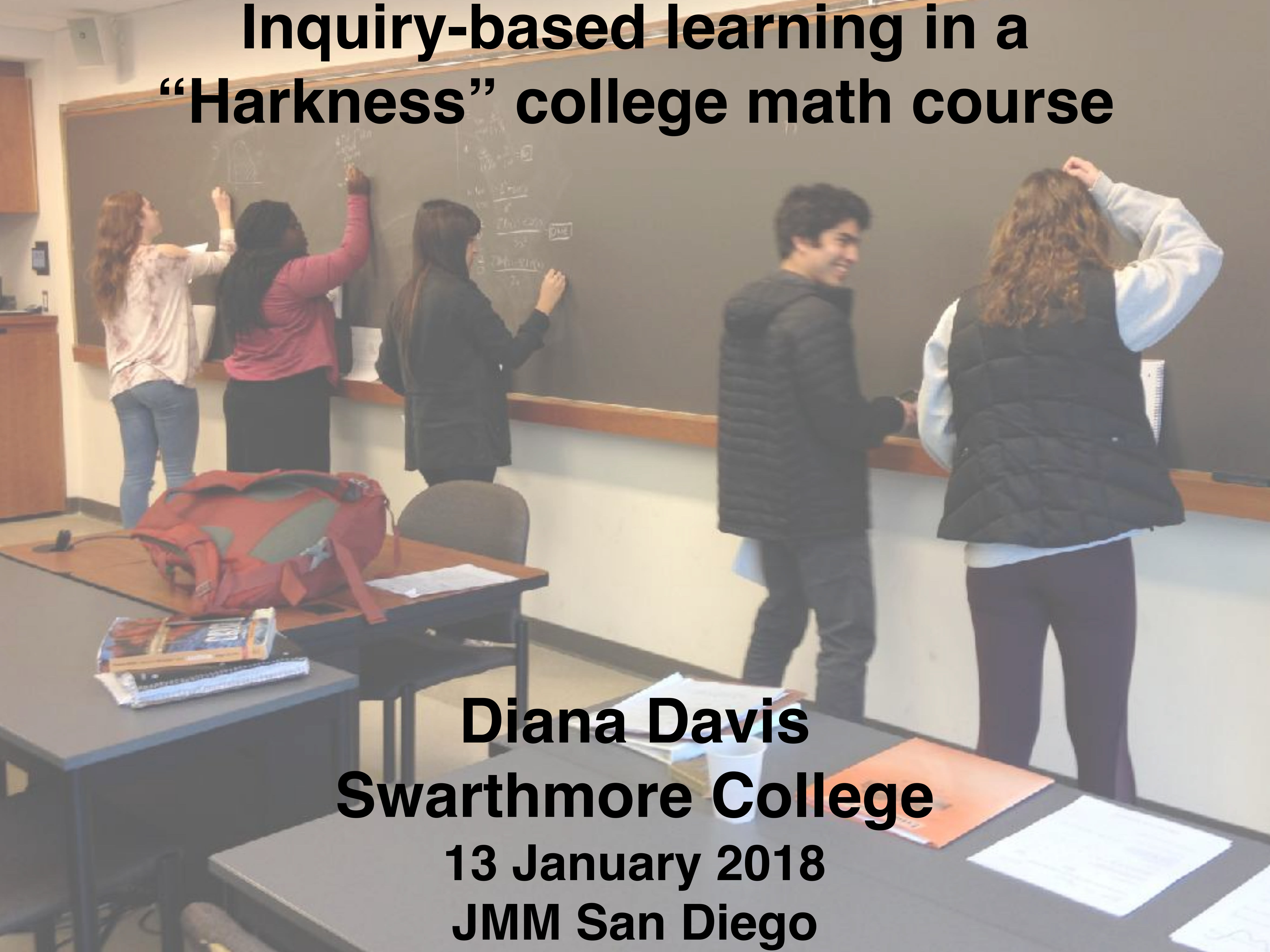
Inquiry-based learning in a "Harkness" college math course



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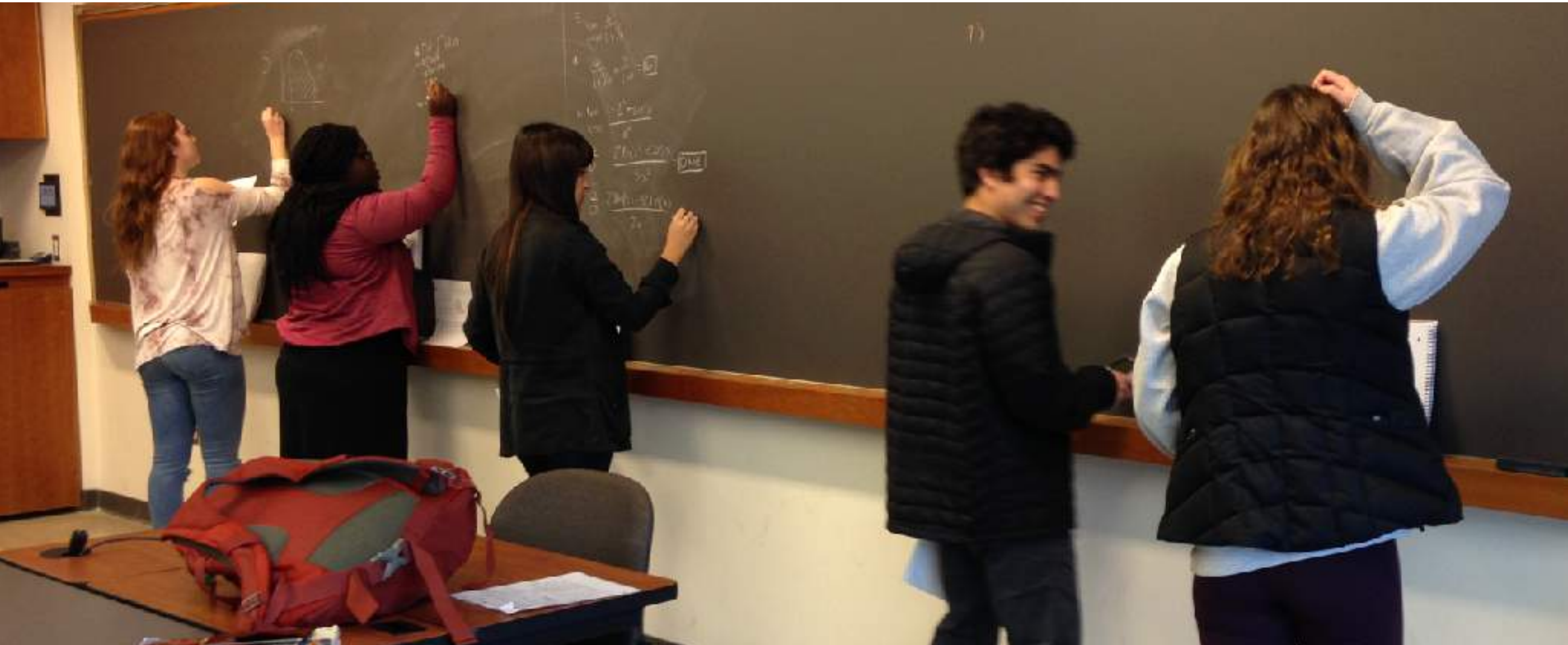
Given the choice between working on a problem that I already know how to solve, or on a problem that I don't know how to solve, yet, I would rather work on the problem I

know how to solve	don't know how to solve

Two aspects to “Harkness” math curriculum (developed at Phillips Exeter Academy HS):

1. **incremental problem-based homework**

2. student-centered discussion classes



1. incremental problem-based homework

Example: Fundamental Theorem of Calculus

Curriculum-writing goal: Hit the FTC from every angle first, so the theorem seems obvious.

FTC problem 10 of 18:

5. (Continuation) The *Fundamental Theorem of Calculus* states that, if the function $f(x)$ is continuous on the interval $[a, b]$, and there is a function $F(x)$ so that $f(x) = F'(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

The function F is called an *antiderivative* for f.

(a) For the function $f(x) = x$, find the function $F(x)$.

(b) Use the FTC to compute $\int_0^2 x\, dx$.

(c) Check that your answer agrees with your function from Page 21 # 7 and your sketch from Page 22 # 7.

1. incremental problem-based homework

Example: Fundamental Theorem of Calculus

Problem 1 - "undoing the derivative"

from PEA

1. Cameron, a student of Calculus, was instructed to find the derivative — with respect to x — of five functions, each expressed in terms of an unknown function y. Below are Cameron's five answers. For each, reconstruct the expression that Cameron differentiated. You will have to write your answers in terms of y (and x), of course. Can you be absolutely sure that your answers agree with the questions on Cameron's assignment?

(a) $2.54 + \dfrac{dy}{dx}$ **(b)** $\dfrac{dy}{dx}\sec^2 y$ **(c)** $\sqrt{5y}\,\dfrac{dy}{dx}$ **(d)** $\dfrac{7}{y^2}\dfrac{dy}{dx}$ **(e)** $(y - \cos x)\left(\dfrac{dy}{dx} + \sin x\right)$

The functions you found are called *antiderivatives* of the functions on Cameron's sheet.

2. (For fun) We're about to learn the Fundamental Theorem of Calculus. Let's explore some other fundamental theorems first.

(a) The *Fundamental Theorem of Arithmetic* says that any number can be uniquely factored into primes. Factor the number 16100. The word *uniquely* means that there's only one way to do it, i.e. that everyone's answer should be the same.

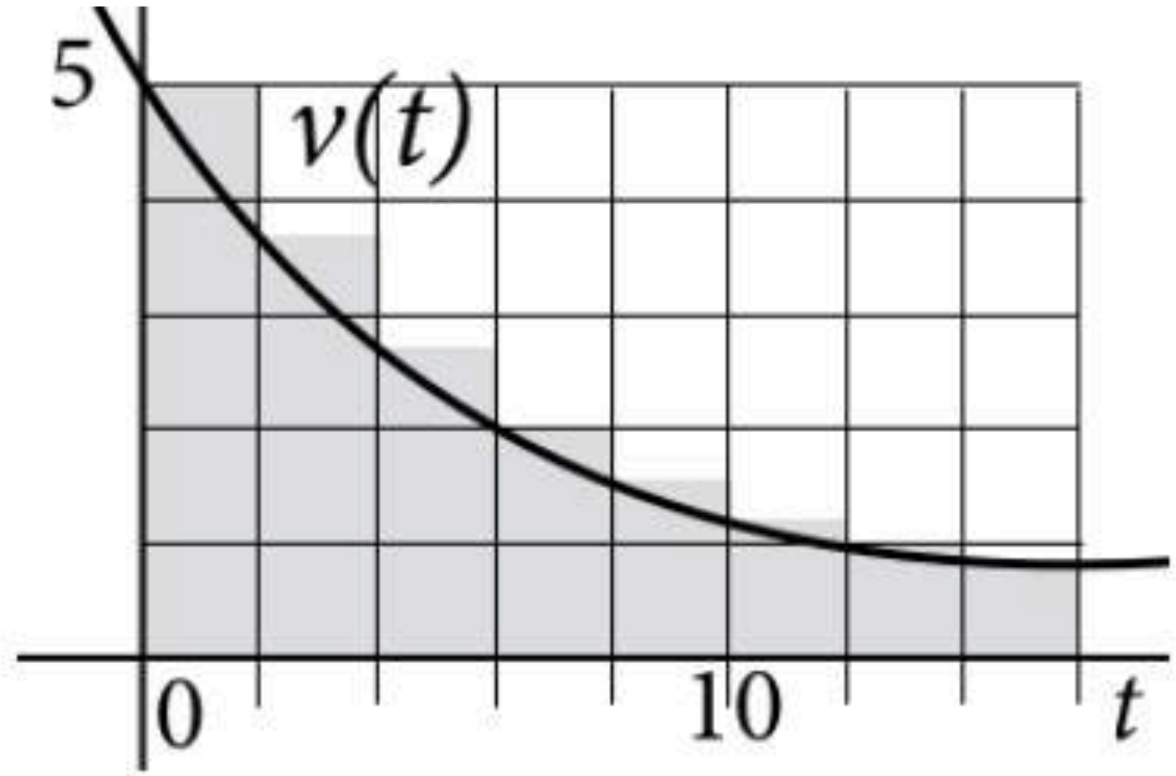
(b) The *Fundamental Theorem of Algebra* says that any polynomial can be factored into terms of the form $(x - a)$, where each a is a complex number (or possibly just a real number). Factor the polynomial $x^3 + 2x^2 - 5x - 6$.

1. incremental problem-based homework

Example: Fundamental Theorem of Calculus

Problem 2 - meaning of area under curve

3. The curve in the figure shows the speed $v(t)$, in meters per second, of a bicycle that is decelerating over the course of 16 seconds. We want to figure out how far the bike traveled during this 16 seconds while it was slowing down.



(a) Estimate the speed of the bike at each of the following times: $t = 0$, $t = 2$, $t = 10$, $t = 16$.

(b) The boxes in the figure suggest a way of estimating the distance traveled. Suppose that, rather than decelerating between $t = 0$ and $t = 2$, the bike had gone its $t = 0$ speed for that entire 2 seconds. Estimate how far would it have traveled during those 2 seconds.

(c) Suppose the bike went its $t = 2$ speed for the entire time from $t = 2$ to $t = 4$. Estimate how far would it have traveled during *those* 2 seconds.

(d) Repeat this calculation for each 2-second interval, and use it to find an estimate for the total distance traveled. Explain how this calculation is related to the boxes in the picture.

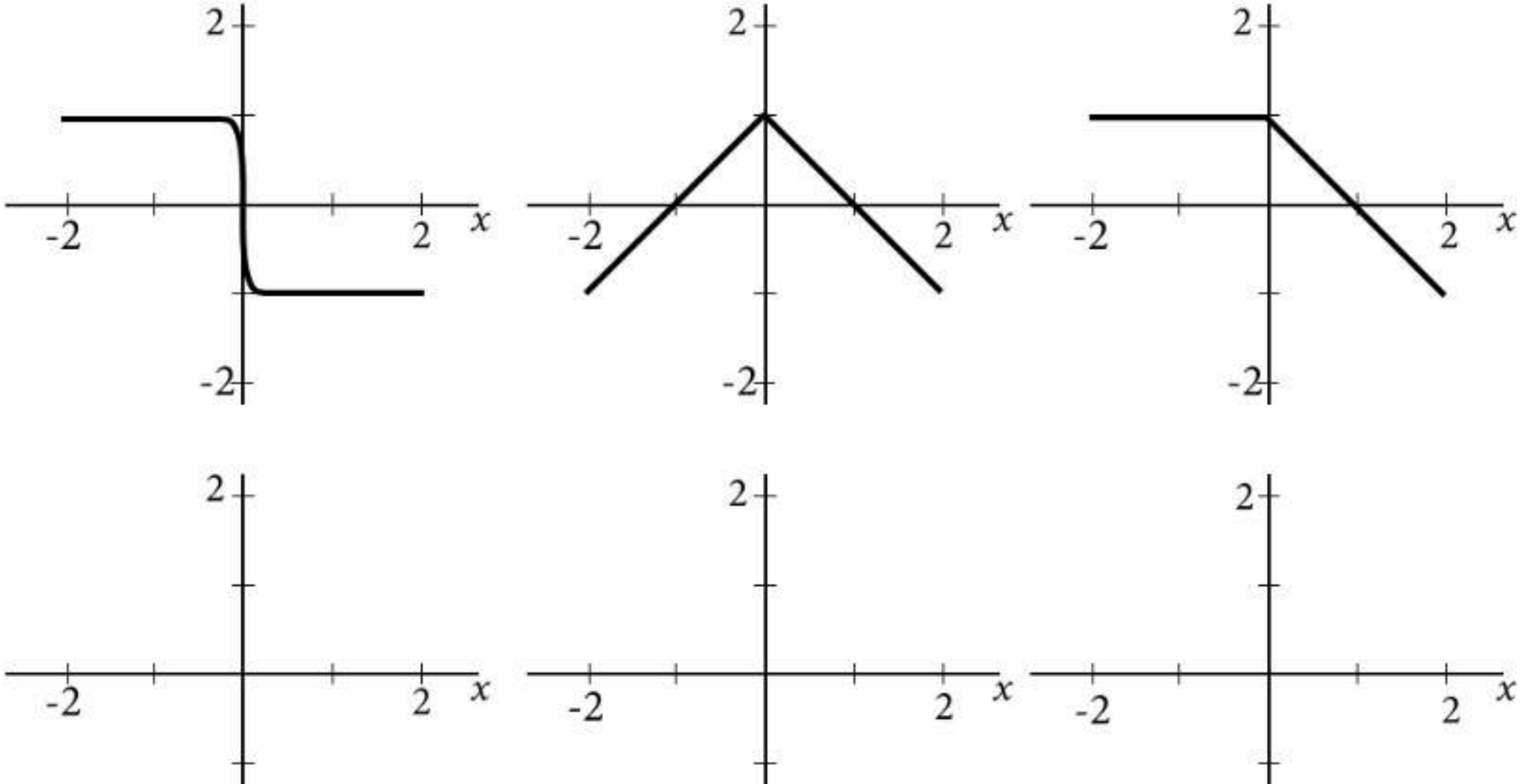
(e) Is your answer to part (d) an overestimate or underestimate of the actual distance?

1. incremental problem-based homework

Example: Fundamental Theorem of Calculus

Problem 3 - sketch *f(x)* given *f'(x)*

5. For each graph of $f'(x)$ below, sketch $f(x)$ on the axes provided.



1. incremental problem-based homework

Example: Fundamental Theorem of Calculus

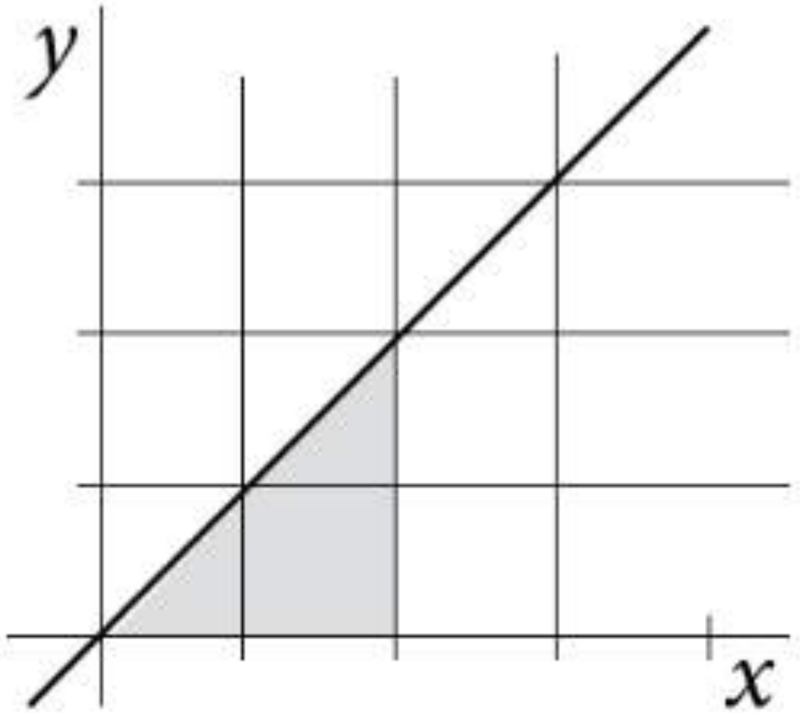
Problem 4 - idea of accumulation function

7. An *accumulation function* is a vacuum-like creature that scoops up area as it goes along, and counts how much it has so far. For example, an accumulation function (which we'll call $F(x)$) for the function $f(x) = x$ scoops up the area under the graph $y = x$. Let's assume that it starts scooping at $x = 0$.

(a) Use the picture to explain why $F(2) = 2$.

(b) Find $F(3)$.

(c) Find $F(k)$ for any positive value k.



1. incremental problem-based homework

Example: Fundamental Theorem of Calculus

Problem 5 - values for Riemann sum

from Hughes-Hallett

5. Using $\Delta x = 1/2$, fill in a table of values (provided to the right) that you could use to estimate $\int_0^2 x^2\,dx$.

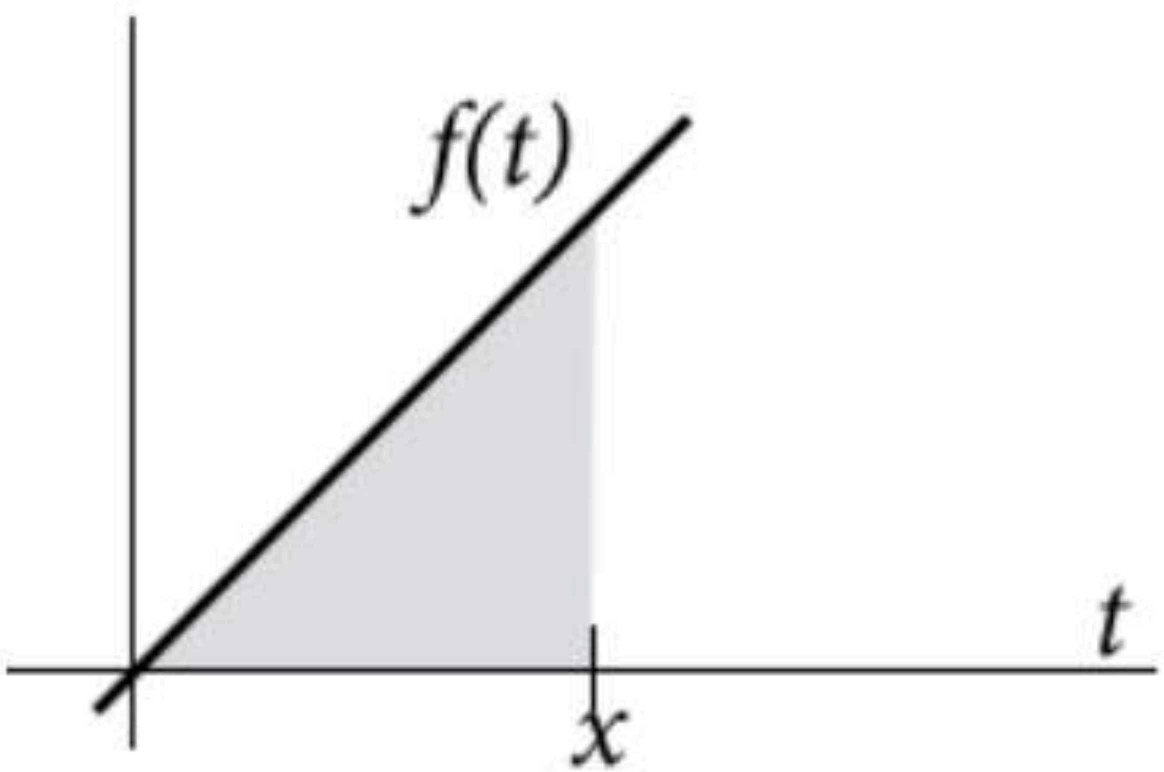
x	0	1/2	1	3/2	2
$f(x)$					

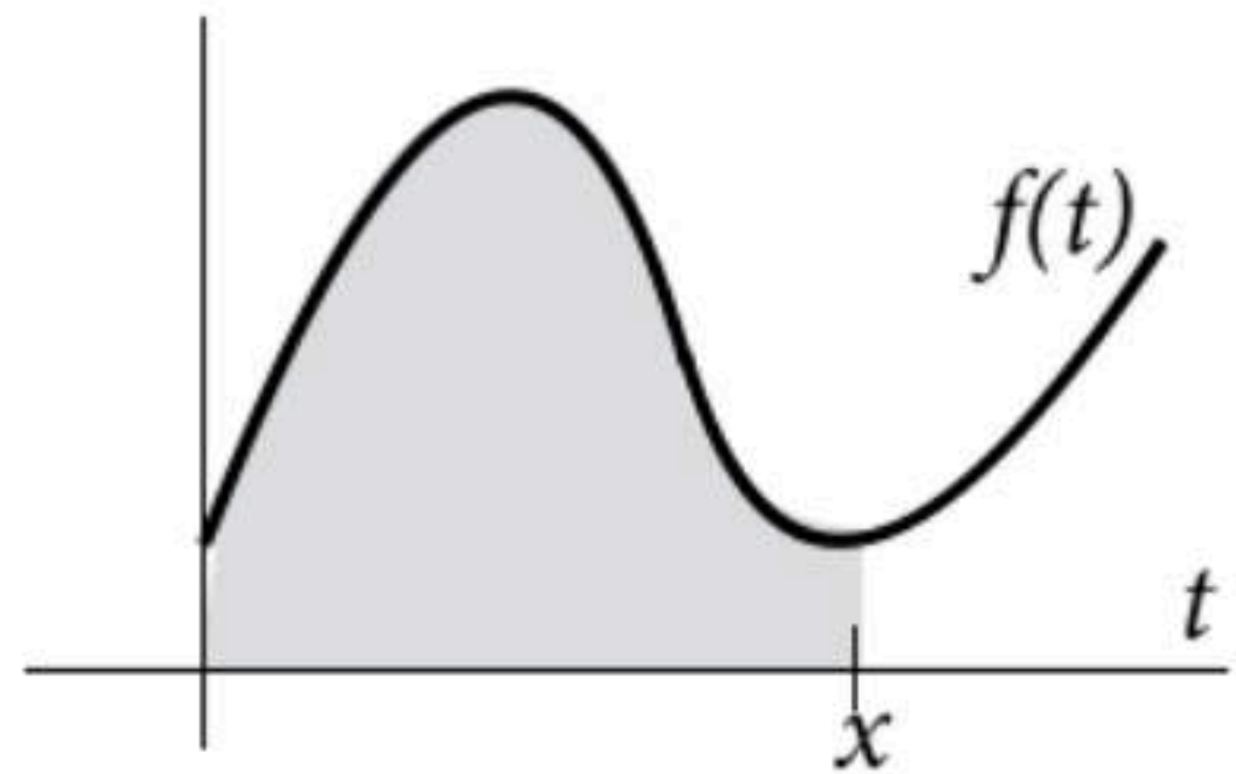
1. incremental problem-based homework

Example: Fundamental Theorem of Calculus

Problem 6 - sketch *F(x)* given *f(x)*

7. For each function $f(t)$ shown below, the related function $F(x) = \int_0^x f(t)\, dt$ is an accumulation function (see Page 21 # 7), adding up all the area under $f(t)$ from $t = 0$ to $t = x$. An example x is shown for each function. Sketch each $F(x)$ on the axes below.





$F(x)$

$F(x)$

1. incremental problem-based homework

Example: Fundamental Theorem of Calculus

Problem 7 - find distance given speeds

from Hughes-Hallett

2. The space shuttle is taking off from Cape Canaveral in Florida, and a NASA observer on the ground is measuring its speed $v(t)$, in meters per second, at intervals of 3 seconds, for the 12 seconds when it is still close enough to do so. The collected data is shown in the table.

t	0	3	6	9	12
$v(t)$	20	23	24	26	31

(a) Make a graph and plot the data points.

(b) Find an upper estimate (you will need to choose between a left sum and a right sum) for the distance traveled by the space shuttle between in the first 12 seconds of flight.

(c) Find a lower estimate of the distance traveled.

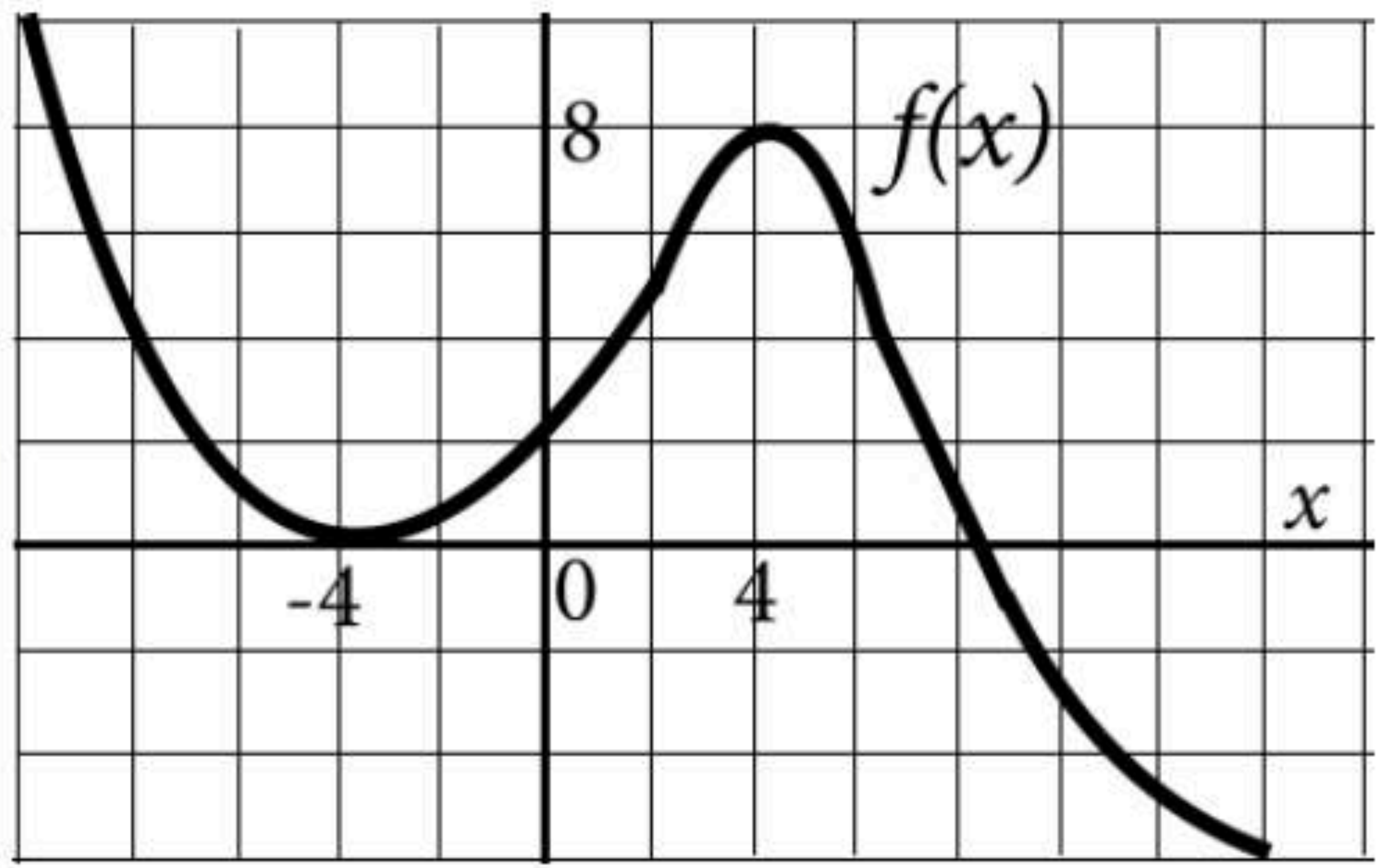
(d) How could you find a more accurate estimate? What do you think the actual distance is?

1. incremental problem-based homework

Example: Fundamental Theorem of Calculus

Problem 8 - integral as accumulated area

3. The picture to the right shows the graph of $f(x)$.



(a) The area of a region below the x-axis is usually taken to be *negative*. Why do you think this is?

(b) Explain the difference between "the area under $f(x)$" and "the area between $f(x)$ and the x-axis." On the graph above, shade in the regions corresponding to each of these descriptions. Then estimate each of these two numbers, for the graph of $f(x)$ shown.

(c) Use the picture to estimate

$$\int_{-8}^{14} f(x)\,dx.$$

1. incremental problem-based homework

Example: Fundamental Theorem of Calculus

Problem 9 - FTC - part I as distance-speed

4. Suppose that the function $f(t)$ gives the velocity of a car at time t.

(a) Explain why the *distance* traveled by the car from time $t = a$ to $t = b$ is given by

$$\int_a^b f(x)\,dx.$$

(b) Suppose that you had a function, let's call it $F(t)$, for the position of the same car at time t. Explain why the distance traveled by the car from time $t = a$ to $t = b$ is $F(b) - F(a)$.

(c) Explain why $F'(t) = f(t)$.

1. incremental problem-based homework

Example: Fundamental Theorem of Calculus

Problem 10 - FTC part I - STATEMENT

5. (Continuation) The *Fundamental Theorem of Calculus* states that, if the function $f(x)$ is continuous on the interval $[a, b]$, and there is a function $F(x)$ so that $f(x) = F'(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

The function F is called an *antiderivative* for f.

(a) For the function $f(x) = x$, find the function $F(x)$.

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1. incremental problem-based homework

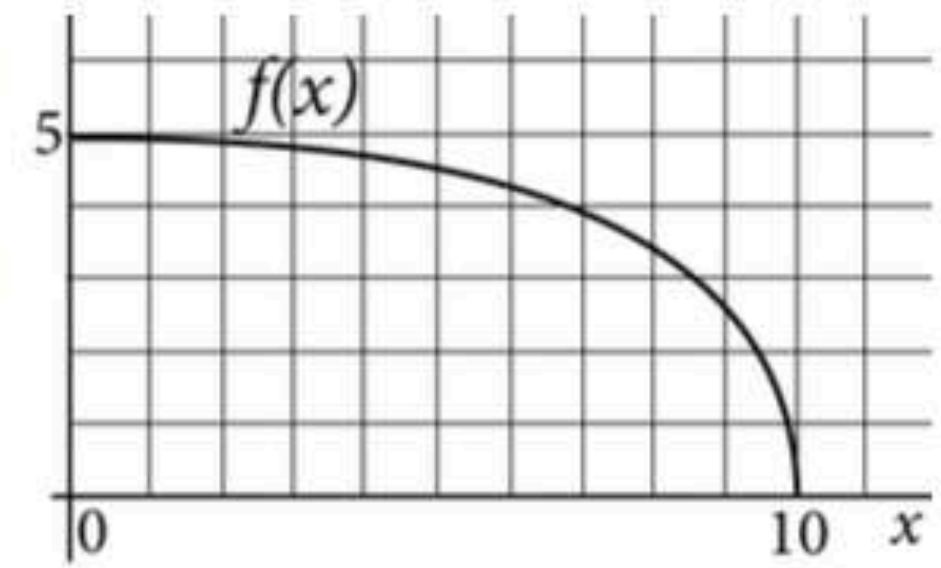
Example: Fundamental Theorem of Calculus

Problems 11-14 - practice FTC part I

1. Use the Fundamental Theorem of Calculus to evaluate $\int_0^2 (4x^3 - 3x^2)\, dx$.

from Hughes-Hallett

2. The value of the integral $\int_0^{10} f(x)\, dx$ is either 30, 40, 50 or 60. Which one is it? Explain.



3. Find an *antiderivative* (a function whose derivative is the given function) for each of the following functions.

(a) $f(x) = 3$ **(b)** $g(w) = \dfrac{1}{w} - w$ **(c)** $p(z) = \dfrac{1}{z^3}$

(d) $h(t) = \sqrt{t}$

from PEA

4. (Continuation) Suppose that you also know that $F(0) = 2$, $G(1) = 3$, $P(1) = 1/2$, and $H(1) = -1/3$, where F, G, P and H are the antiderivatives of f, g, p and h, respectively. Find $F(x)$, $G(w)$, $P(z)$ and $H(t)$.

1. incremental problem-based homework

Example: Fundamental Theorem of Calculus

Problem 15 - information about *f* from *f'*

from Hughes-Hallett

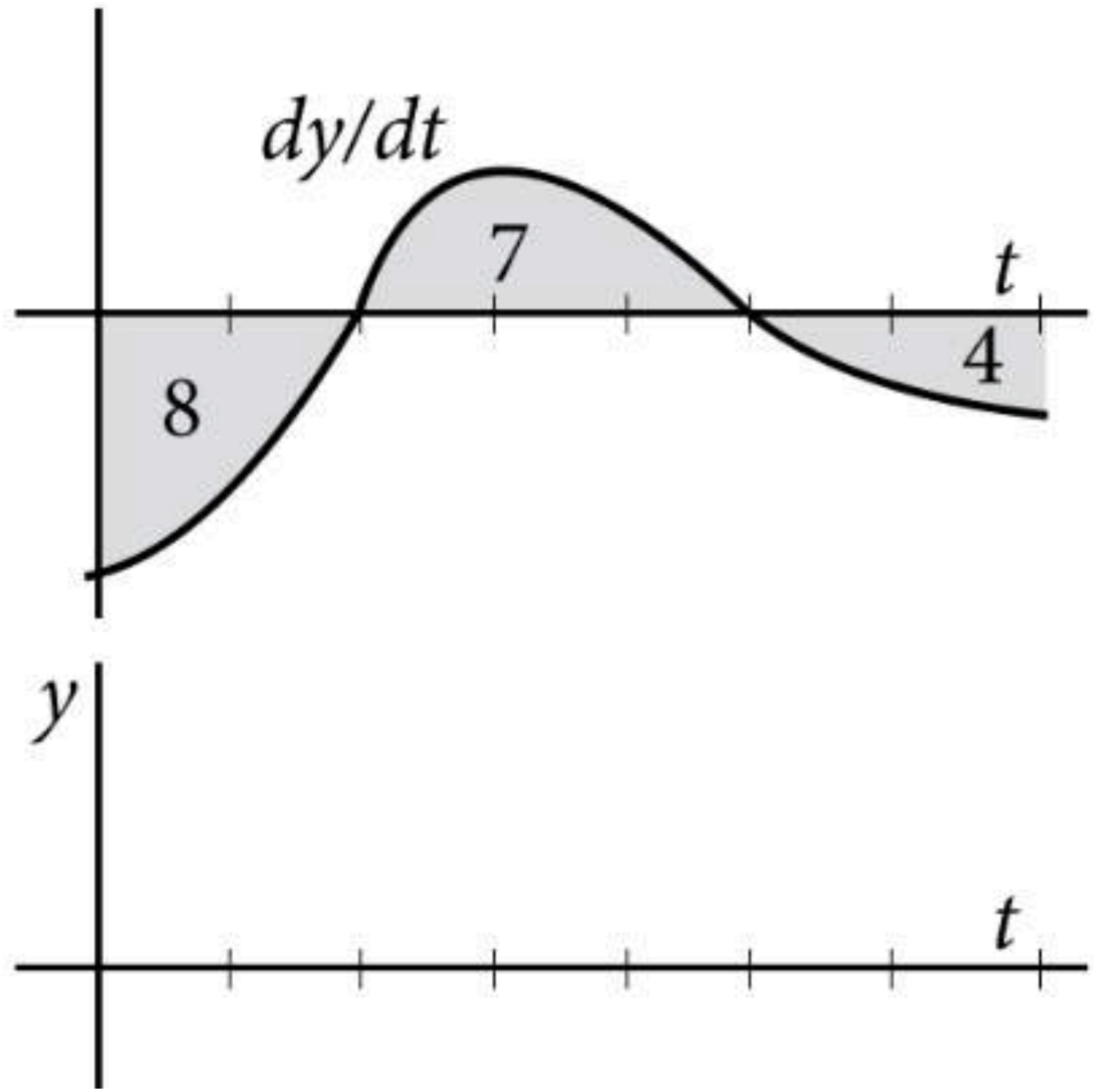
5. The graph of dy/dt is shown to the right. The area of each region is as indicated. Suppose that $y = 3$ when $t = 0$, i.e. $y(0) = 3$.

(a) Which y-values can you determine? Determine them.

(b) Where (which t-values) are the maxima, minima and inflection points of $y(t)$?

(c) dy/dt has a maximum at $t = 3$. What happens at the corresponding point on the graph of $y(t)$?

(d) Use all of this information to sketch an accurate graph of $y(t)$.



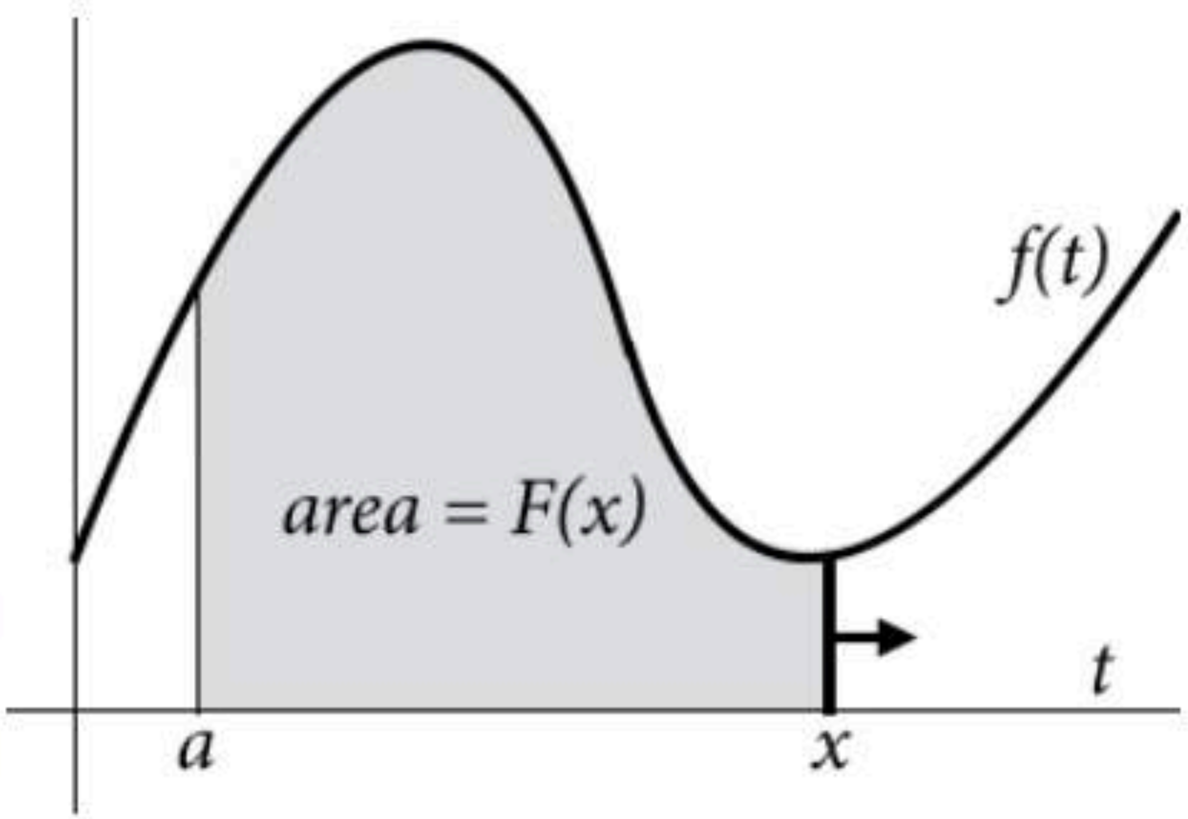
1. incremental problem-based homework

Example: Fundamental Theorem of Calculus

Problem 16 - understand FTC part II (!!!)

3. Let F be an accumulation function for $f(t)$, scooping up area starting at some fixed t-value a and ending at some t-value x:

$$F(x) = \int_a^x f(t)\, dt.$$



(a) Explain the difference between the meaning of the variable t and the meaning of the variable x in the above equation.

(b) Suppose we want to find the rate of change $F'(x)$ of the function F. Explain why

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}.$$

(c) Mark $x+h$, h, $F(x+h)$ and $F(x+h) - F(x)$ on the picture above (assume h is a small positive number).

(d) Using an argument using the picture above and what you marked on it in (c), and involving the statement "height is area divided by width," explain why $F'(x) = f(x)$.

1. incremental problem-based homework

Example: Fundamental Theorem of Calculus

Problem 17 - prove FTC part II

4. (Continuation) The FTC actually has two parts. The First Fundamental Theorem of Calculus (which we already saw) states that if $f(t)$ is continuous on $[a, b]$ and $F'(t) = f(t)$, then $\int_a^b f(t)\, dt = F(b) - F(a)$. The Second Fundamental Theorem of Calculus states that, under the same hypotheses (i.e. in the same situation),

$$F(x) = \int_a^x f(t)\, dt$$

is an antiderivative of f, and that

$$\frac{d}{dx}\int_a^x f(t)\, dt = f(x).$$

(a) Use problem 3(d) to explain why $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\, dt = f(x)$.

(b) Explain why it is not possible to compute the definite integral $\displaystyle\int_0^x \cos(t^2)\, dt$.

(c) Nonetheless, compute $\dfrac{d}{dx}\displaystyle\int_0^x \cos(t^2)\, dt$.

1. incremental problem-based homework

Example: Fundamental Theorem of Calculus

Problem 18 - understand what *f'* tells us

1. The graph of of $f'(x)$ is shown to the right.

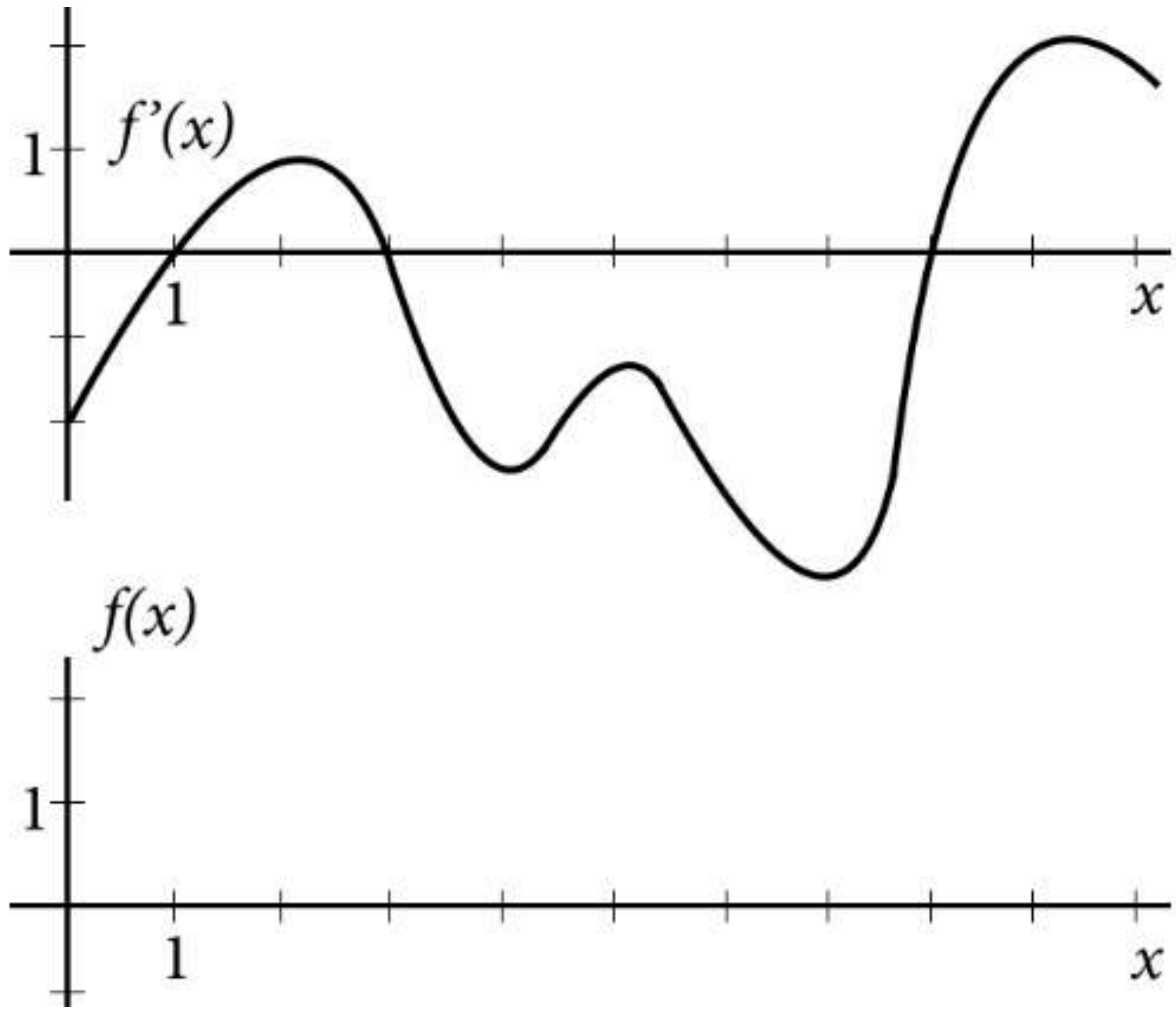
(a) Find the x-coordinates of any local maxima and minima.

(b) Find the x-coordinates of any inflection point(s).

(c) Determine the interval(s) over which the function is decreasing.

(d) Determine the interval(s) over which the function is concave-up.

(e) Suppose $f(0) = 1$. Sketch a graph of $f(x)$ that is as accurate as you can make it.



1. incremental problem-based homework

Math 15

trigonometric derivative

1. Find the derivative of $\arcsin x$, using the same method we used for $\arctan x$.

2. Consider the curve $e^t = 0.03t + 0.97$.

linearization

(a) Explain why there is a solution near $t = 0$.

(b) Replace e^t by its linearization, and use this to find a (approximate) solution for t.

3. We have previously shown that the Power Rule applies to functions of the form $f(x) = x^k$ where k is an integer, or the number $1/2$. Let's prove it for all rational powers, so for

$$y = x^{m/n}$$

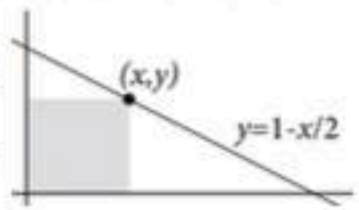
Power Rule

where m, n are integers.

(a) Raise each side of the equation above to a power to eliminate fractions in the exponents.

(b) Use implicit differentiation to find $\frac{dy}{dx}$, and simplify your answer to a familiar form.

4. Suppose that you wish to maximize the area of a rectangle, which has one corner at the origin and the opposite corner on the line $y = 1 - x/2$, as shown.



(a) Write an expression for the area of the rectangle, which is a function of x and y.

(b) Rewrite the expression for the area, in terms of just x.

(c) Use your differentiation skills to find the dimensions of the rectangle that maximize its area!

optimization (with guidance)

(d) Choose a different point on the line, find the area of the corresponding rectangle, and confirm that it is less than the area of the one you found in (c).

5. Find the value of c so that the function $f(x) = x \cdot e^{cx}$ has a critical point at $x = 5$.

6. Sketch a possible graph of $f(x)$, given that:

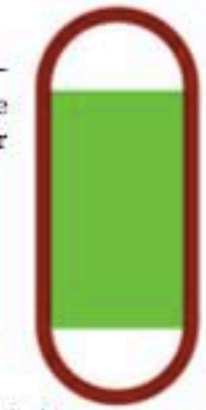
critical points

- $y' = 0$ for $x = 0$ and $x = 10$
- $y' > 0$ for $x < 10$
- $y' < 0$ for $x > 10$
- $y'' = 0$ for $x = 0$ and $x = 5$
- $y'' < 0$ for $x < 0$ and for $x = 10$
- $y'' > 0$ for $0 < x < 5$

graph sketching

Math 15

optimization application



7. A 400-meter running track is made of two parallel straightaways, connected by semicircular curves, as shown. Suppose that you want to choose the dimensions of the track to maximize the area of the rectangular soccer field at its center. Let's call the length of the straightaway x.

(a) What range of x-values makes physical sense?

(b) Find the perimeter of one curved end, in terms of x.

(c) Use part (b) to find the width of the field, in terms of x.

(d) Find the area of the field, in terms of x.

(e) Use your skills to determine how long should the straightaways should be!

8. Find the derivatives of the following functions:

(a) $L(t) = \ln(1 + t^2)$

derivative practice

(b) $T(x) = \arctan(4x)$

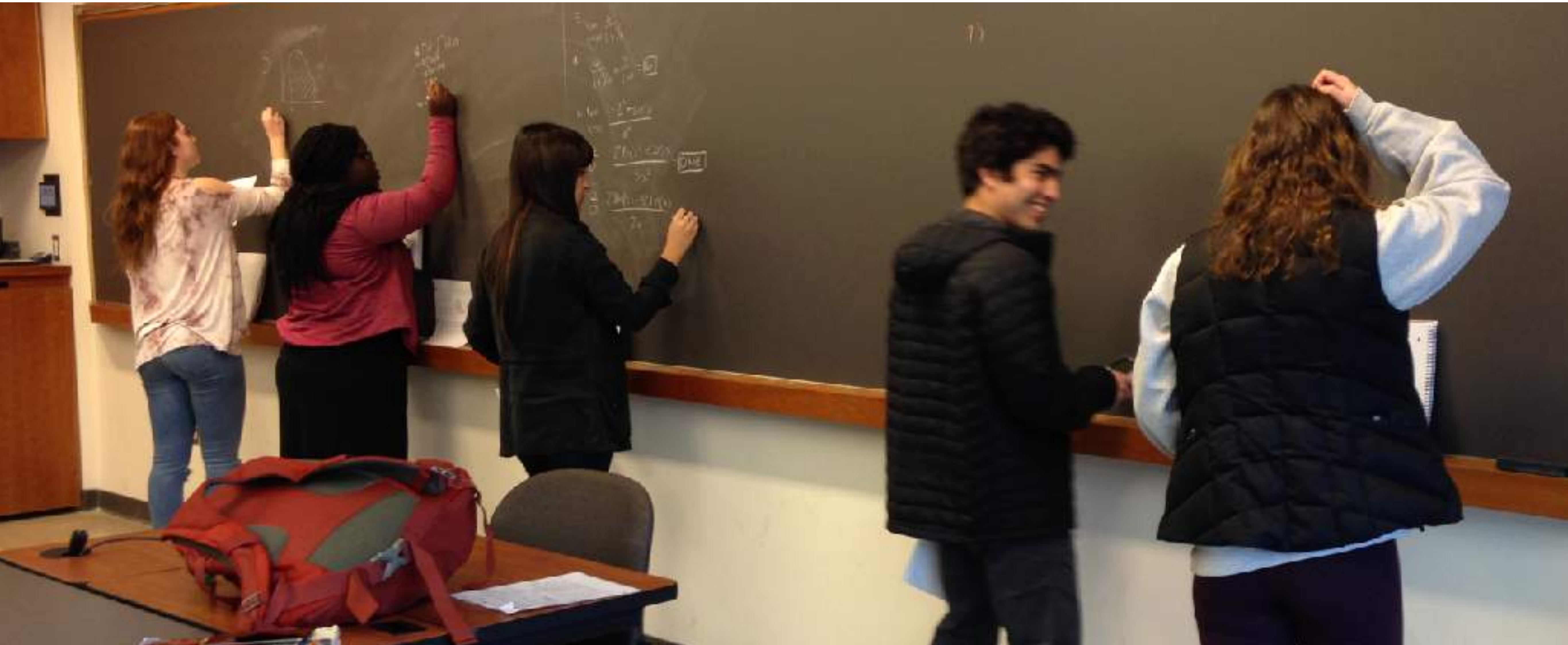
9. Sam says to Cam, "if $f''(0) = 0$, then f has an inflection point at $x = 0$." Cam says, "I don't think that's correct, but I'm having trouble thinking of a counterexample." Who is right – Sam or Cam? Find evidence to support your claim.

inflection points

Example: This night's homework includes 9 problems on 8 different topics

Two aspects to “Harkness” math curriculum (developed at Phillips Exeter Academy):

1. incremental problem-based homework
2. **student-centered discussion classes**



2. student-centered discussion classes

- Students are randomly assigned into groups of about 8
- Each group is responsible for writing a **correct solution to each problem** on the board and making sure **everyone understands** it
- When they are ready, I choose a student for each problem, to stand at the board and explain the solution.

2. student-centered discussion classes

- Part 1: Students write up a solution to whichever problem they choose, often in collaboration with others;
- Part 2: Students sit down, explain solutions to each other one by one, ask questions, correct errors, and discuss;
- Part 3: I choose (with an agenda) a student to explain each problem.

2. student-centered discussion classes

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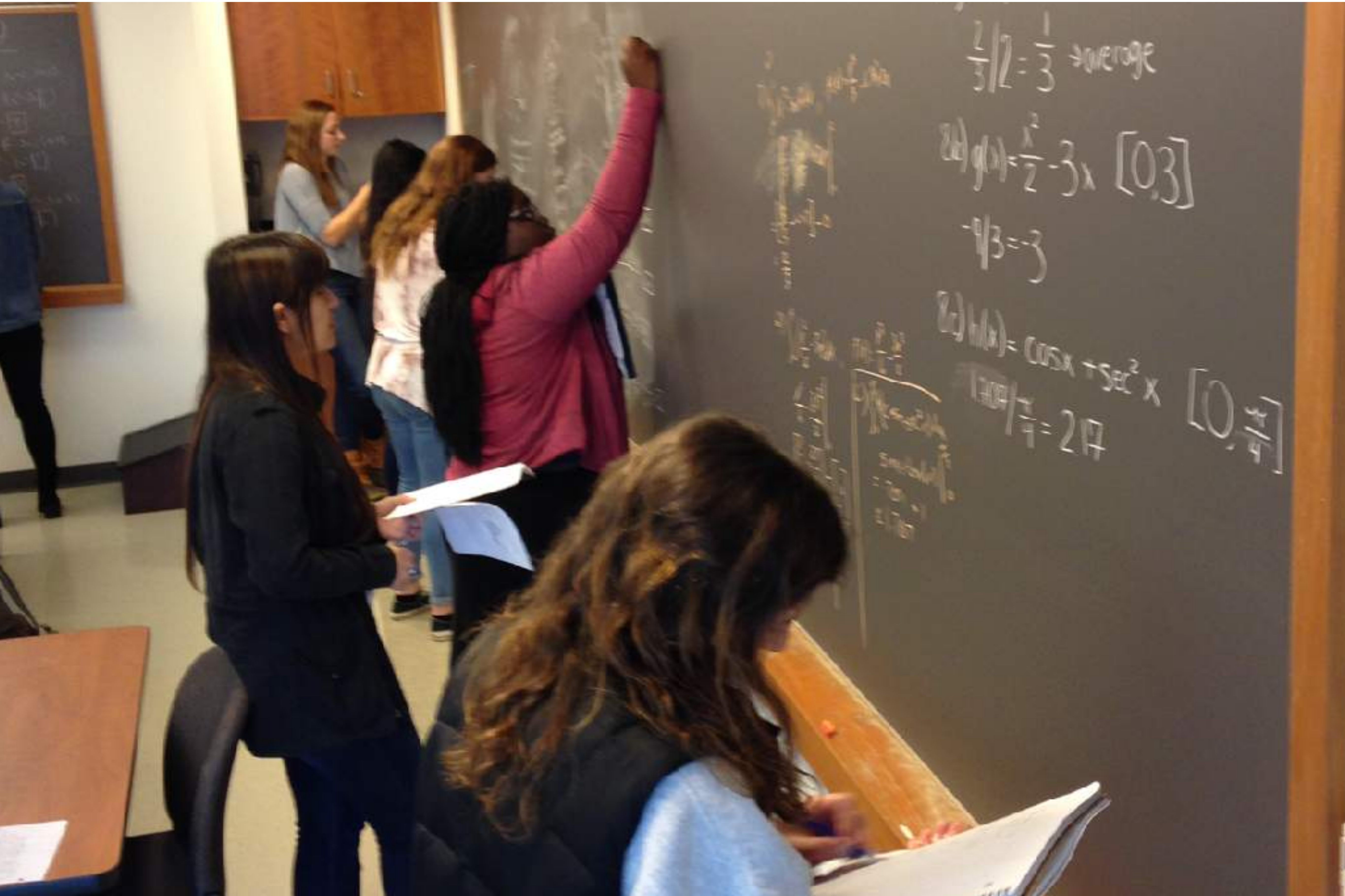
2. student-centered discussion classes



2. student-centered discussion classes

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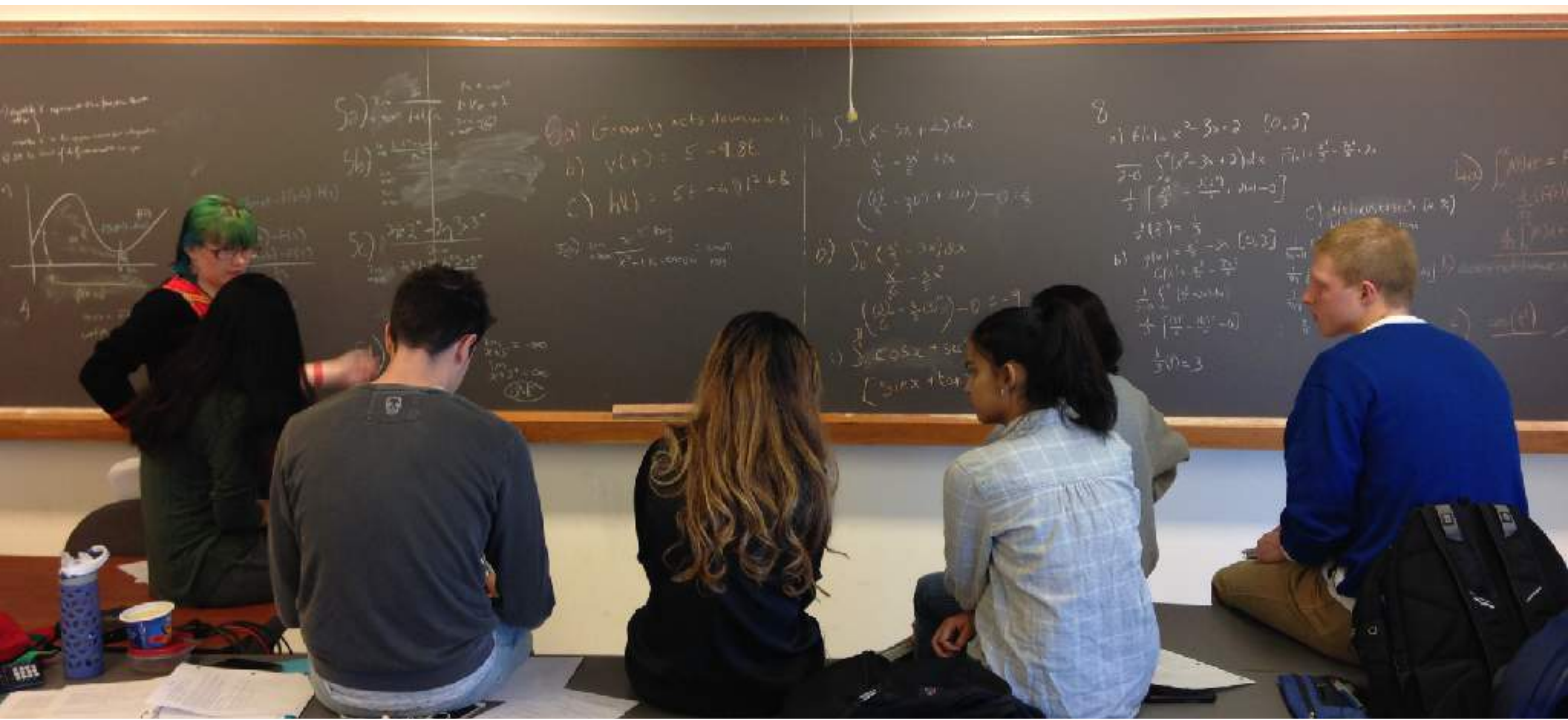
- Part 2: Students sit down, explain solutions to each other one by one, ask questions, correct errors, and discuss



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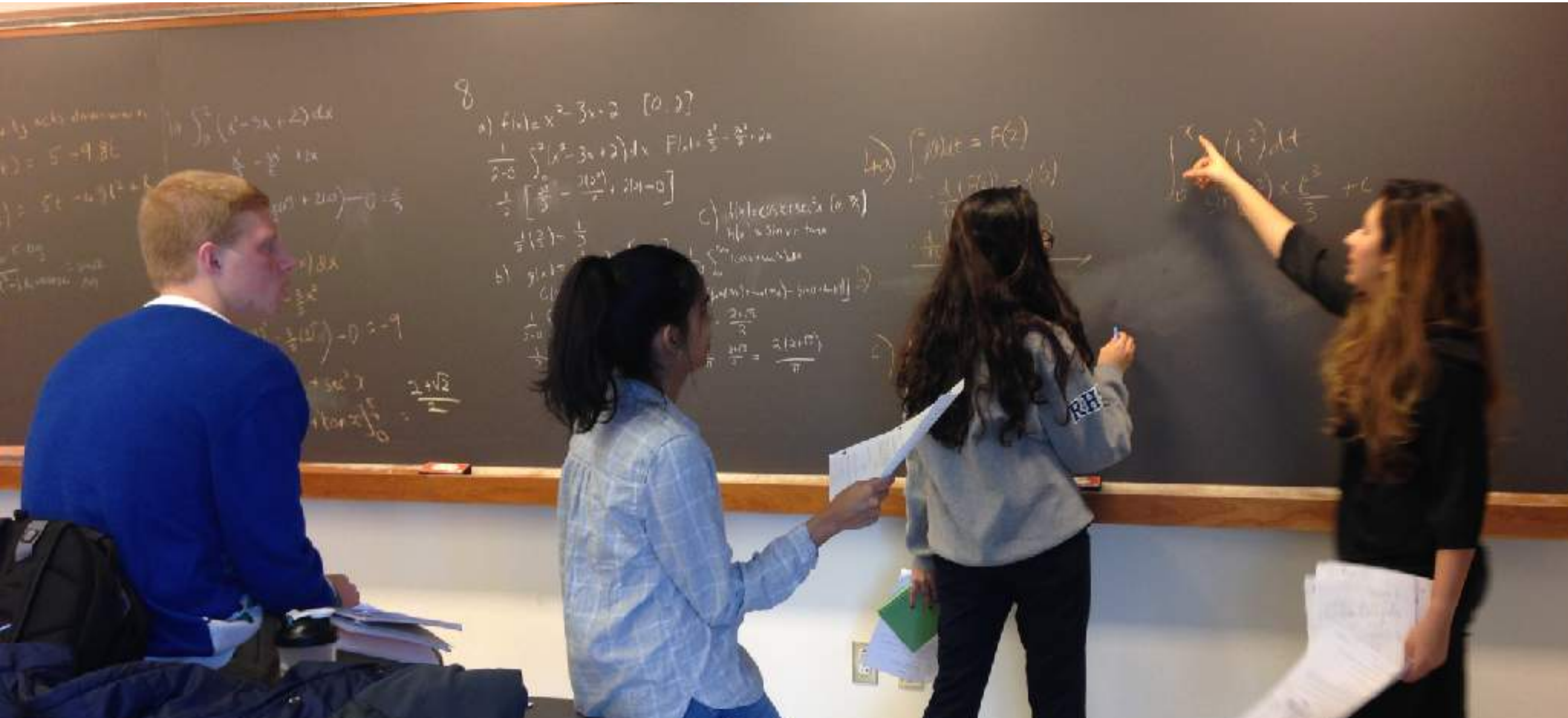

2. student-centered discussion classes

- Part 2: Students sit down, explain solutions to each other one by one, ask questions, correct errors, and discuss

2. student-centered discussion classes

- Part 3: I choose (with intention) a student to explain each problem.
  - Choose a student who did a problem one way, to explain it a different way
  - Choose a student lacking confidence, to explain the hardest problem

Summary of Harkness math instruction

1. incremental problem-based homework
   - ideas & skills are developed through problems (no lectures)
   - a given topic is developed over the course of several weeks
   - many topics developed simultaneously

2. student-centered discussion classes
   - students explain solutions to each other and ask each other questions
   - instructor interjects only when needed

Given the choice between working on a problem that I already know how to solve, or on a problem that I don’t know how to solve, yet, I would rather work on the problem I

know how to solve	don’t know how to solve

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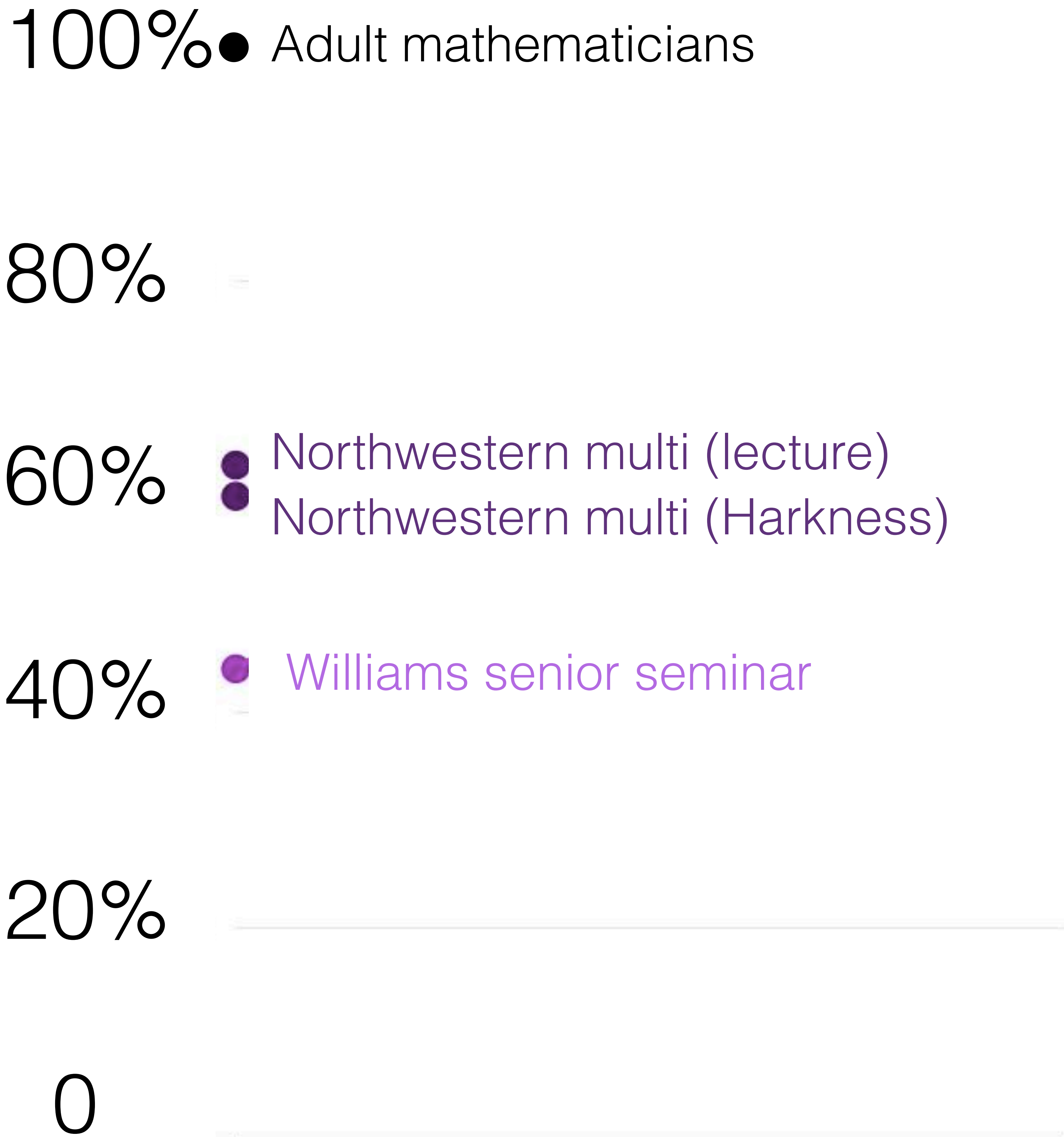

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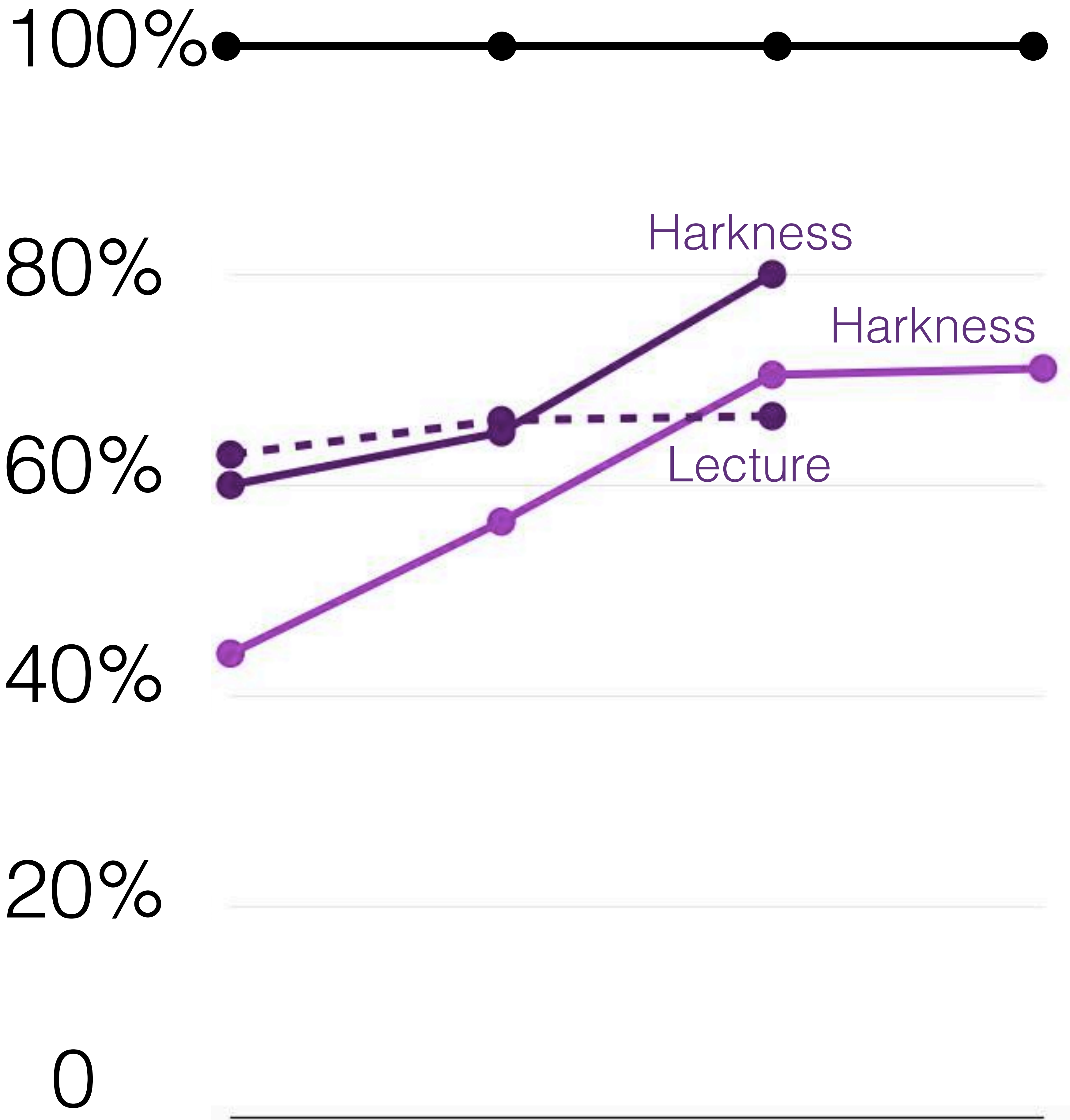

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For more resources:

- My web site has curricula I have written for:
  - Single-variable calculus
  - Multivariable calculus w/o linear algebra
  - Multivariable calculus with linear algebra
  - Senior seminar on billiards and geometry
- Phillips Exeter Academy teaching materials:
  - https://www.exeter.edu/mathproblems
- My article in PRIMUS:
  - *Inquiry-based learning in a first-year honors course*
  - Available at arxiv.org/abs/1606.08834