

Interval exchange transformations from tiling billiards

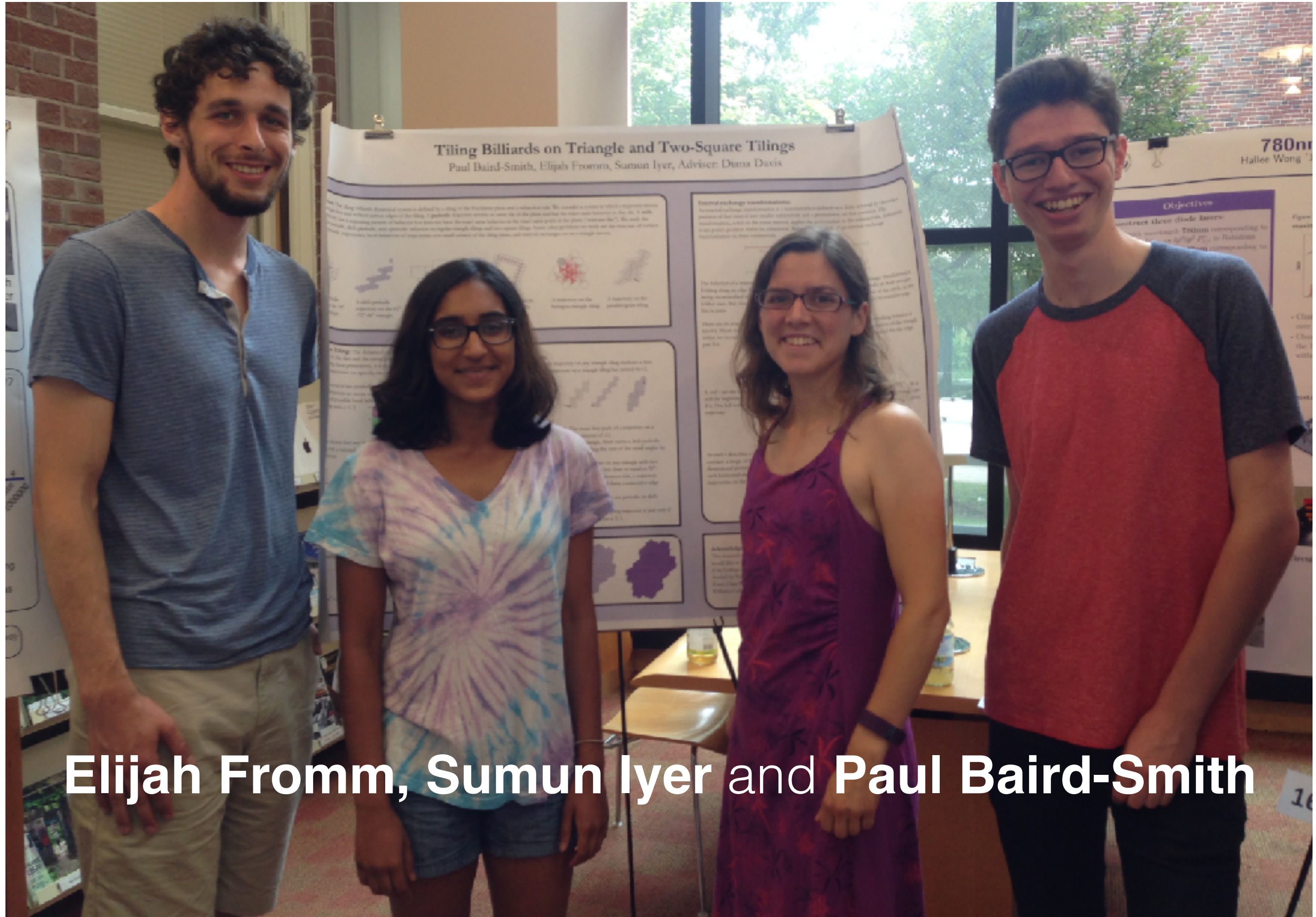
Diana Davis
Swarthmore College

13 January 2018
JMM, San Diego

Joint work with:

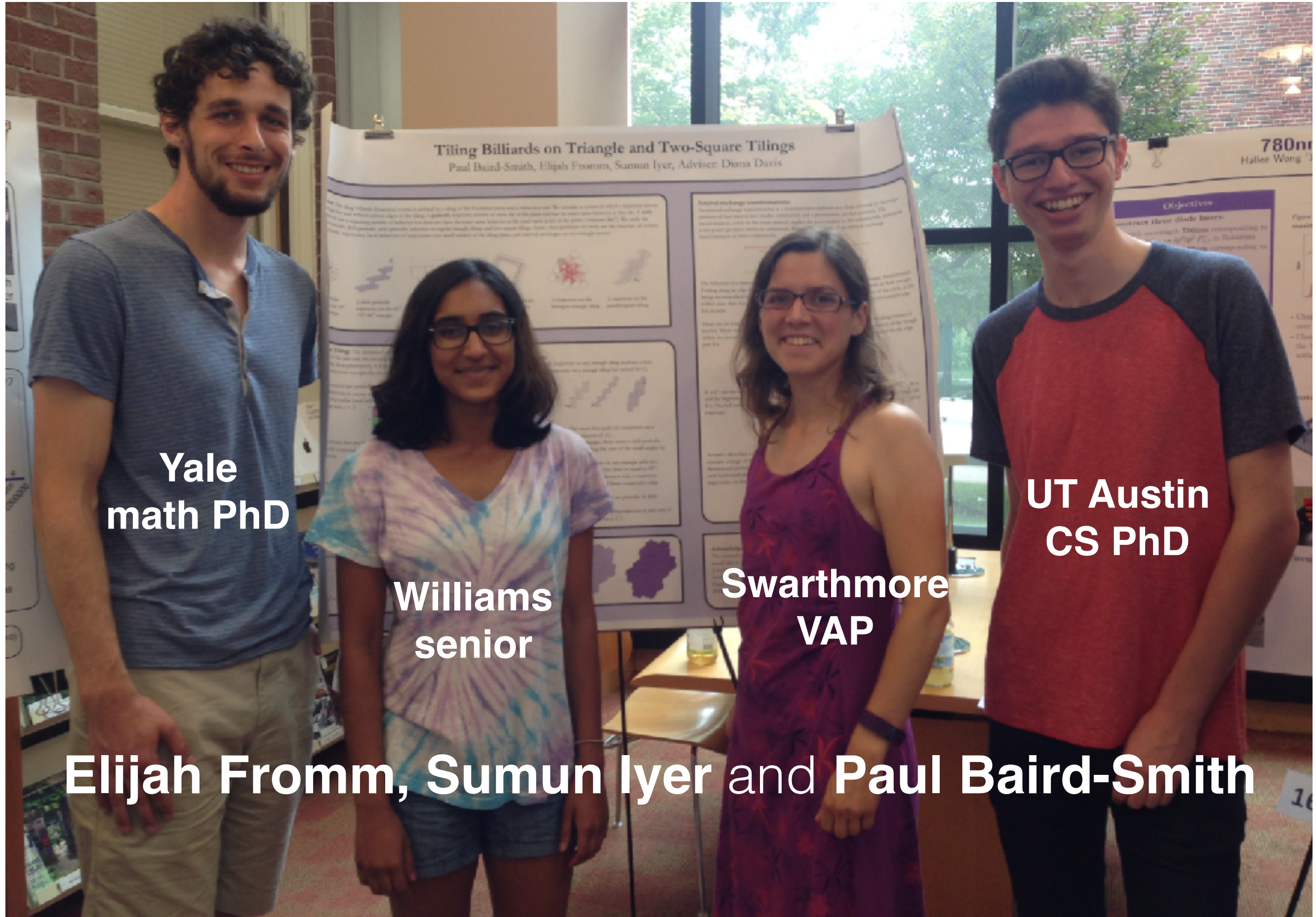
Paul Baird-Smith, Elijah Fromm and Sumun Iyer
SMALL 2016

SMALL 2016 - Billiards group



Elijah Fromm, Sumun Iyer and Paul Baird-Smith

SMALL 2016 - Billiards group



**Yale
math PhD**

**Williams
senior**

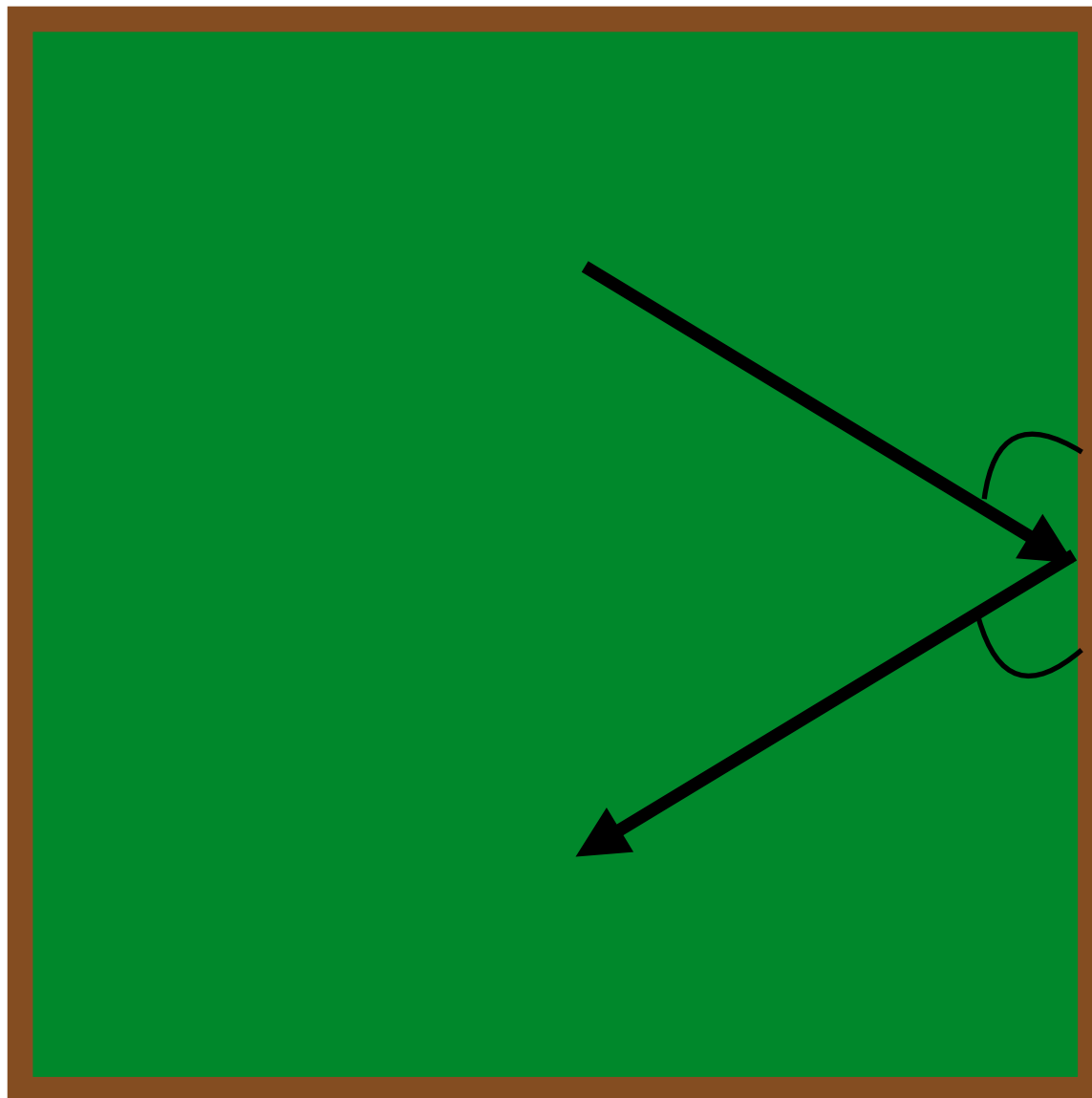
**Swarthmore
VAP**

**UT Austin
CS PhD**

Elijah Fromm, Sumun Iyer and Paul Baird-Smith

Standard (Inner) Billiards:

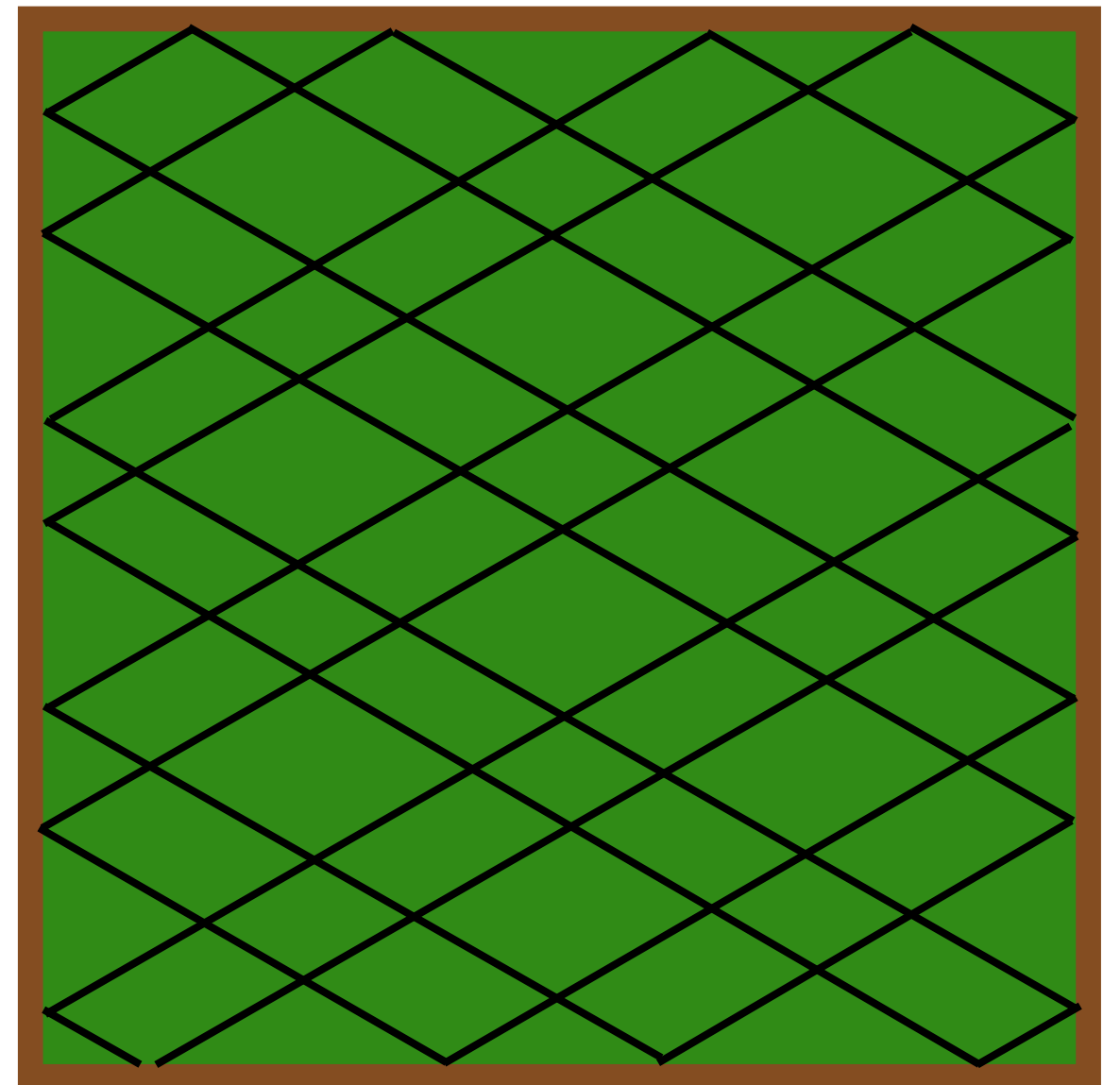
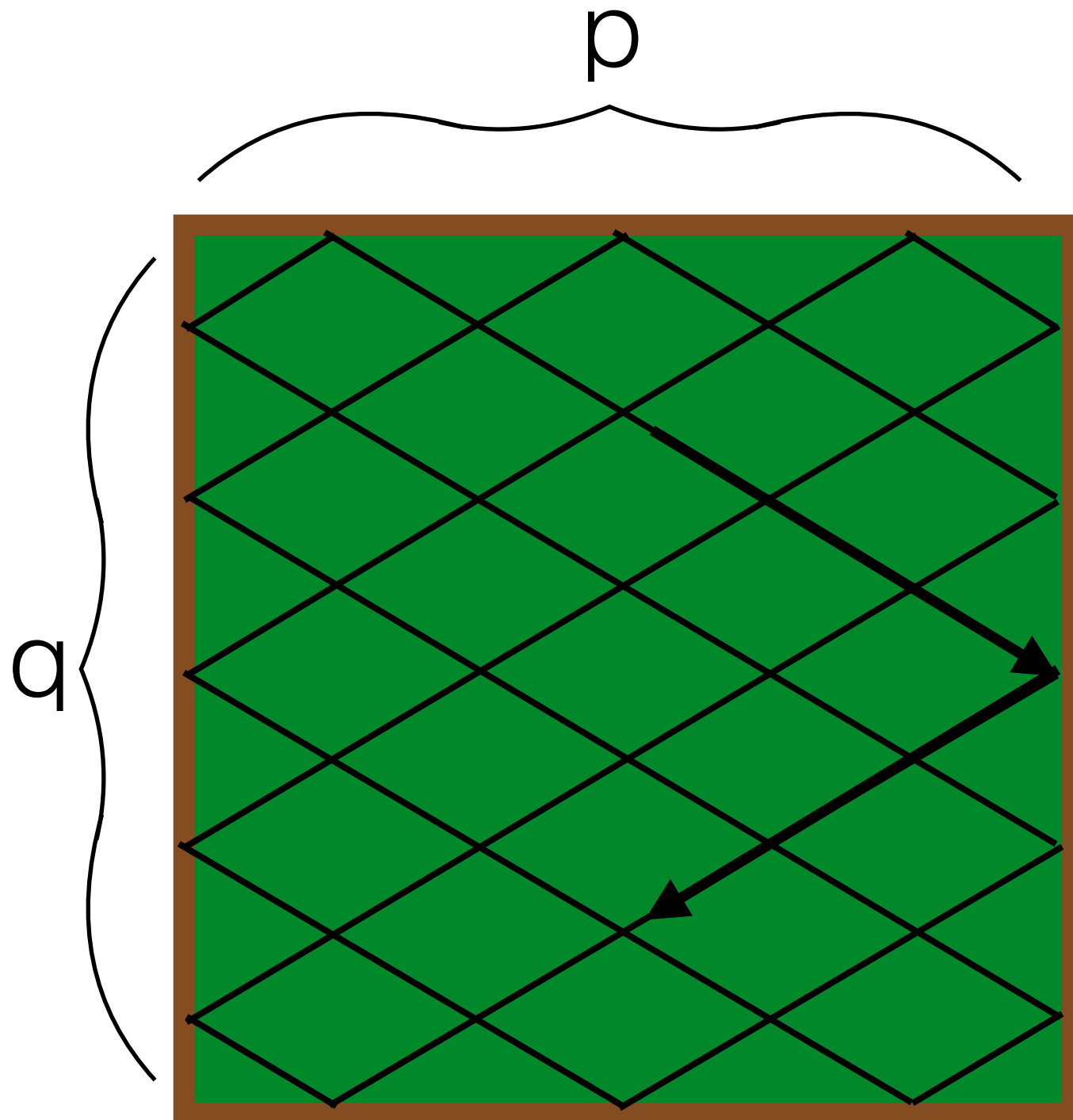
trajectory is reflected
against the edge of the
table, preserving angles



In the square billiard table, a trajectory with:

rational slope p/q
is periodic, with
period $2(p+q)$

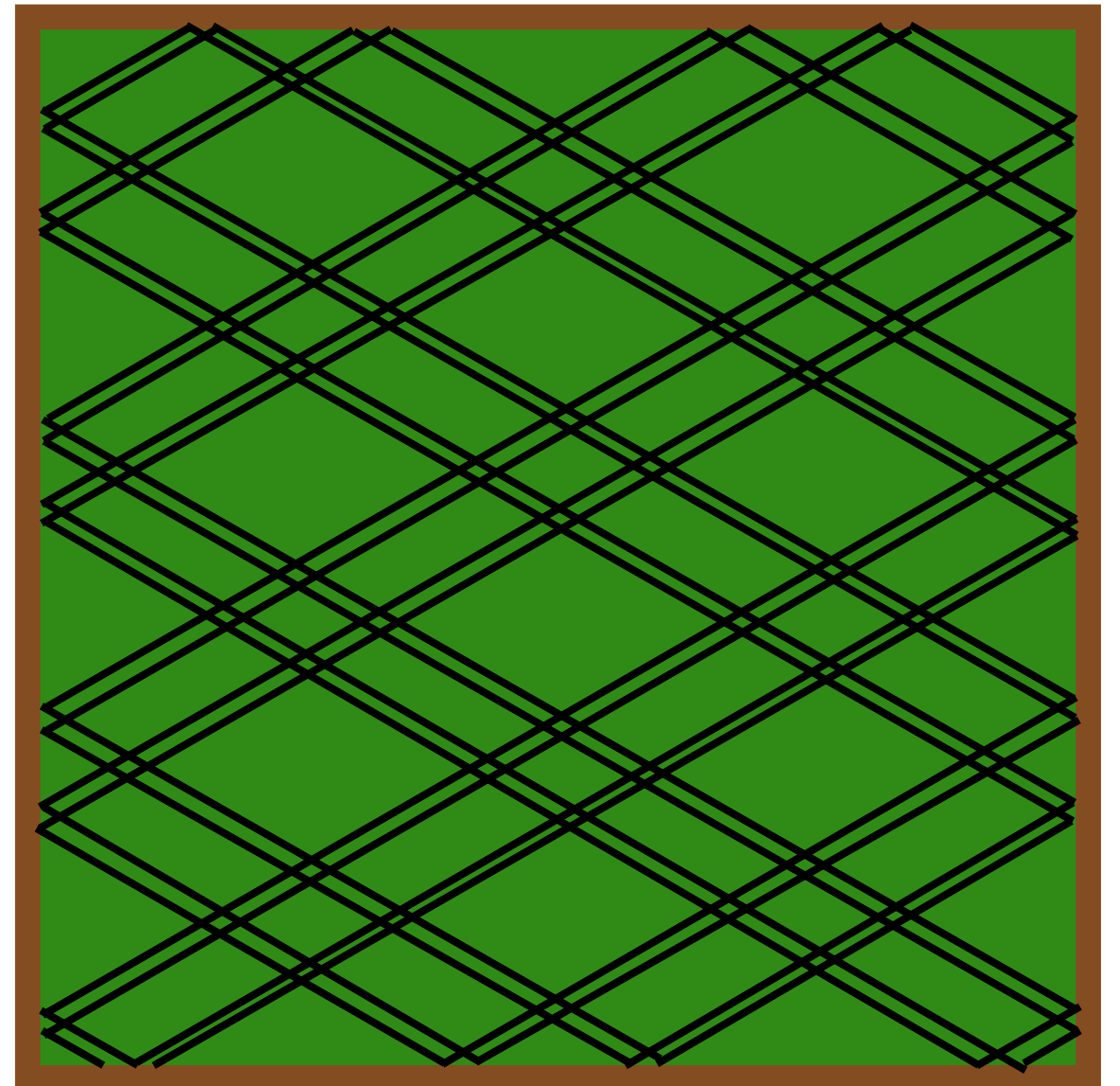
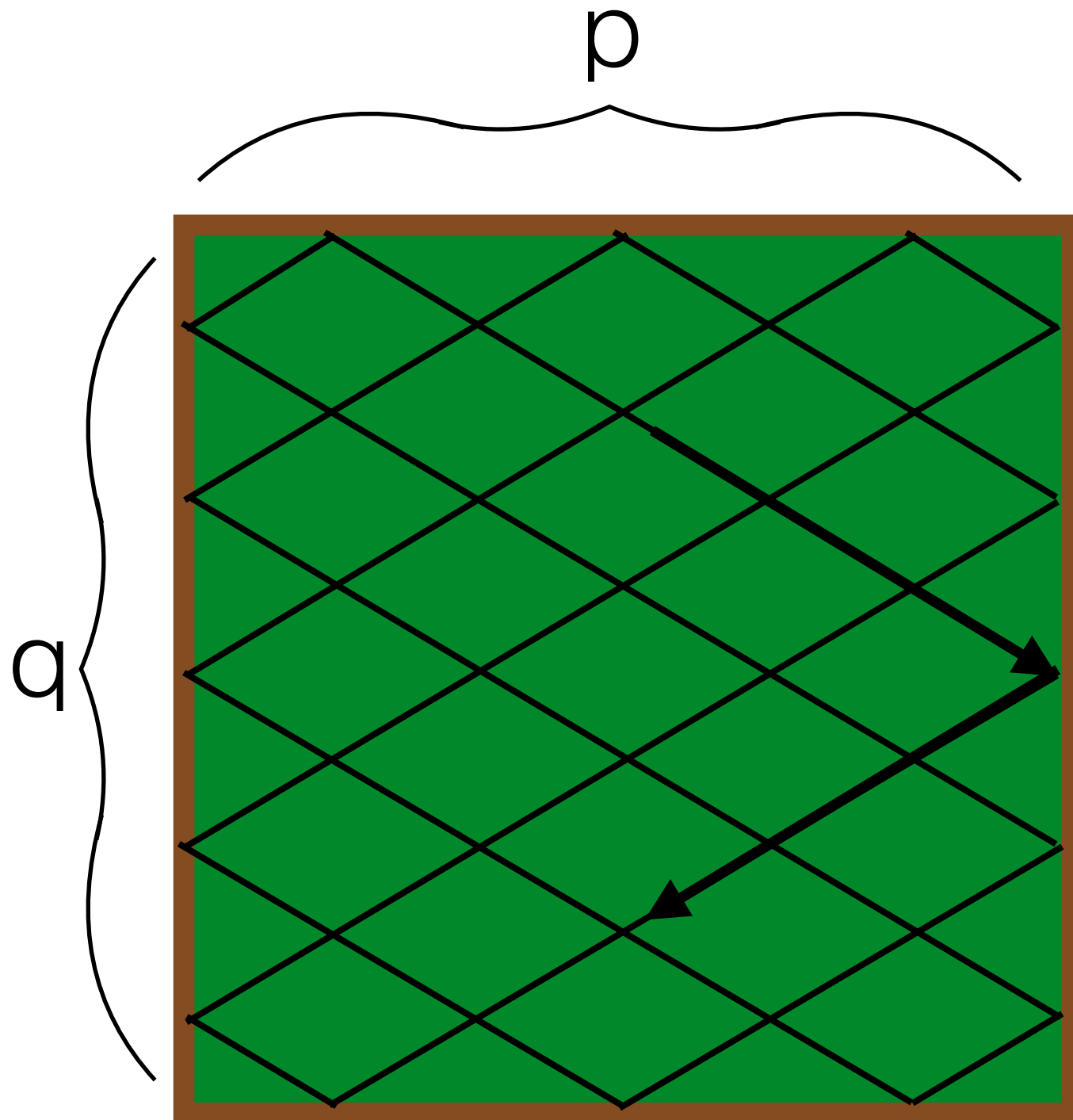
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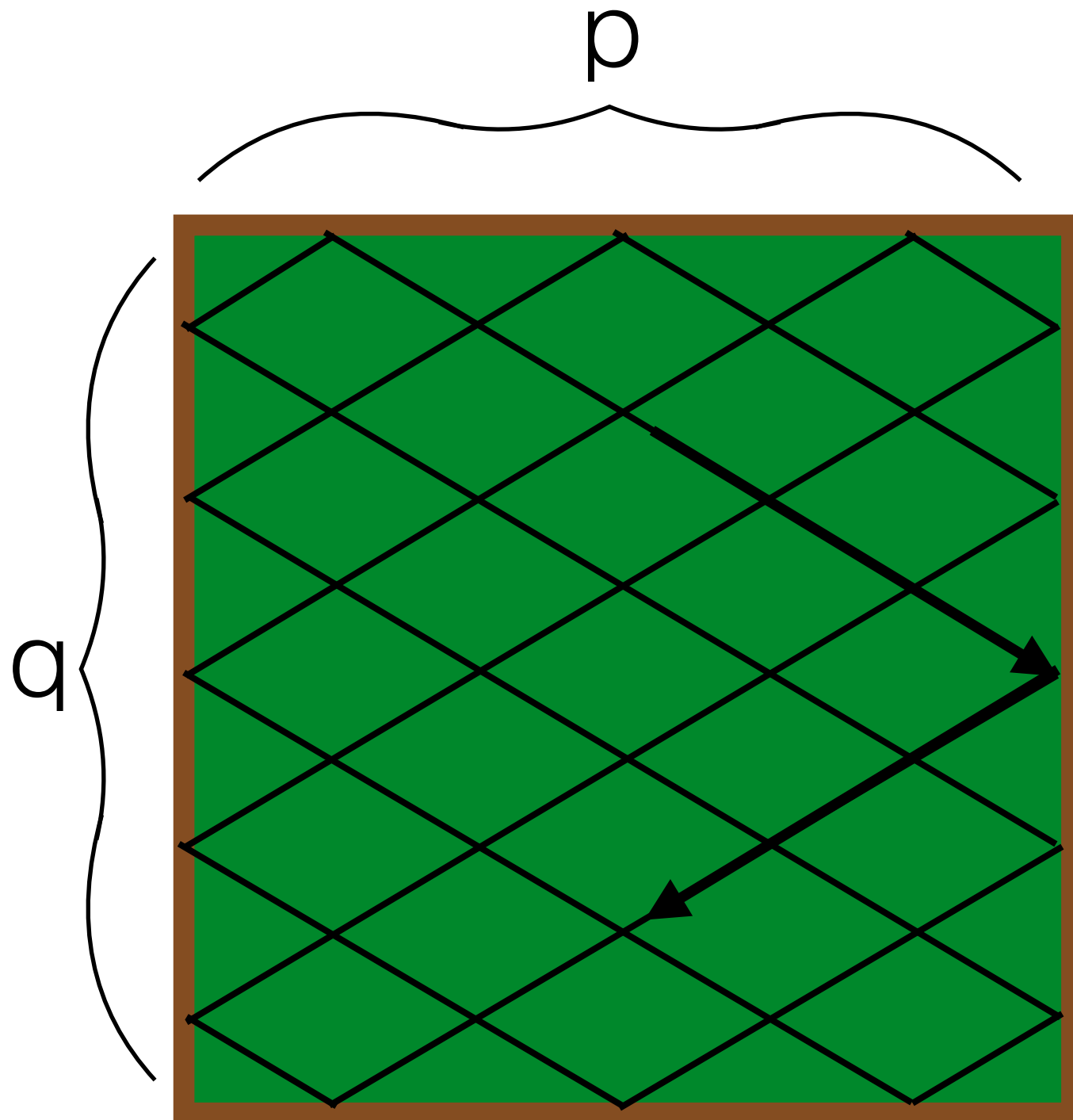
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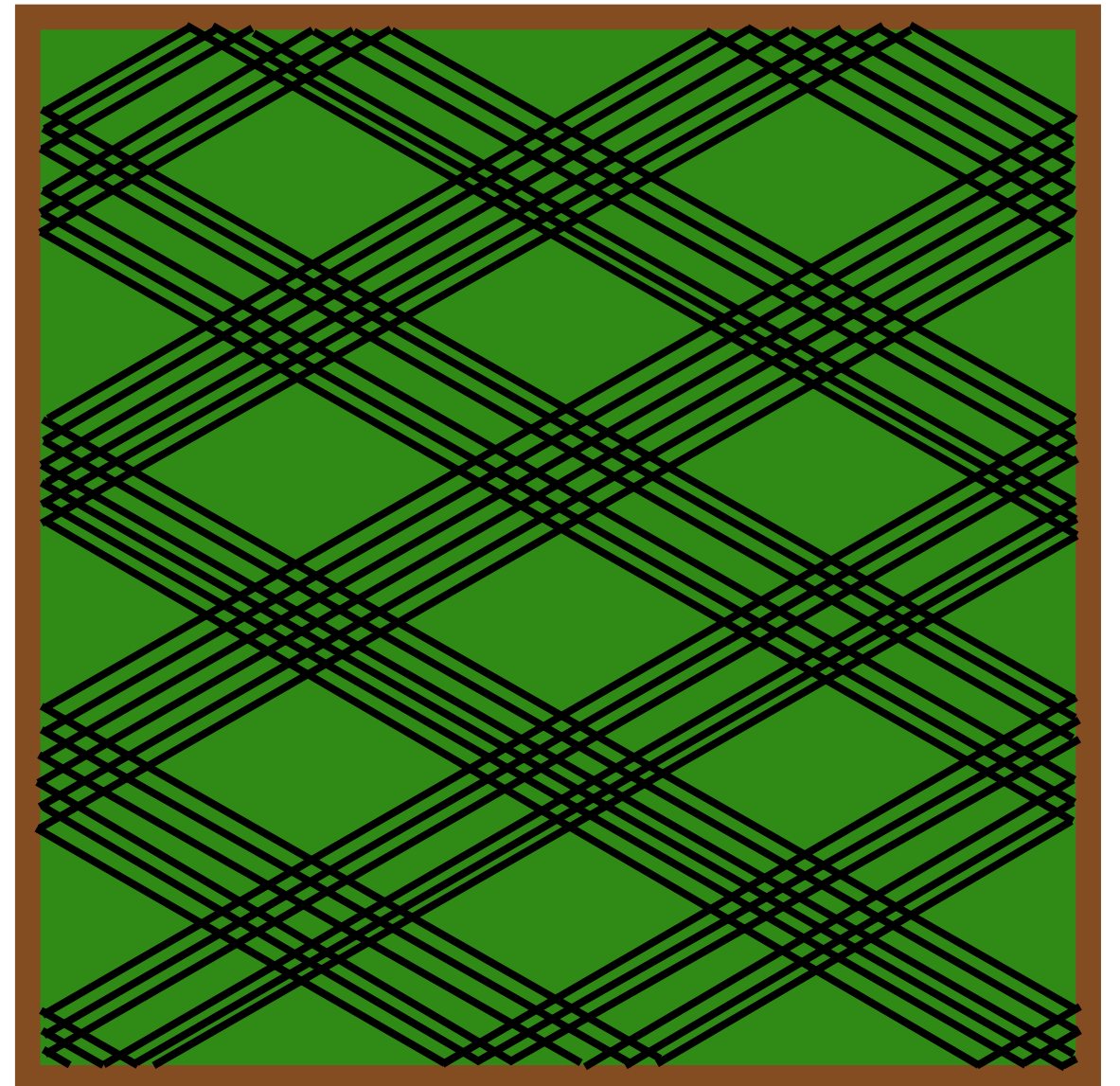


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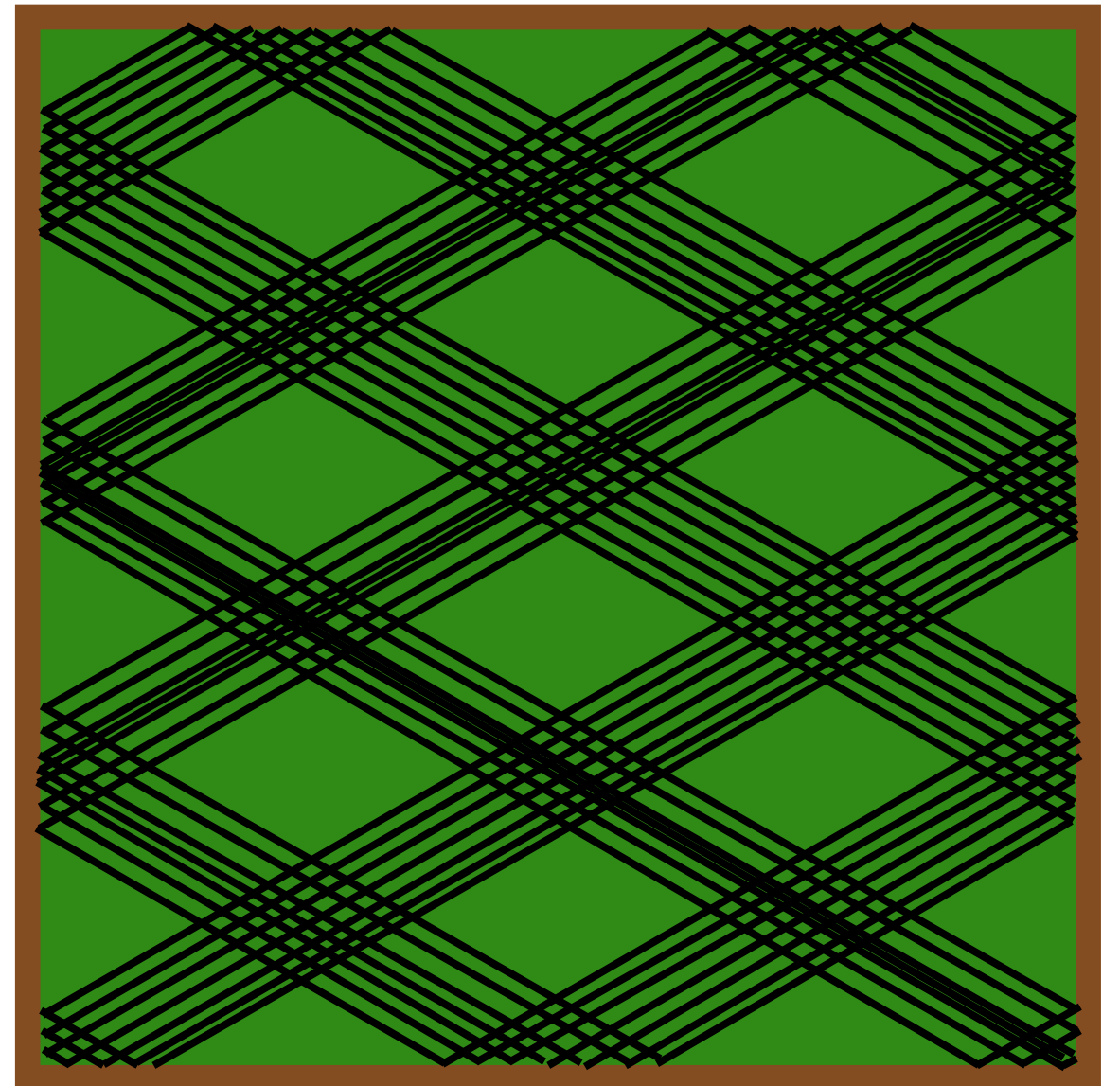
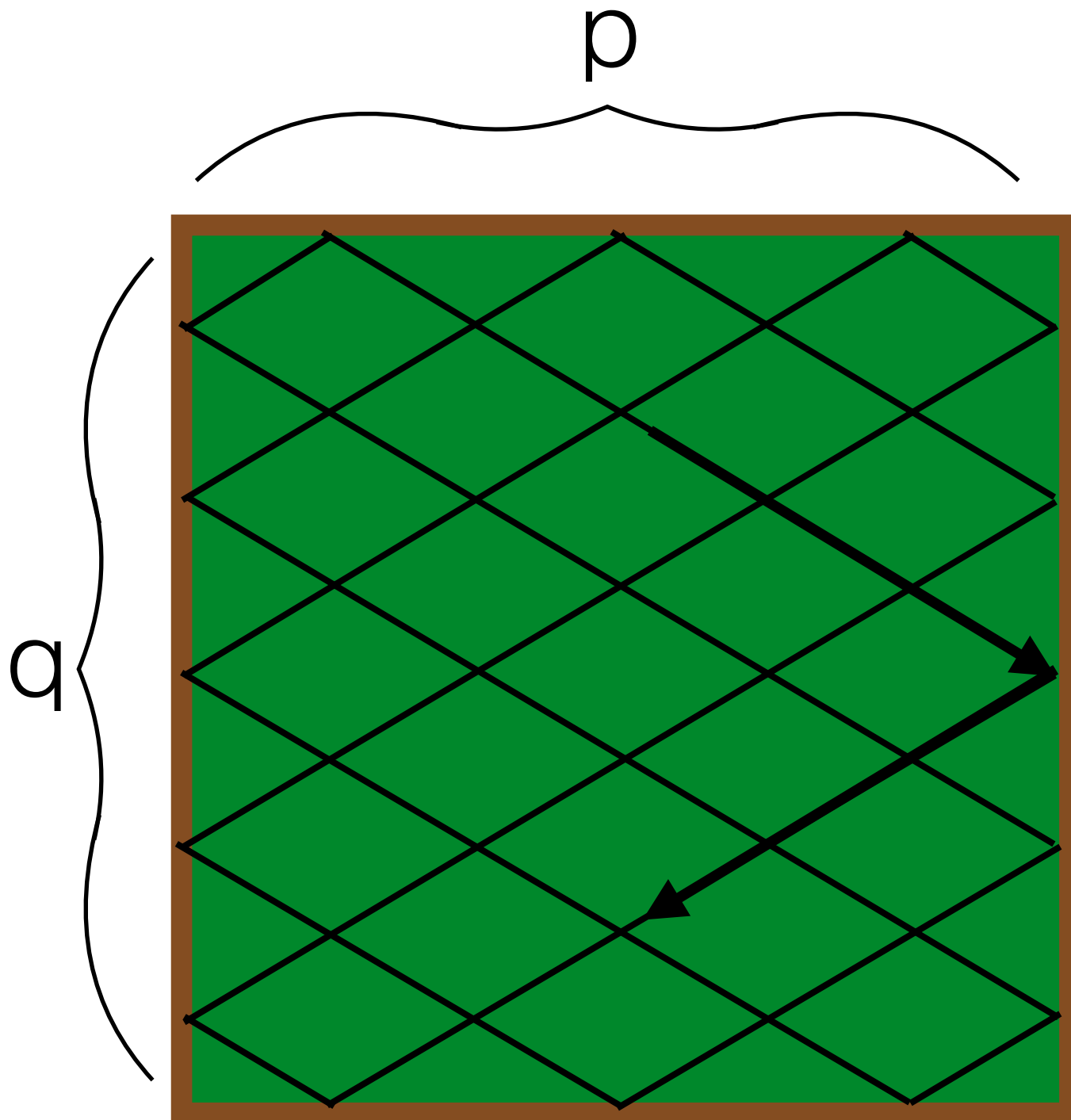
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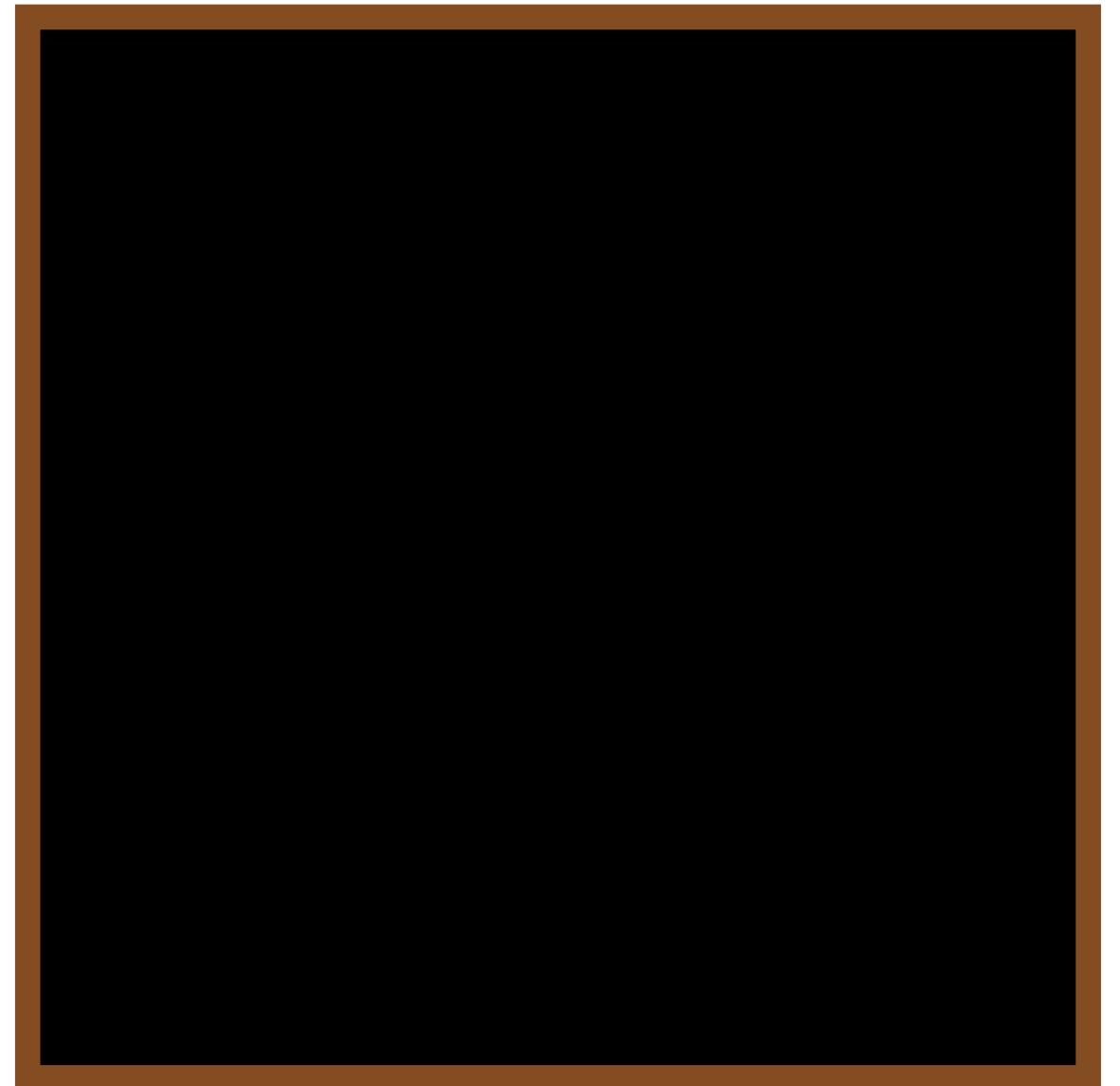
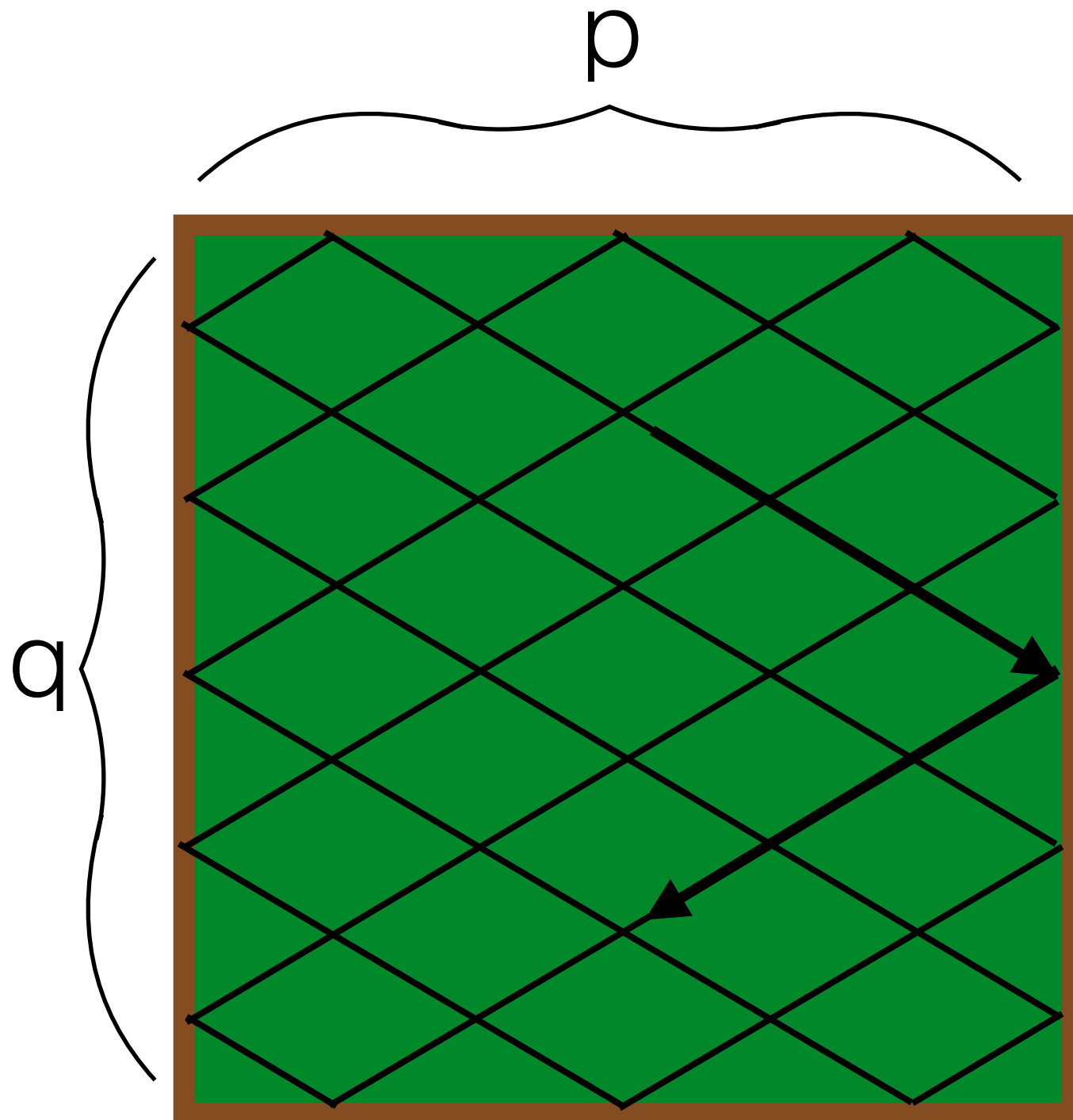
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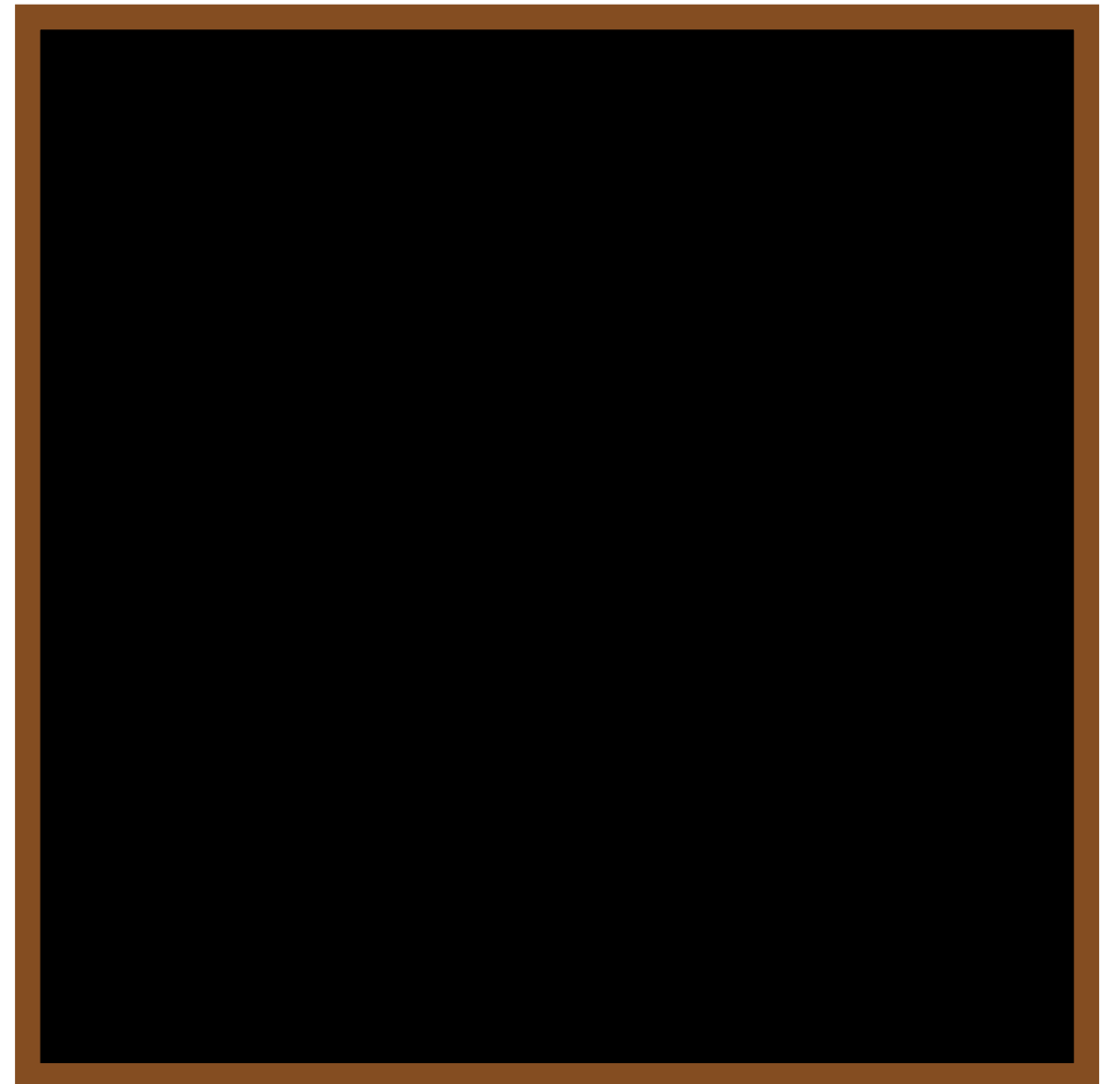
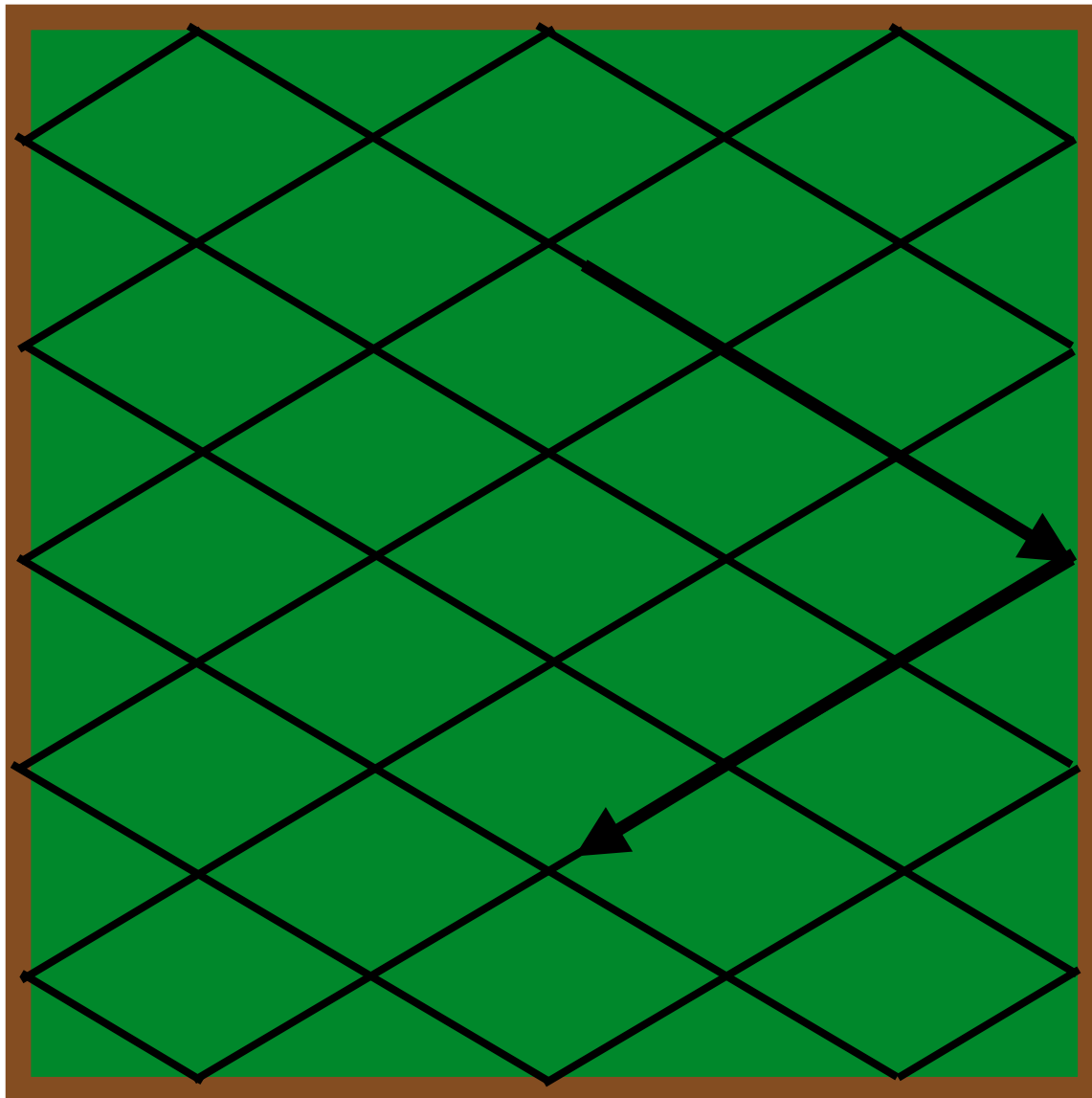
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Standard (Inner) Billiards:

- A trajectory is **dense** with probability 1
- Behavior is **unstable**



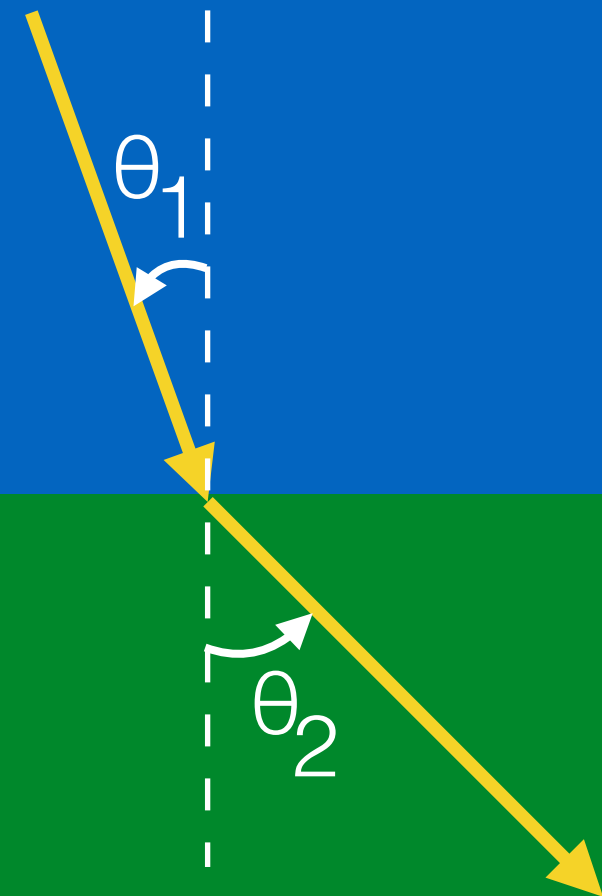
Snell's Law:

governs the refraction of a beam of light passing from one material to another

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{k_1}{k_2}$$

standard material,
positive index of
refraction k_1

standard material,
positive index of
refraction k_2



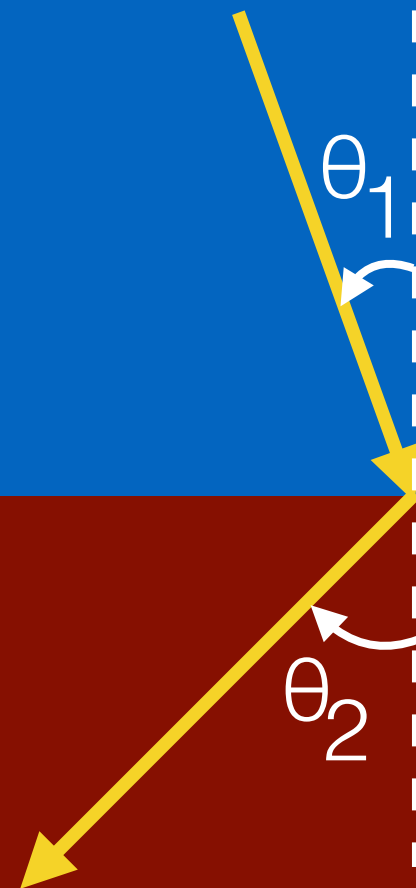
Snell's Law:

governs the refraction of a beam of light passing from one material to another

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{k_1}{k_2}$$

standard material,
positive index of
refraction k_1

metamaterial,
negative index of
refraction k_2

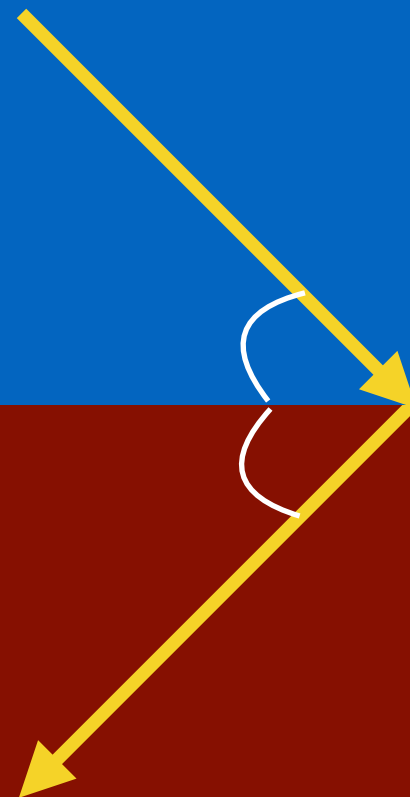


Snell's Law:

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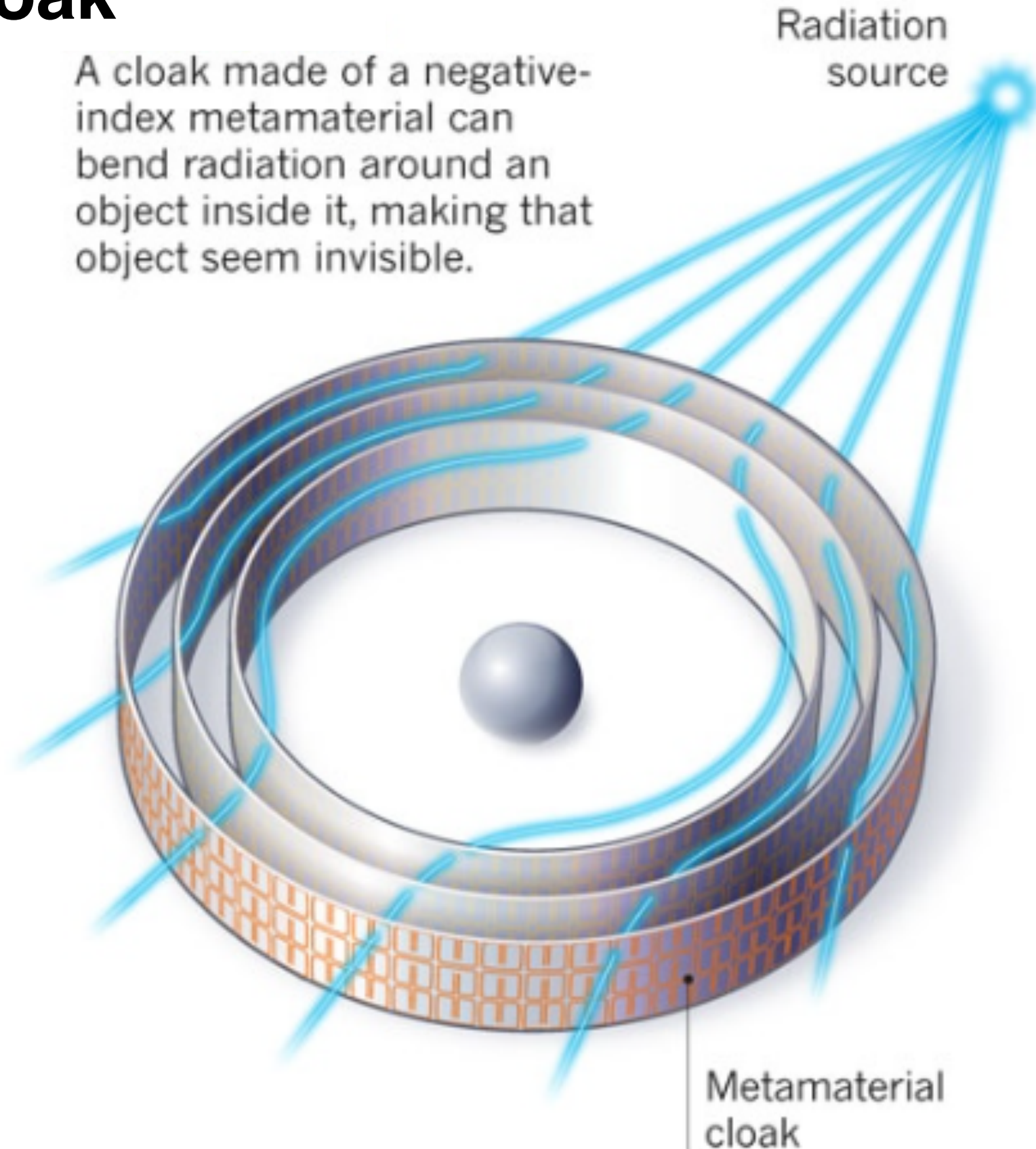
materials with
equal and opposite
indices of
refraction



Applications:

1. Invisibility cloak
2. Perfect lens

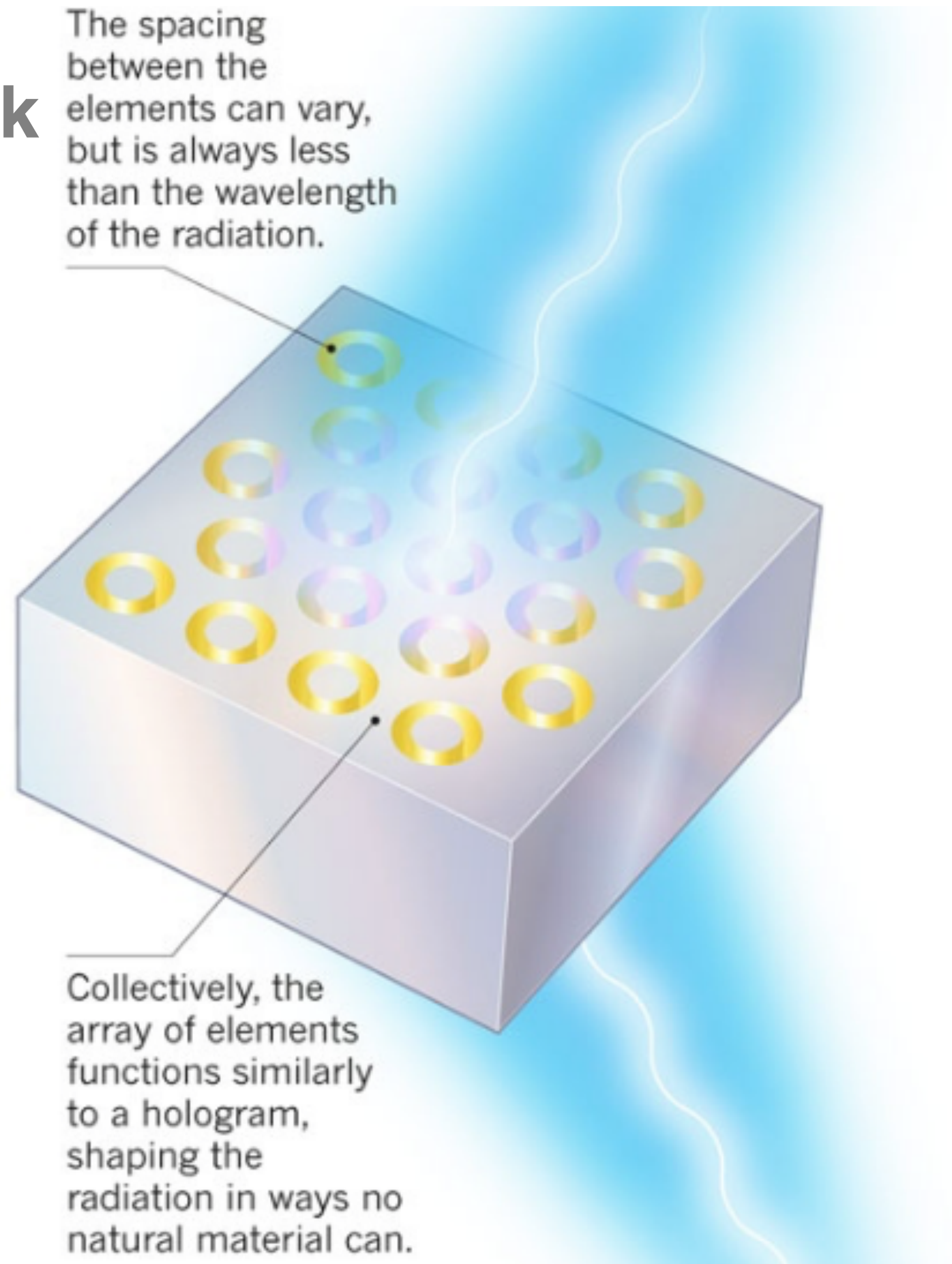
A cloak made of a negative-index metamaterial can bend radiation around an object inside it, making that object seem invisible.



Applications:

1. Invisibility cloak

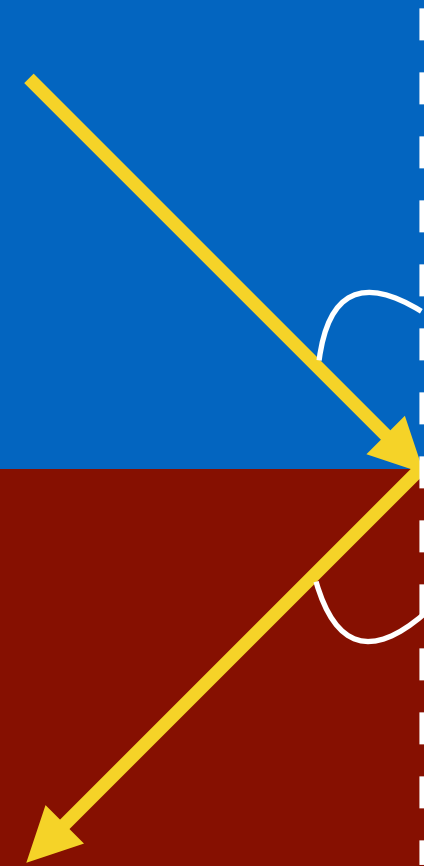
2. Perfect lens



Tiling Billiards:

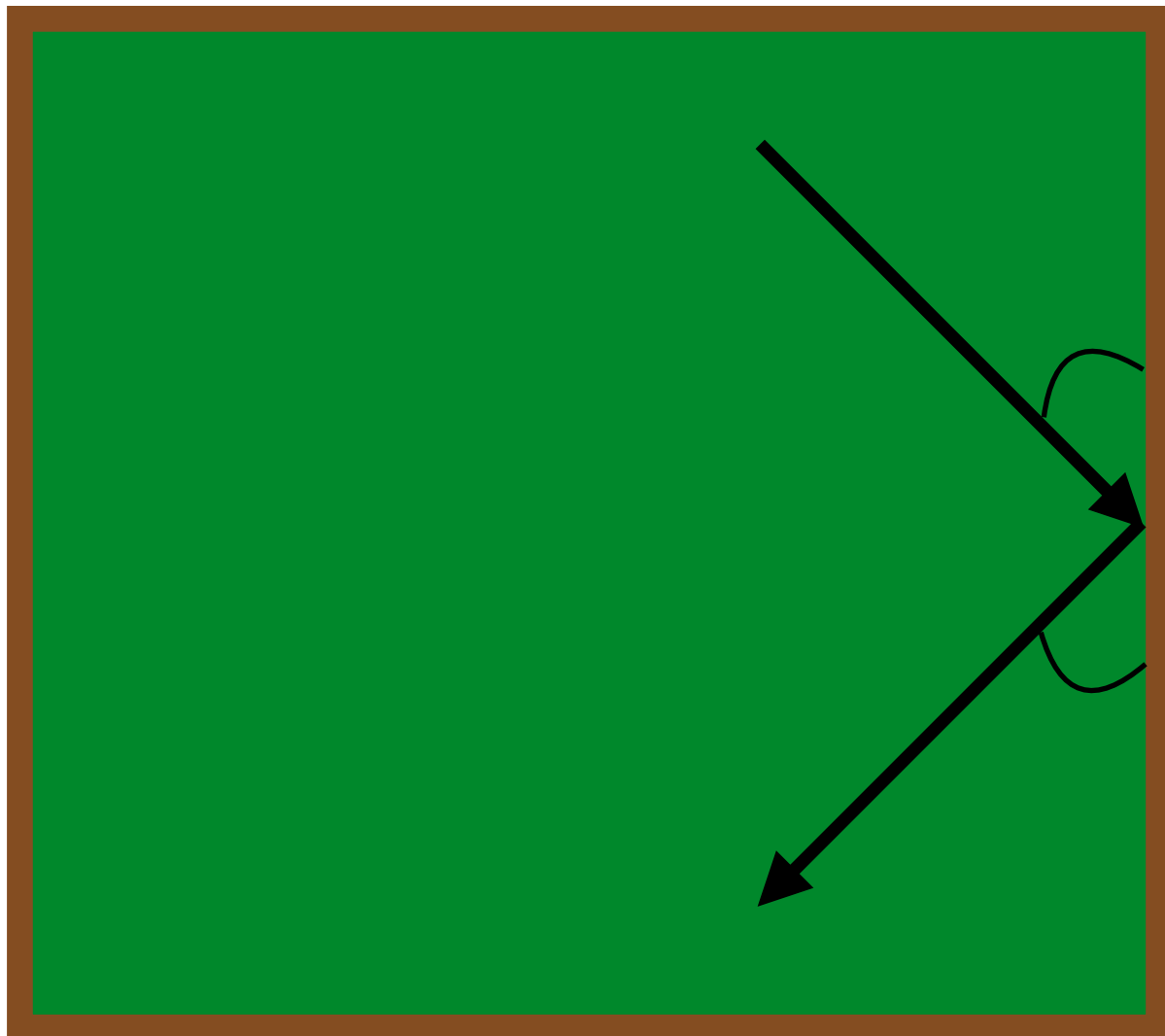
A dynamical system where light refracts through a planar tiling by materials with equal and opposite alternating indices of refraction

materials with
equal and opposite
indices of
refraction



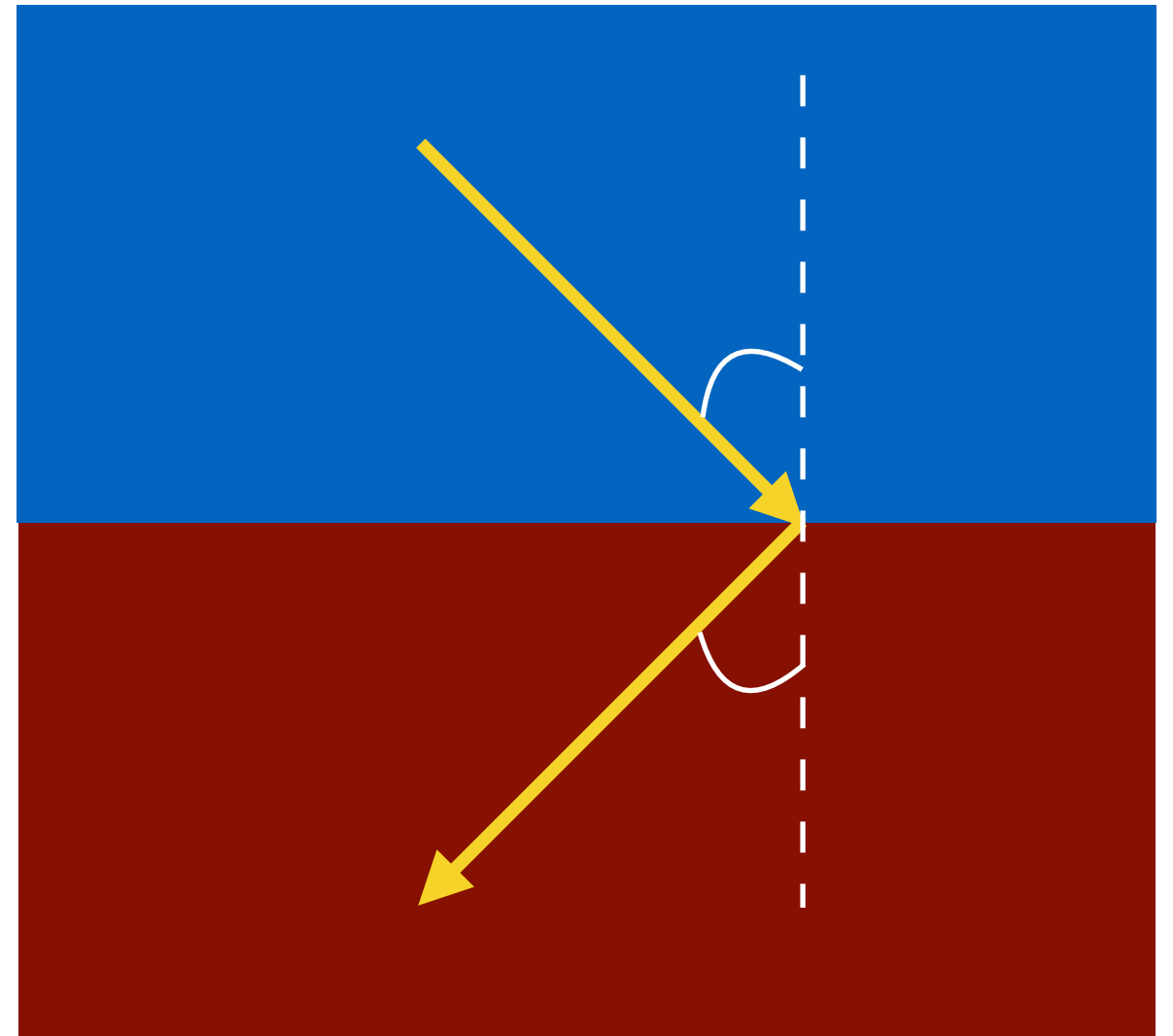
Standard Billiards:

trajectory is reflected
against the edge of the table,
preserving angles



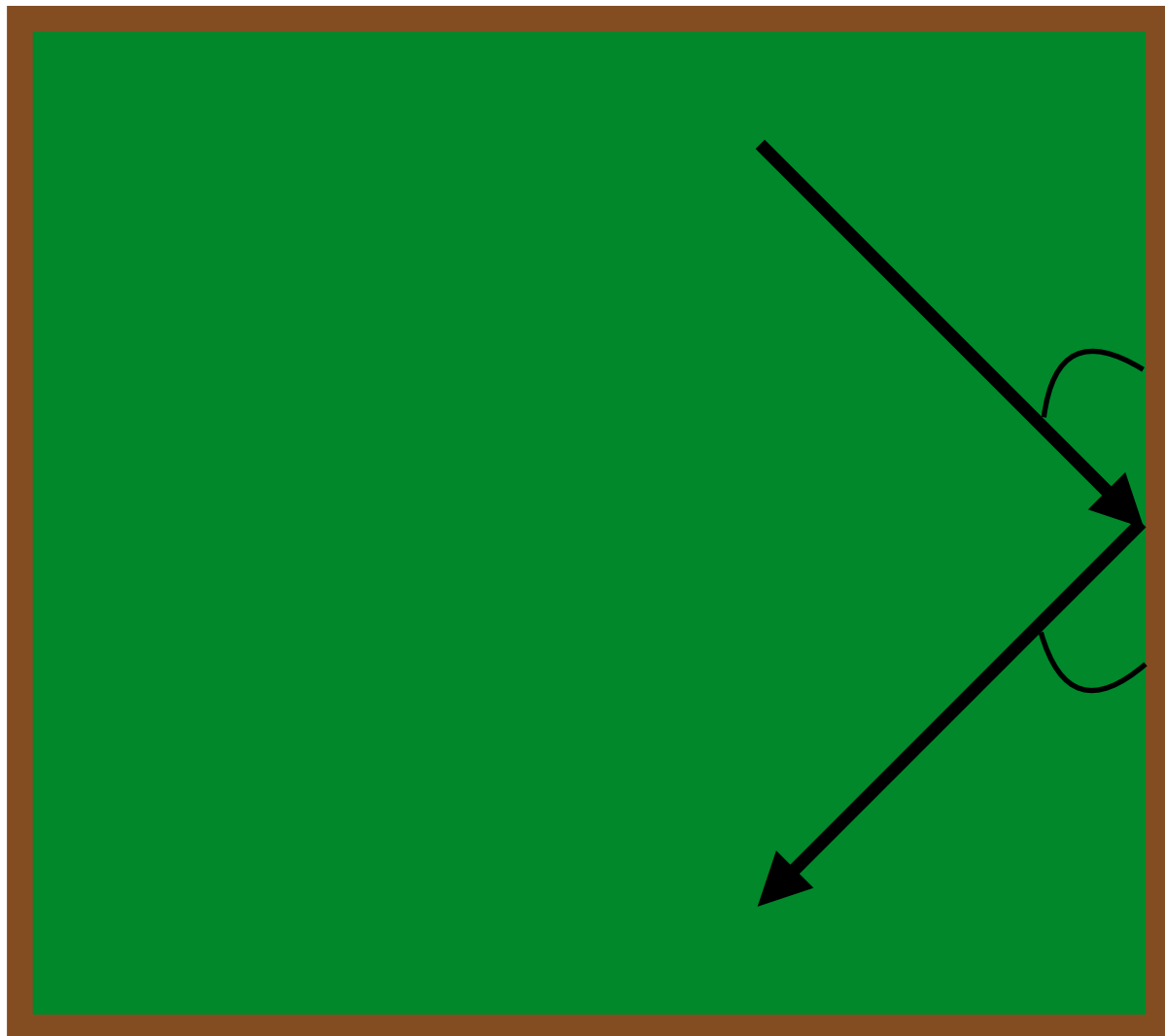
Tiling Billiards:

trajectory is reflected
across each edge of the
tiling, preserving angles



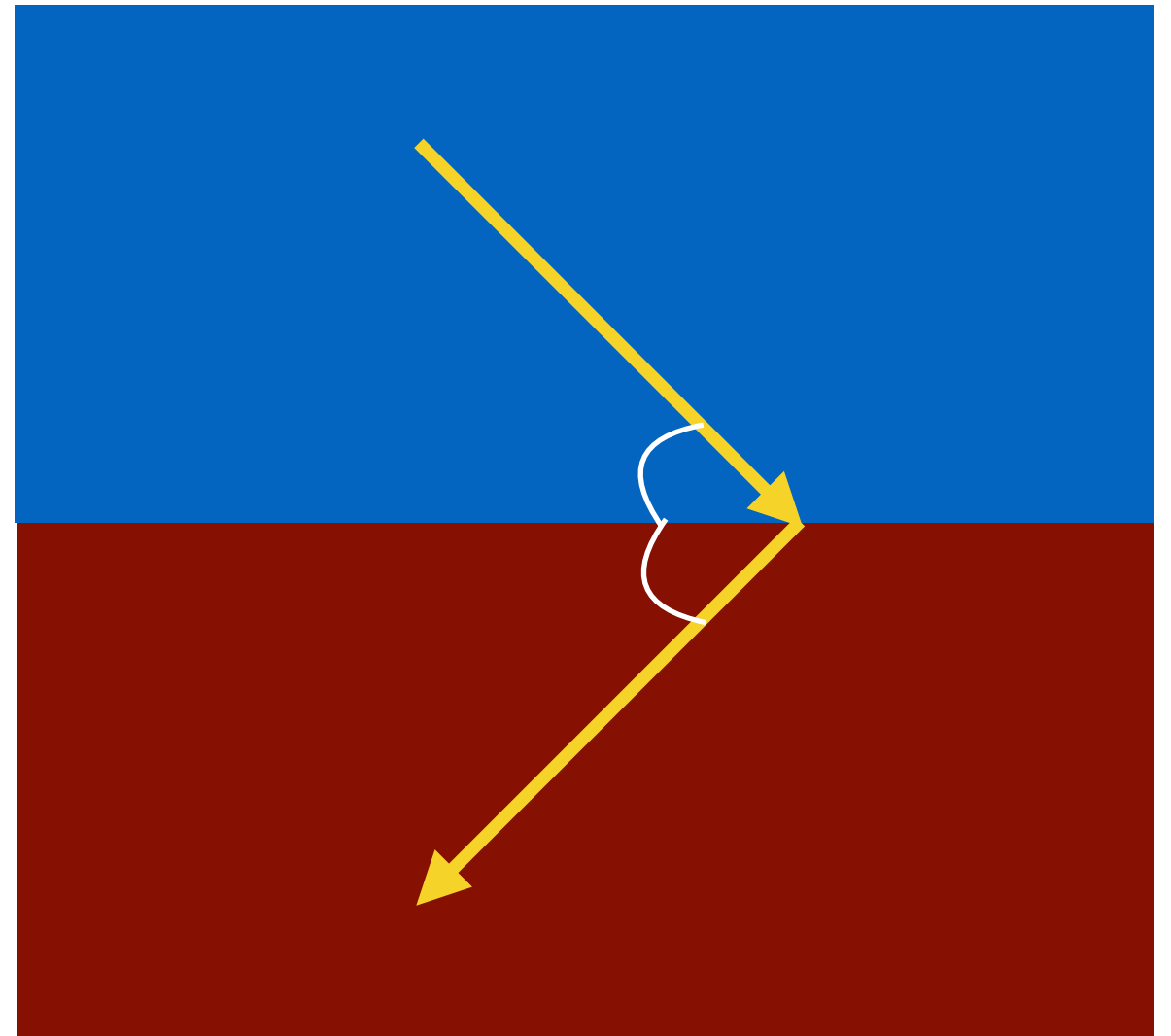
Standard Billiards:

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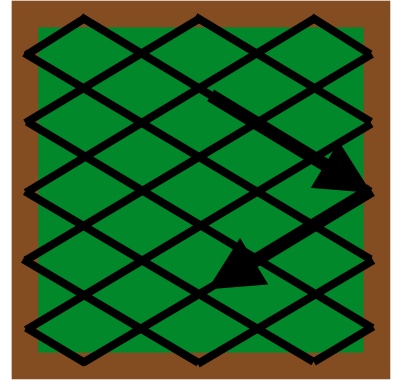
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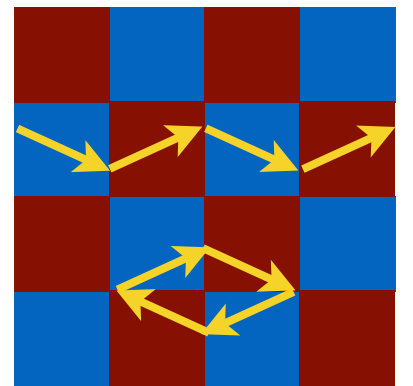
Standard (Inner) Billiards:

- A trajectory is **dense** with probability 1
- Behavior is **unstable**



Tiling Billiards on triangle tilings:

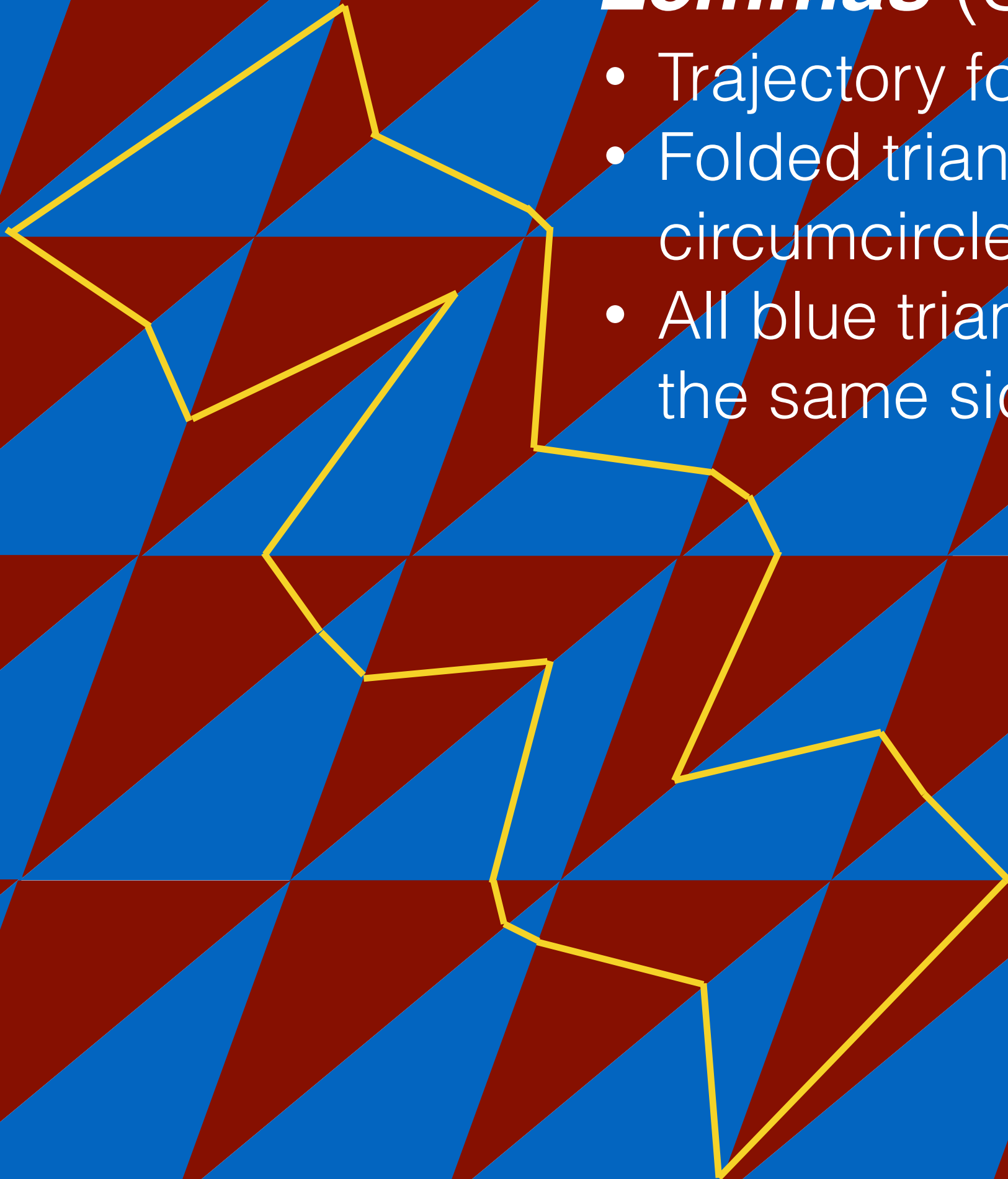
- Most trajectories are **periodic**
- Trajectories are very **stable**



Burning question: What causes periodicity and stability in tiling billiards on triangle tilings?

Lemmas (SMALL '16):

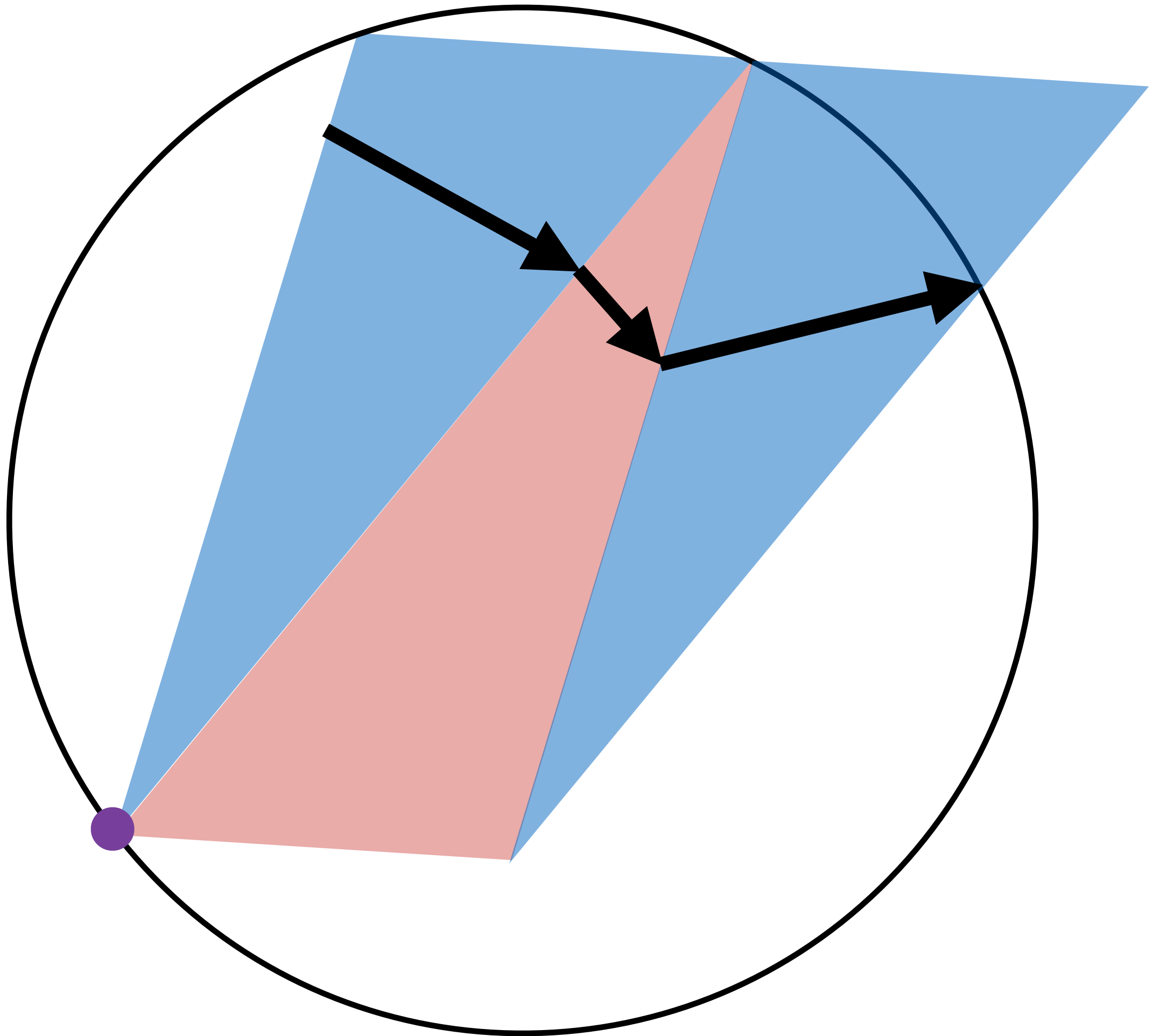
- Trajectory folds to a single line
- Folded triangles share a circumcircle
- All blue triangles end up on the same side

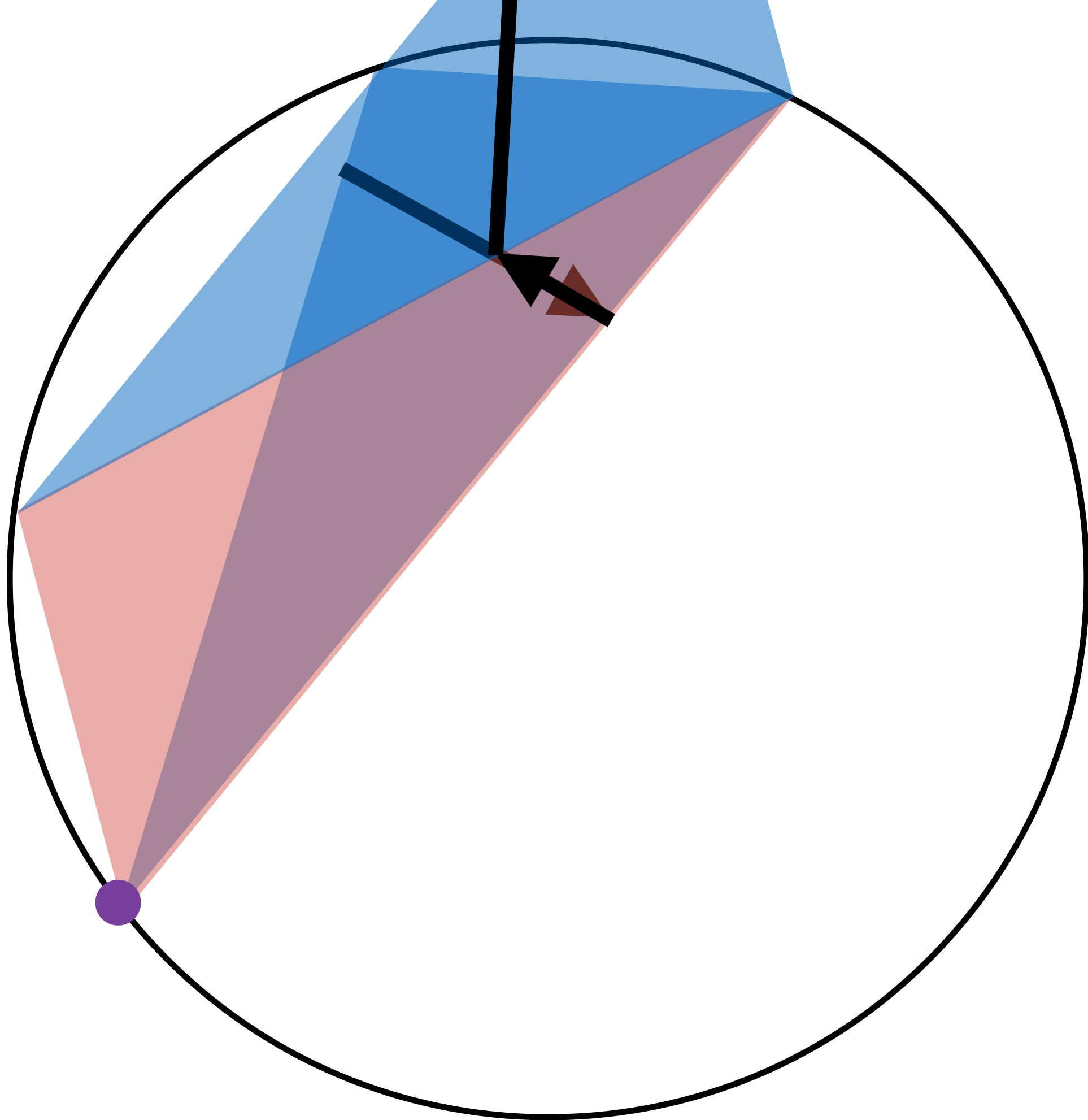


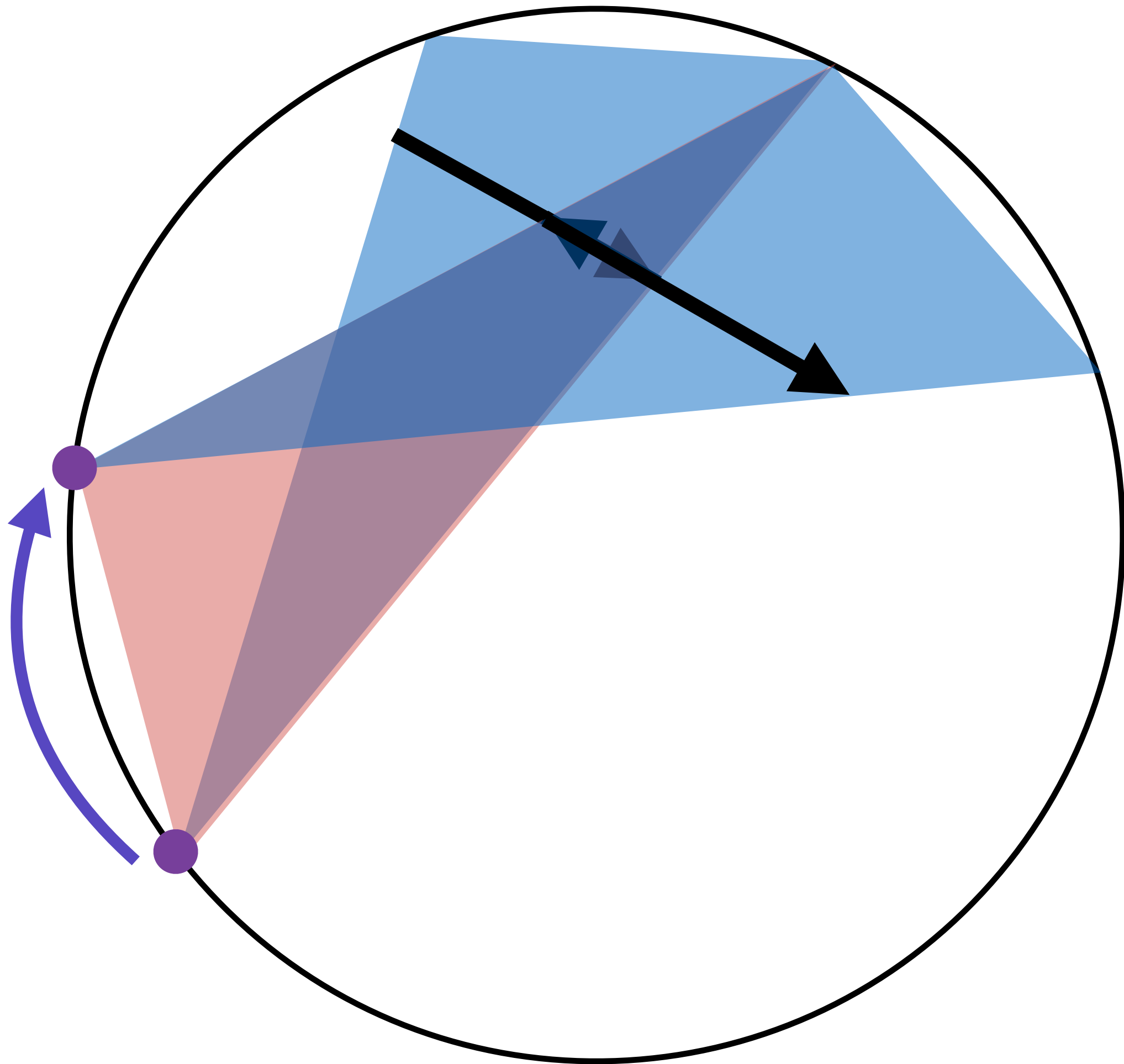
A yellow line trajectory starts in the upper left and moves through a grid of triangles, which are colored in a repeating pattern of blue and dark red. The trajectory consists of several connected line segments that change direction at various points, illustrating a path through the triangular lattice. The text is located in the upper right quadrant of the image.

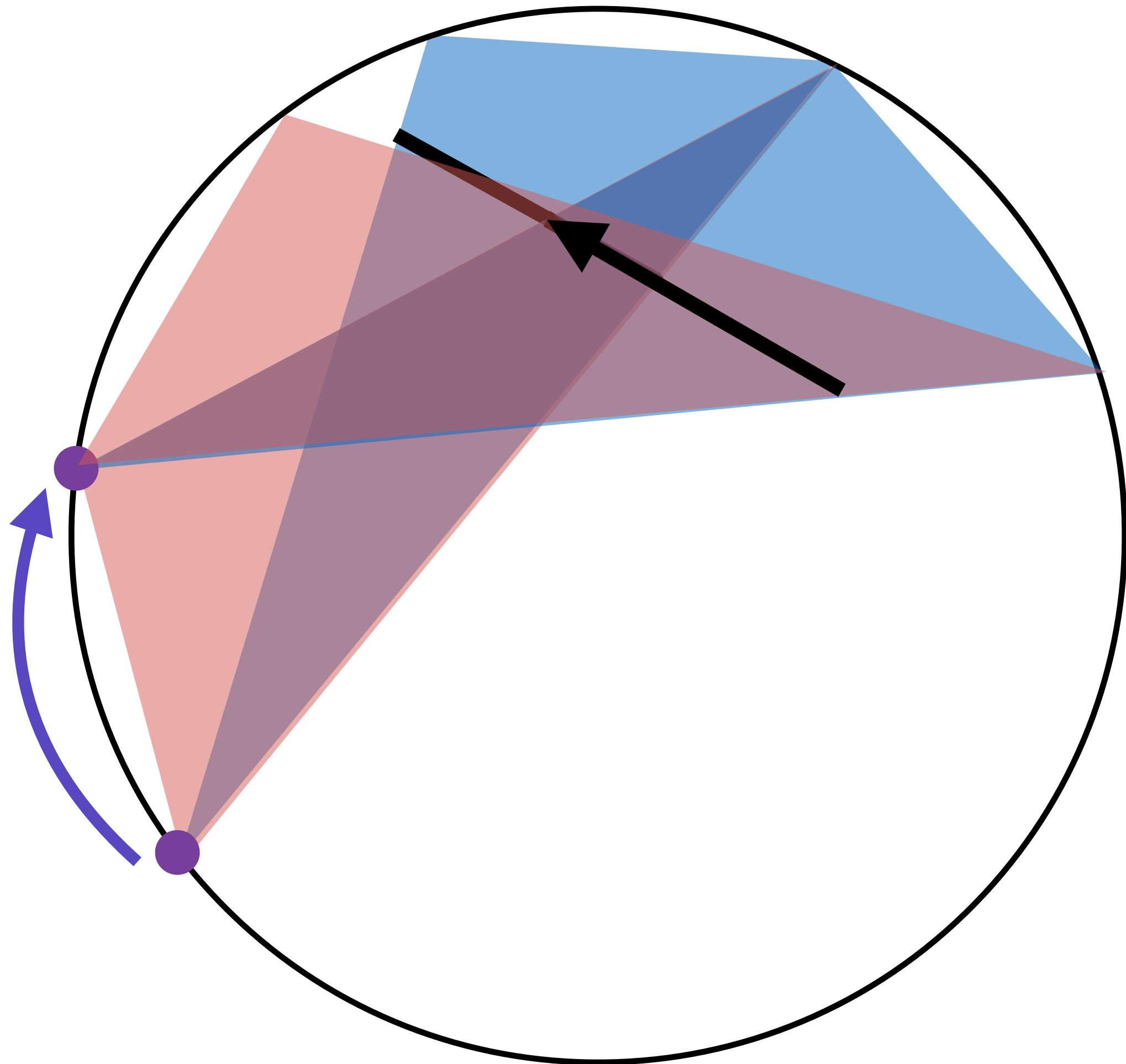
Insight (SMALL '16):

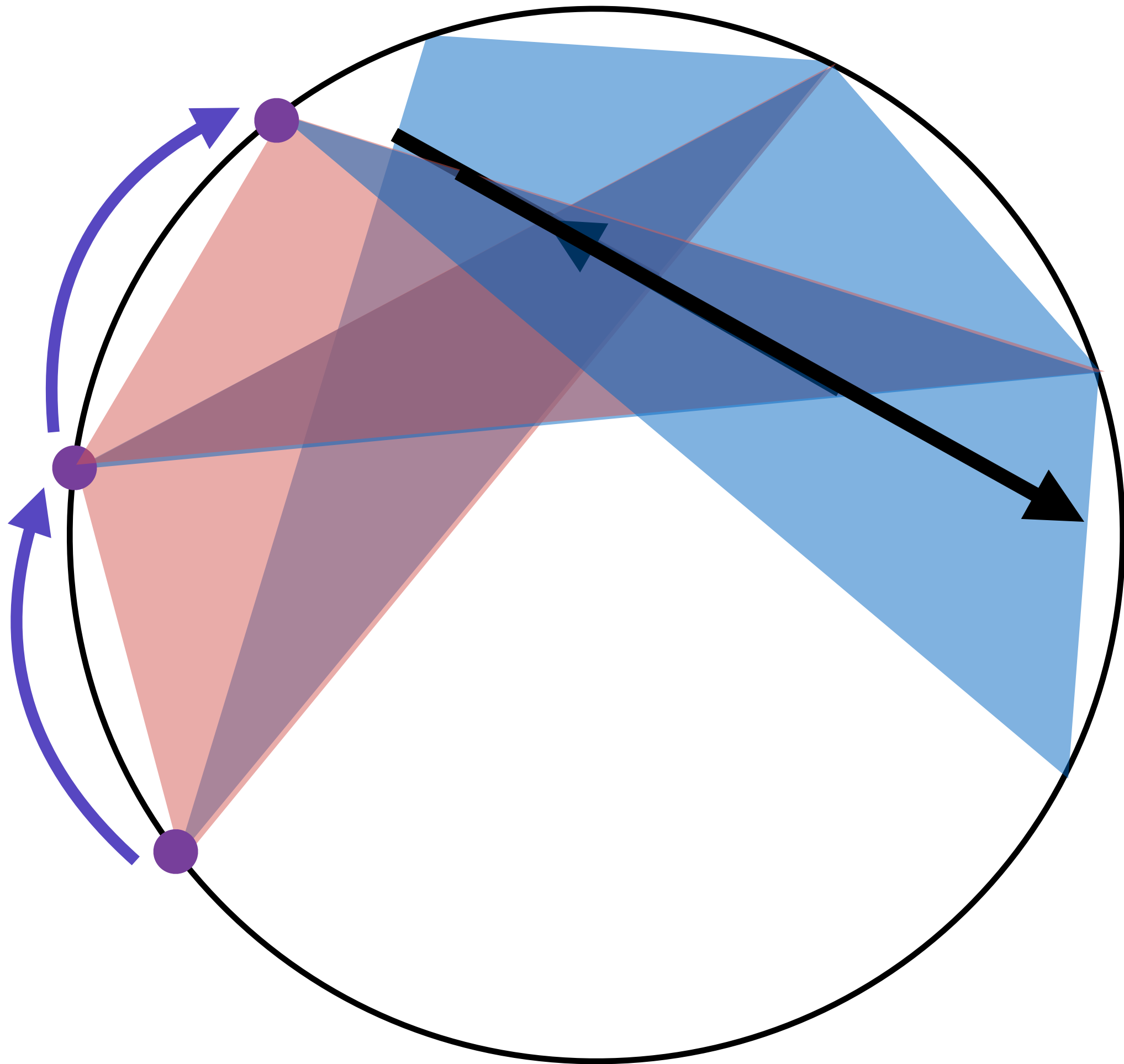
- Fix trajectory; triangle moves
- Keep track of favorite vertex
- This yields a 1-dimensional system, in fact an Interval Exchange Transformation (IET).

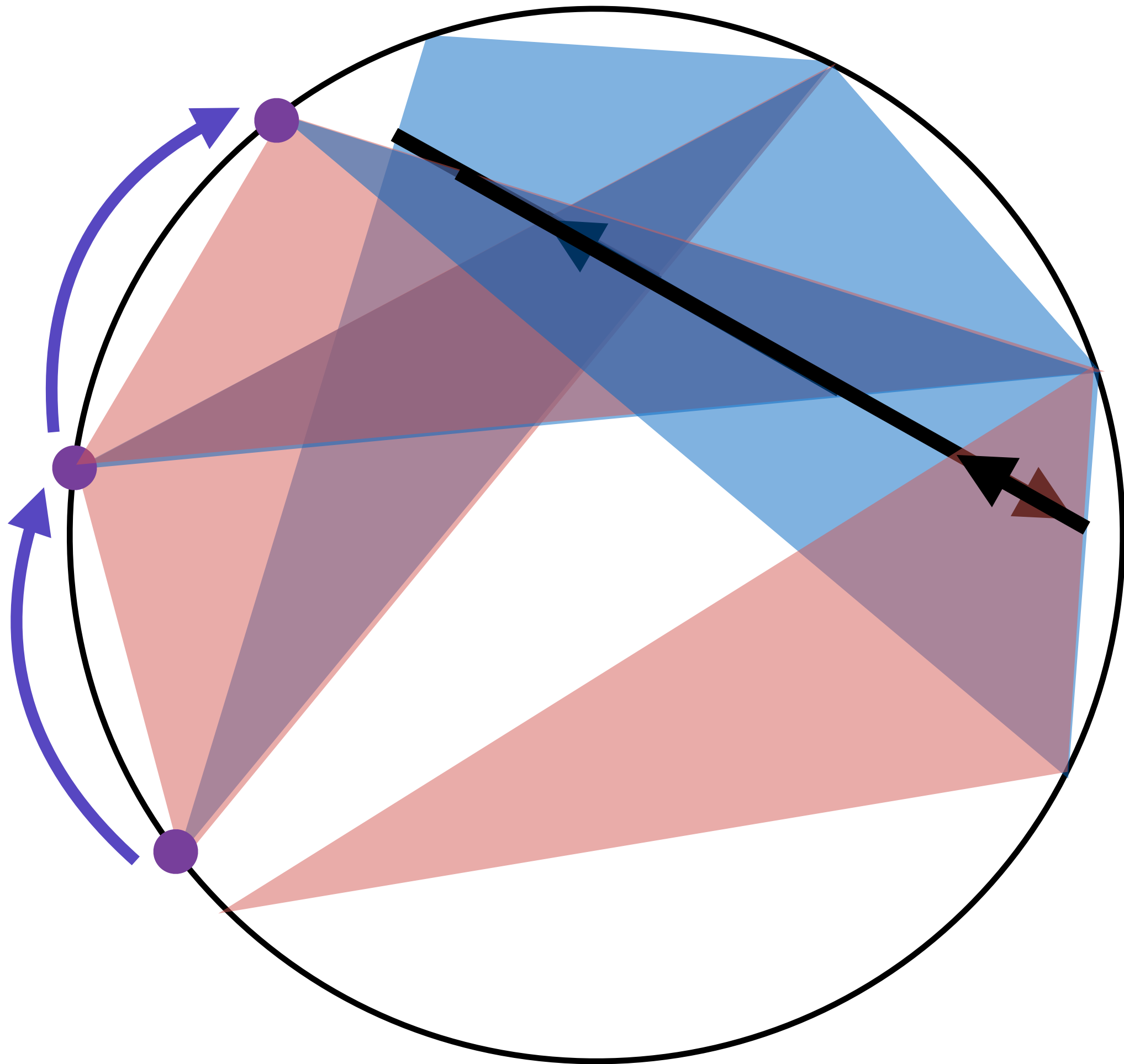


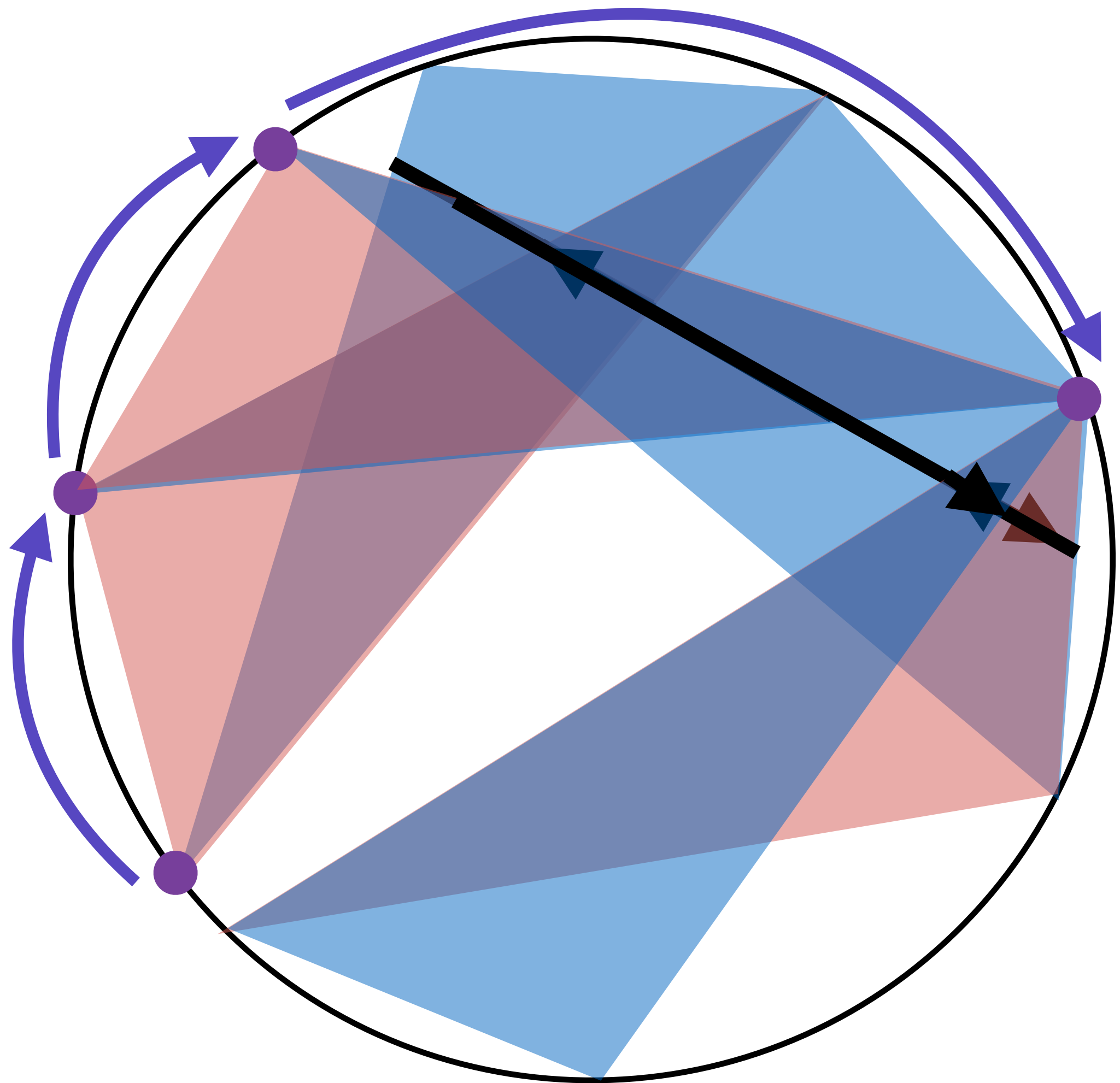


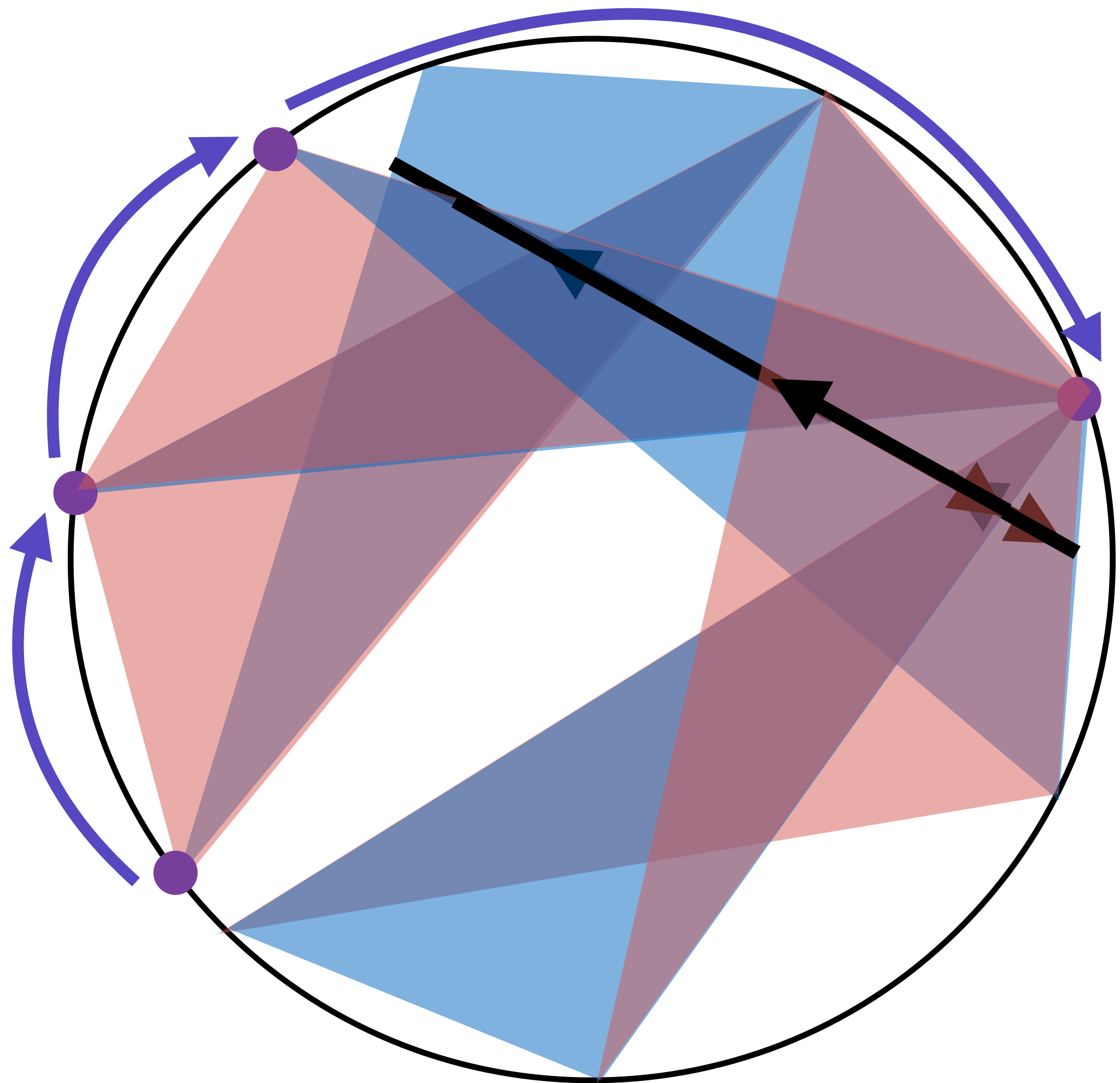


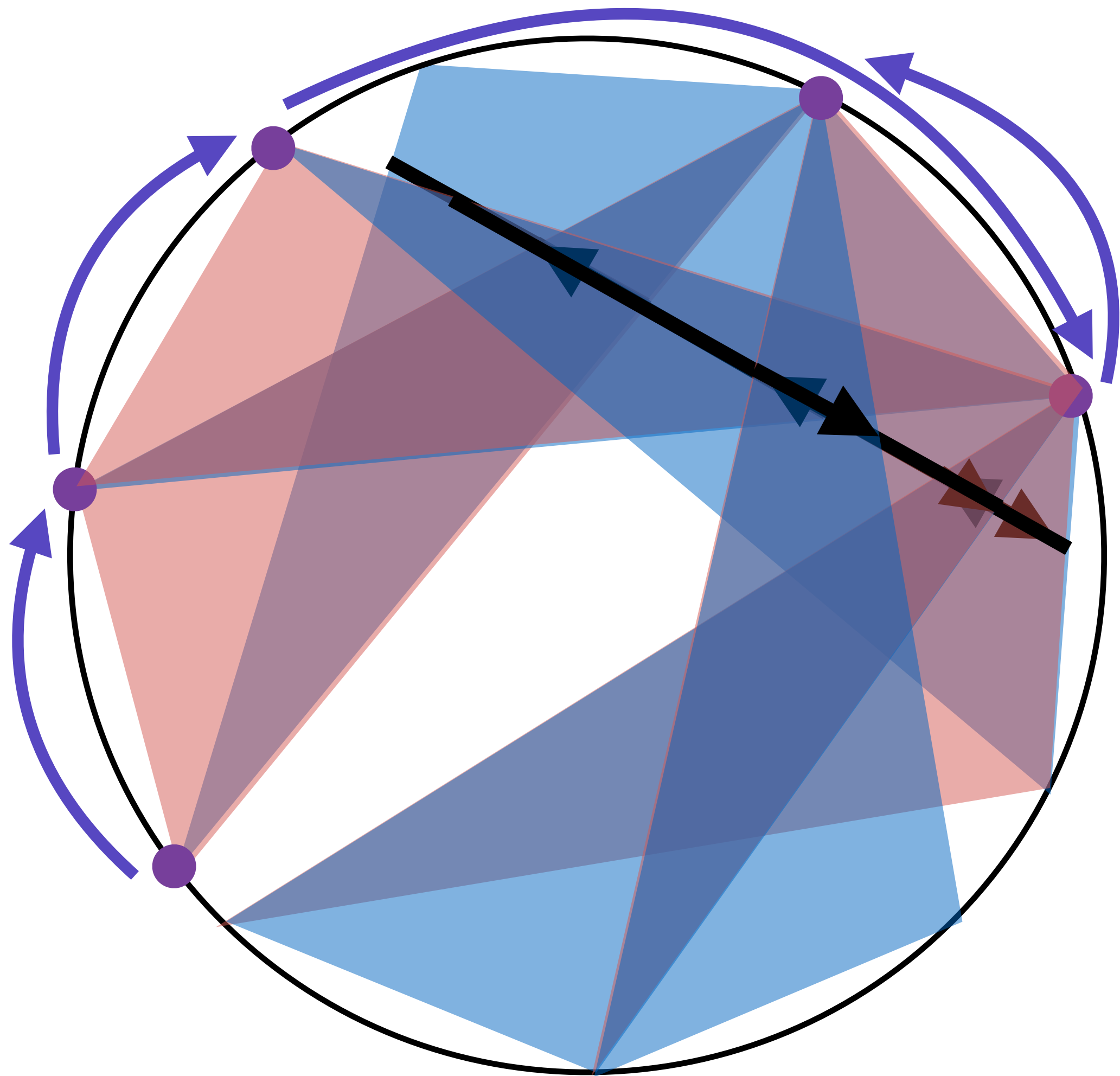












Let X be the location of the identified vertex, and τ the angle subtended by the trajectory chord.

Our IET is defined by:

$$X' = \begin{cases} \tau + 2\beta - X & \text{if } 0 < X < 2\beta \\ \tau + 2\beta - 2\gamma - X & \text{if } 2\beta < X < 2\beta + 2\gamma \\ \tau - 2\gamma - X & \text{if } 2\beta + 2\gamma < X < 2\pi, \end{cases}$$

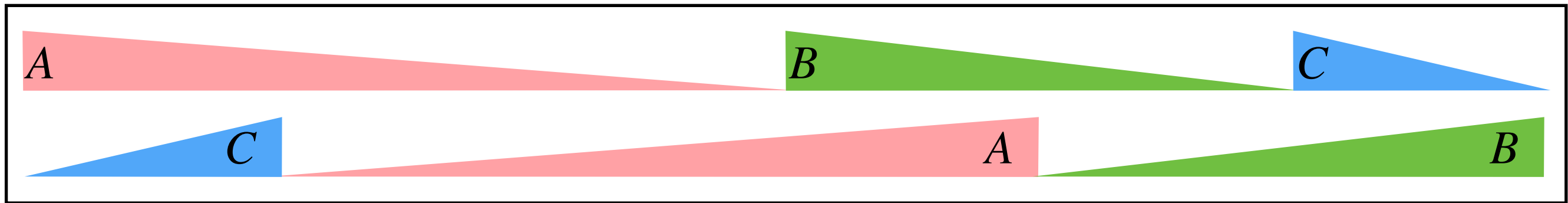
an orientation-reversing IET.

The diagram shows a circle with a grey outline. Inside the circle, there are three points marked with purple dots on the circumference. These points are connected by a thick purple arc that follows the outer edge of the circle, with arrows indicating a counter-clockwise direction. From each purple dot, a grey line segment extends towards the interior of the circle. These three grey lines meet at a central point, dividing the interior into three regions. Each region is filled with a semi-transparent color: one is light blue, one is light red, and one is light purple. The overall effect is a tiling of the circle's interior by triangles. The text "Tiling Billiards on triangle tilings:" is overlaid on the top left of the diagram.

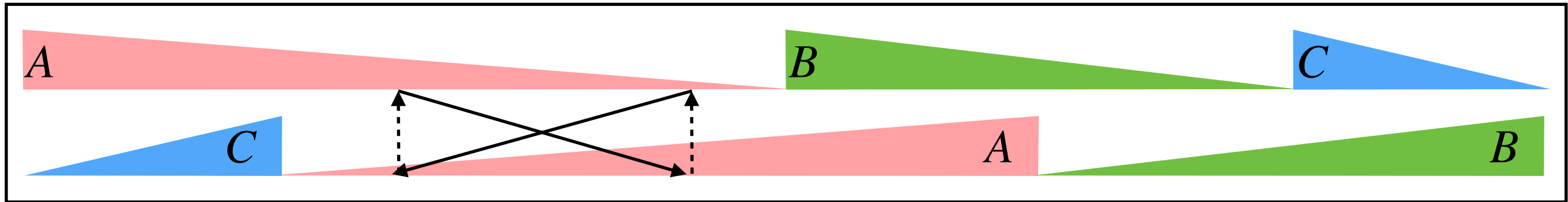
Tiling Billiards on triangle tilings:

- Give a 3-IET on the circle
- Interval lengths: $2\alpha, 2\beta, 2\gamma$
- Shift transformations: based on $\alpha, \beta, \gamma, \tau$
- Are orientation-reversing (“fully flipped”)

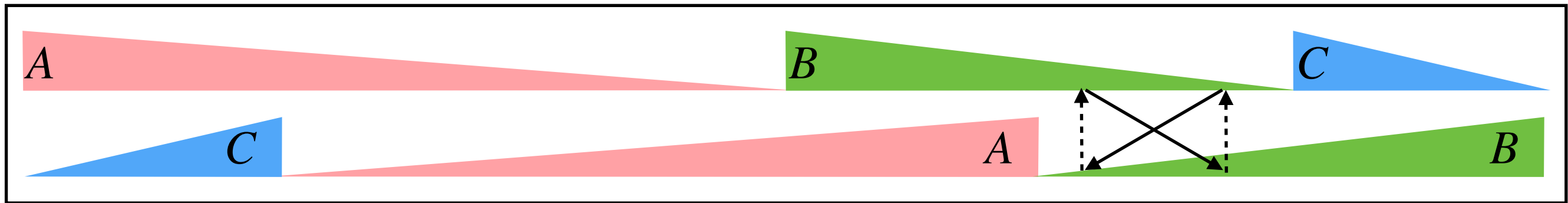
Why flipped IETs are stable & periodic



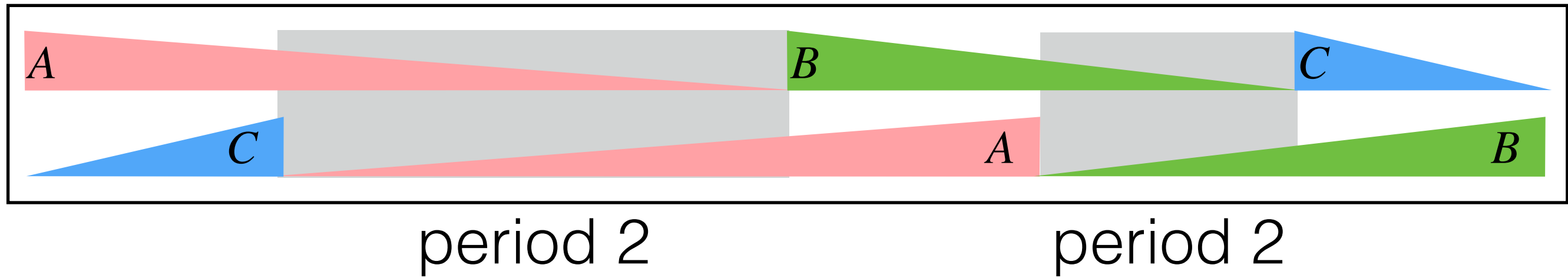
Why flipped IETs are stable & periodic



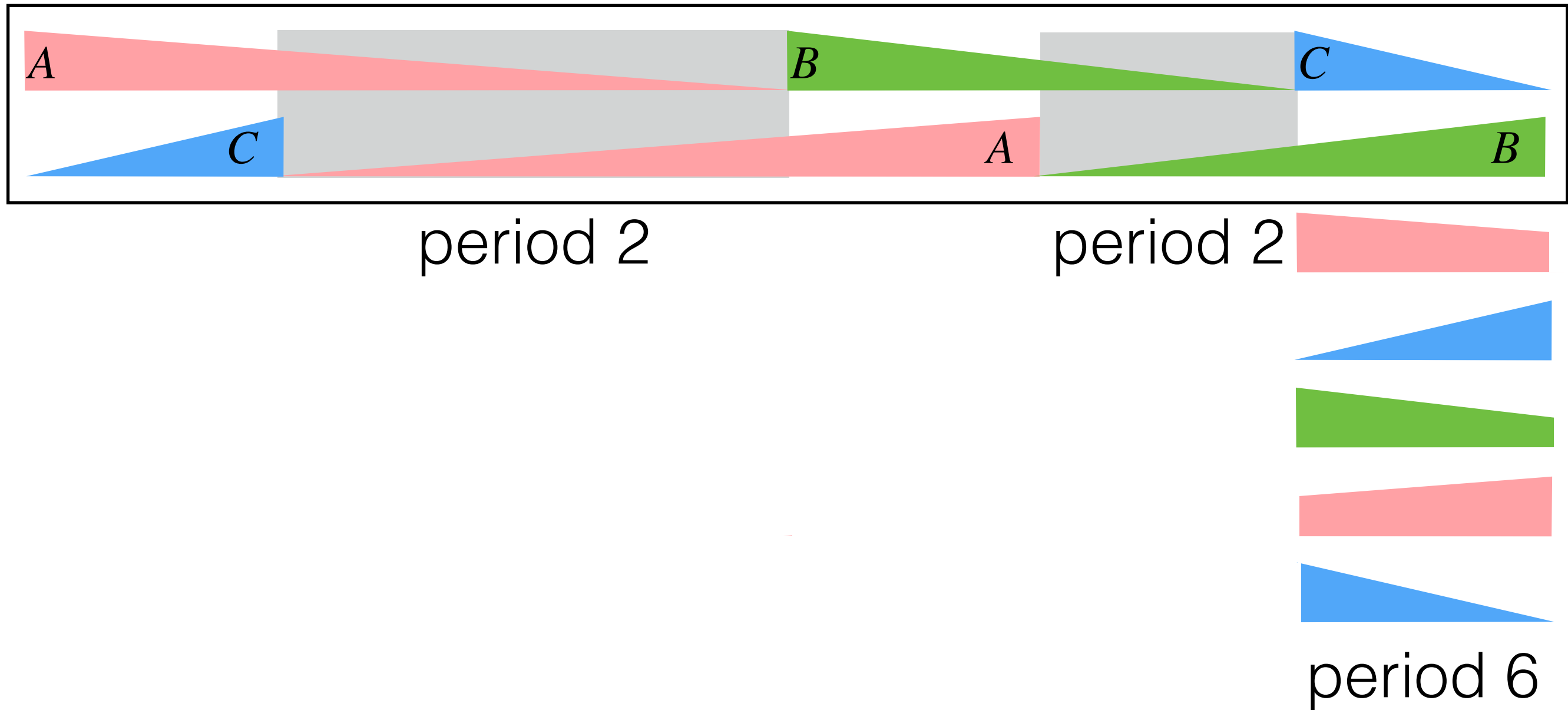
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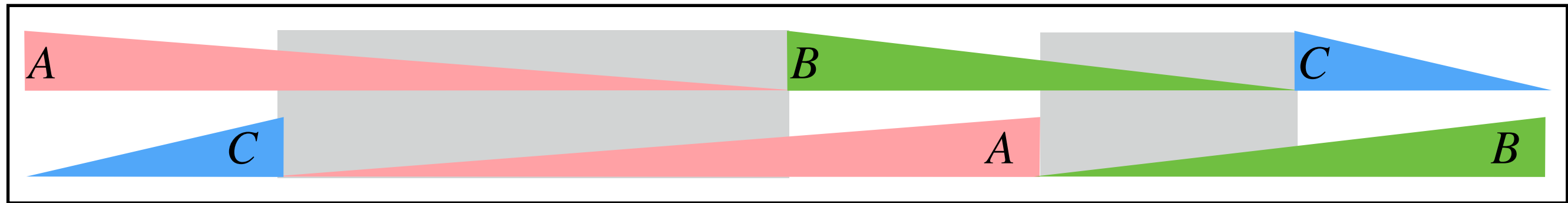
Why flipped IETs are stable & periodic



Why flipped IETs are stable & periodic



Why flipped IETs are stable & periodic



period 2



period 2



period 6

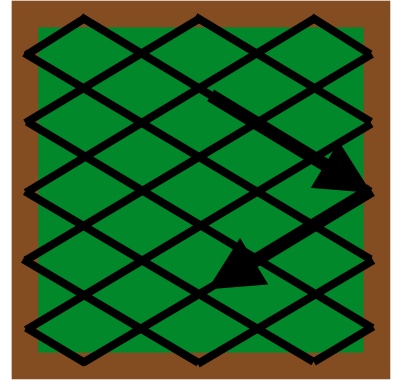
period 6

period 6

Everything is flipped periodic, every point is stable, periods of form $4n+2$

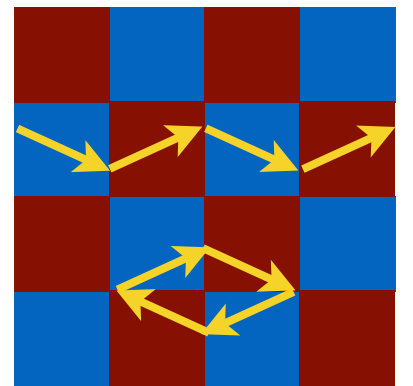
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Tiling Billiards on triangle tilings:

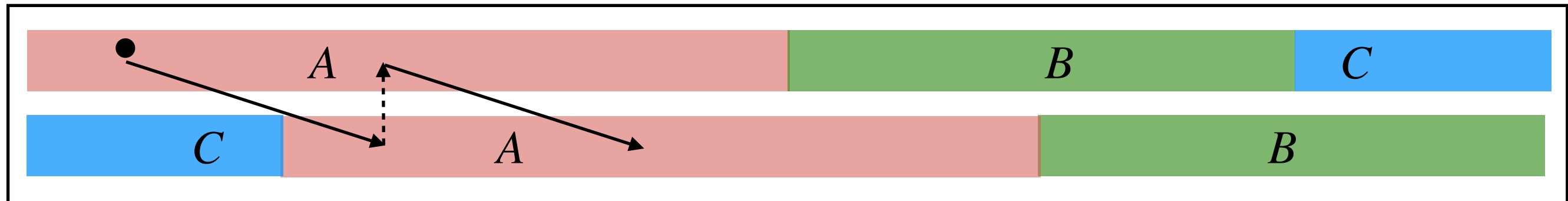
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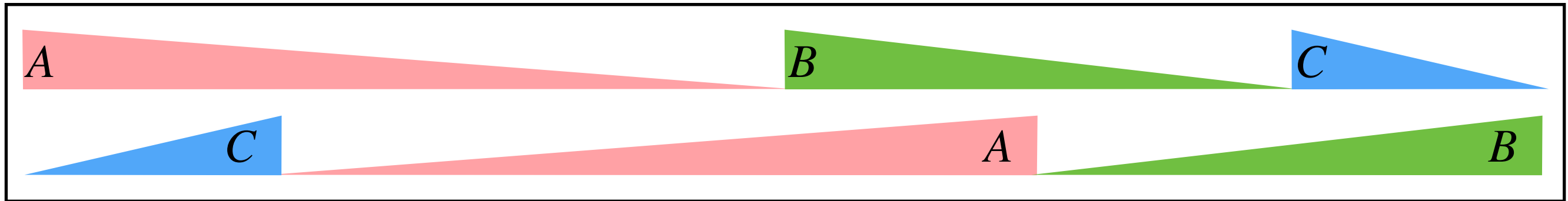
Burning question: What causes periodicity and stability in tiling billiards on triangle tilings?

➔ They correspond to fully flipped IETs.

Comparison to non-flipped IETs



If $|AB|$ and $|C|$ are irrationally related, every point is aperiodic (rotation).

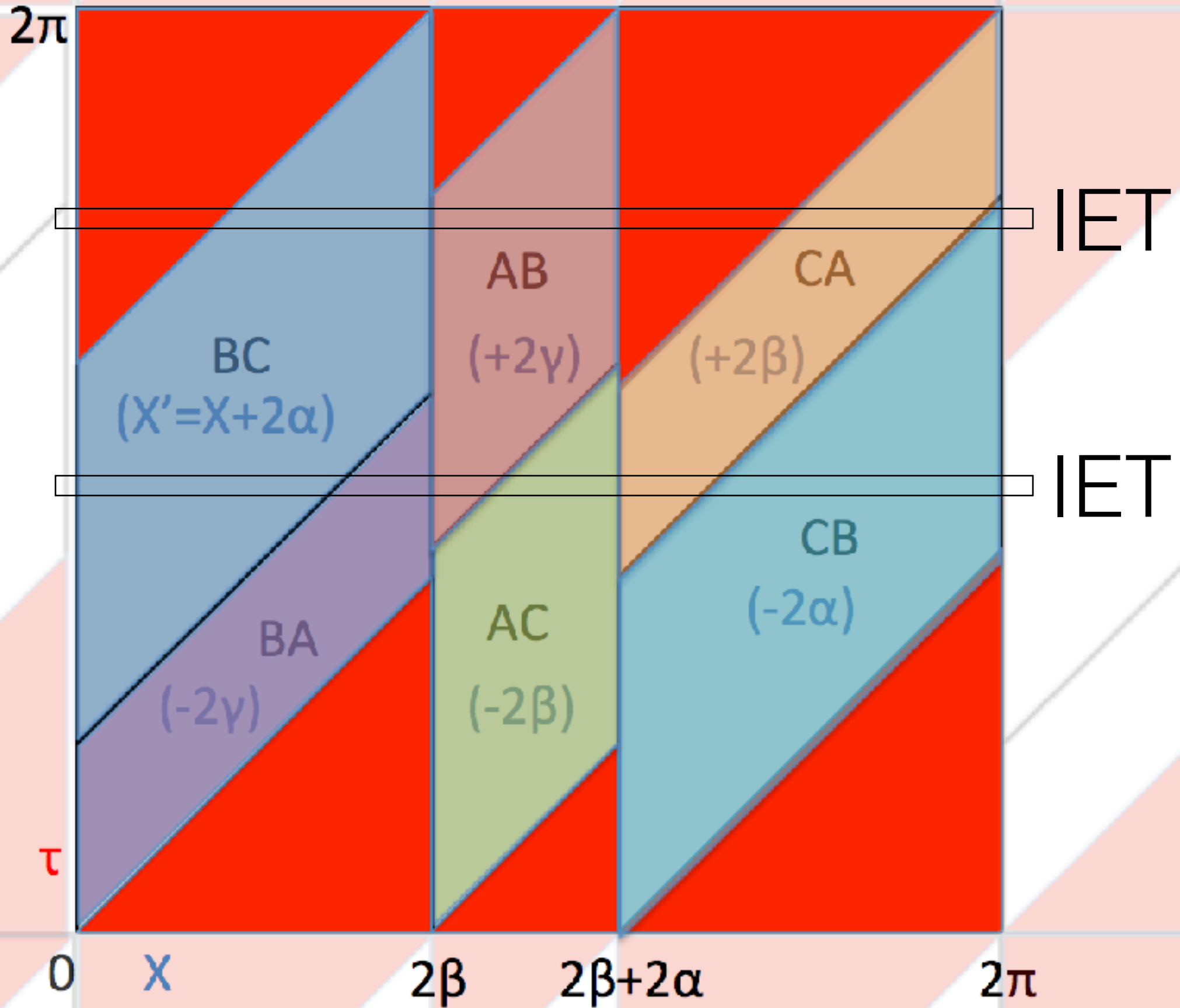


Tiling billiards corresponds to
orientation-reversing 3-IET

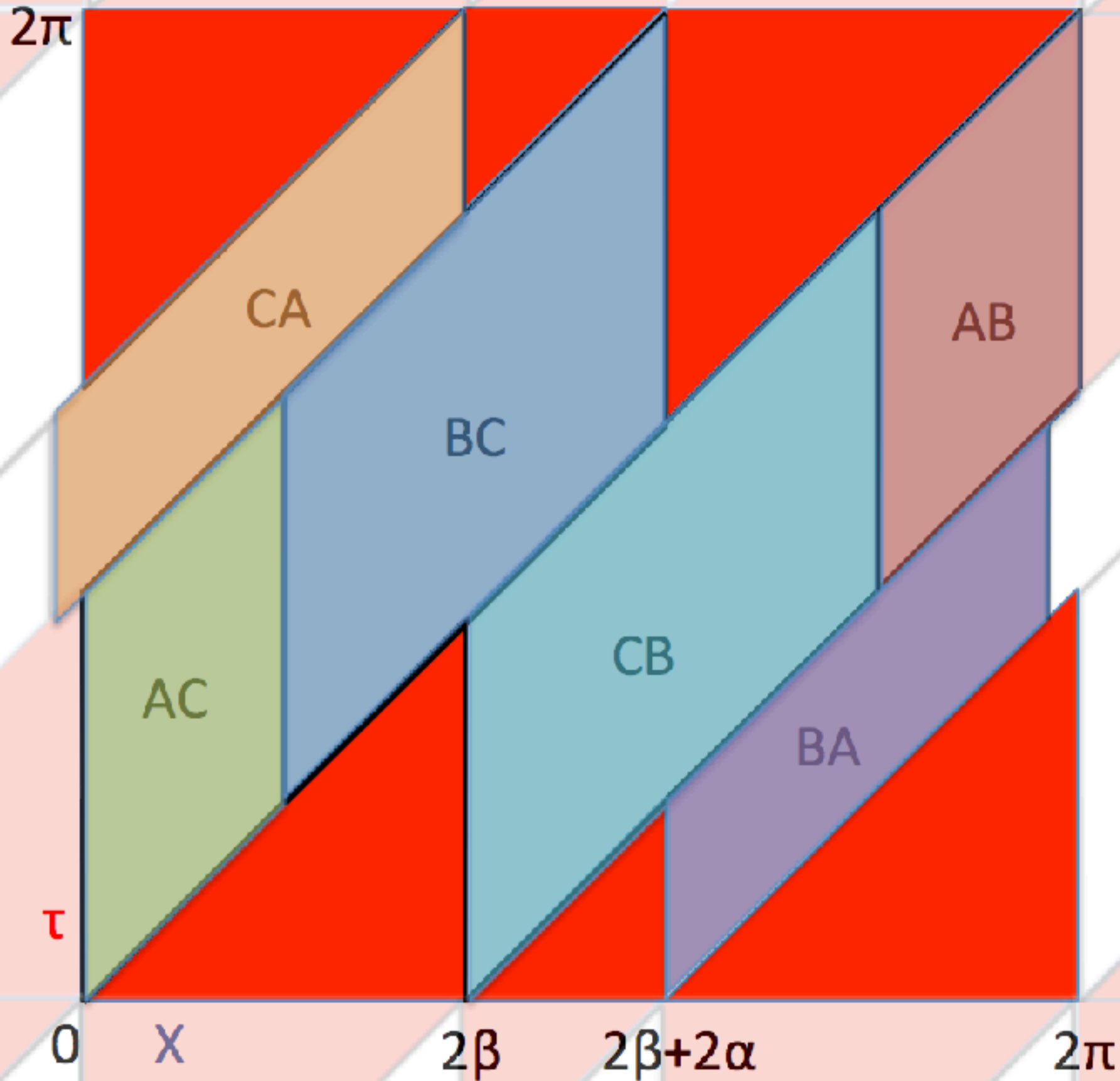
Idea:

- Use the **square** of the 3-IET
- Get an orientation-**preserving** 6^* -IET
- Stack all of them into a PET

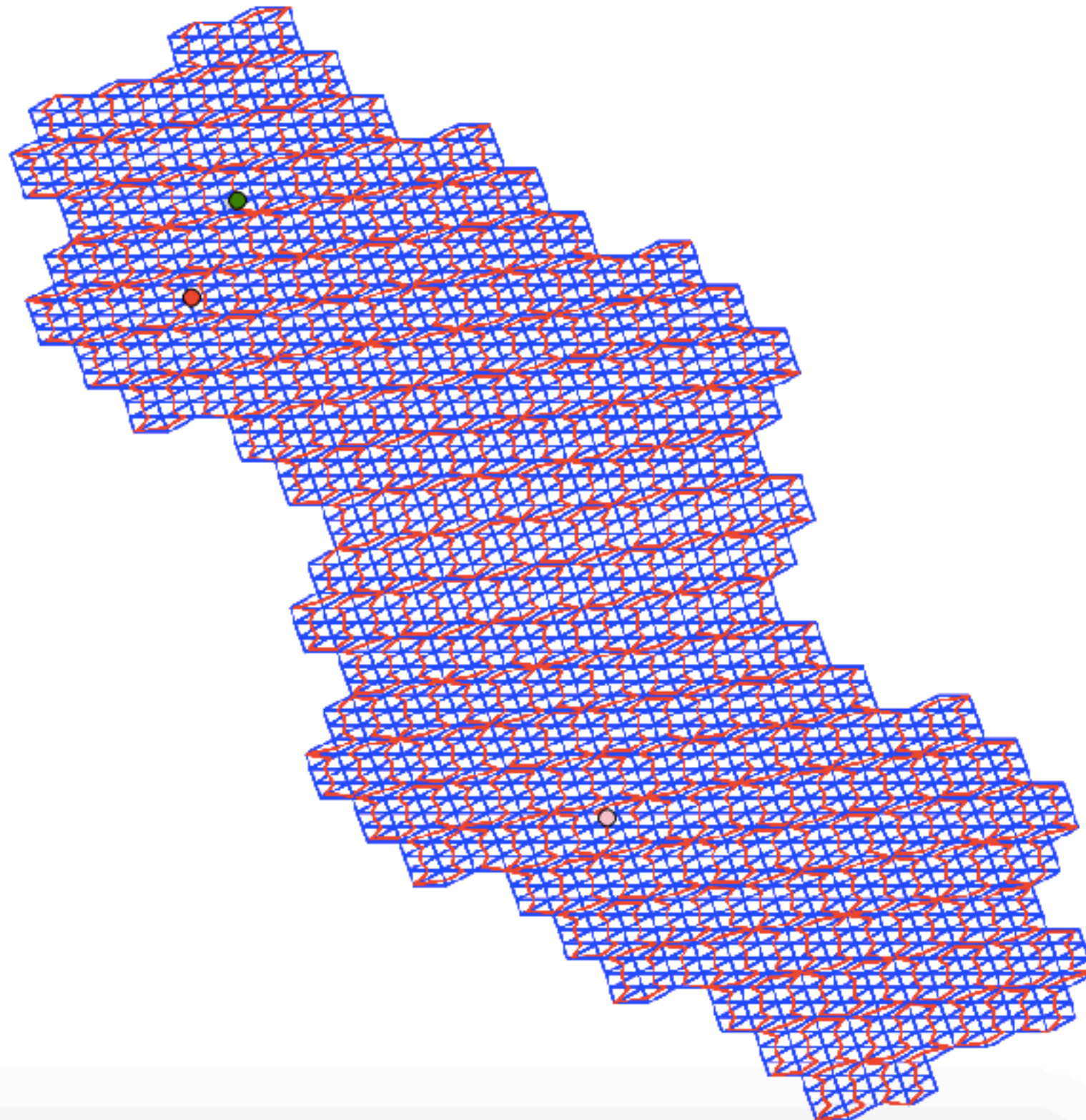
Tiling billiards PET: stack of IETs



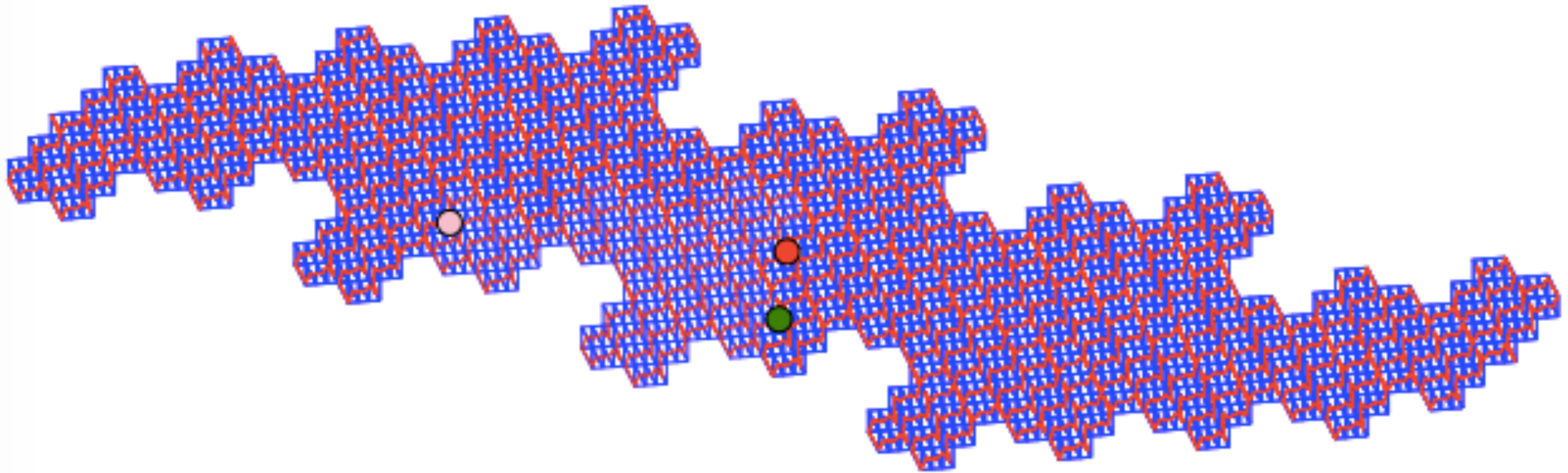
Tiling billiards PET: stack of IETs



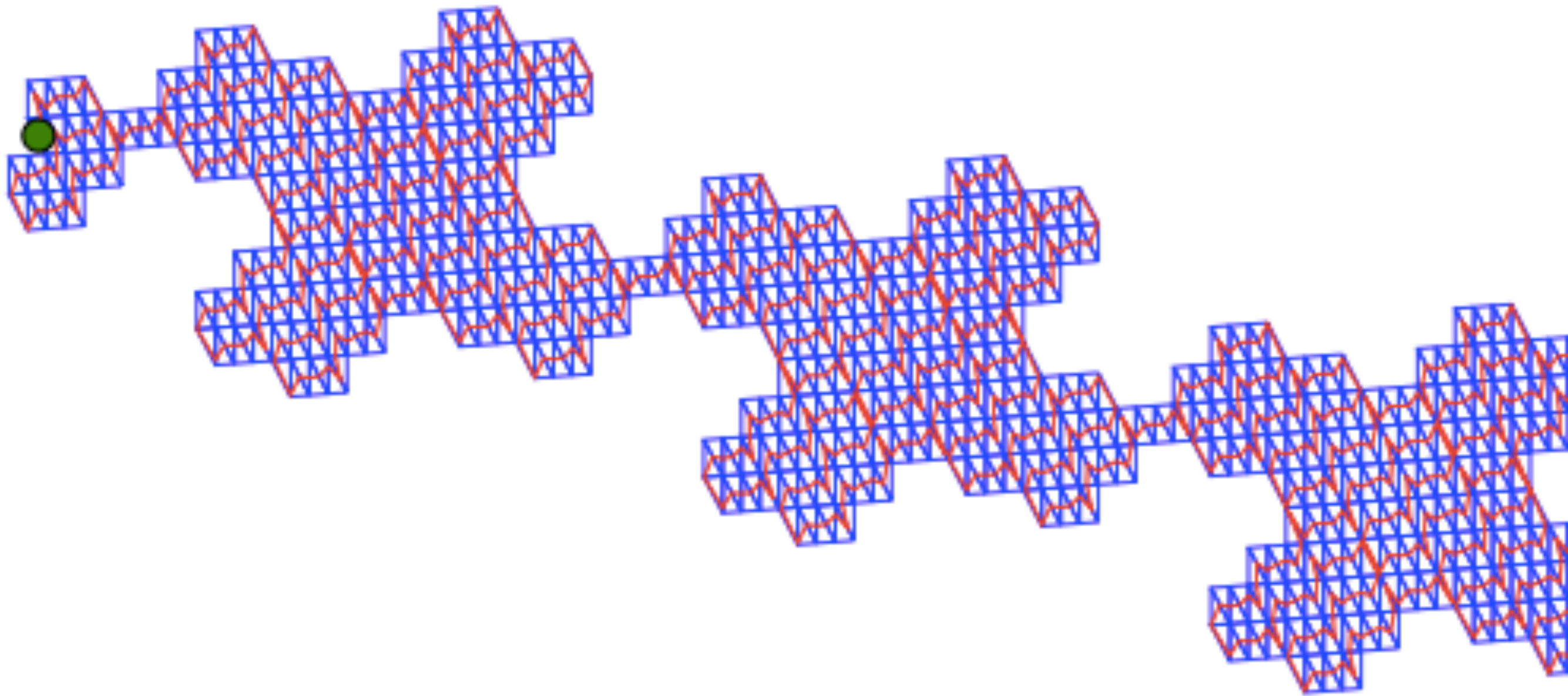
Visit the zoo: Billiard trajectories on triangle tilings



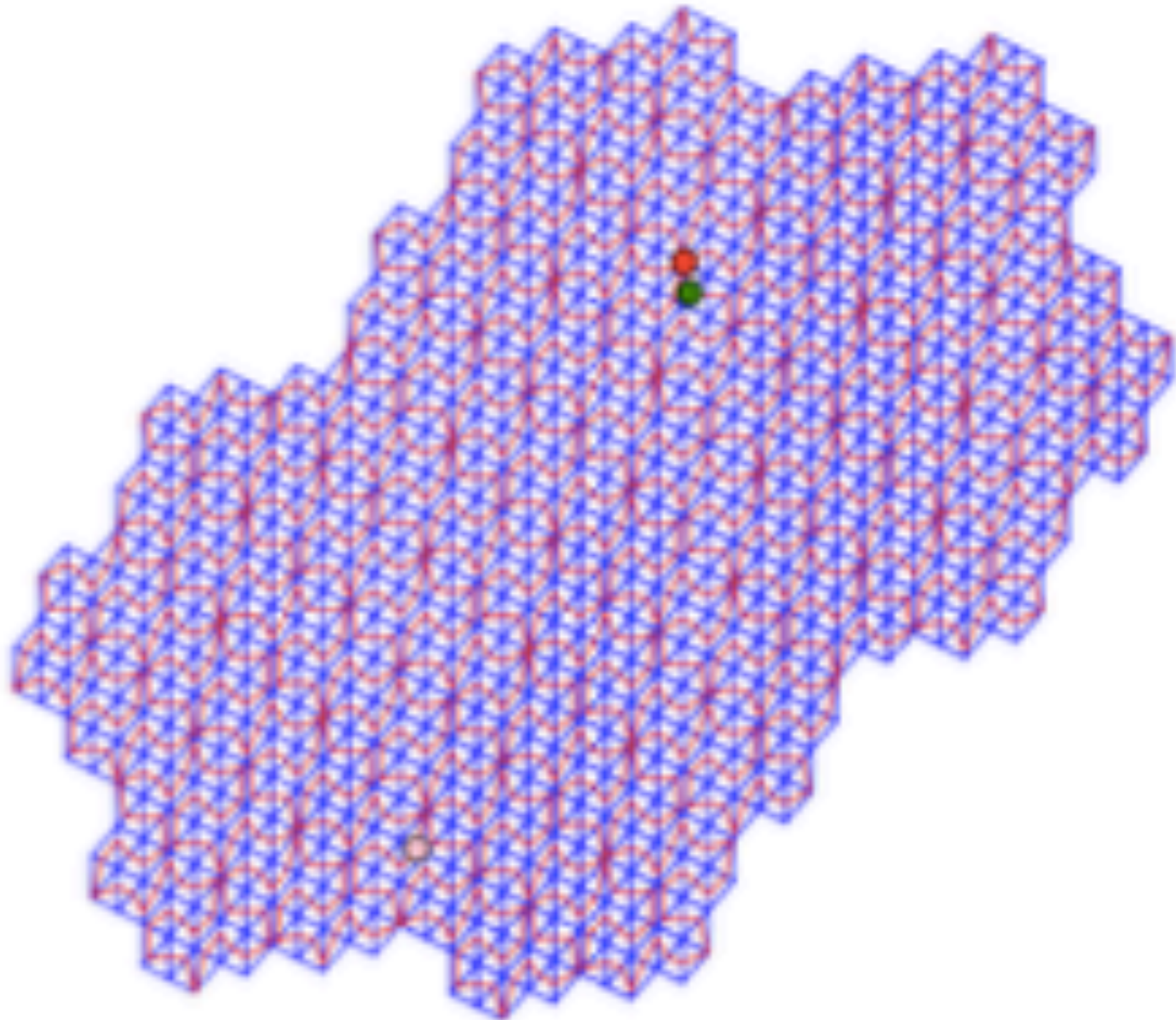
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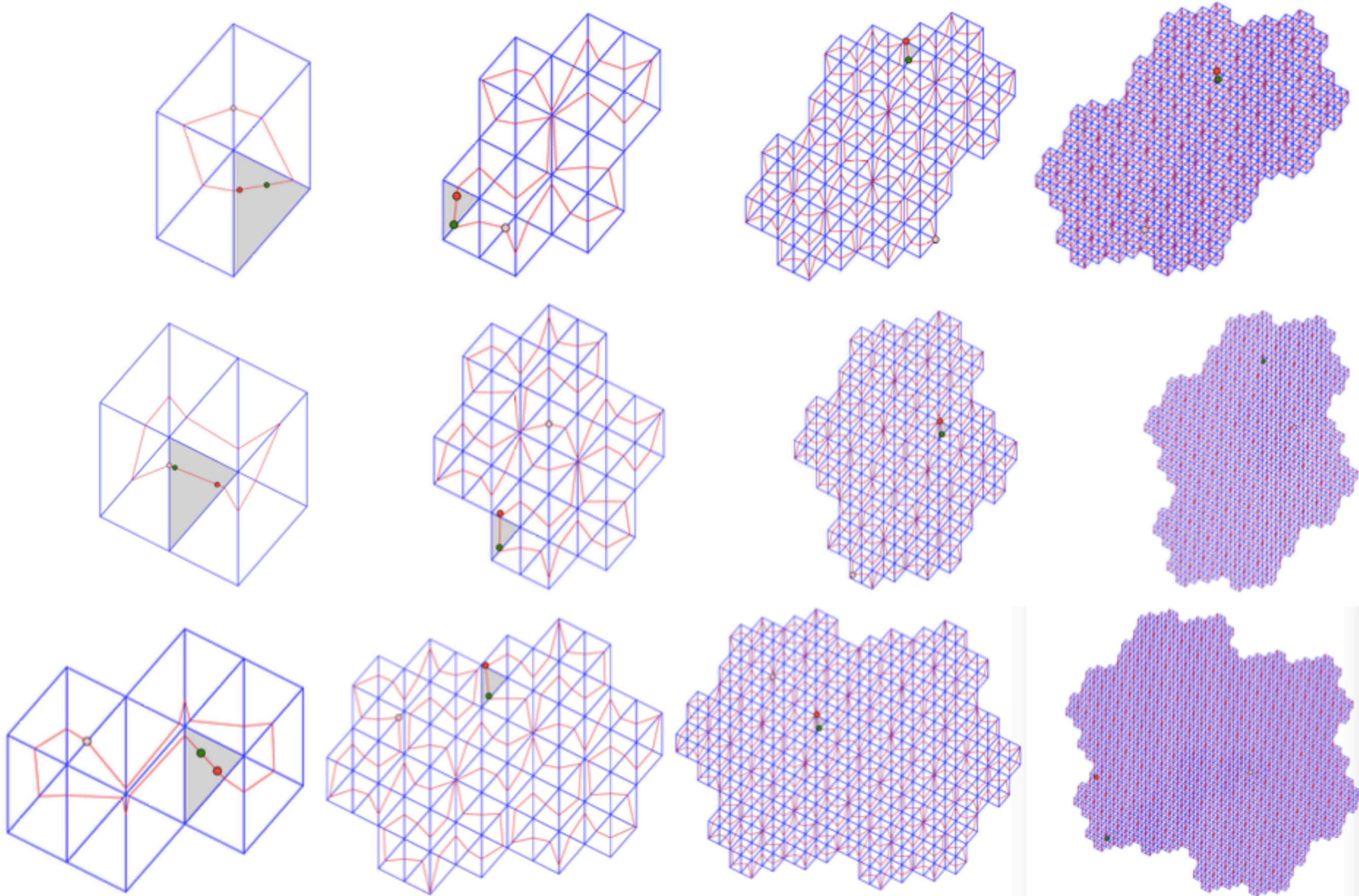
Visit the zoo: Billiard trajectories on triangle tilings



The Rauzy fractal as a billiard trajectory



The Rauzy fractal as a billiard trajectory!



Future work:

- Show that we actually get fractals as the limit of billiard trajectories
- Completely understand fully flipped IETs (service to community)
- Other tilings!