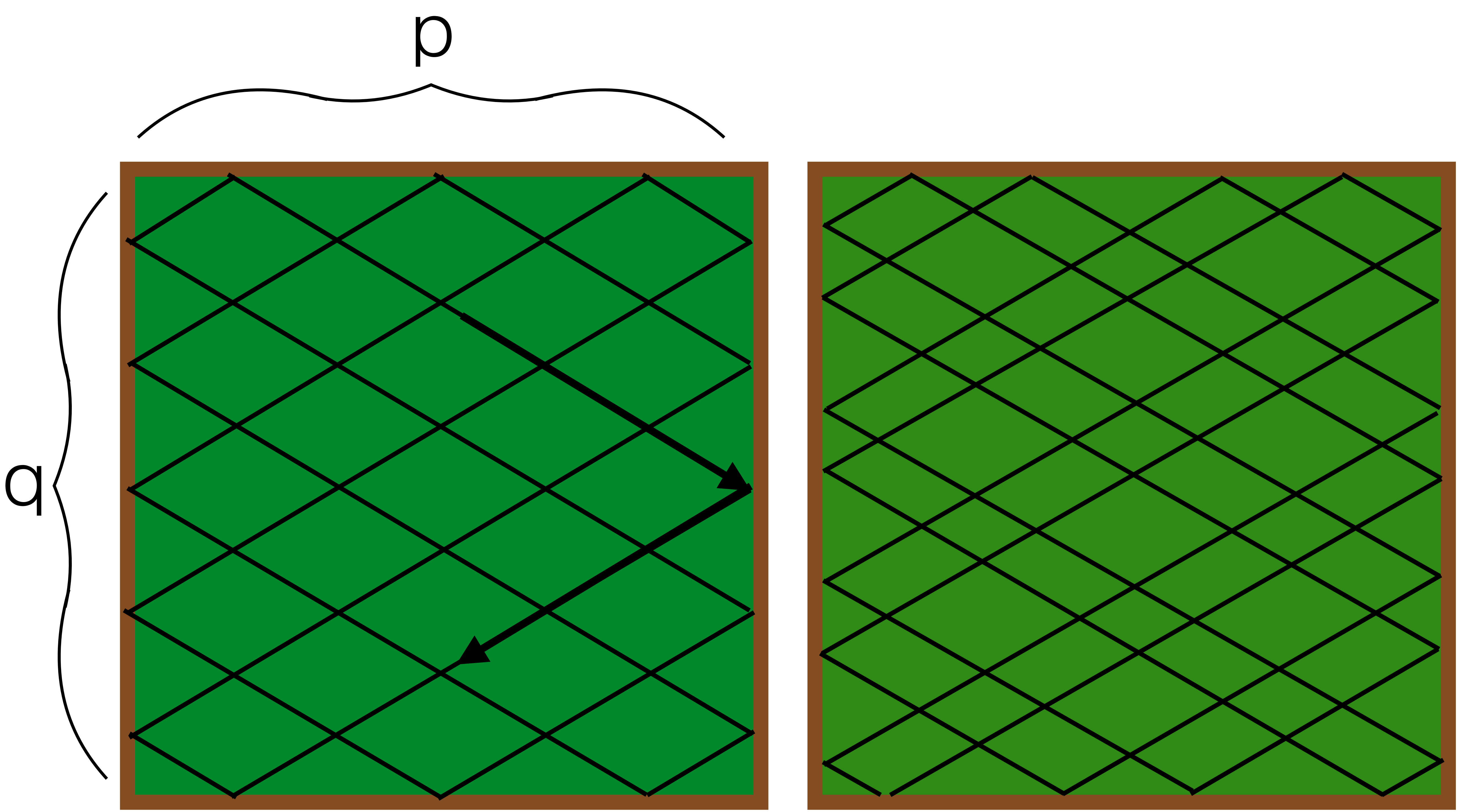
trajectory is reflected against the edge of the table, preserving angles



In the square billiard table, a trajectory with:

rational slope p/q is periodic, with period 2(p+q)

irrational slope is dense, filling in the entire table.



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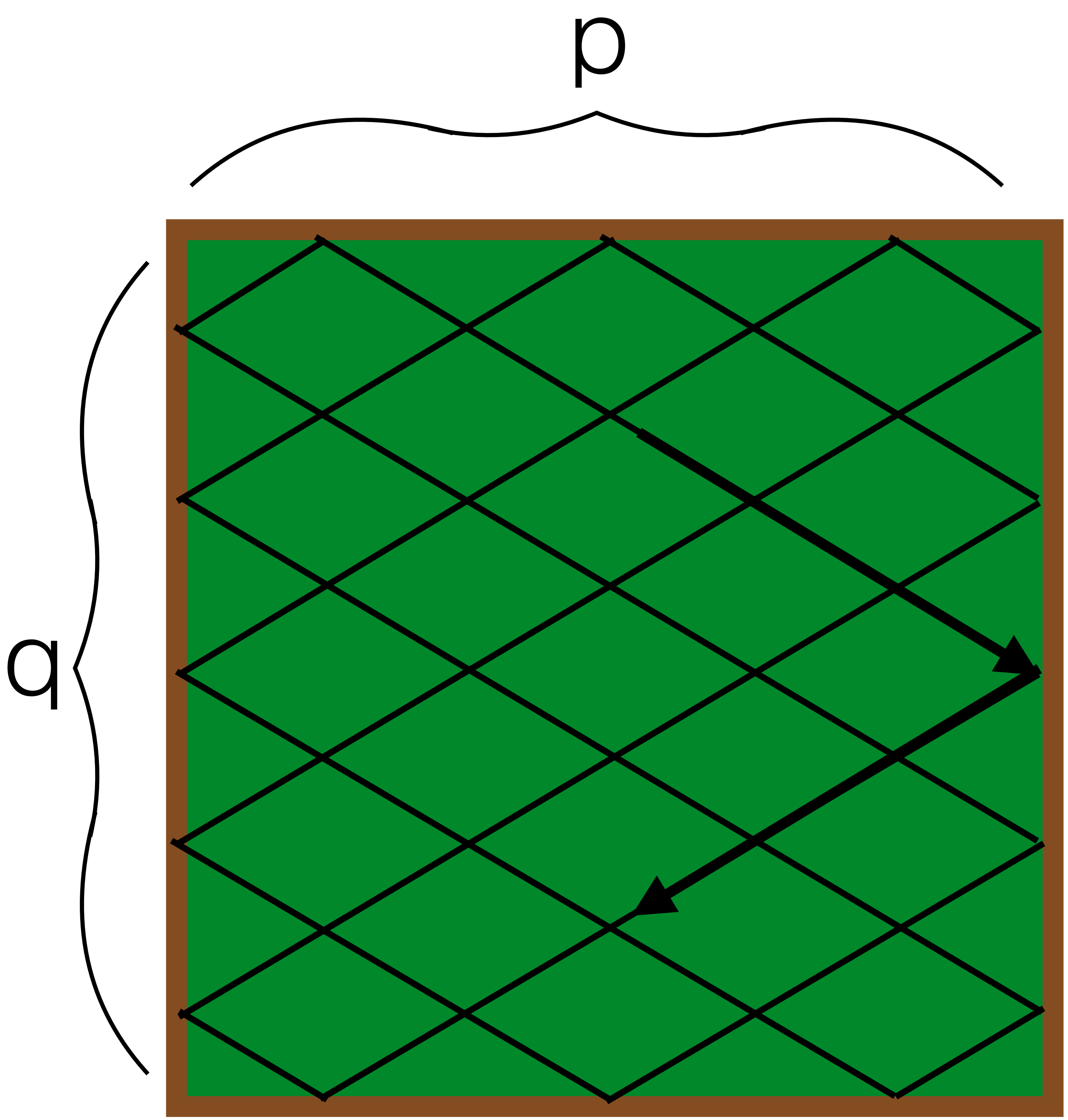

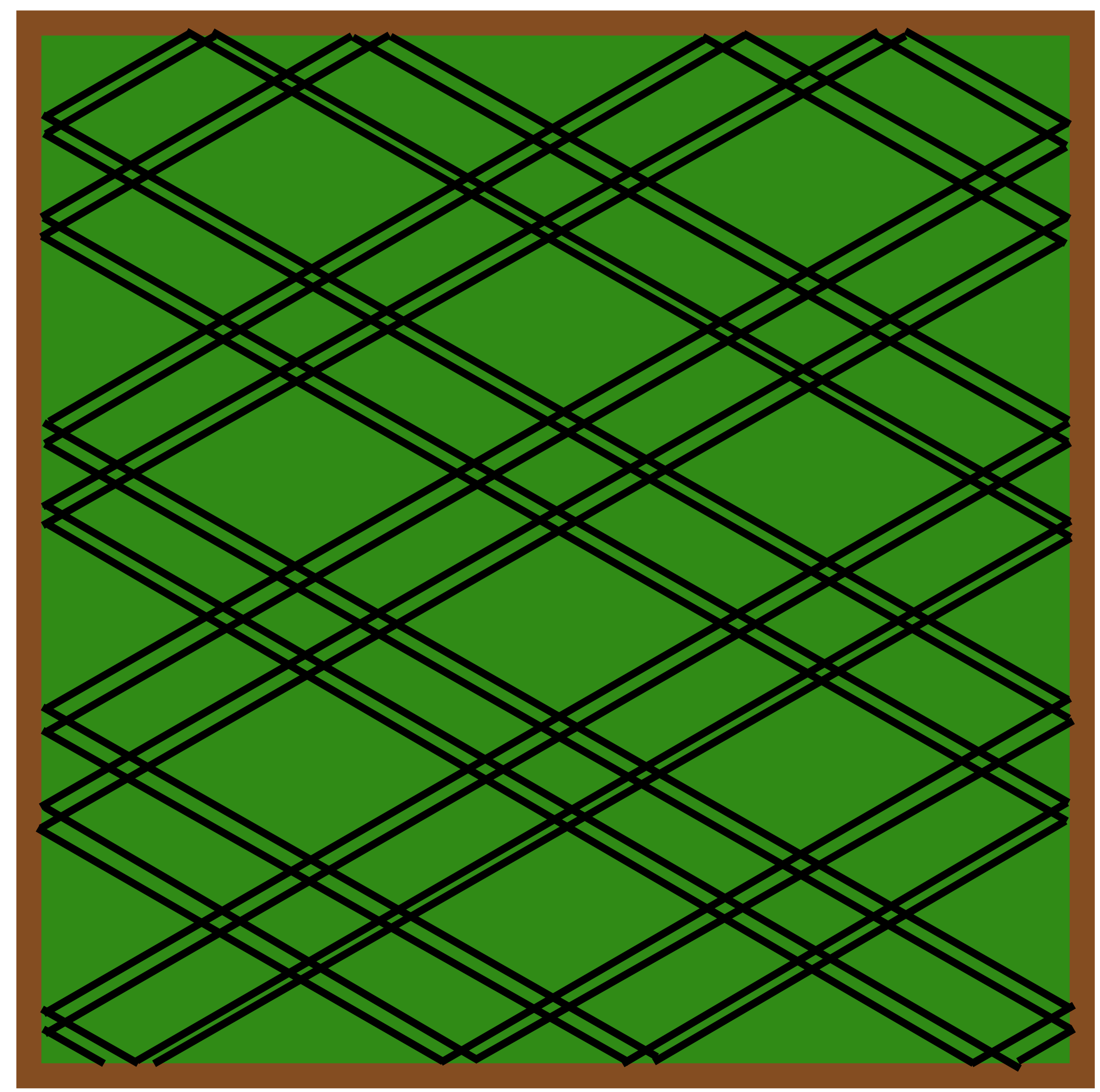

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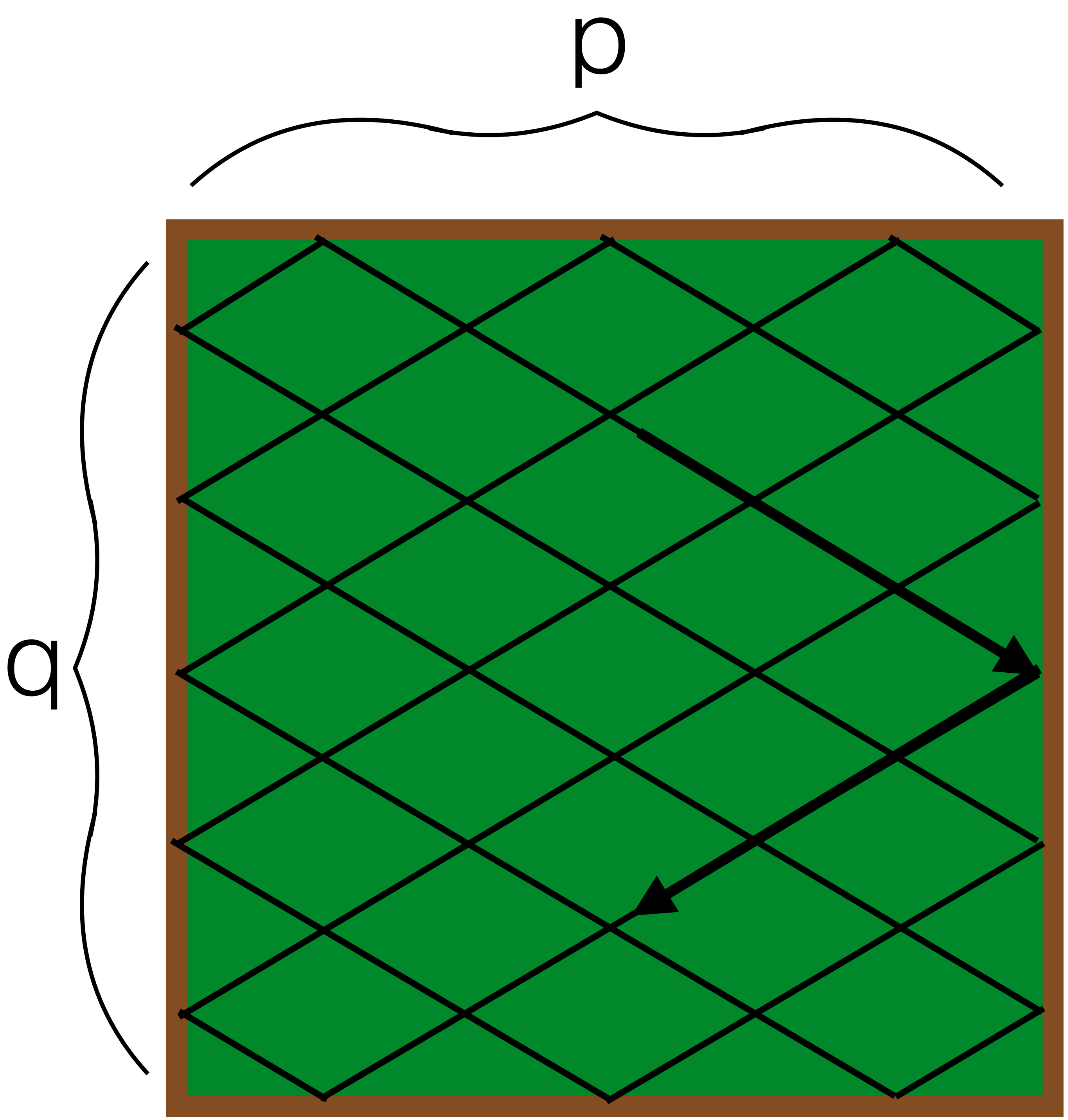

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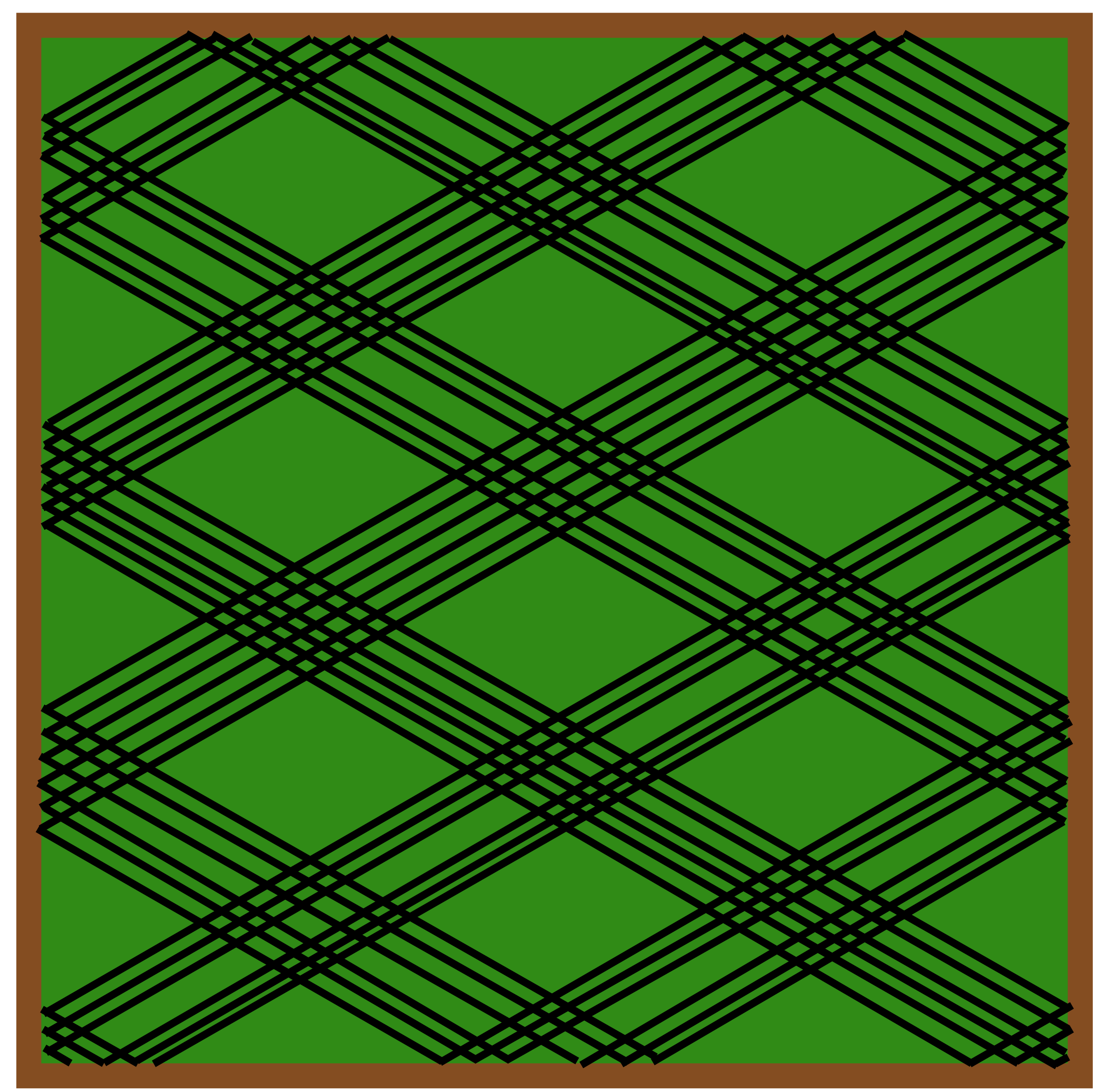

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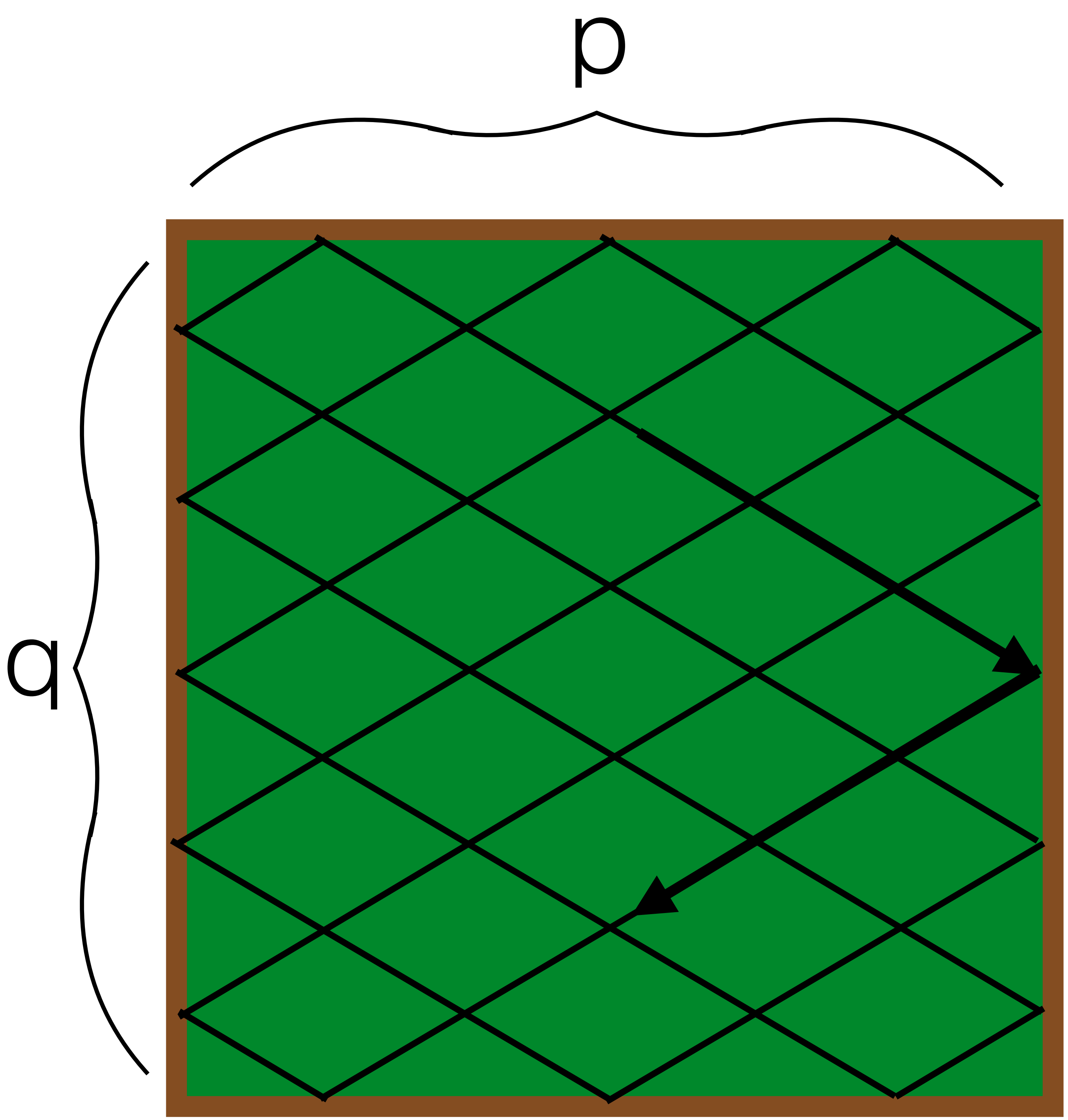

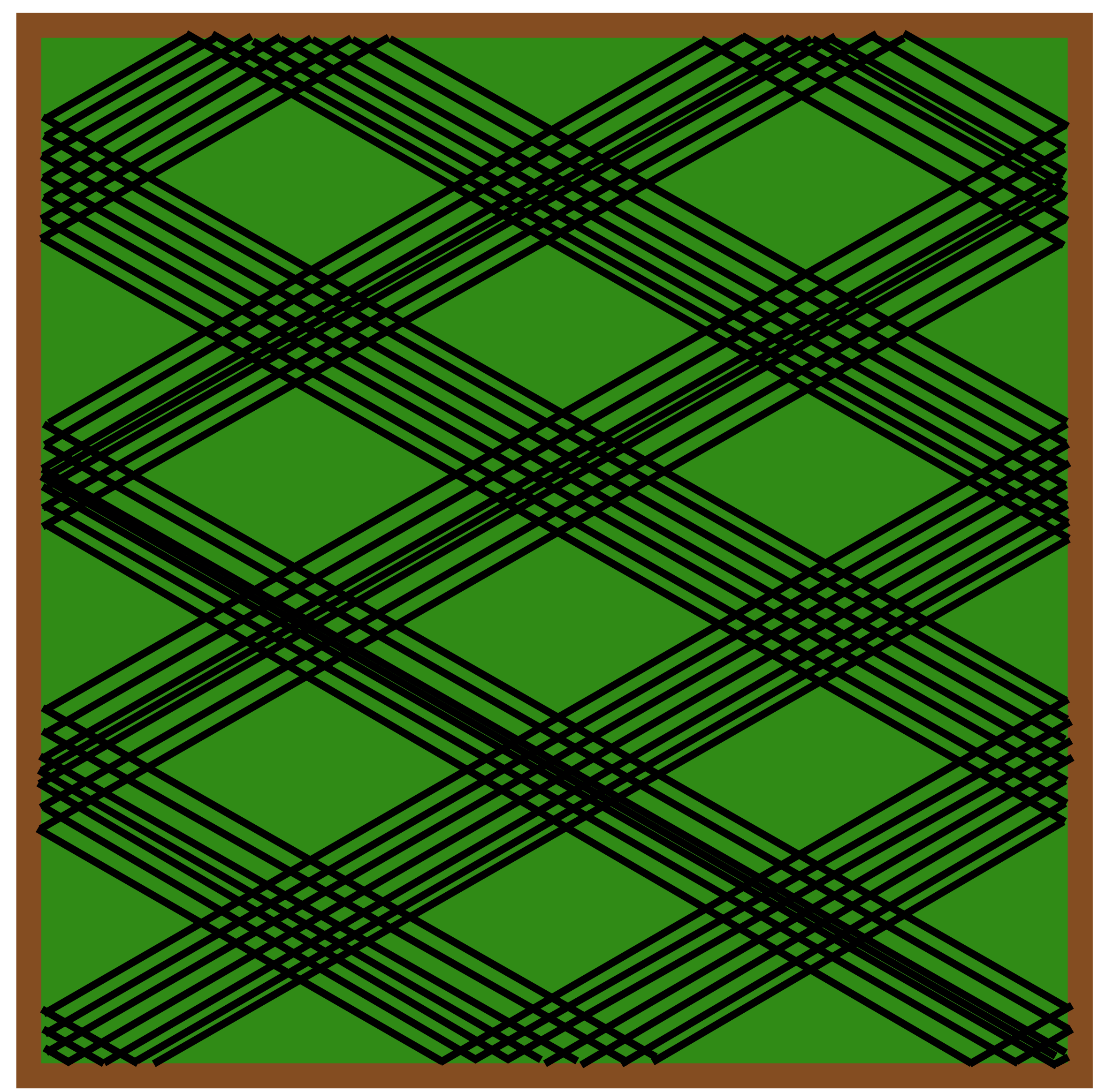

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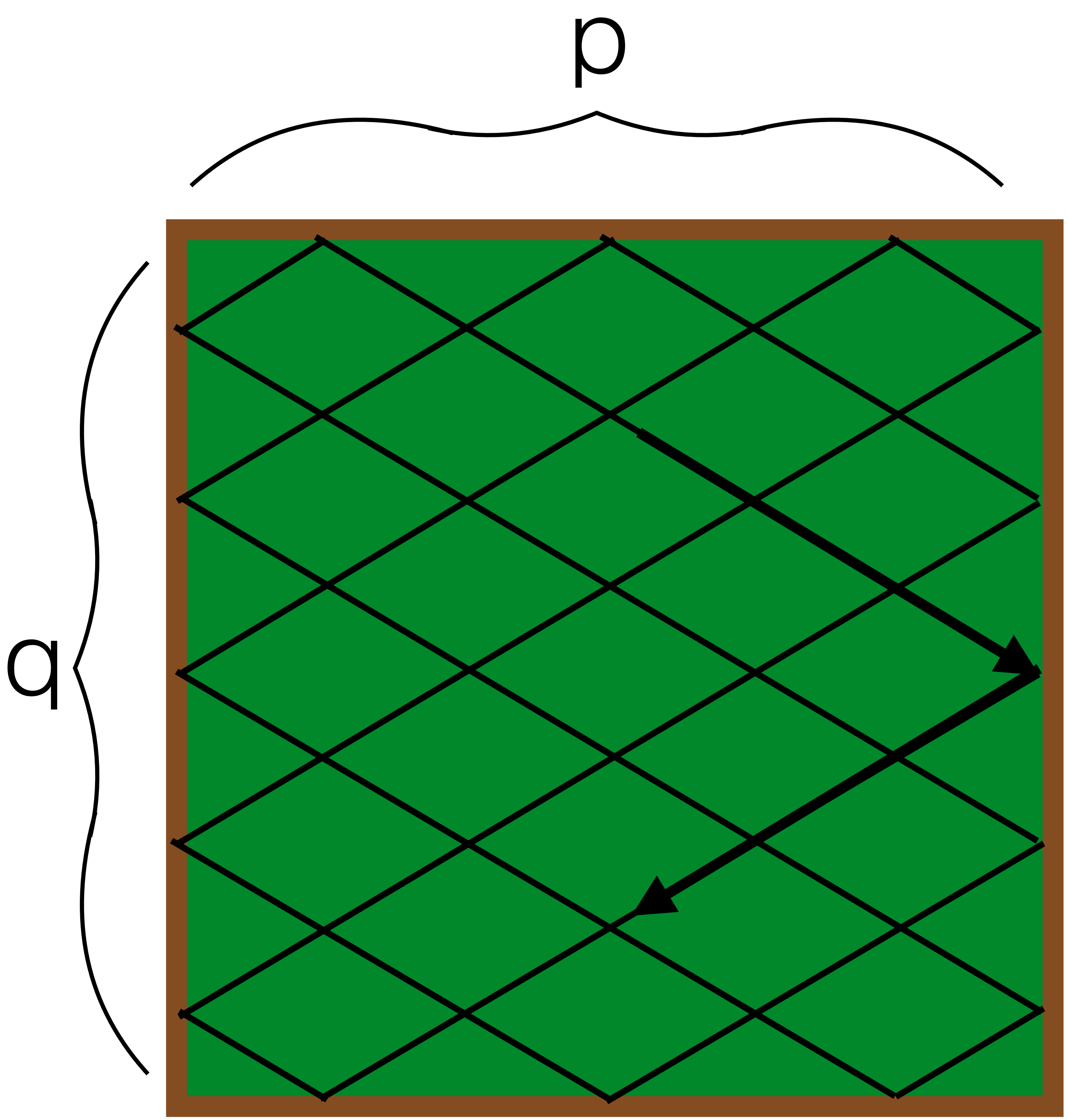
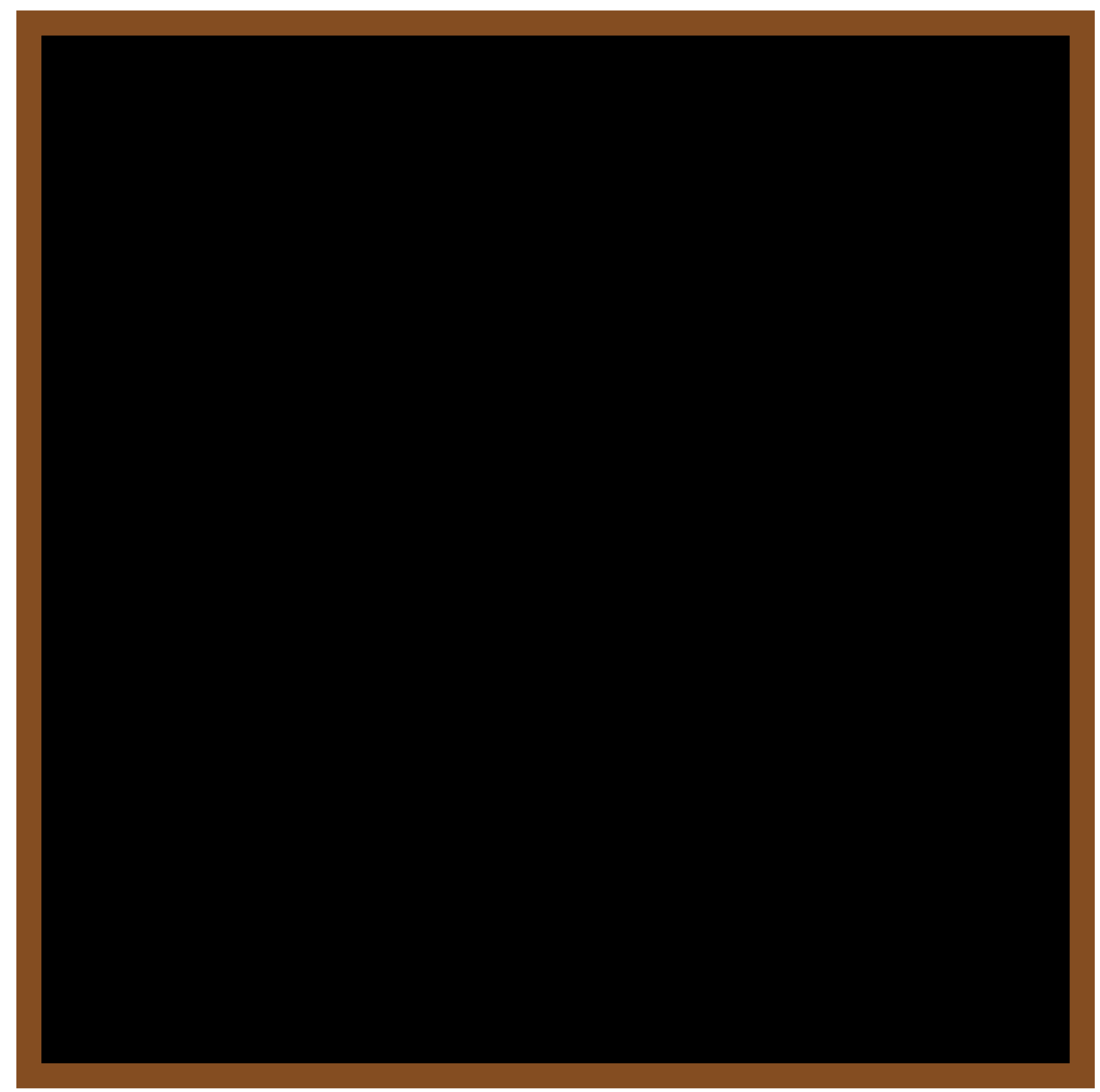
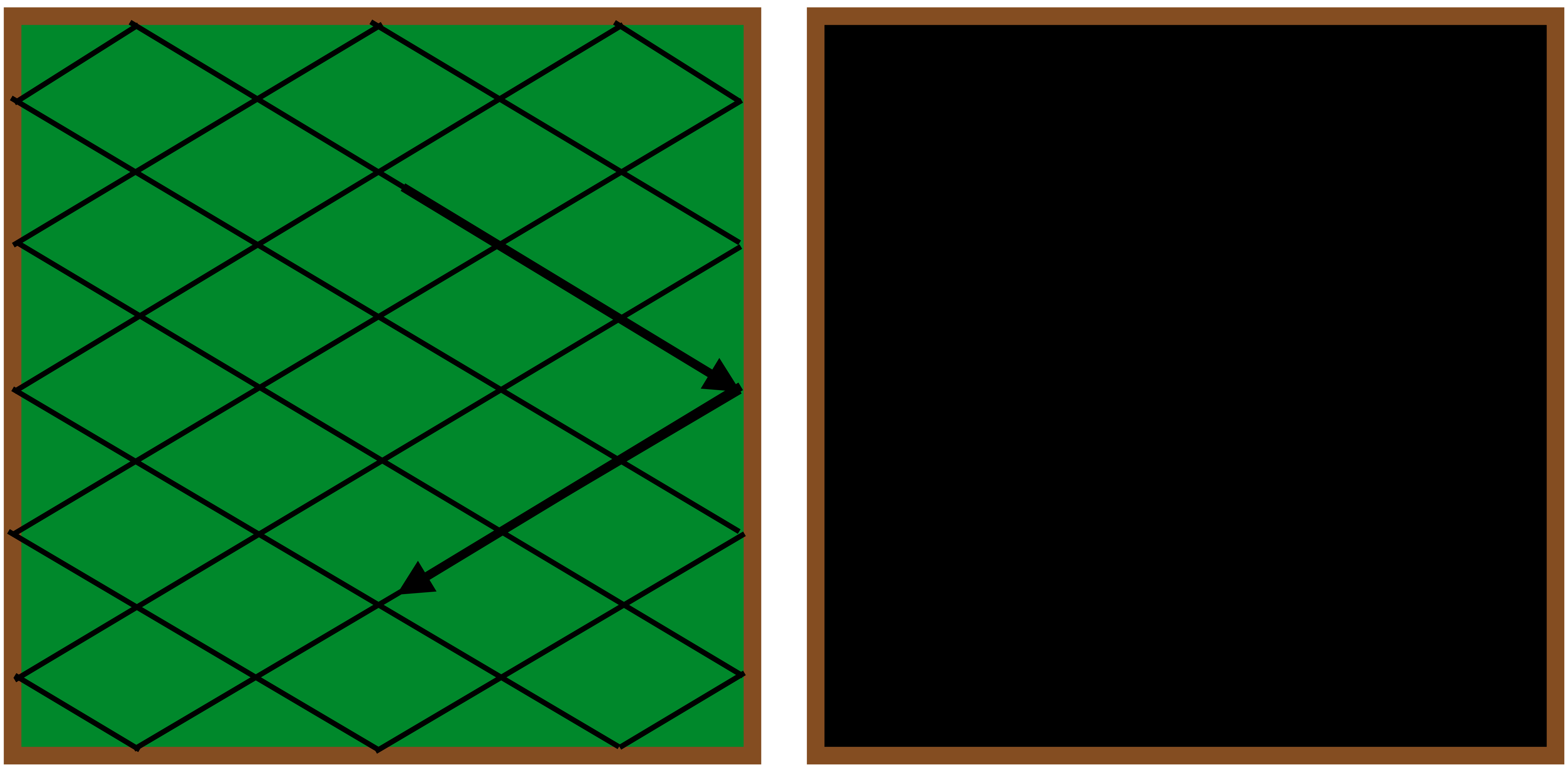




Standard (Inner) Billiards:

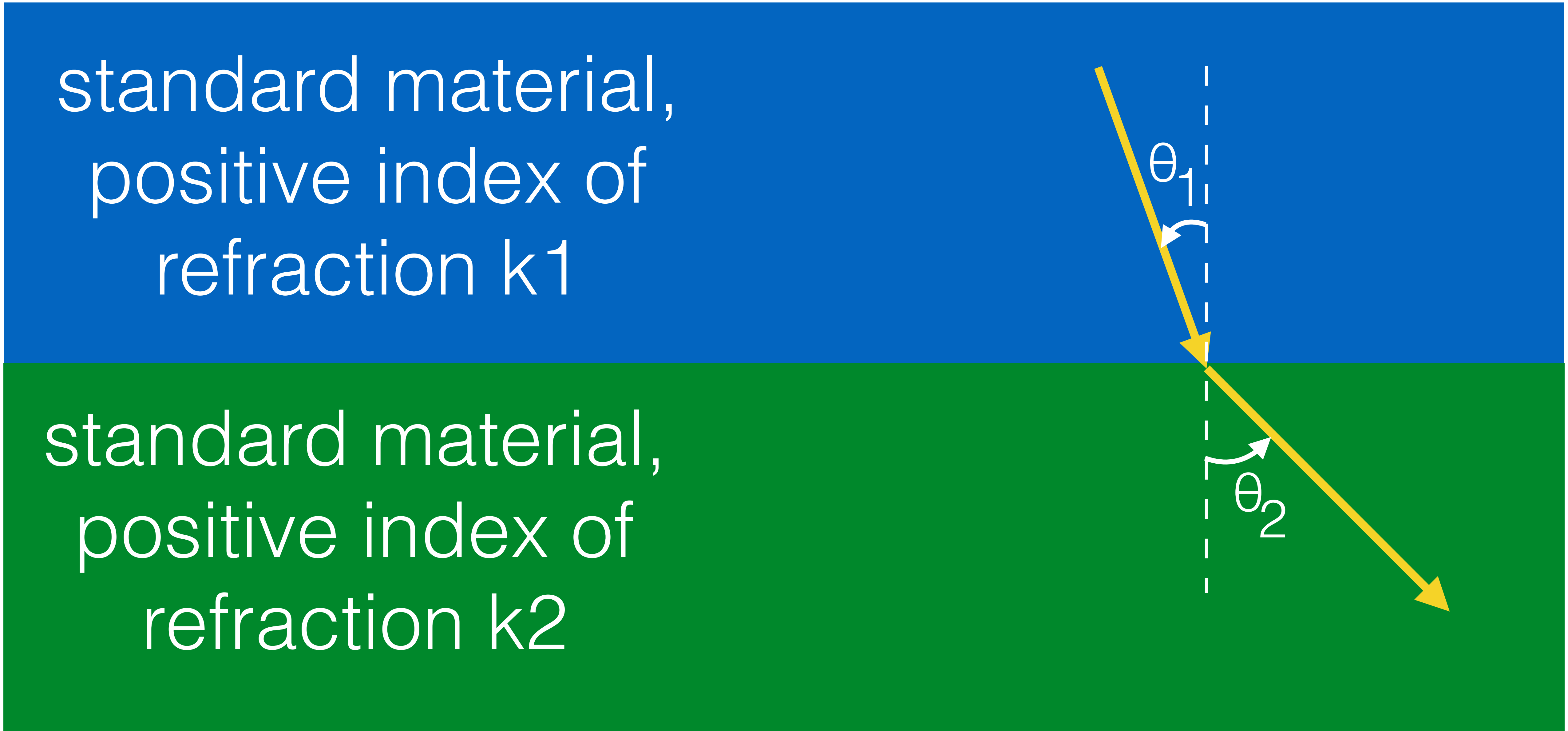
- A trajectory is **dense** with probability 1
- Behavior is **unstable**



Snell's Law:

governs the refraction of a beam of light passing from one material to another

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{k1}{k2}$$



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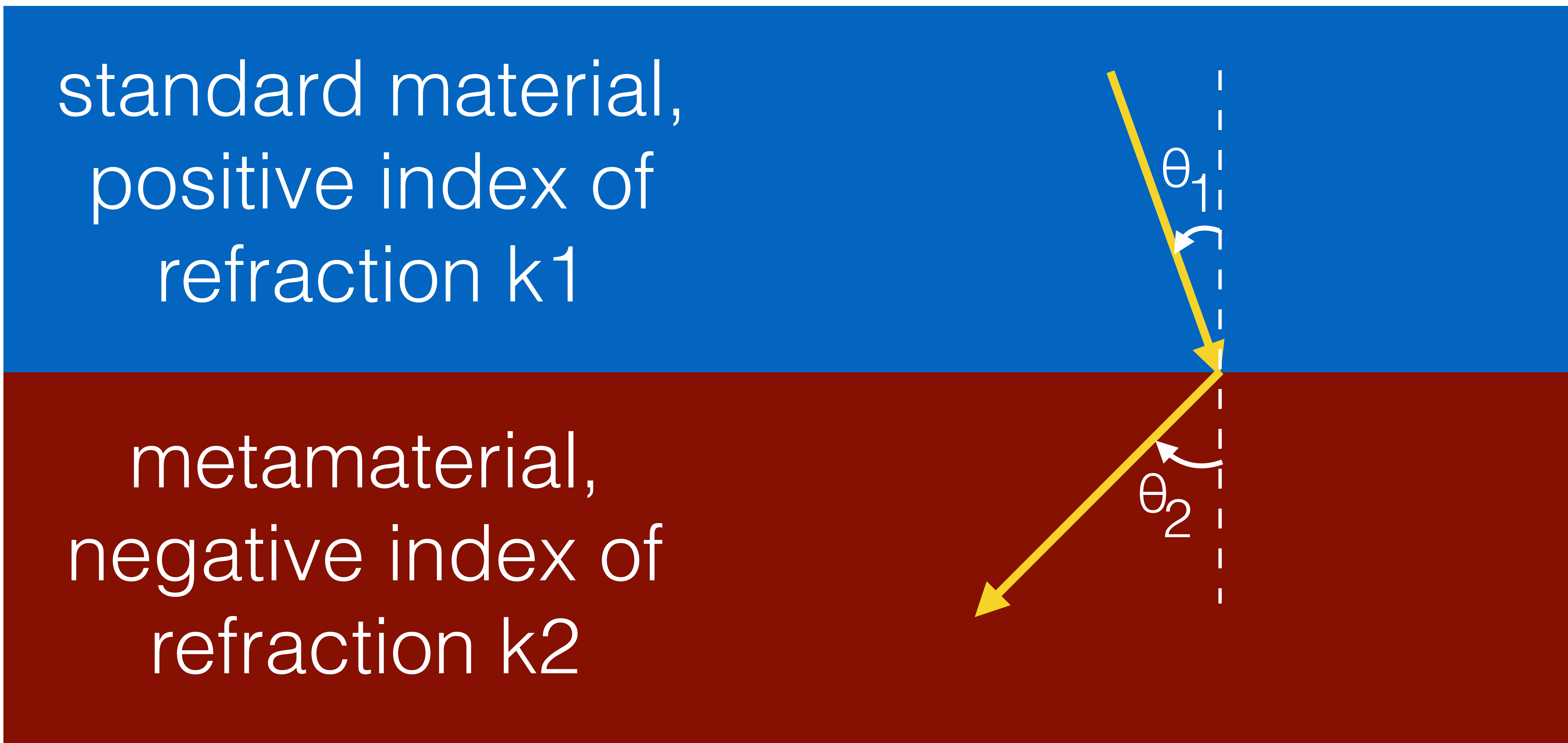

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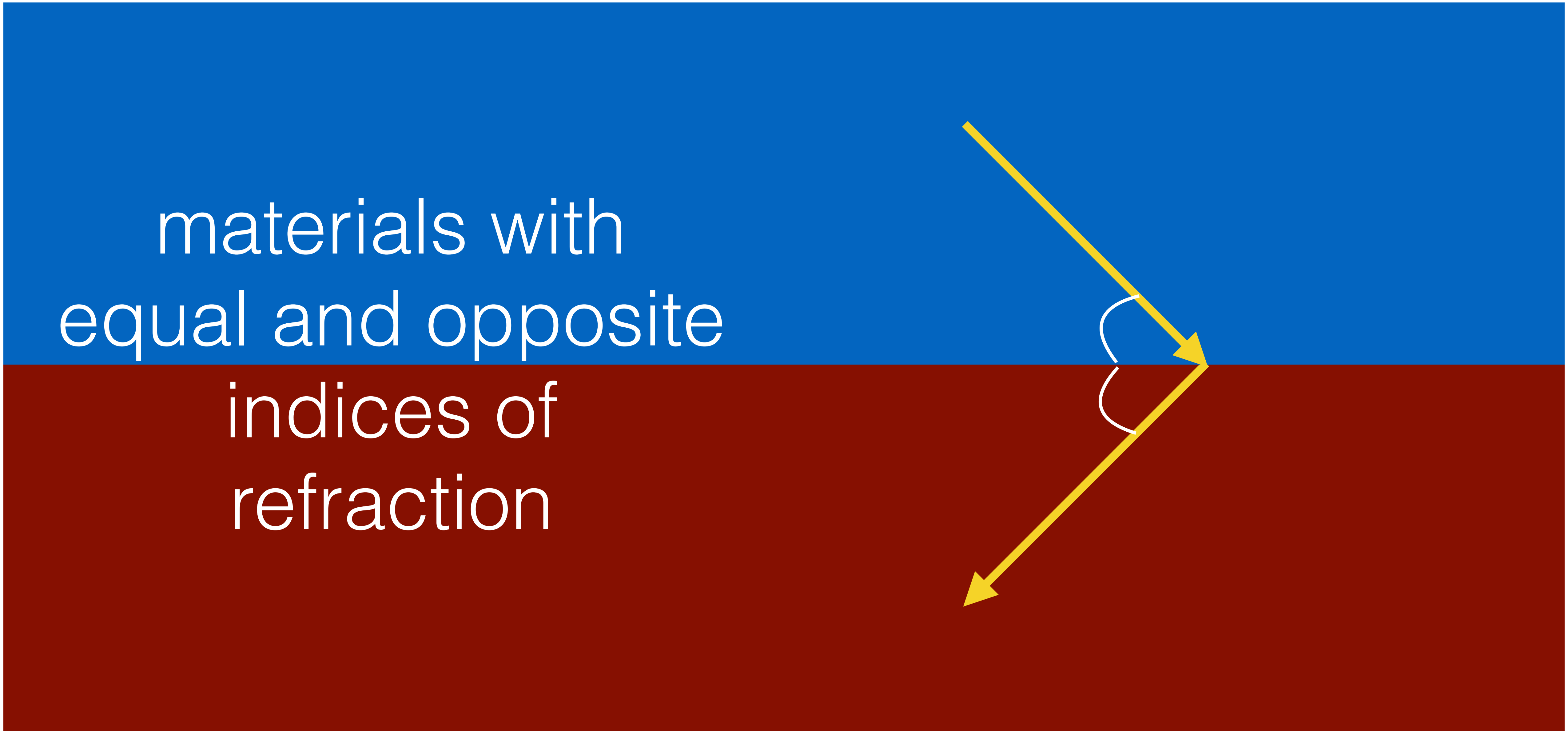
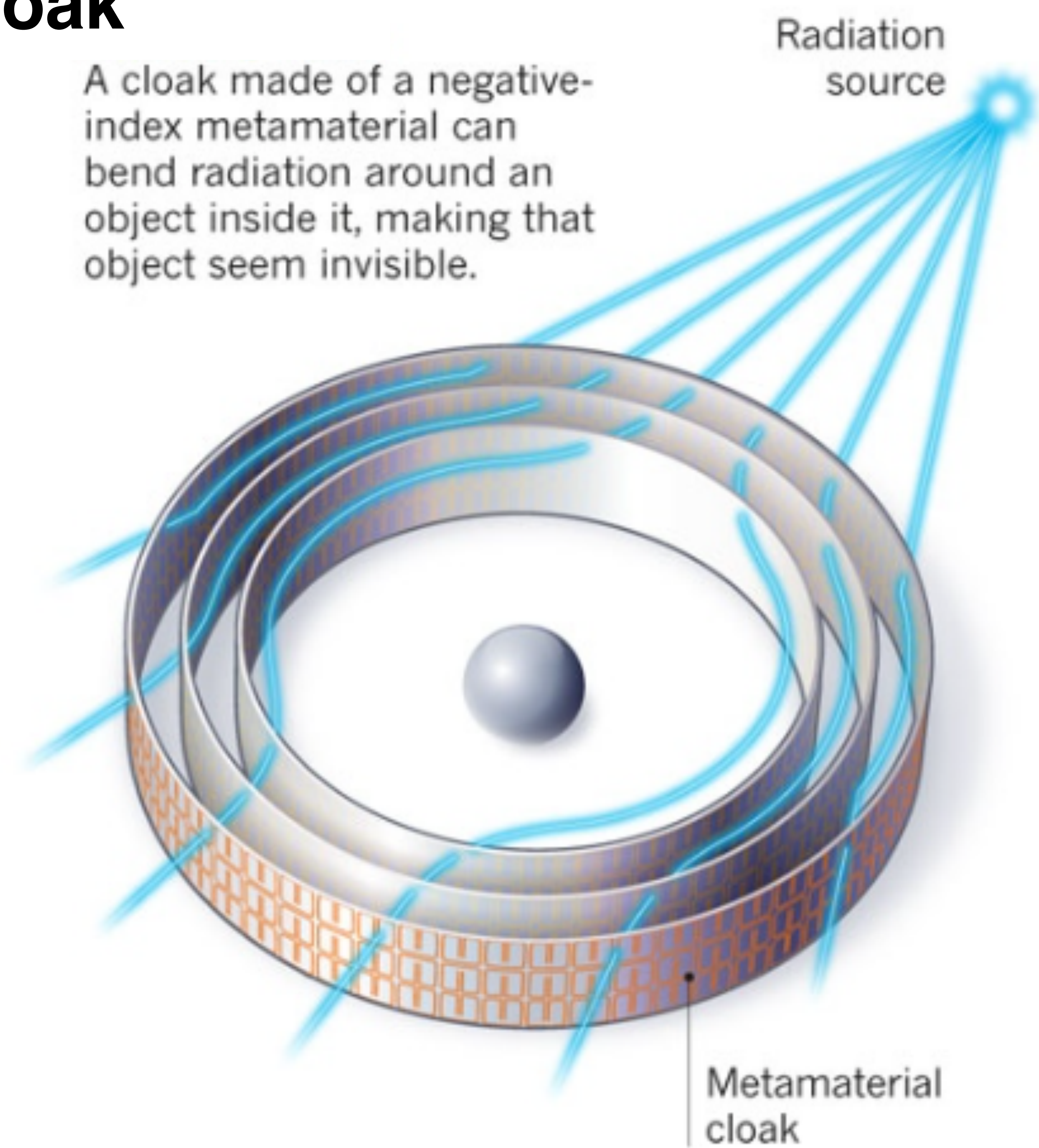


Applications:
1. Invisibility cloak
2. Perfect lens



Graphic by Jasiek Krzysztofiak / *Nature*

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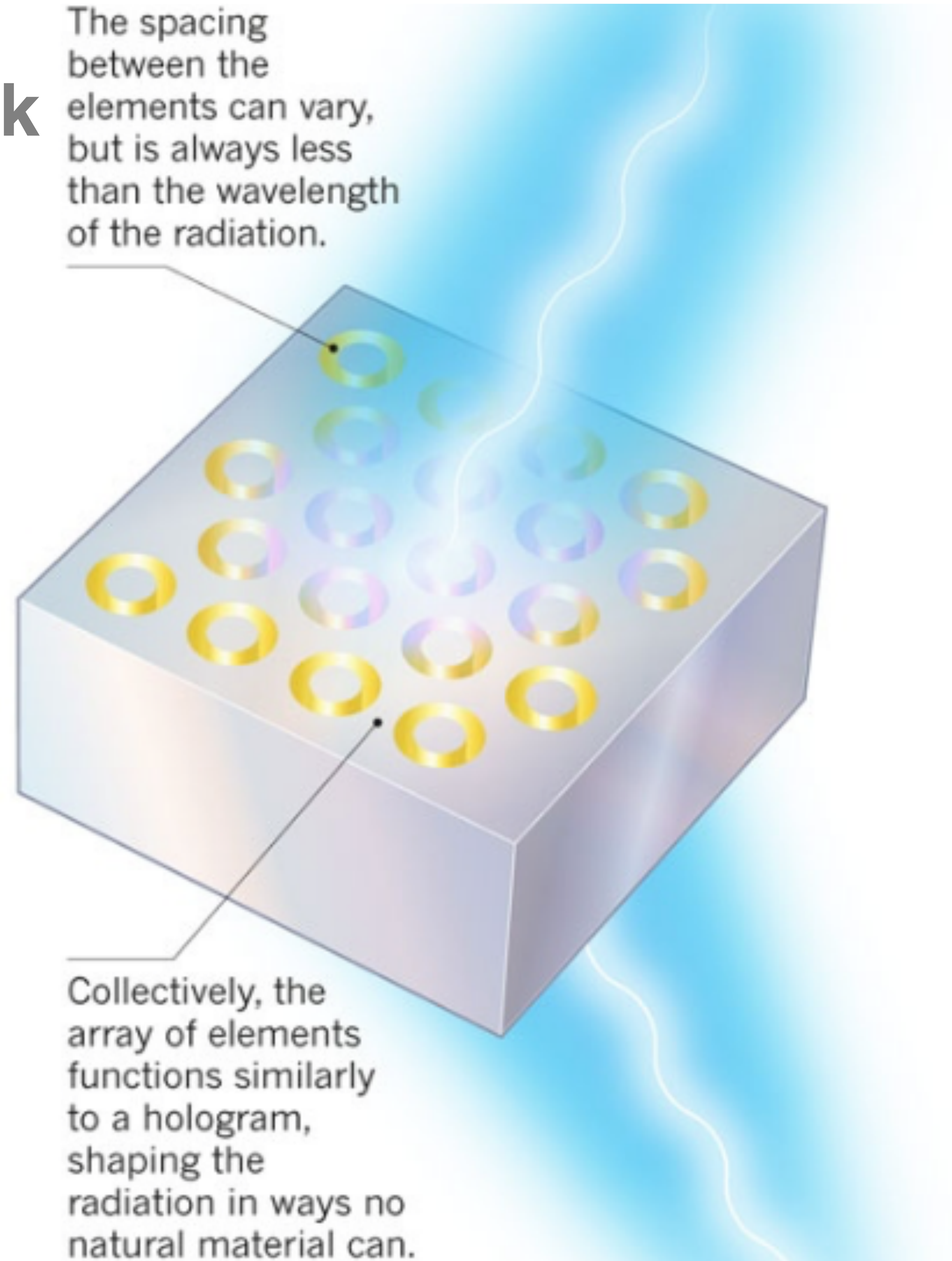

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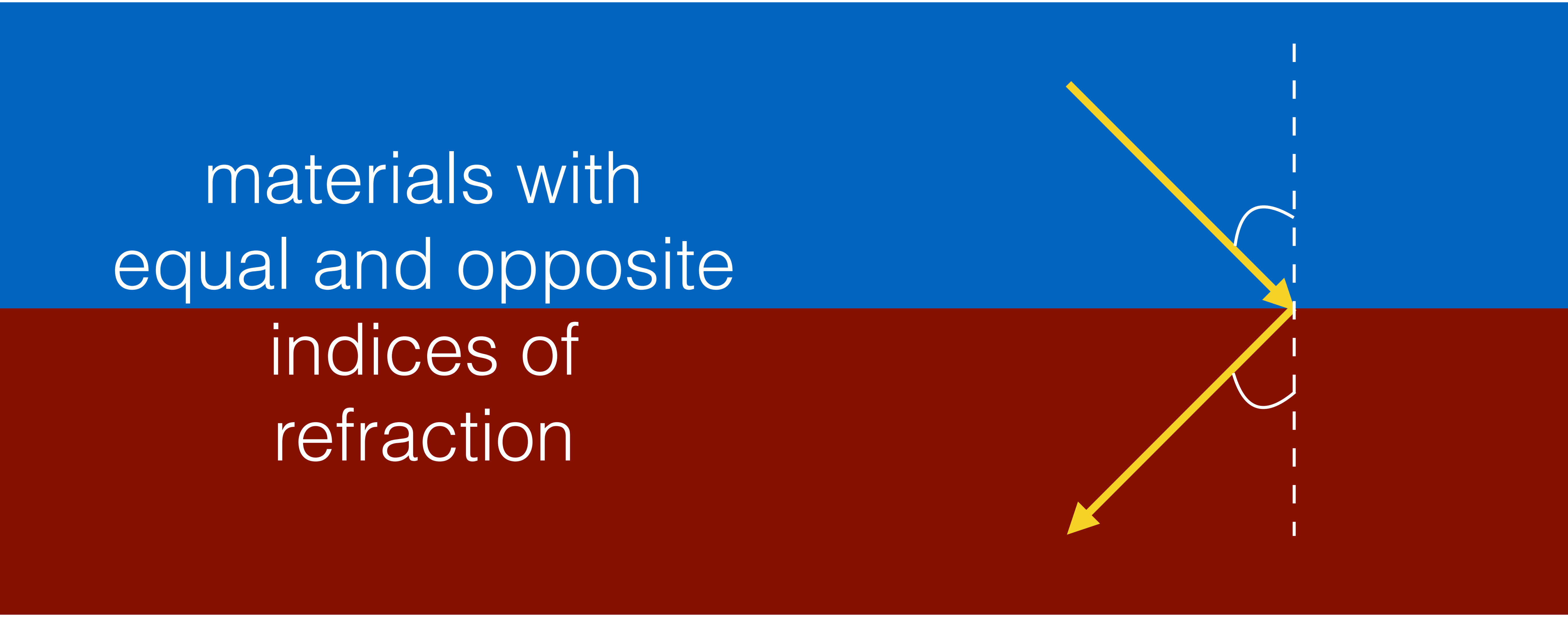
Tiling Billiards:

A dynamical system where light refracts through a planar tiling by materials with equal and opposite alternating indices of refraction



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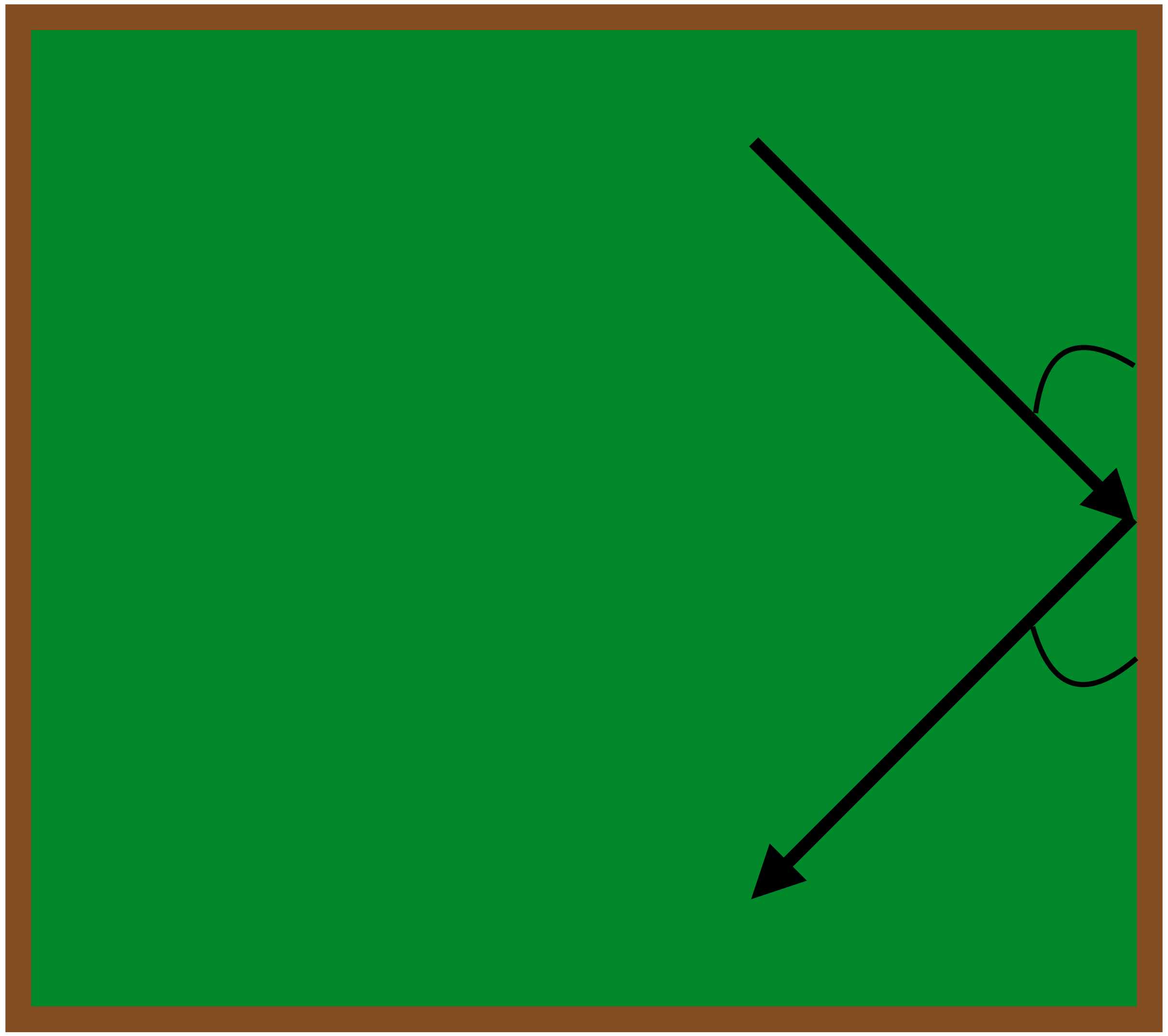
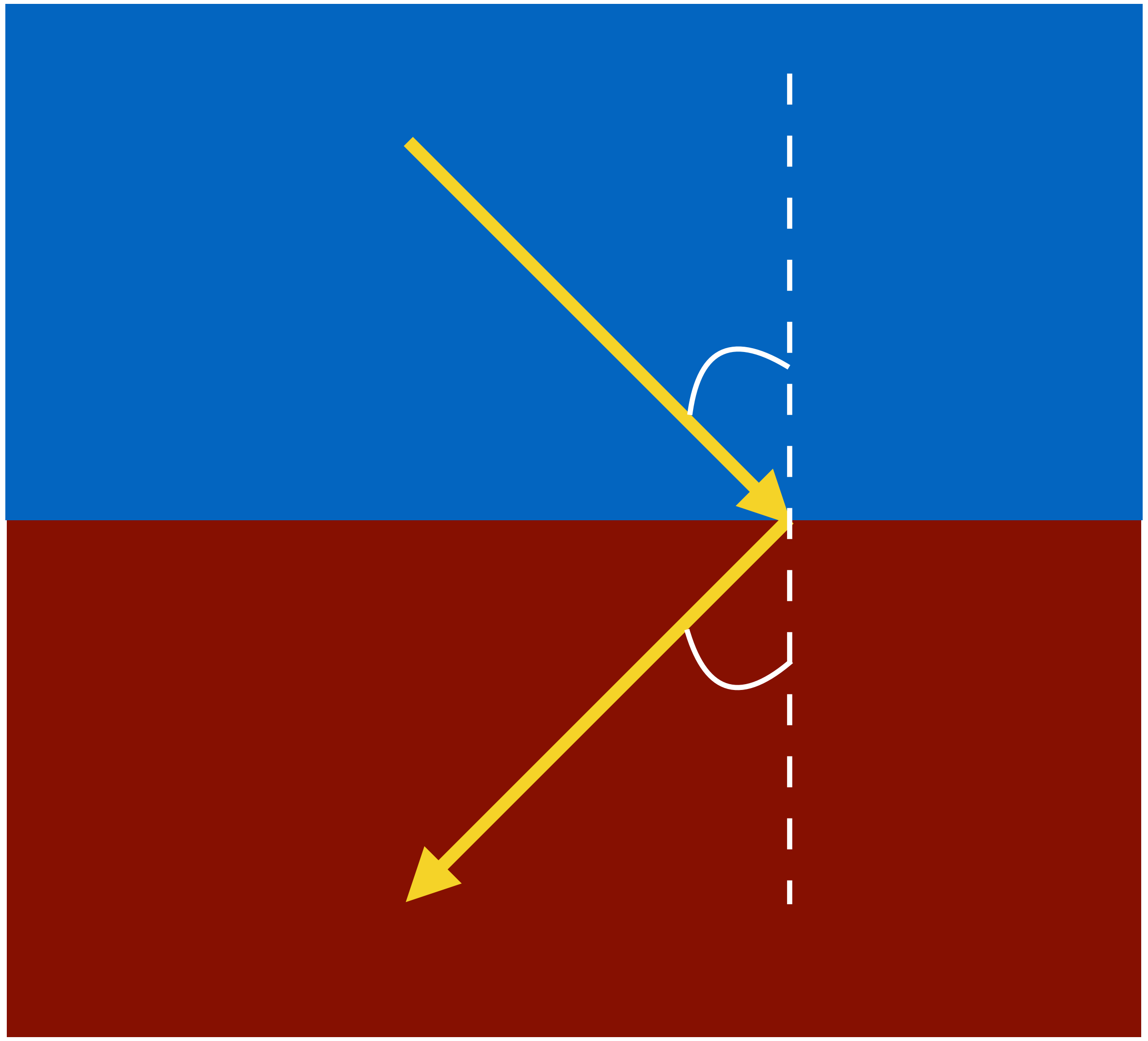
trajectory is reflected **against** the edge of the table, preserving angles



Tiling Billiards:

trajectory is reflected **across** each edge of the tiling, preserving angles



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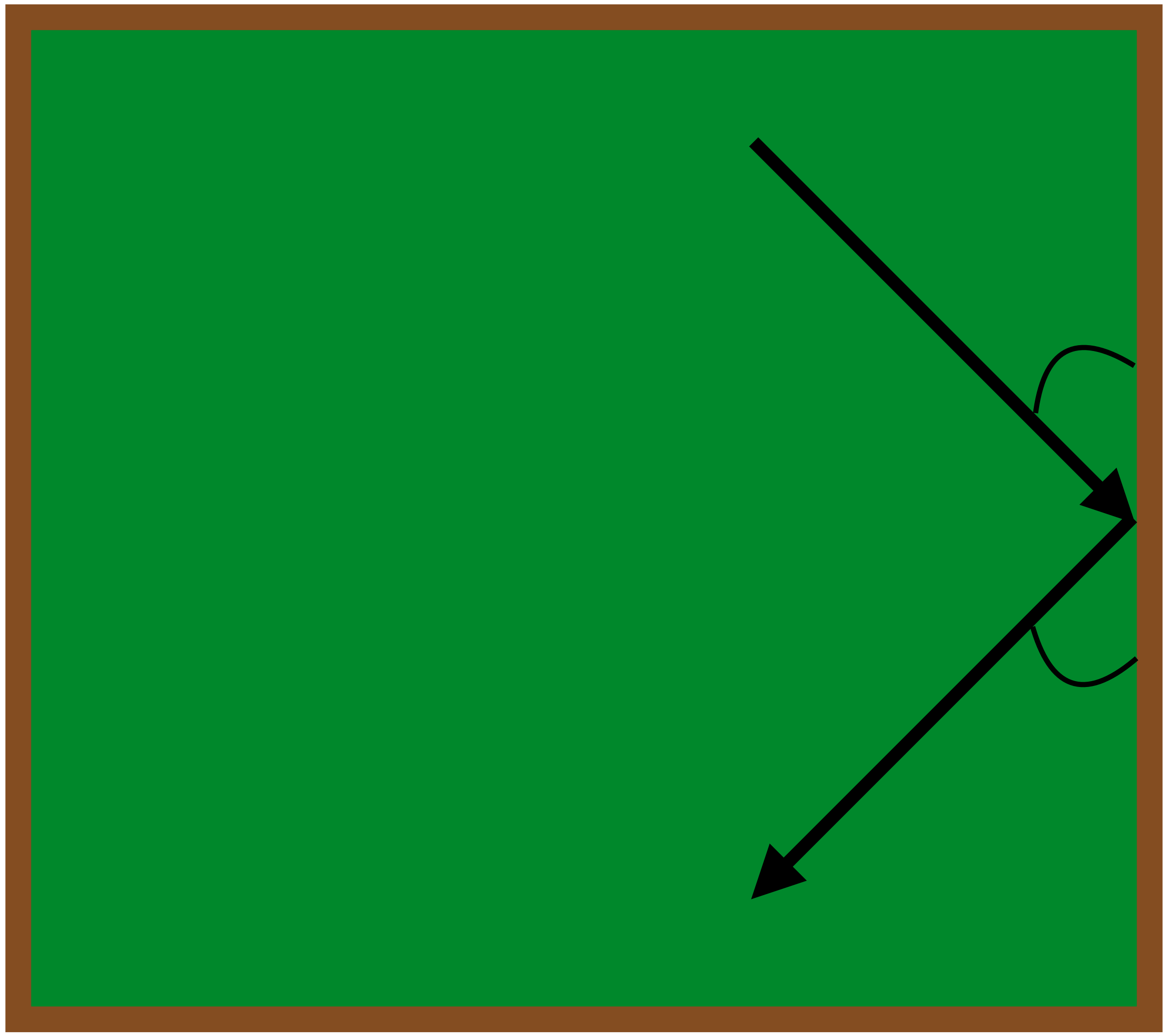

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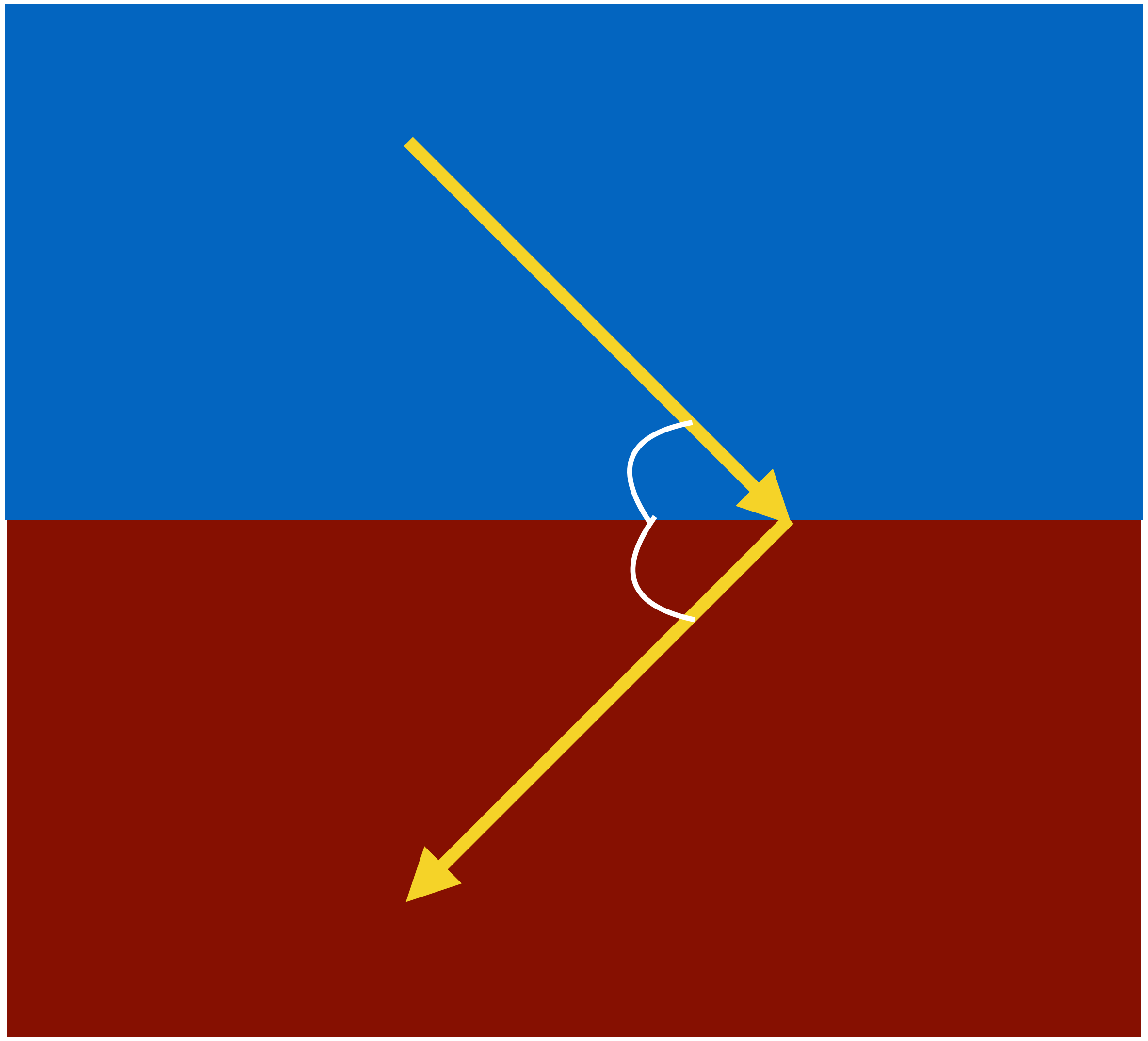
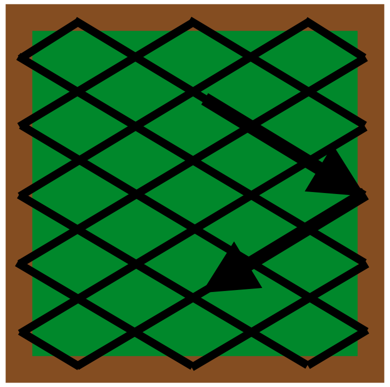
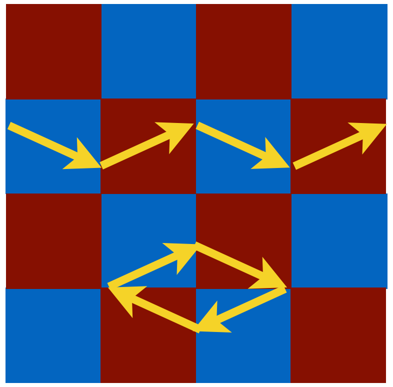


Standard (Inner) Billiards:



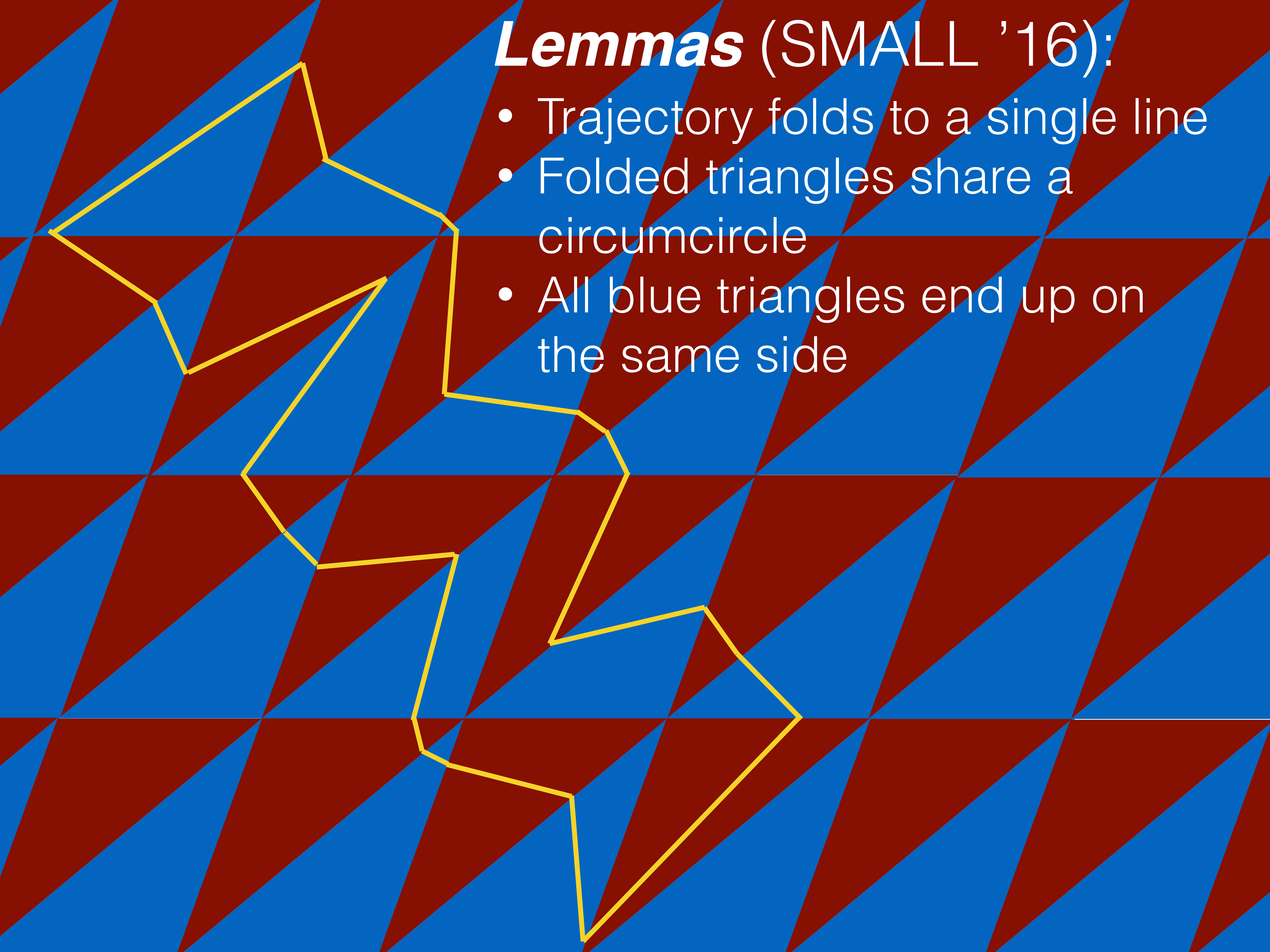
- A trajectory is **dense** with probability 1
- Behavior is **unstable**

Tiling Billiards on triangle tilings:



- Most trajectories are **periodic**
- Trajectories are very **stable**

Burning question: What causes periodicity and stability in tiling billiards on triangle tilings?

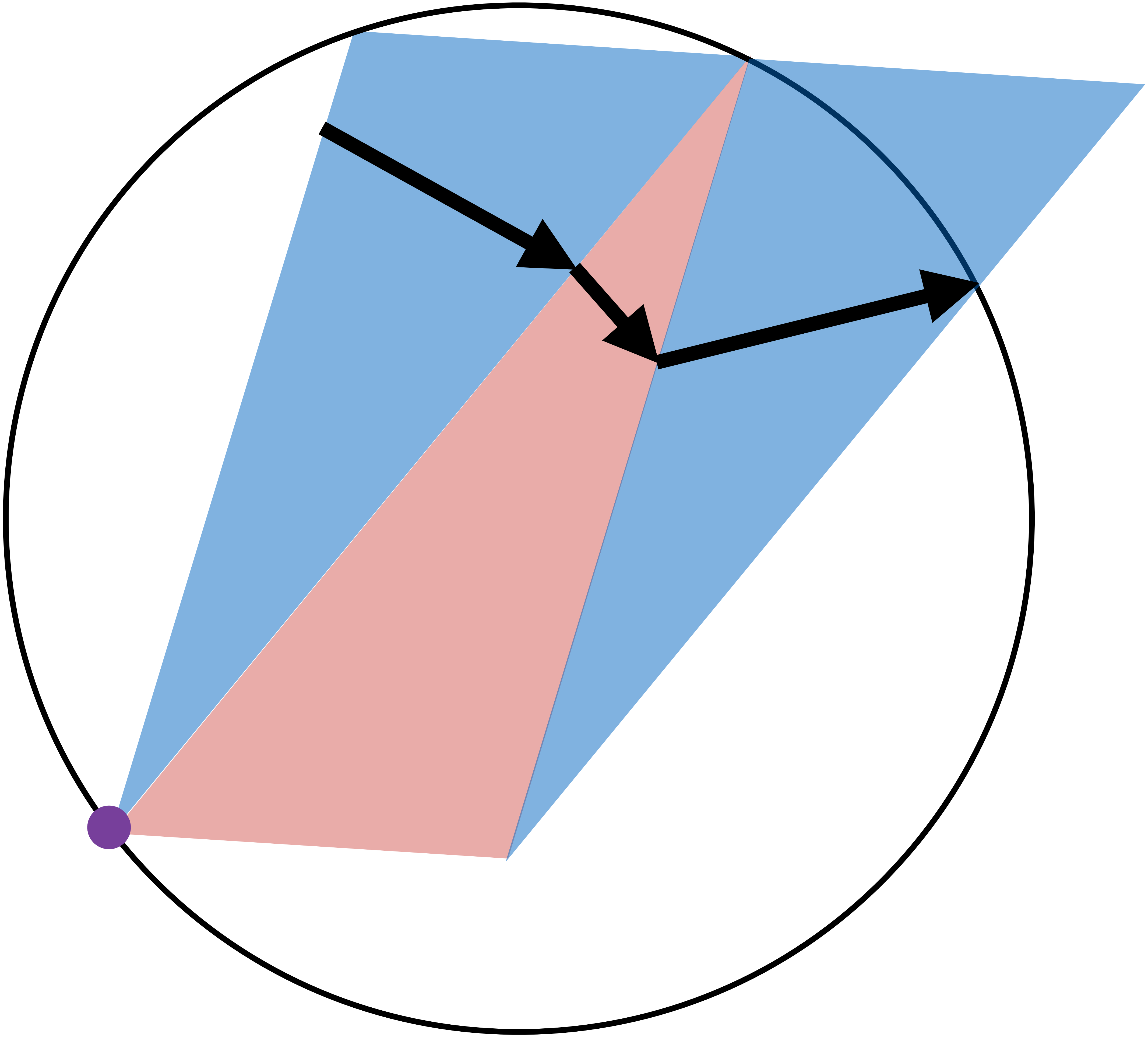


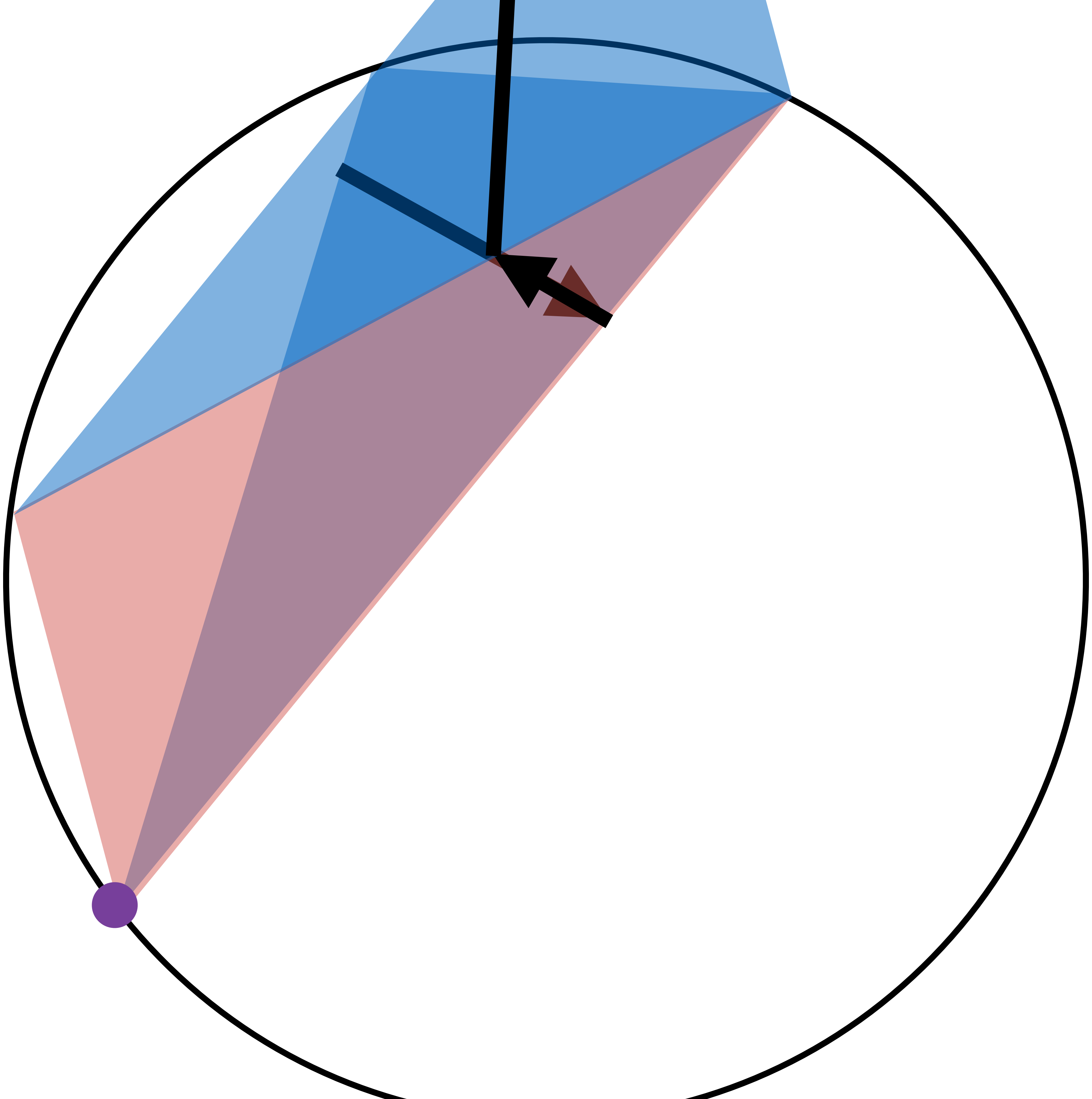
Lemmas (SMALL ’16):

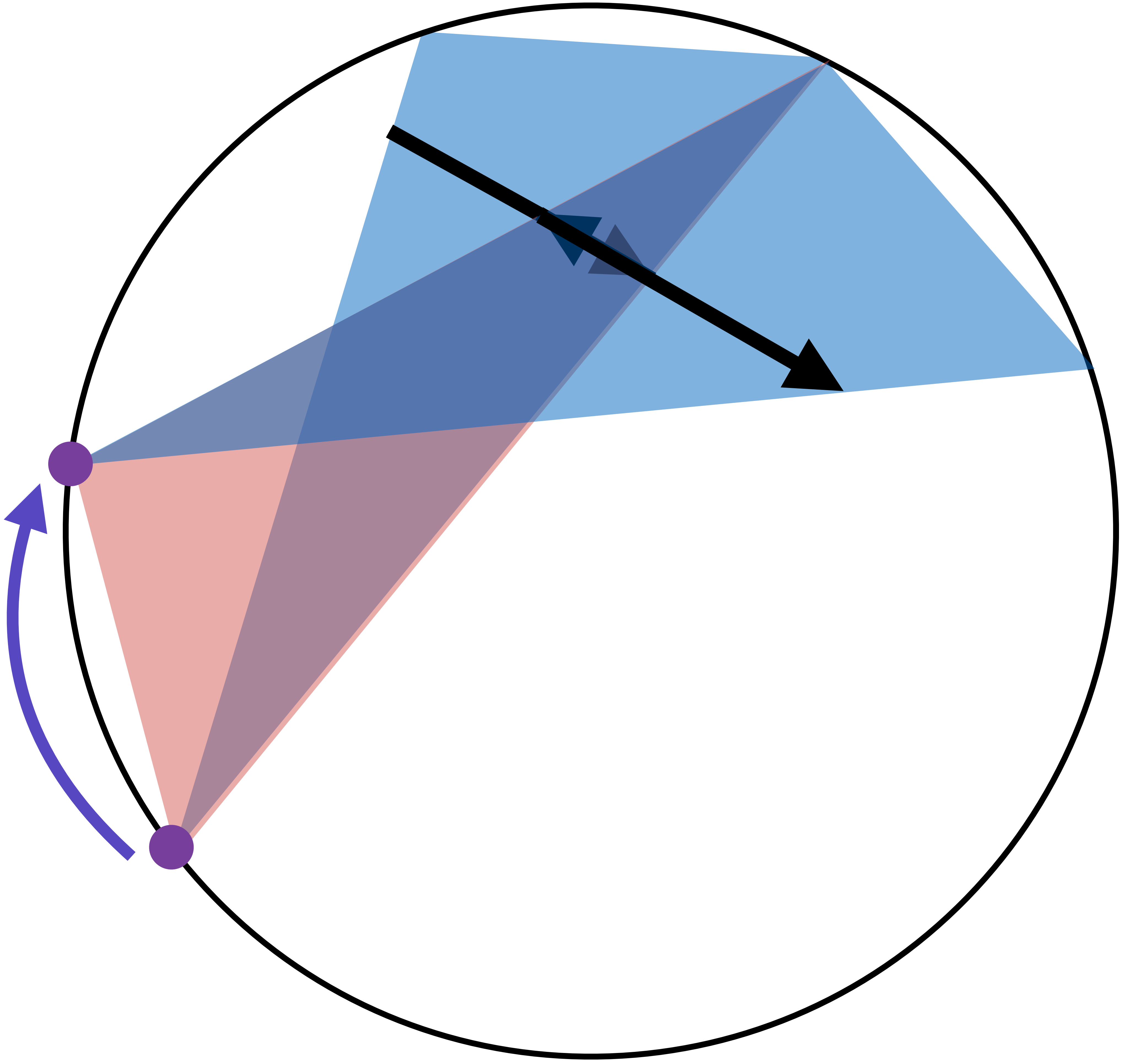
- Trajectory folds to a single line
- Folded triangles share a circumcircle
- All blue triangles end up on the same side

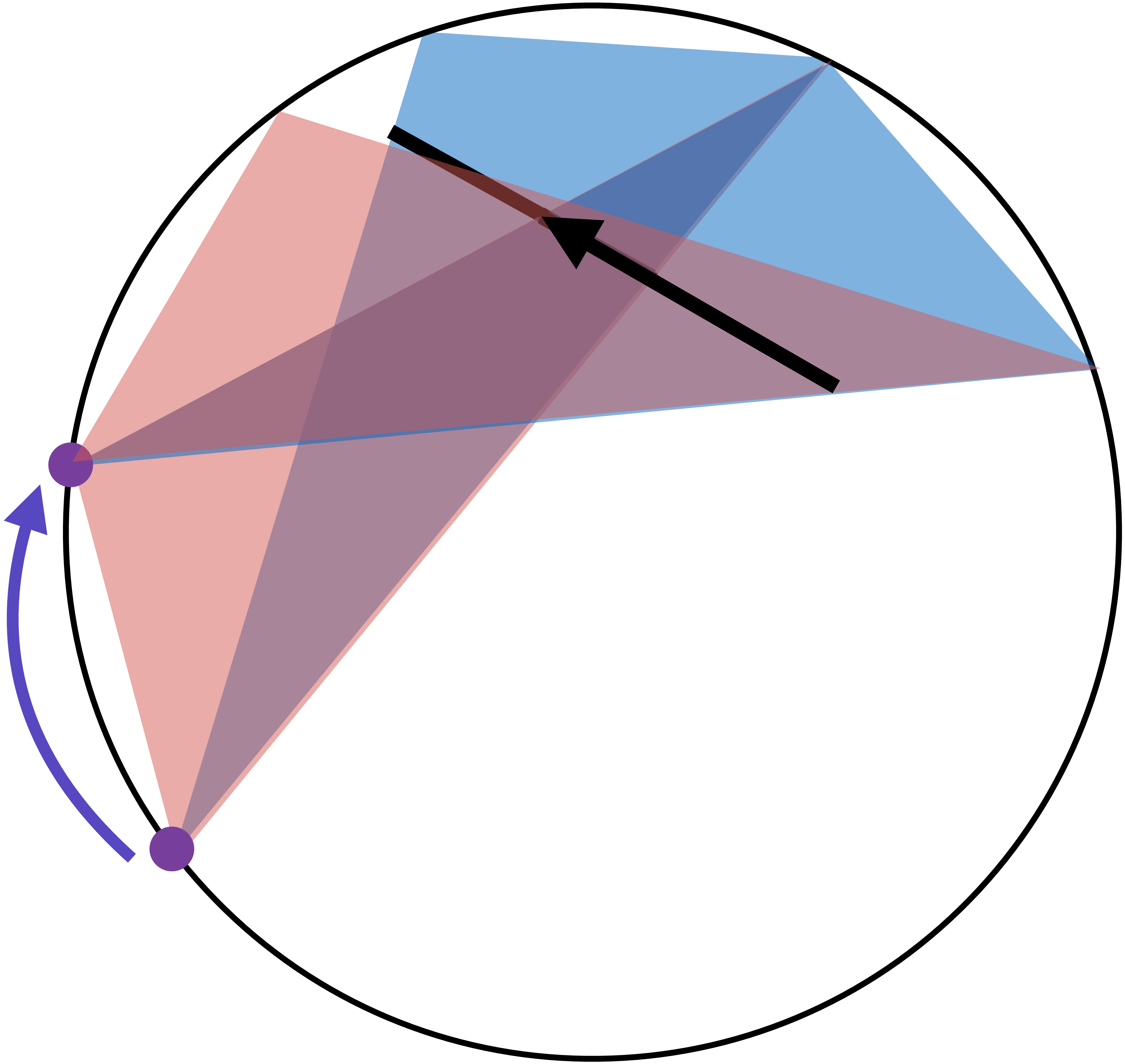
Insight (SMALL '16):

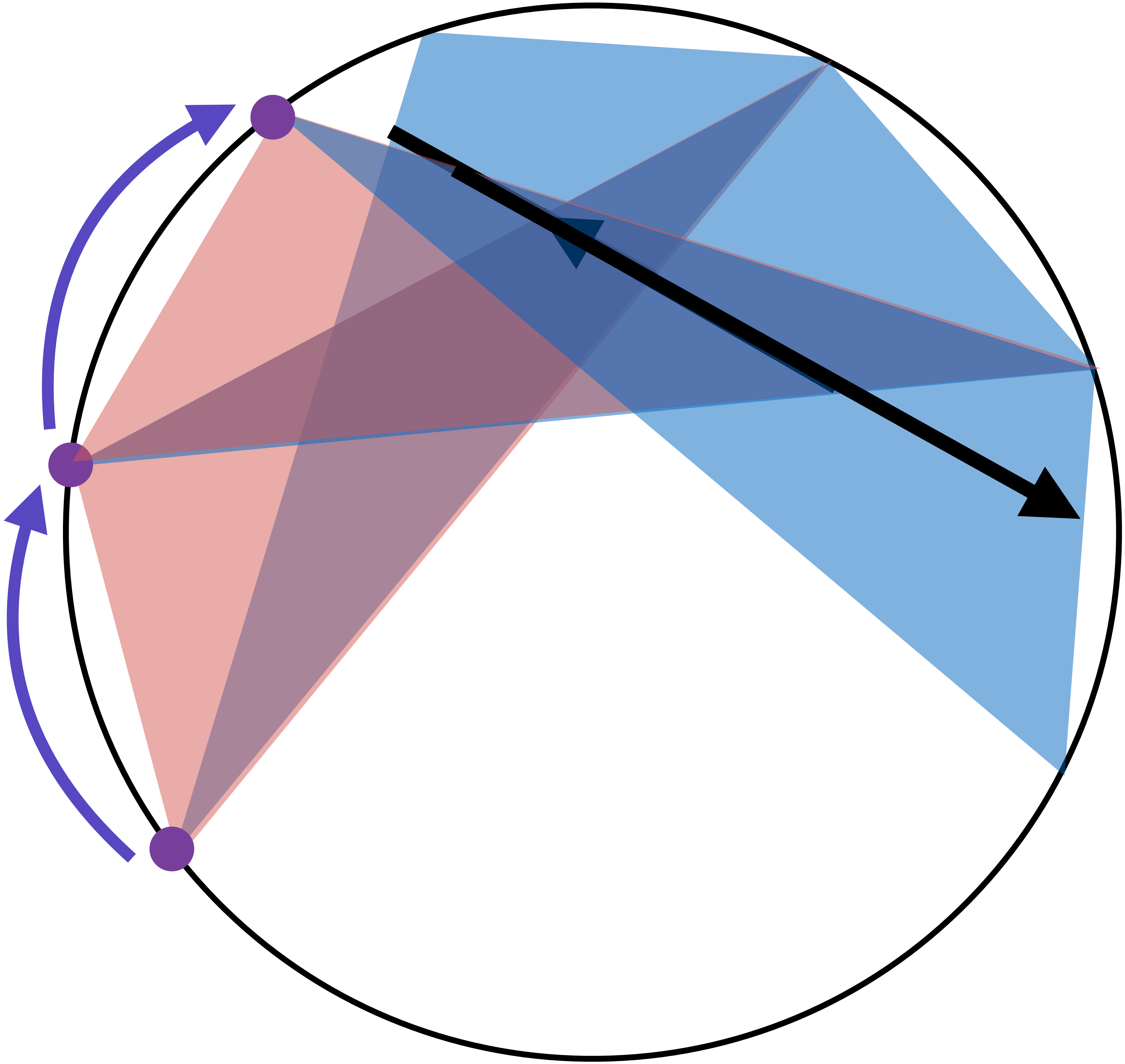
- Fix trajectory; triangle moves
- Keep track of favorite vertex
- This yields a 1-dimensional system, in fact an Interval Exchange Transformation (IET).

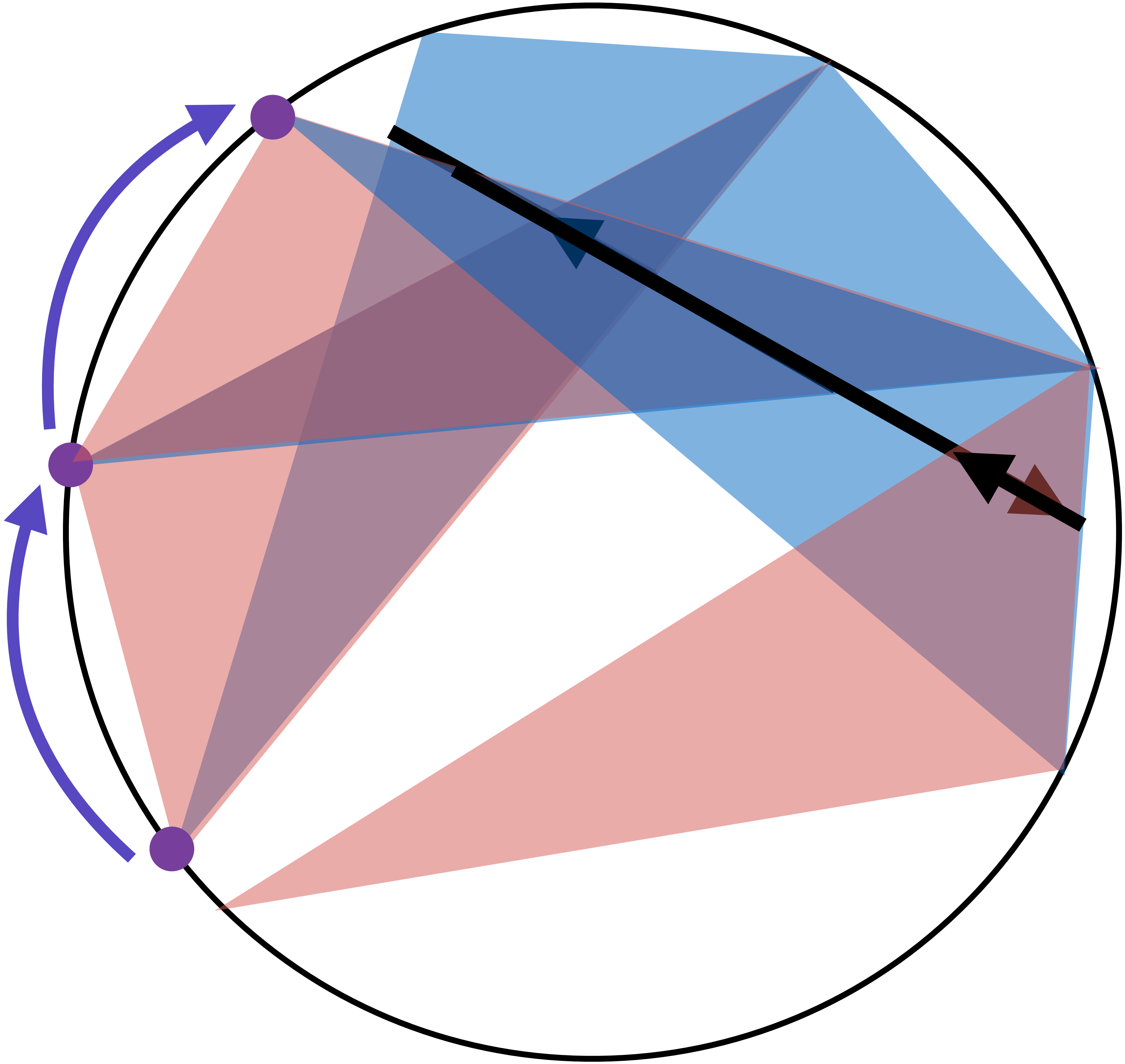


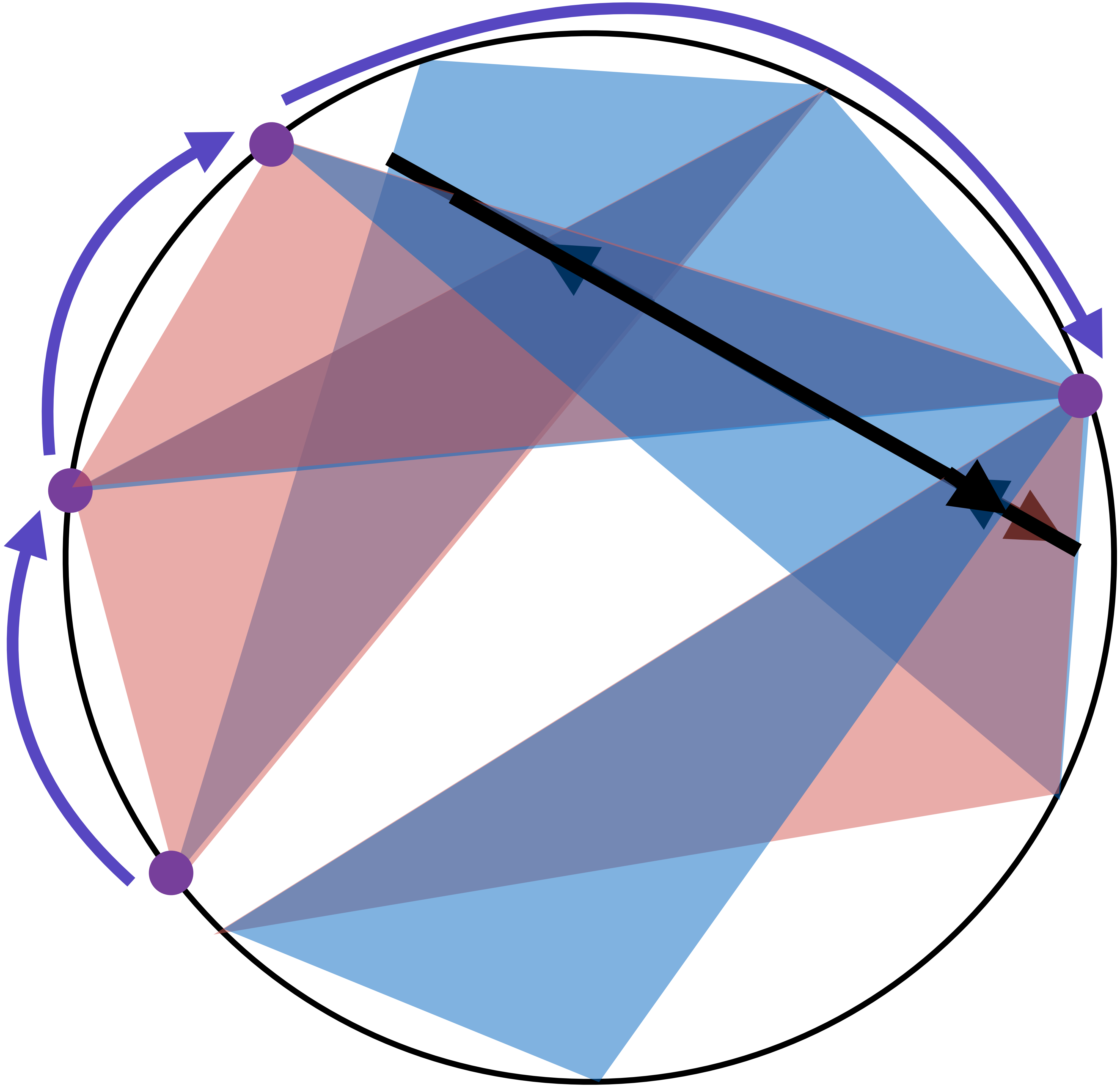


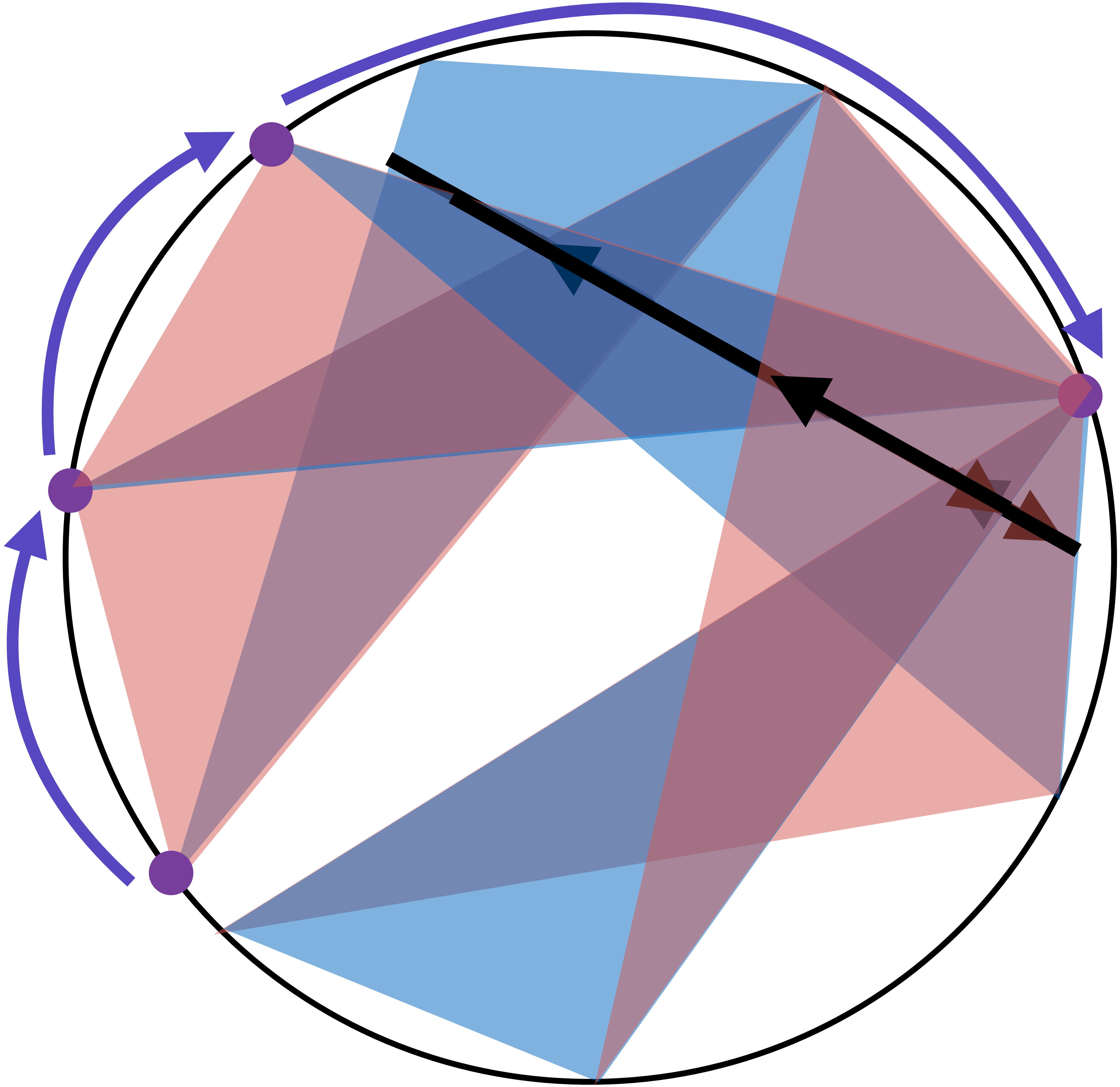


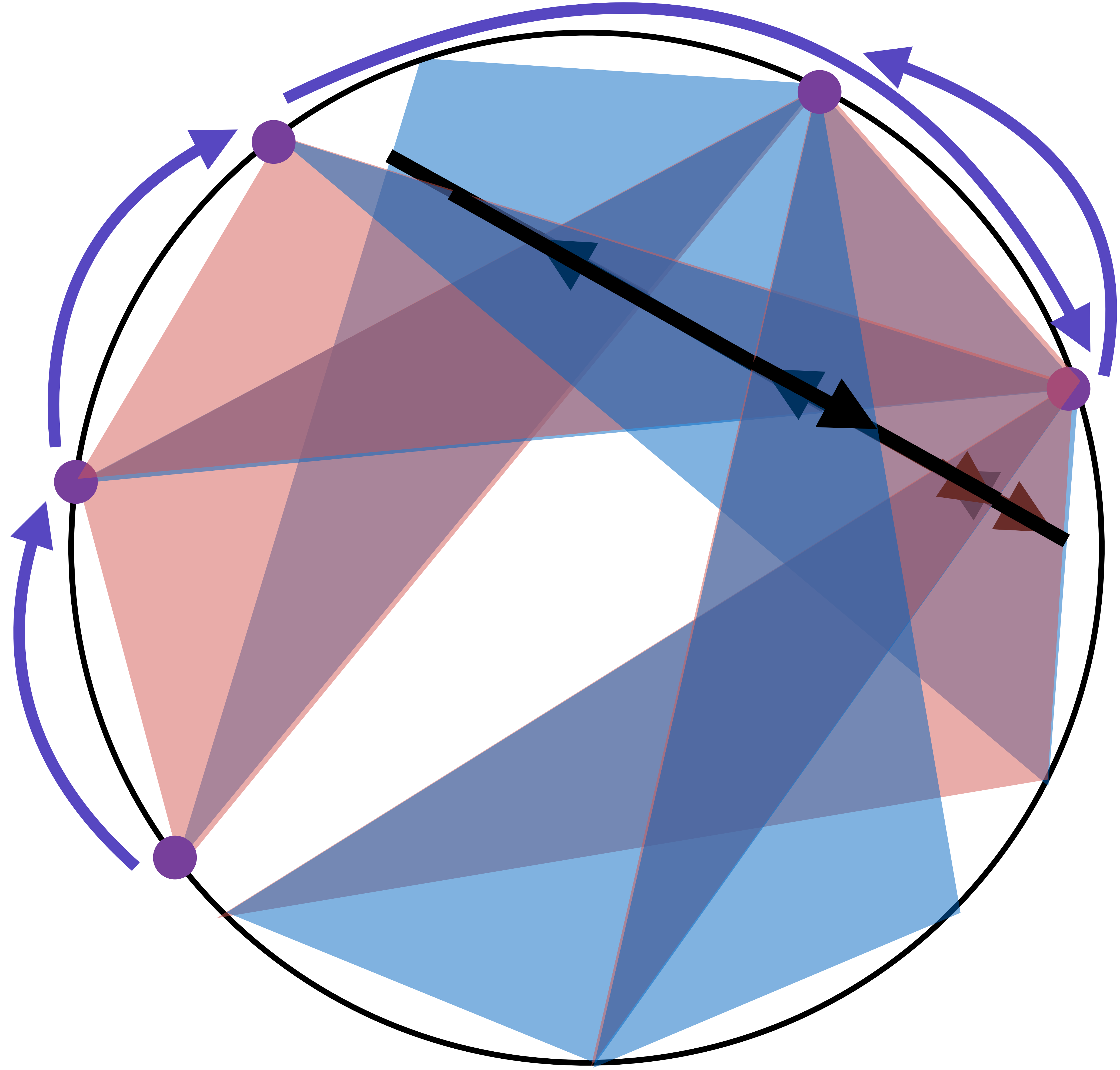




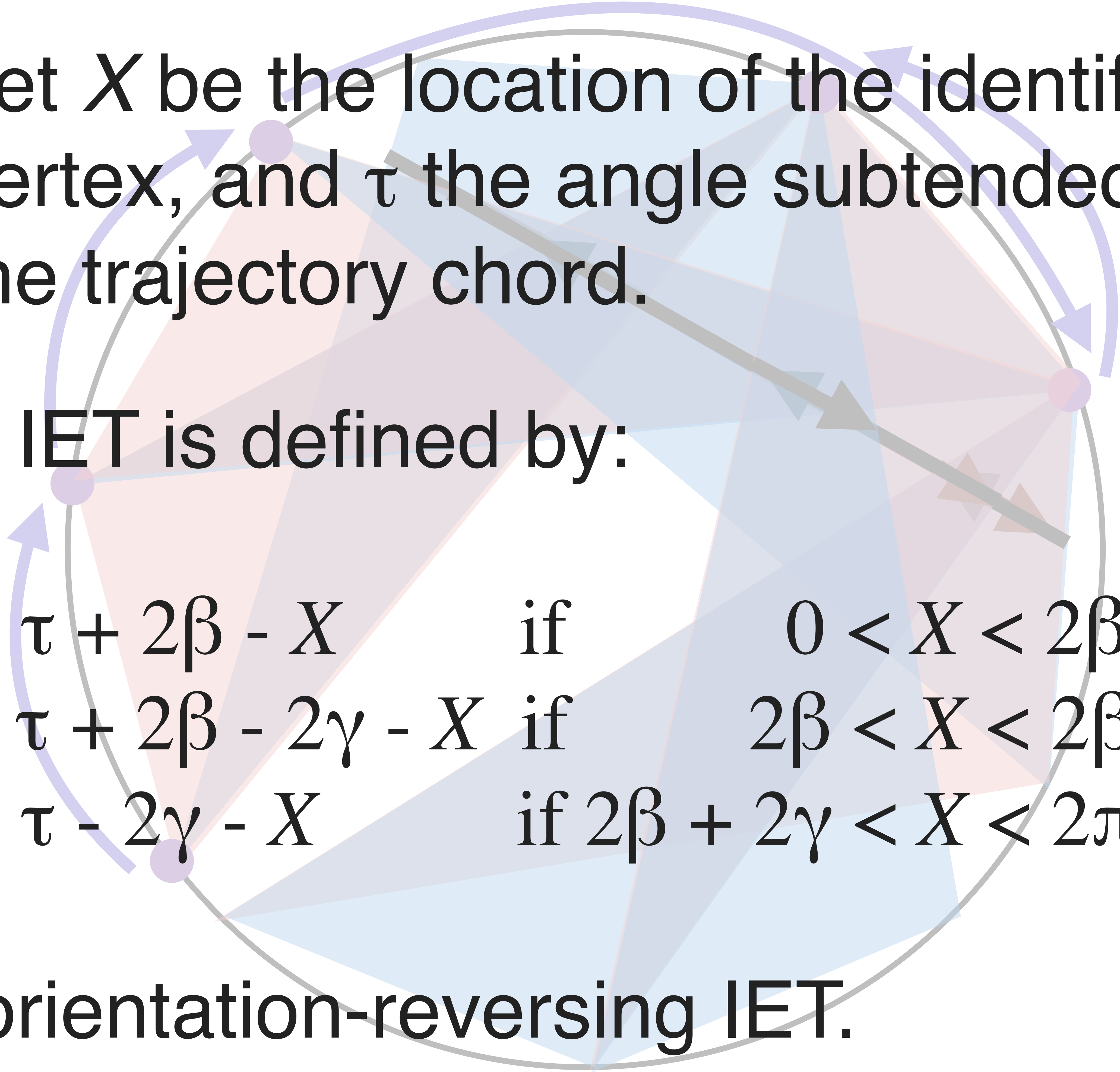








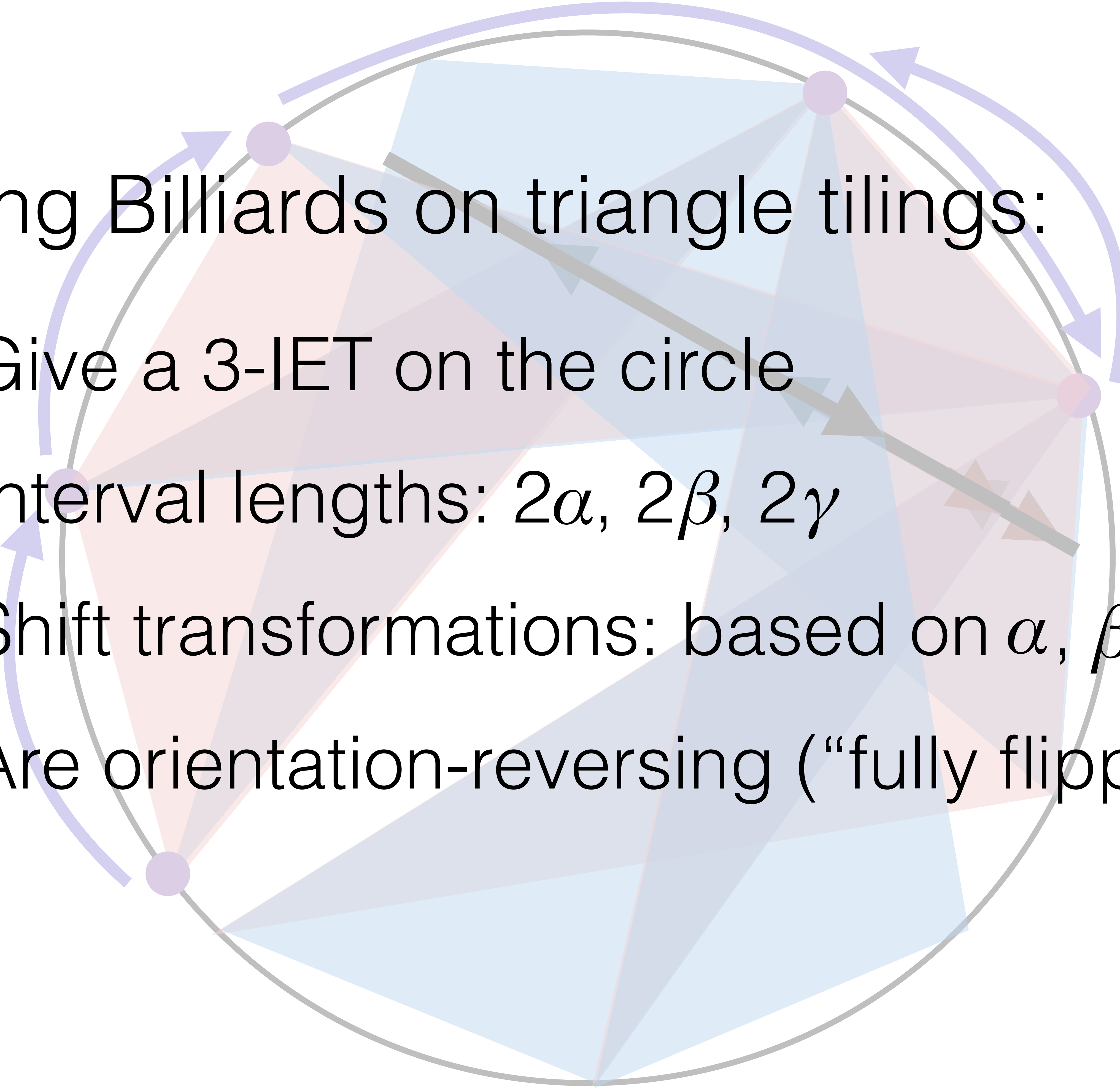
Let X be the location of the identified vertex, and τ the angle subtended by the trajectory chord.



Our IET is defined by:

$$X' = \begin{cases} \tau + 2\beta - X & \text{if } \quad 0 < X < 2\beta \\ \tau + 2\beta - 2\gamma - X & \text{if } \quad 2\beta < X < 2\beta + 2\gamma \\ \tau - 2\gamma - X & \text{if } \quad 2\beta + 2\gamma < X < 2\pi, \end{cases}$$

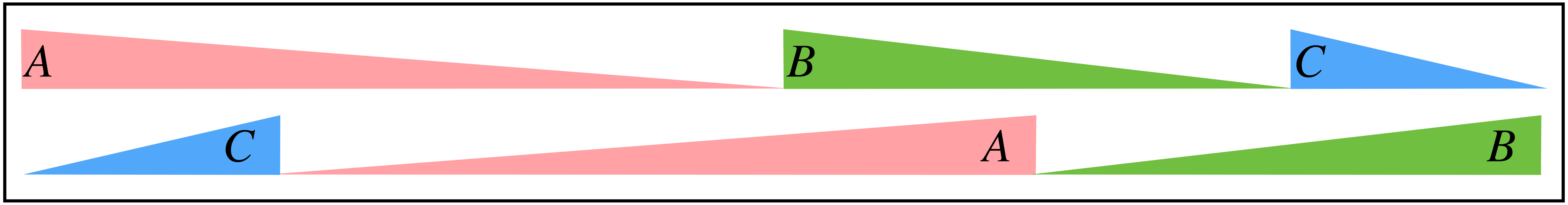
an orientation-reversing IET.



Tiling Billiards on triangle tilings:

- Give a 3-IET on the circle
- Interval lengths: 2α, 2β, 2γ
- Shift transformations: based on α, β, γ, τ
- Are orientation-reversing (“fully flipped”)

Why flipped IETs are stable & periodic



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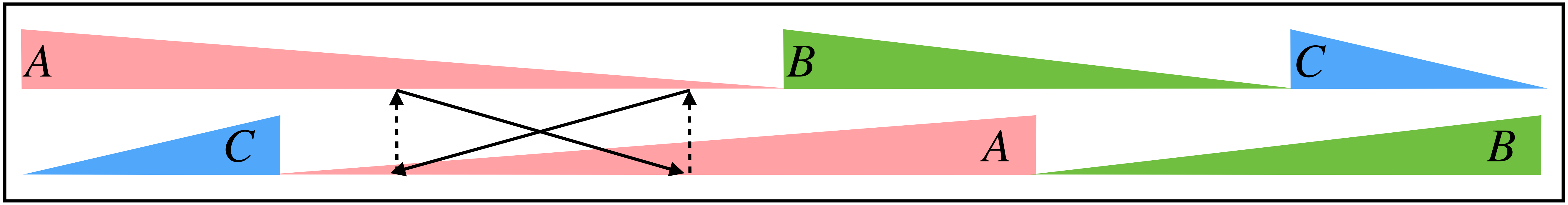

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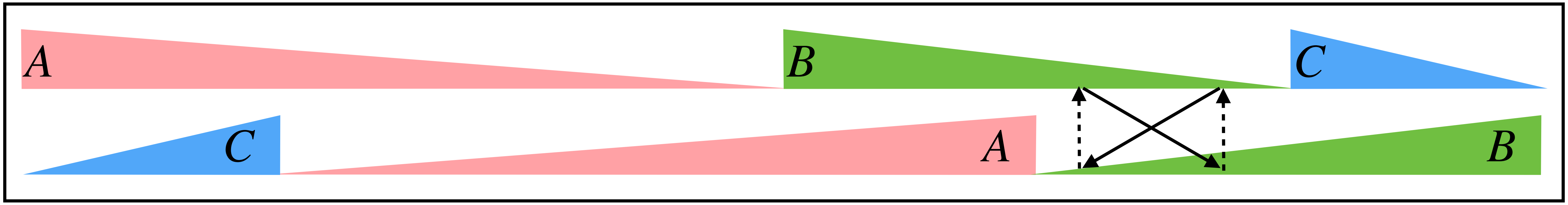

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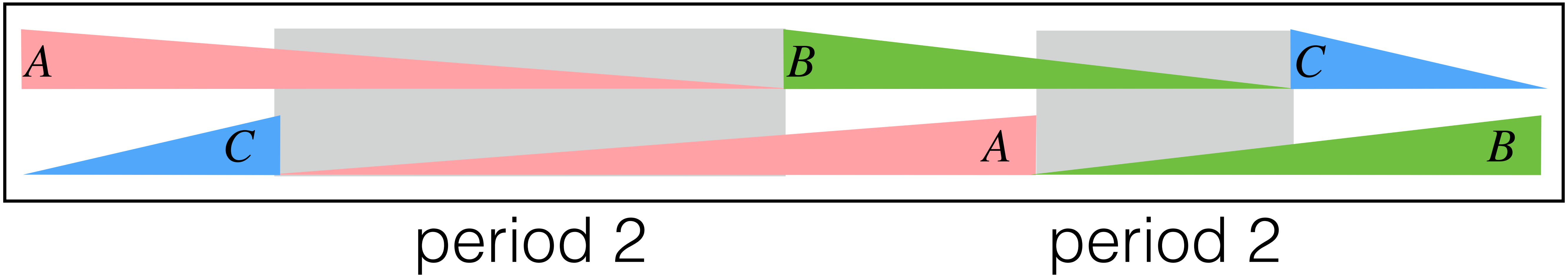

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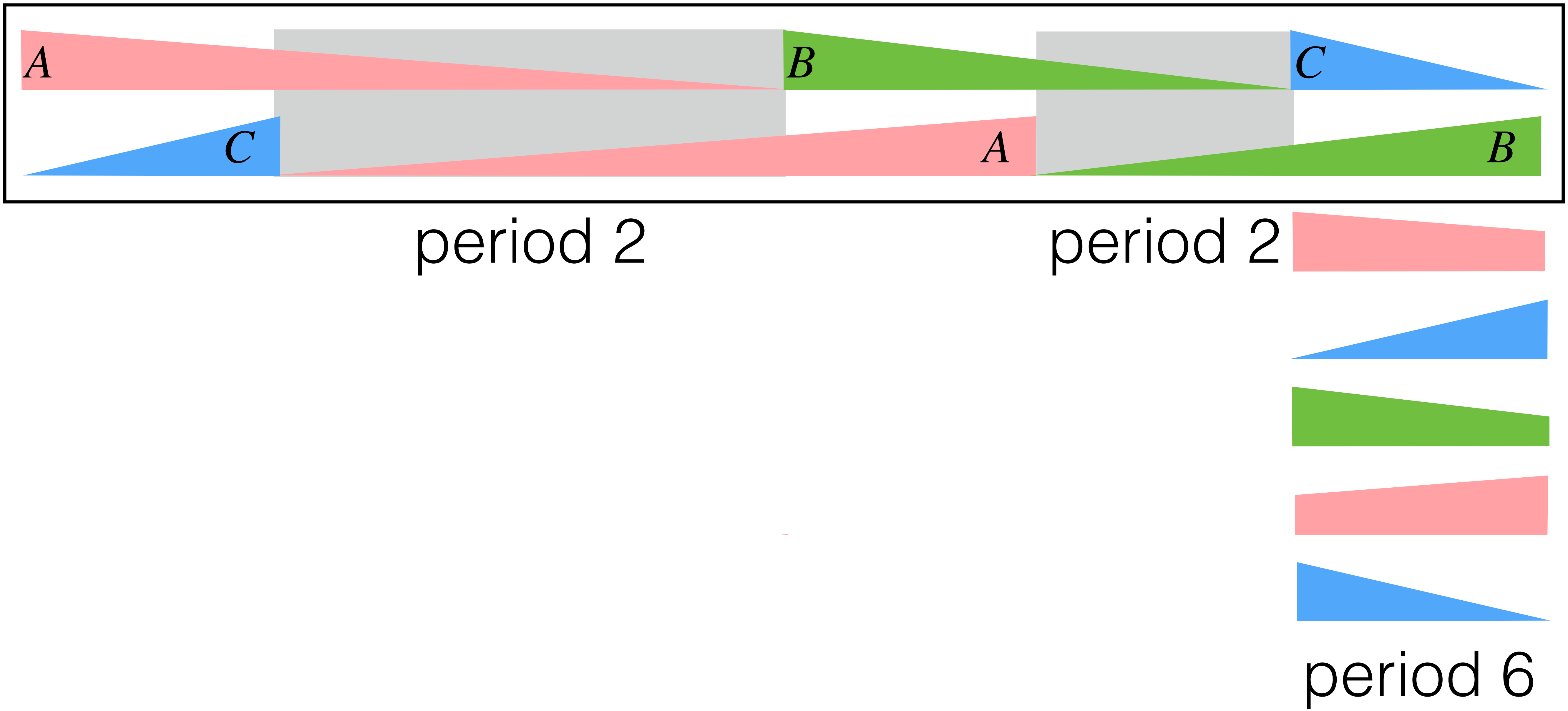

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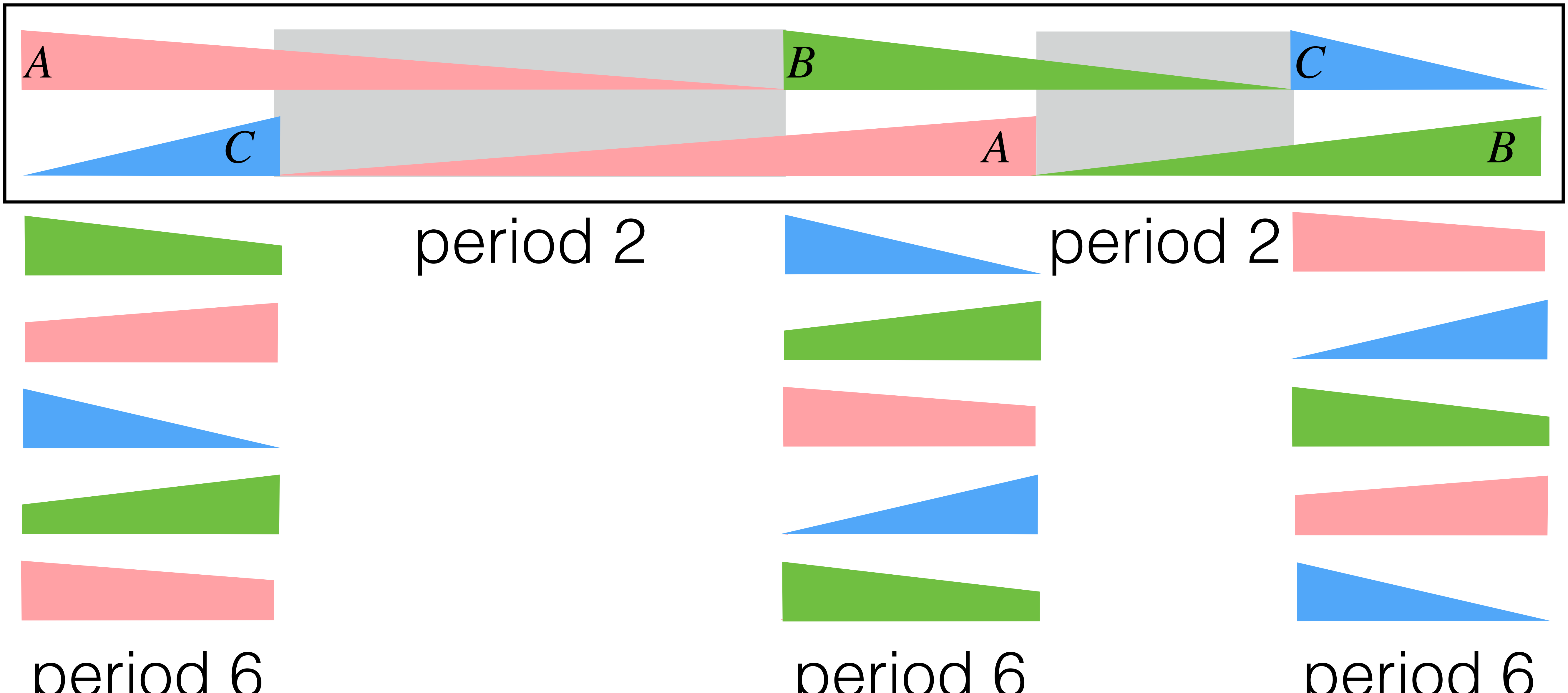

Everything is flipped periodic, every point is stable, periods of form 4n+2

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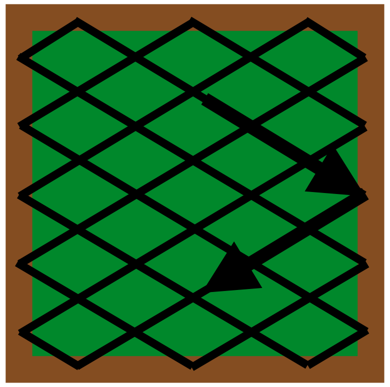

Tiling Billiards on triangle tilings:

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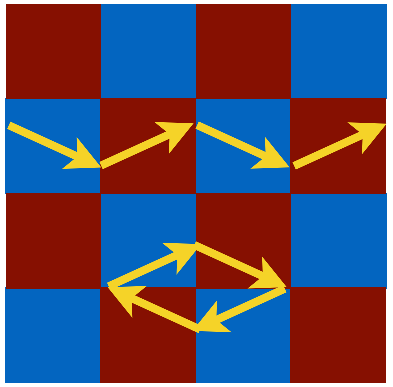
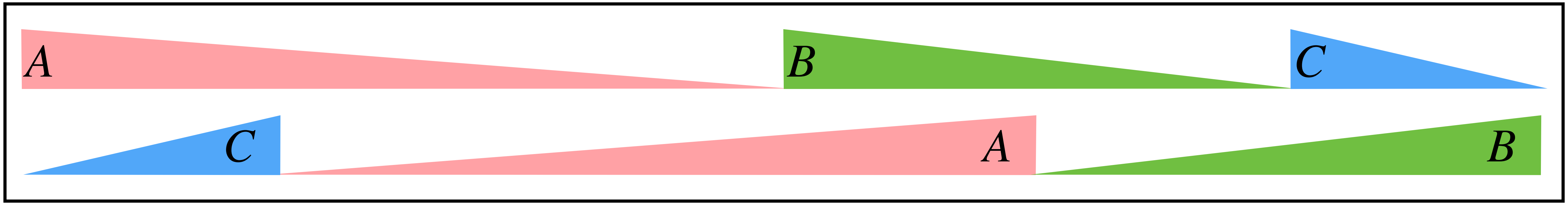


Burning question: What causes periodicity and stability in tiling billiards on triangle tilings?

➔They correspond to fully flipped IETs.

Comparison to non-flipped IETs

If |AB| and |C| are irrationally related, every point is aperiodic (rotation).

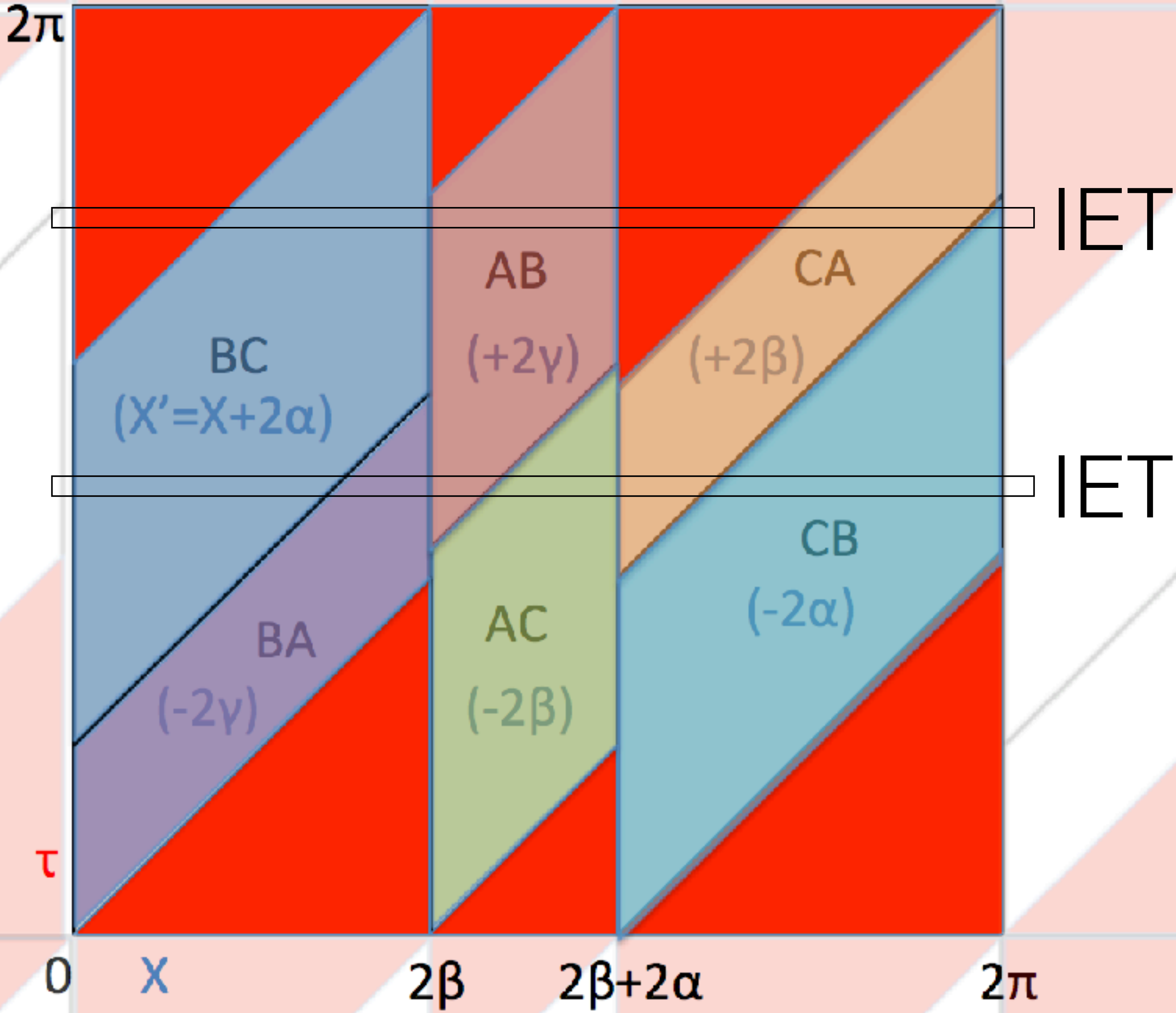


Tiling billiards corresponds to orientation-reversing 3-IET

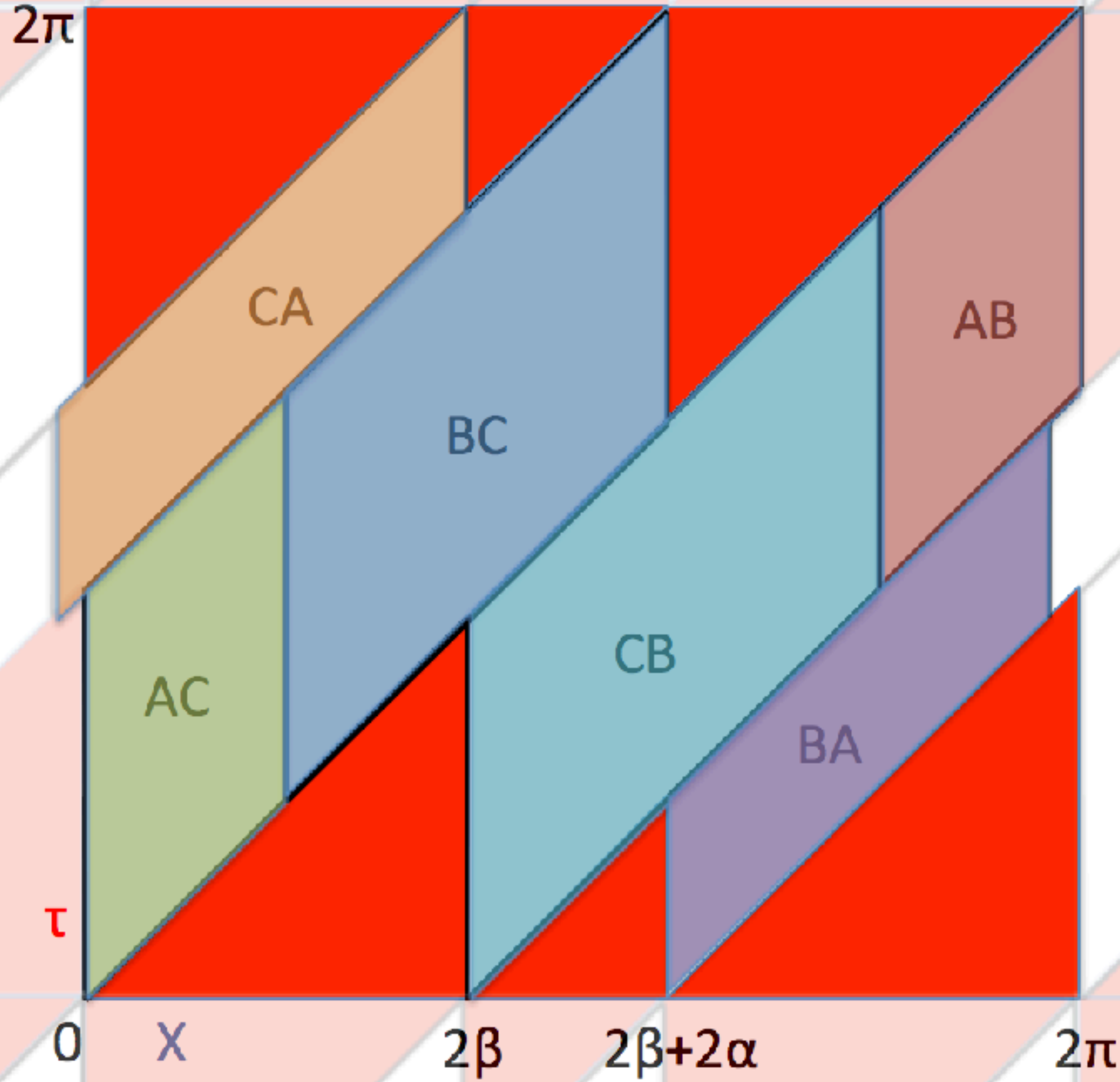
Idea:

- Use the **square** of the 3-IET
- Get an orientation-**preserving** 6*-IET
- Stack all of them into a PET

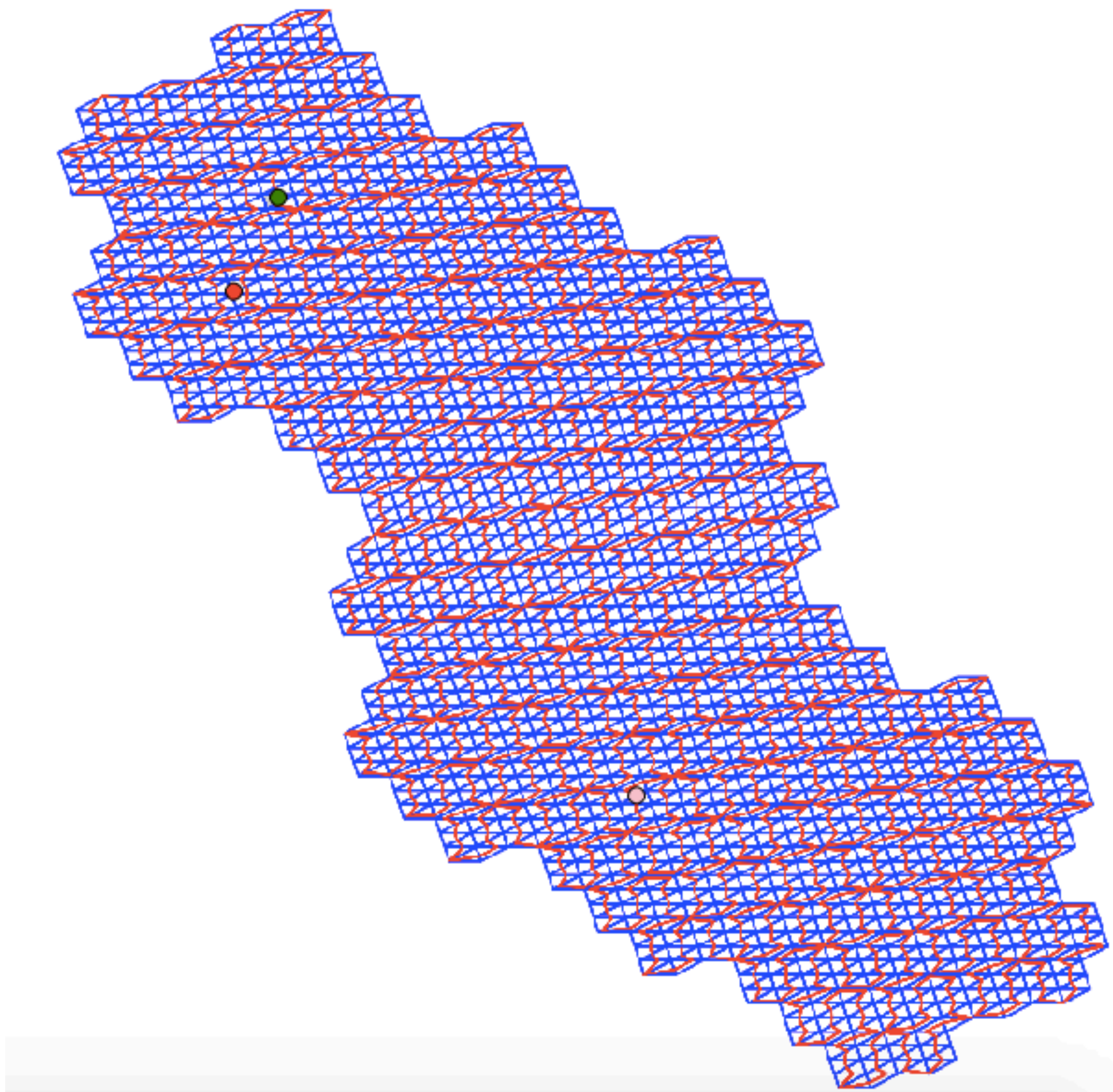
Tiling billiards PET: stack of IETs



Tiling billiards PET: stack of IETs



Visit the zoo:
Billiard trajectories on triangle tilings



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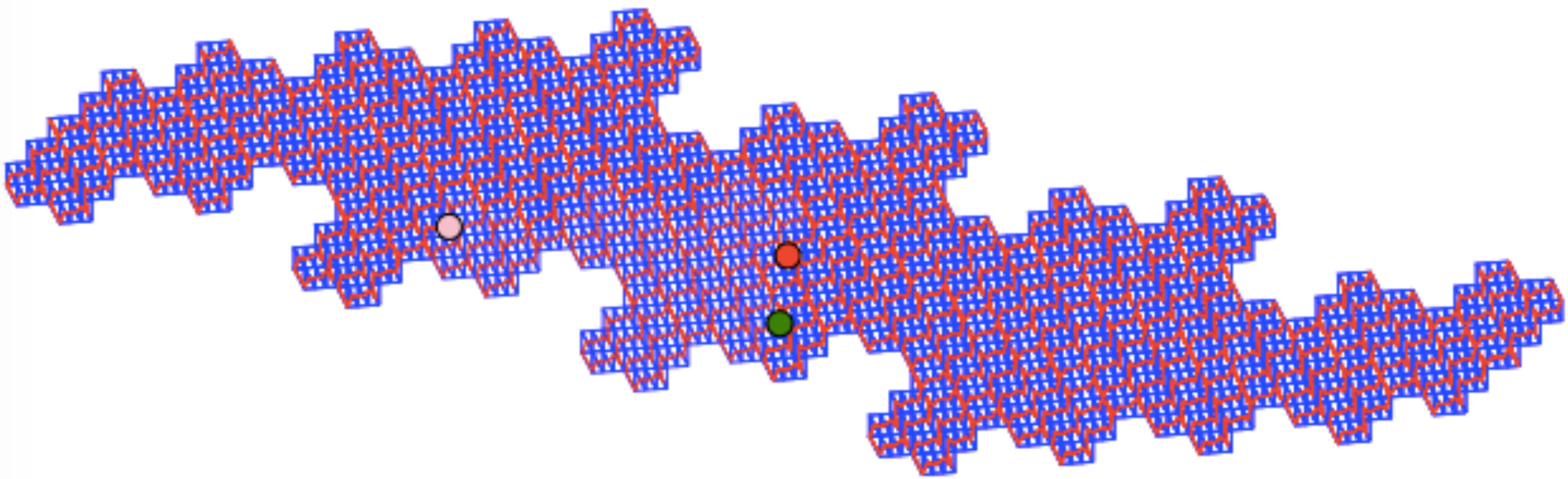

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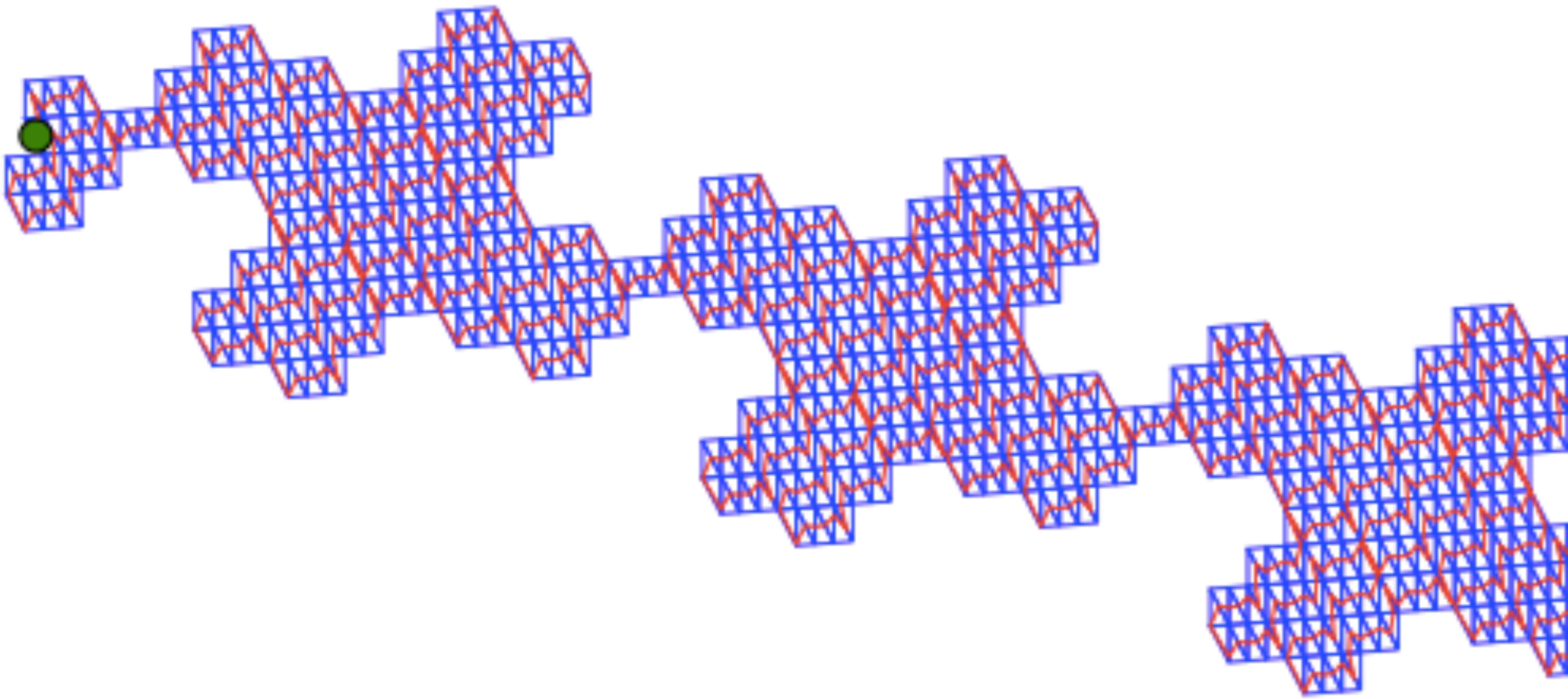
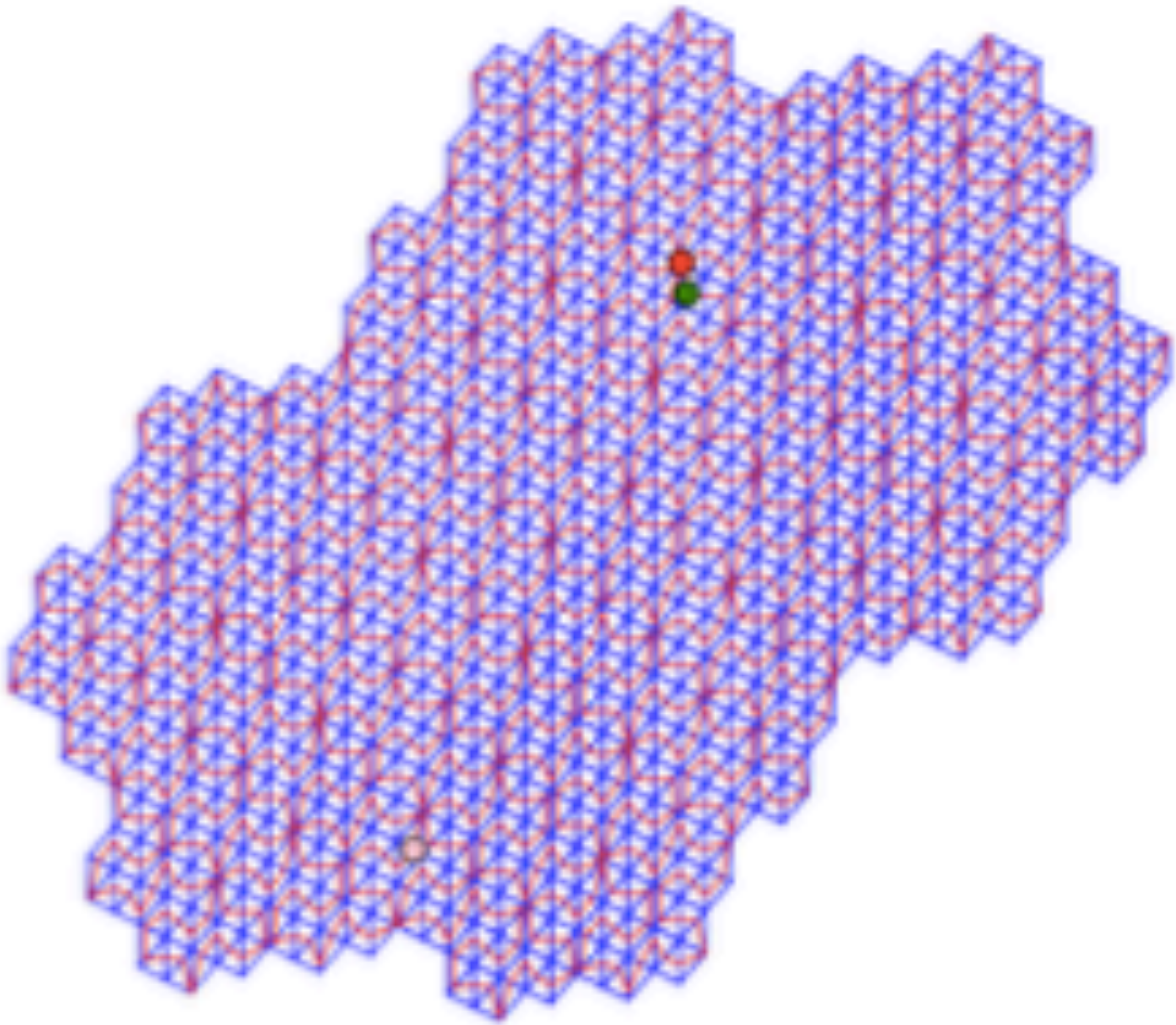
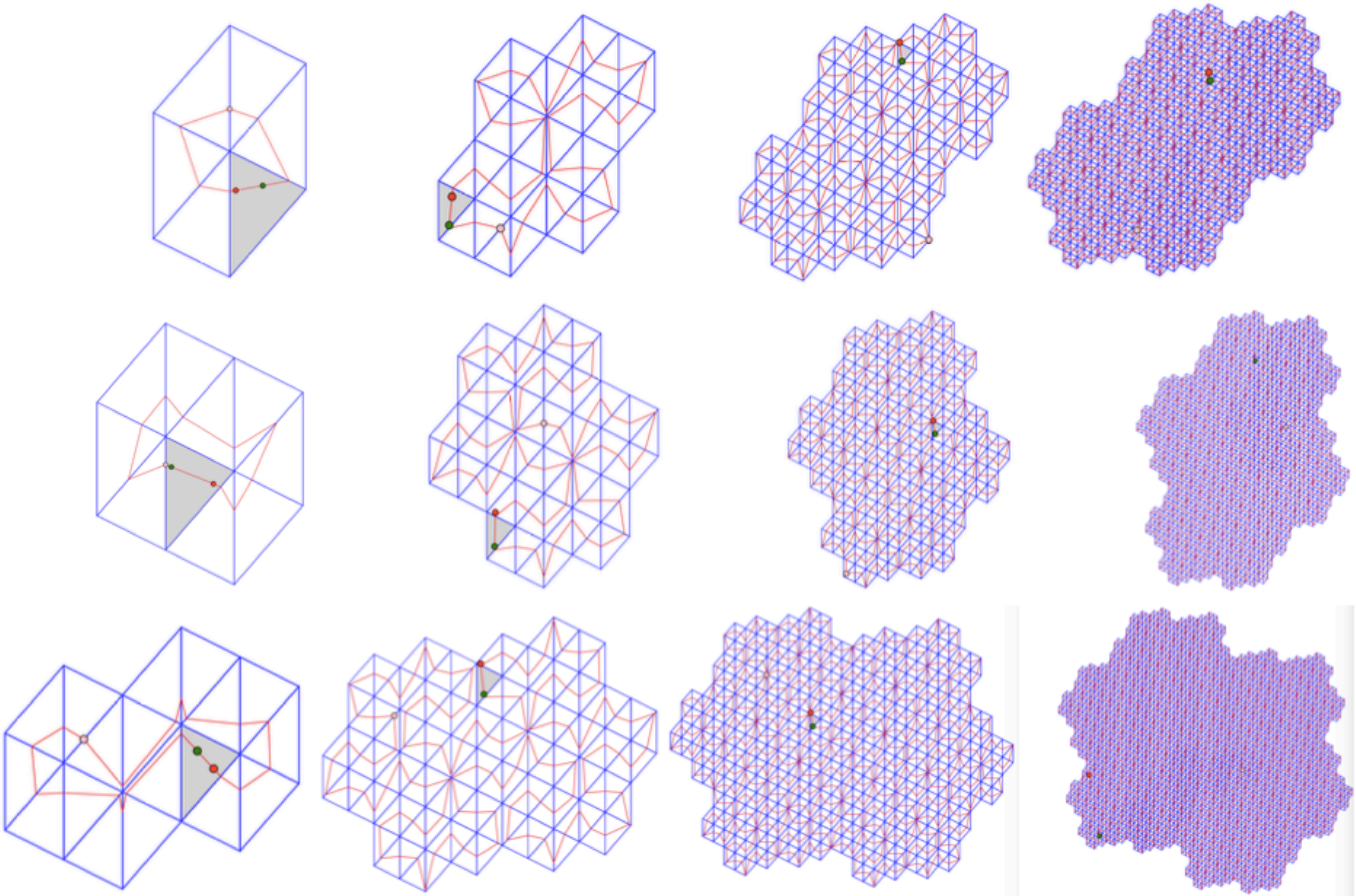


The Rauzy fractal as a billiard trajectory



The Rauzy fractal as a billiard trajectory!



Future work:

- Show that we actually get fractals as the limit of billiard trajectories
- Completely understand fully flipped IETs (service to community)
- Other tilings!