Gerrymandering with Math

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Exeter, NH
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The problems in this text

I (Diana Davis) wrote these problems for a discrete mathematics course at Swarthmore College in Fall 2018. The entire book of problems is much larger, and you can find it on my academic web site. The method of instruction used with these problems is based on the curriculum at Phillips Exeter Academy, a private high school in Exeter, NH. There were no lectures; students worked on problems for homework, and then spent class time discussing their solutions.

Topics explored in this course were graph theory, map coloring, apportionment, districting, gerrymandering methods and metrics, voting methods, algorithms, and proof writing. This document contains only the gerrymandering problems. Most of the problems about graph theory and apportionment were taken directly from the *Discrete Mathematics* book, written by Rick Parris and other members of the Phillips Exeter Academy Mathematics Department. This document contains only gerrymandering problems, plus a few other problems that I wrote; the Exeter problems are omitted, but are freely available on the Exeter web site. If you create your own text using these problems, please give appropriate attribution, as I am doing here.

It worked well to study apportionment and map coloring before studying gerrymandering, because students became familiar with maps, and worked on cutting 435 into 50 integer pieces, before trying to cut a map into pieces. Voting methods also tied in nicely with these topics, so I have included some problems on voting methods as well.

One downside of starting the course with graph and map coloring is that, when students thought about graphs that represent maps, they thought of vertices as regions, with edges connecting adjacent regions. So when I wanted to introduce the idea of a random walk on the graph of districting plans, students had a lot of trouble grasping what such a graph represents, because they wanted vertices to be districts and edges to represent adjacency.

To counter this, I introduced (using Exeter problems) *word graphs*, where vertices are words, and an edge connects two words if you can get from one to the other by changing one letter. These are similar to graphs of districting plans, where vertices are plans, and an edge connects two plans if you can get from one to the other by changing the assignment of one district. So every time I gave a problem about the graph of districting plans, I preceded it by a problem about word graphs, to get students in that mindset.

Students had to type up and hand in two proofs per week. Problems with bold numbers were eligible for this.

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1. *A word graph.* Consider a graph whose vertices are English words, with an edge connecting two vertices if it is possible to transform one into the other by changing one letter. Example word graphs are given below for the words JESS, PEST, JEST and also for the words ALY, AMY, ANY.

(a) Create the graph for the words EAT, PAN, PAT, RAN, RAT, RUN, RUT.

A graph is *connected* if it is possible to get from any vertex to any other by following edges of the graph. A graph is *planar* if it is possible to draw it in such a way that no edges cross.

(b) Is your word graph connected? Is it planar?

Gerrymandering.[1] Once we have decided how many Congressional representatives a state gets – for instance, Washington has 10 – we then need to do *districting* of each state. For example, Washington is cut up into 10 geographic regions, which each elect a representative. How should we cut it up? There are some laws for how the the geometry of the districting must be done:

(1) Districts should have equal population.

(2) Districts should be as compact as possible.

(3) Each district should consist of a single contiguous piece.

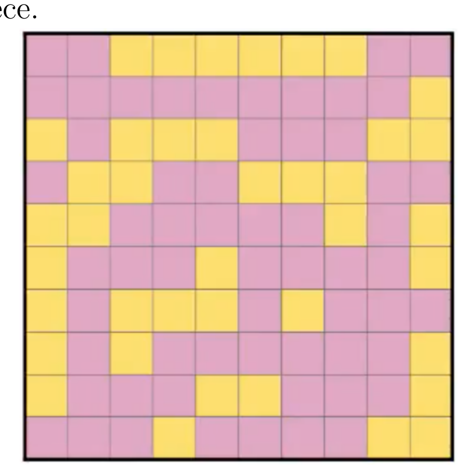
There are also non-geometric rules:

(4) Districts should respect county and city boundaries.

(5) Districts should respect communities of interest.

(6) Districts must comply with the Voting Rights Act.

We will begin with the geometric rules.



2. The 10×10 grid to the right represents 100 equal-populatoin towns in a state, each of which uniformly supports either the pink (60 towns) or the yellow (40 towns) party. This state gets 10 Congressional representatives, each of whom belongs to one of the two parties.

(a) How many representatives should each party get? What range of outcomes would you accept as "fair"?

[1]This section, including the figures, is taken from Moon Duchin's lecture "Political Geometry" at the 2018 Joint Mathematics Meetings, available at `https://www.youtube.com/watch?v=VddL0evo7QY`

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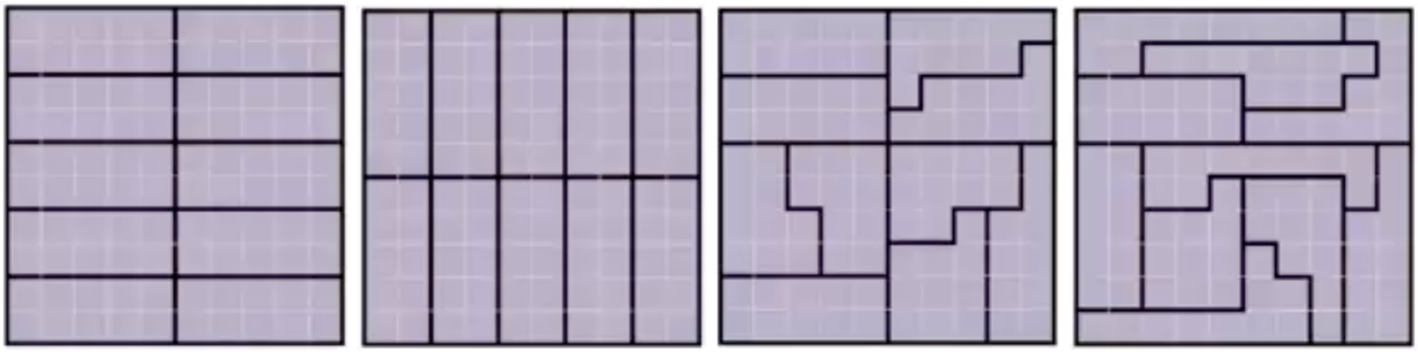
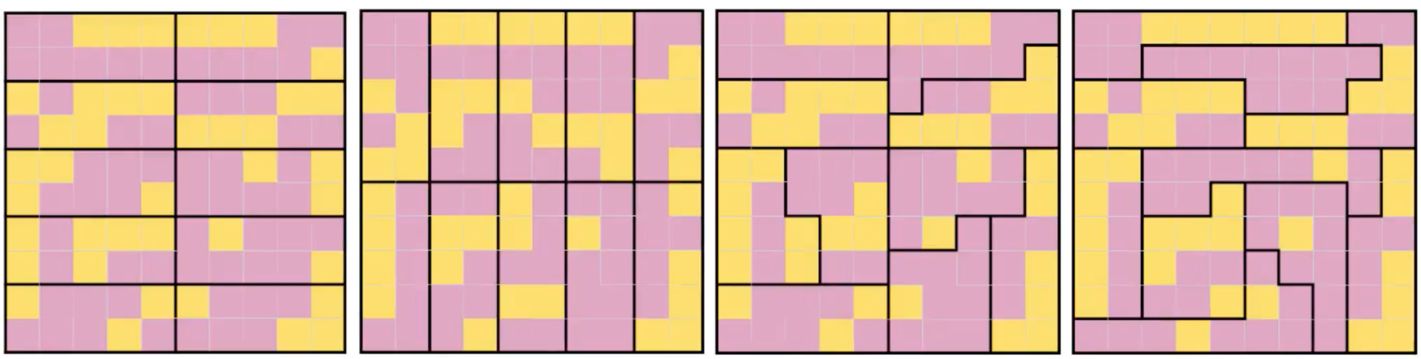
(b) Below are four proposed districting plans, for dividing the state into 10 districts. For each map, explain why it satisfies rules (1), (2) and (3) listed above. Which one seems the most "fair," based on the district shapes?



(make sure you finish part **(b)** before moving on)

(c) Below, each of the four districting plans has been overlaid onto the state map. A party *wins* a district if it has the most votes (squares of its color) in that district. Determine how many representatives each party wins, using each of the four maps.



(d) Do any of the maps seem to be designed to yield a particular outcome? If so, explain how the map has been engineered, or *gerrymandered*, to make this outcome happen.

Gerrymandering with Math

Homework for Monday

Try your hand at real-life gerrymandering.

(a) Go to the web site `districtr.org`, scroll down to the map, click on New Hampshire, and select the left purple button for "2 Congressional districts built out of 2020 VTDs."

(b) Using the paintbrush, paint the state in blue until the population bar gets to the "Ideal" line. Then click "Lock already-drawn districts" at the top, switch to yellow paint, and paint the entire rest of the state in yellow. These are your two Congressional districts.

(c) Click the "Evaluation" tab and click to open "Election details." From the dropdown menu, choose "2016 U.S. Senate Election." Notice that the overall vote share is 50.07% Democratic and 49.93% Republican. Close!

(d) Look at "By District" – these are for the blue and yellow districts *that you drew*. Which party won each of your two regions?

Time to gerrymander! Your goal is to make two districts where the populations of the districts are within 1% of ideal, *and*:

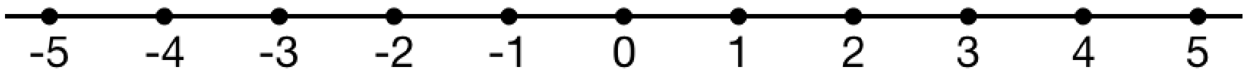
- If Democrats won one of your districts and Republicans won the other: your job is to create a map where Democrats win *both* districts.
- If Democrats won both of your districts: your job is to create a map where Republicans safely win one district, by getting at least 53% of the vote in that district.

(e) But where do the Democrats and Republicans live? Click the "Data Layers" tab and under "Statewide Elections," from the dropdown menu, choose "2016 U.S. Senate Election." Then check the box below for "Show election results." The red or blue tint tells you how that town voted.

(f) Unclick the "Lock already-drawn districts" box and use the paintbrush to modify your districts! When you get a winning map, click "Save" at the top and record the URL of your mapsterpiece: https://districtr.org/plan/____________

Gerrymandering with Math

1. *A random walk.* Stand at 0 on a number line and flip a coin: if it comes up heads, step one unit to the left; if tails, step one unit to the right.



(a) Using an actual coin, do a 10-step random walk and write down where you land: ______

(b) Repeat this exercise 9 more times, yielding a total of 10 numbers between -10 and 10. Record the outcomes of your 10 random walks (we will pool everyone's data in class).

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

2. Your friend did a 100-step random walk by flipping a coin 100 times.

(a) "I ended on 11," your friend says. What is your response?

(b) "I ended on 96," your friend says. What is your response?

3. In the maps we looked at yesterday, it seemed that the fourth map had been *gerrymandered* in favor of the yellow party. It would be nice to have a mathematical measurement to support this claim. People have argued that "bizarre shapes," or *non-compact* districts, indicate gerrymandering. Here are some ways of measuring *compactness* that have been proposed. For each state, you calculate the measure for each district, and then take the average over all the districts.

(1) *Total perimeter.*

(2) *Skew:* W/L, where W is the district's shortest diameter and L its longest.

(3) *Isoperimetric:* $16A/P^2$, where A is the district's area and P is its perimeter.

(4) *Square Reock:* A/S, where A is the district's area and S is the area of the smallest square that contains the district.

Using the maps from yesterday, calculate each of these numbers for the

(a) first and second (which are the same), **(b)** third, and **(c)** fourth map.

(d) Which measure do you think is the best for measuring compactness and lack thereof?

Historically, many people brought forward lawsuits alleging gerrymandering based on the "bizarre shapes" of the districts, using metrics like the ones above to justify the presence of gerrymandering. Some of them went all the way to the U.S. Supreme Court, where Justice Kennedy famously refused to endure "endless beauty pageants" based on districts' shapes. We need a new and different method. (Spoiler: random walks and outlier analysis!)

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Gerrymandering with Math

Geometric ways of measuring partisan gerrymandering, such as the Skew, Isoperimetric and Square Reock measures, tend to actually measure districts' *compactness*. It is arguably more relevant to directly assess whether a districting plan favors one political party over the other. Many ways of measuring this have been proposed; we will study two of them: *partisan symmetry* and *efficiency gap*.

Partisan symmetry. This measure uses the *seats-votes* curve, which has the *vote share* along the x-axis and the *seat share* along the y-axis. We'll follow convention, and use the Republican vote share. One election gives you exactly one data point on this curve. To construct the rest of it, political scientists use the assumption of *uniform partisan swing.*

4. In 2016, the percentages of Republican voters in New Mexico's three congressional districts were 34.9%, 62.7%, and 37.6%. Refer to the picture at the bottom of the page.[2]

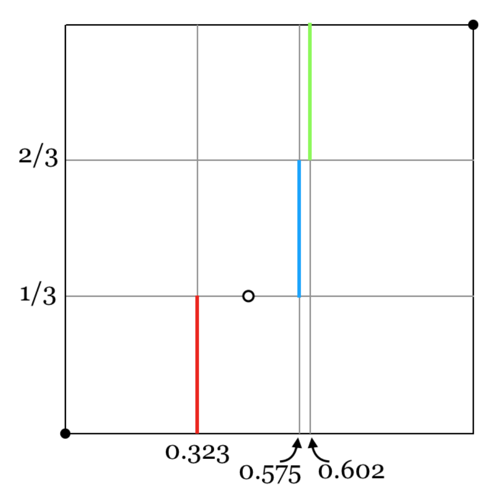
- Assuming an equal number of voters in each district, the Republican vote share was $(34.9 + 62.7 + 37.6)/3 = 45\%$, and the Republican seat share was 1/3, so the outcome of this election was $(0.45, 0.33)$. This is the white dot in the picture.
- Assuming "uniform partisan swing" means that we add an equal number of percentage points to each district until one of them goes over (or under) 50%. First, we add 12.4% so that the third district hits 50%. This yields the simulated election $(47.3, 75.1, 50)$ for a vote share of $(47.3 + 75.1 + 50)/3 = 57.5\%$ and a seat share of 2/3, meaning that the seat share jumps from 1/3 to 2/3 at a vote share of 0.575 (the blue line).
- Adding 2.7 more percentage points so the first district goes over 50% yields the simulated election $(50, 77.8, 52.7)$ for a vote share of 60.2% and a seat share of 3/3, meaning that the seat share jumps from 2/3 to 1 at a vote share of 0.602 (the green line).
- Going the other way from the true election results until all of the districts are under 50% requires subtracting 12.7 percentage points, yielding the simulated election results $(22.2, 50, 24.7)$, for vote share 32.3% and seat share 0/3, meaning that the vote share jumps from 0 to 1/3 at a vote share of 0.323 (the red line).

(a) Explain why the points $(0, 0)$ and $(1, 1)$ must both be on *every* seats-votes curve.

(b) Explain why the seats-votes curve is horizontal with 0 seat share between votes shares of 0 and 0.323, and horizontal with 1/3 seat share between vote shares of 0.323 and 0.575. Fill in and explain the other two horizontal parts of the seats-votes curve.

(c) Based on the seats-votes curve, do you think that New Mexico's districting plan favors Republicans, Democrats, or neither? Explain.

(d) Do you think that the assumption of uniform partisan swing is plausible? Explain.

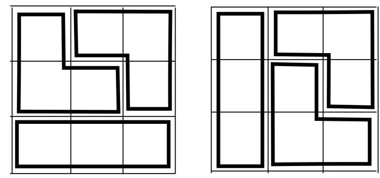


[2]This problem is from Moon Duchin's course Math of Social Choice, Math 19-02 at Tufts University.

Gerrymandering with Math

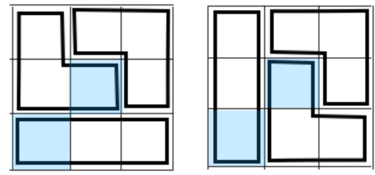
Homework for Tuesday

Consider a state consisting of 9 towns in a 3×3 block. Your job is to divide them into 3 districts, each consisting of 3 towns, where towns within a district must meet along an edge. Two possible ways of doing this are shown. Draw pictures of *all* possible ways of doing this. How many are there?



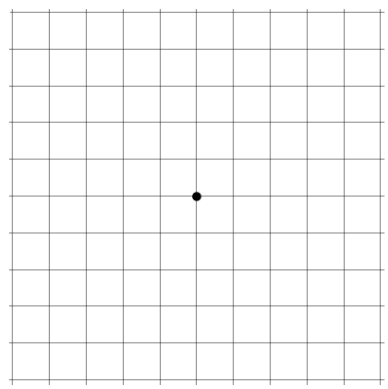
Gerrymandering with Math

1. (Continuation) By flipping which districts the two blue squares are assigned to, we can turn the districting plan on the left into the one on the right. Create a graph, where the vertices are districting plans of this 3×3 block, and an edge connects two plans if, as in this example, a swap of two squares (any two) transforms one into the other. Is it connected? Is it planar?



Record your graph here for future use:

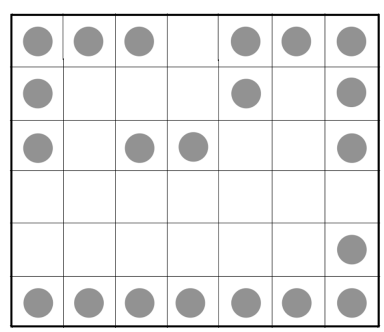
2. *A random walk in the plane.* Start at $(0, 0)$ on the integer grid in the plane, and move one unit up, down, right, or left, each with probability 1/4. You can do this by assigning each number 1,2,3,4 to a direction, and rolling a die to get random numbers (just ignore 5 or 6 when you get them). Do a 30-step random walk in the plane, and plot the entire path on the grid (or on a larger grid if necessary).



3. (Continuation) Suppose you did a random walk *forever.* What do you think is the probability that you would return, at least once, to your starting point, for a random walk on

(a) the integers, **(b)** the integer grid in the plane? **(c)** the integer grid in 3-space?

2. Here is a state consisting of 42 towns, of which 21 vote for the "Blank" party and 21 vote for the "Dot" party. Your job is to divide it into 6 regions, each consisting of 7 squares.[3]



(a) *Gerrymander for proportional representation:* Divide it so the Blanks win (have a majority) in three regions, and the Dots have a majority in the other three.

(b) *Gerrymander for Blank:* Divide it so that Blank wins as many regions as possible.

(c) *Gerrymander for Dot:* Divide it so that Dot wins as many regions as possible.

(d) The two classic gerrymandering techniques are *packing* your opponents into a few districts that they win by a lot, and *cracking* the remaining opponents so that they don't have quite enough in any other district to win. Above, did you pack? did you crack?

[3]This map is from *A formula goes to court: Partisan gerrymandering and the efficiency gap*, by Mira Bernstein and Moon Duchin, May 2017, available at `https://arxiv.org/abs/1705.10812`.

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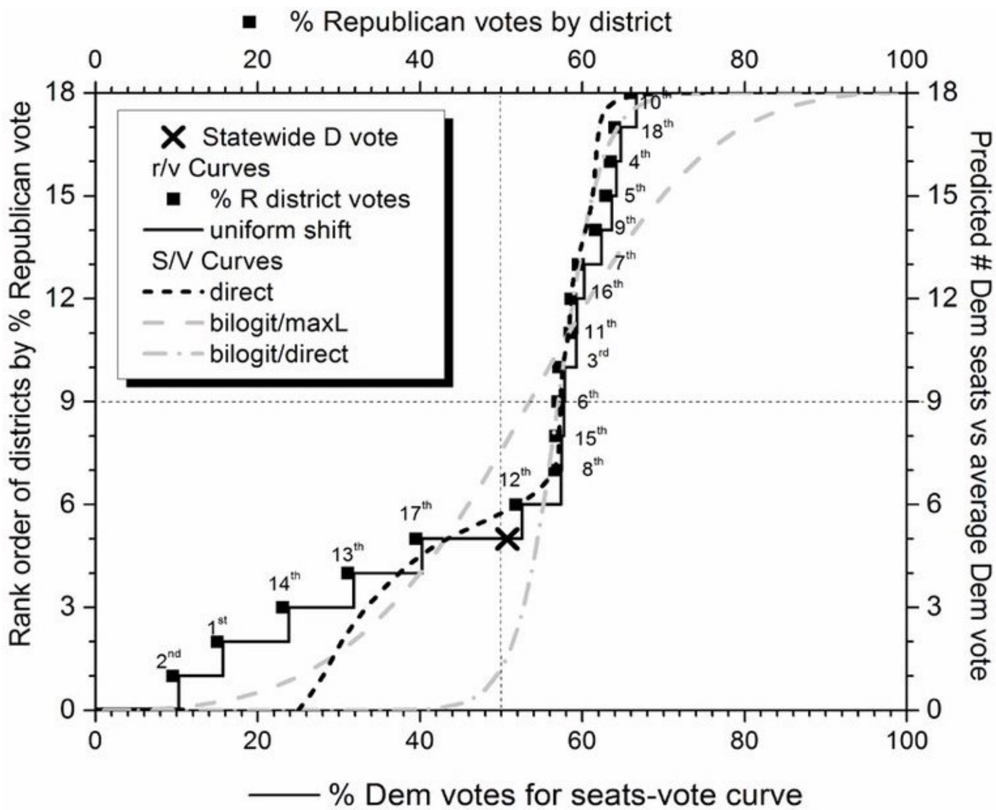
5. The chart below is from the paper *Measures of Partisan Bias for Legislating Fair Elections* by John F. Nagle. You can ignore everything except the bold stair-step curve, which is the seats-votes curve for Pennsylvania's 2012 Congressional election. Here, the x-axis is the Democratic vote share, and the y-axis is the Democratic seats (here *seats*, rather than seat share).



(a) The district that tips over 50% Democratic votes is marked for each stair step. Which districts are packed, and which are cracked? In favor of which party?

(b) The seats-votes curve does *not* pass through the central point $(0.5, 0.5)$ – or in this case, $(0.5, 9)$ – indicating there is partisan bias. In favor of Republicans or Democrats? Explain.

Two measurements in a seats-votes curve have particular meaning.

(c) Consider the horizontal distance from the central point to the curve (near the note "6th"). This is called the *mean-median* score. What is its meaning?

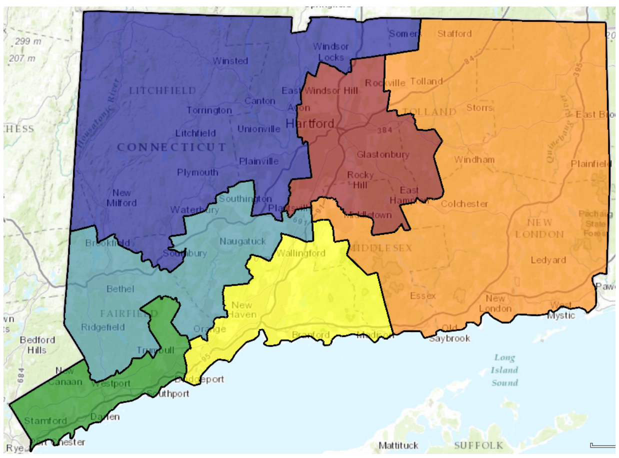
(d) Consider the vertical distance from the central point to the curve (near the "X"). This is called the *partisan bias* score. What is its meaning?

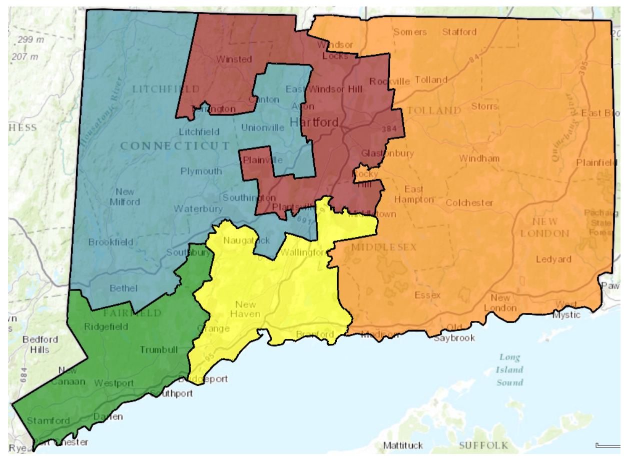
Gerrymandering with Math

Homework for Wednesday

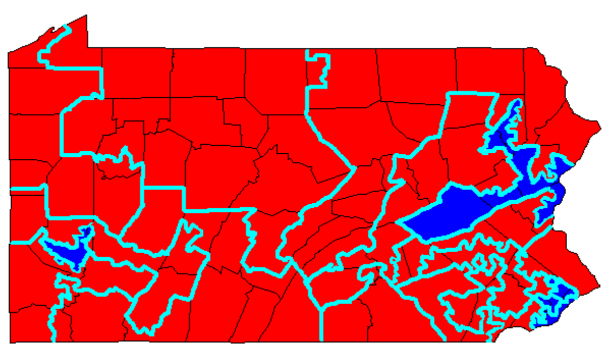
Let's take a look at maps from *your states*. Why do you think they are the way they are? Do you think there is gerrymandering? How would you go about proving it?

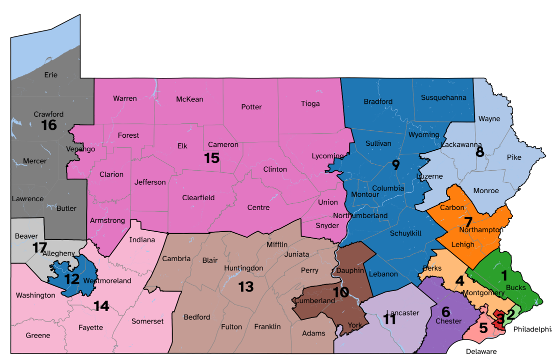
Connecticut. Below is the map that was in place until 2002 (left picture) with 6 districts, and the map that has been in place since 2003 (right picture) with 5 districts. Comment on what you notice.





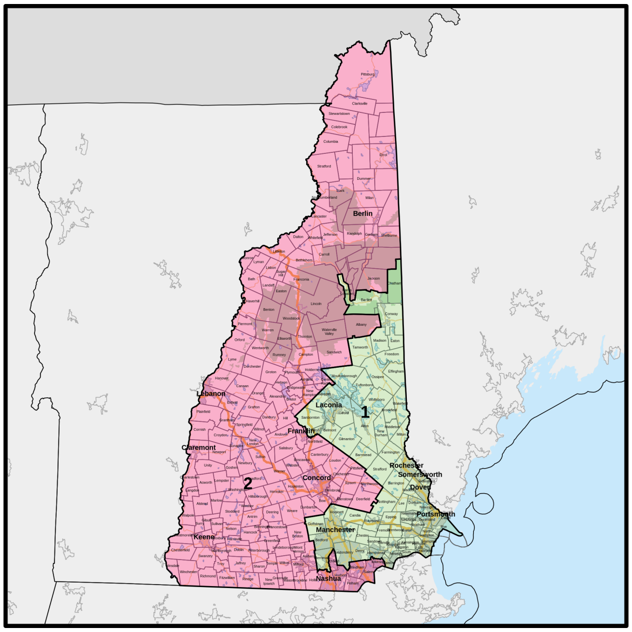
Pennsylvania. The districting plan up until 2019 managed to turn a 50-50 electorate into 13 seats for Republicans and 5 for Democrats (left picture). The state supreme court determined that the map was unconstitutional (the governor brought in Moon Duchin to analyze the situation!) but the legislature could not agree on a new map, so they had to bring in a "special master" to draw nonpartisan lines (right picture). Like Connecticut, Pennsylvania also lost one district, from 18 to 17, following the 2020 Census.

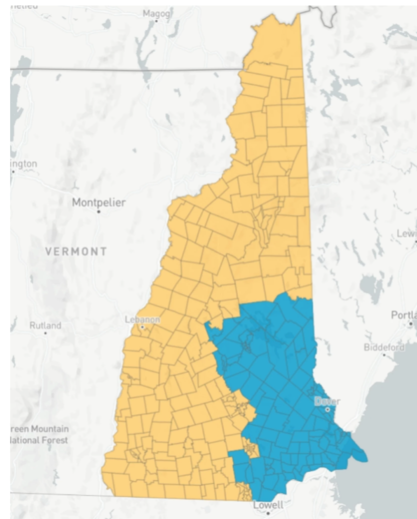




Gerrymandering with Math

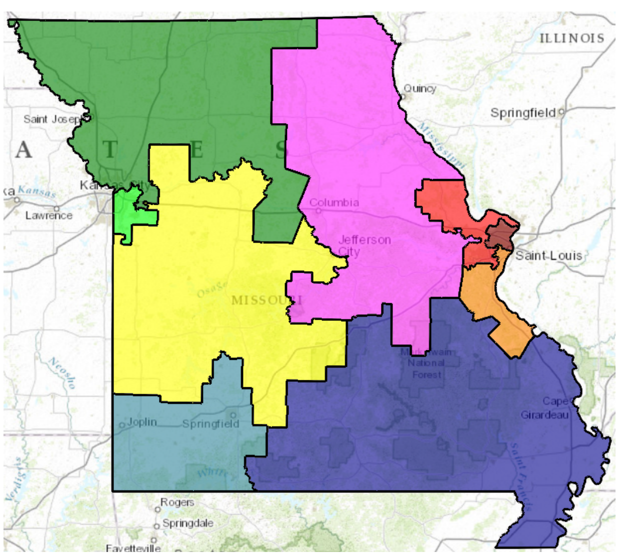
New Hampshire. The districting plan shown below left is essentially how it's been for many decades. Why is it like this? The proposed map on the right – well, you can read the caption. (They used Districtr just like we did!)

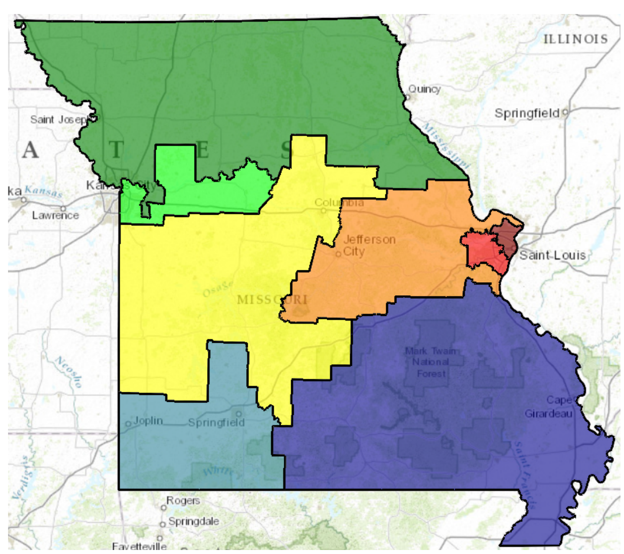




The map approved by lawmakers, but opposed by Gov. Chris Sununu, proposed moving Manchester, Concord and Nashua into the same district.

Missouri. Like Connecticut and Pennsylvania, Missouri also lost a district, going from 9 to 8 after then 2010 Census. Can you comment on how things have changed from the left picture (map up to 2013) to the right (map after 2013)?



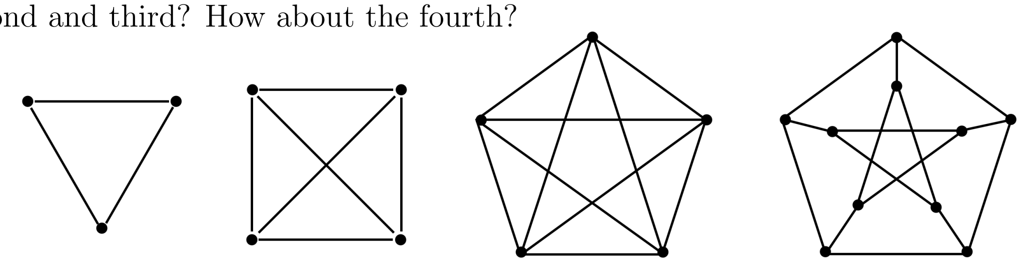


Gerrymandering with Math

1. *Word graphs*: As before, the vertices are words, and an edge connects two vertices if you can transform one word into the other by changing one letter.

(a) Make a graph for the word list *LAP, LIP, TAP, TIP, TIN, TOP, TON*.

(b) Can you find a word list whose word graph matches the first graph below? How about the second and third? How about the fourth?



2. *A random walk on a graph of districting plans.* Yesterday, you created a graph of all possible districting plans for our very simple 9-town, 3-district state, where two plans are connected if you can go from one to the other by flipping two towns' district assignments.

(a) Take a 30-step random walk on this graph: Assign numbers 1, 2, 3 or 1, 2, 3, 4 to the edges emanating from each vertex, choose a starting point, roll a die to get a random number, make a step accordingly, and repeat.

(b) There are two types of districting plans (2 of one and 8 of the other); does the proportion of how often you land on one or the other depend on what type you start on?

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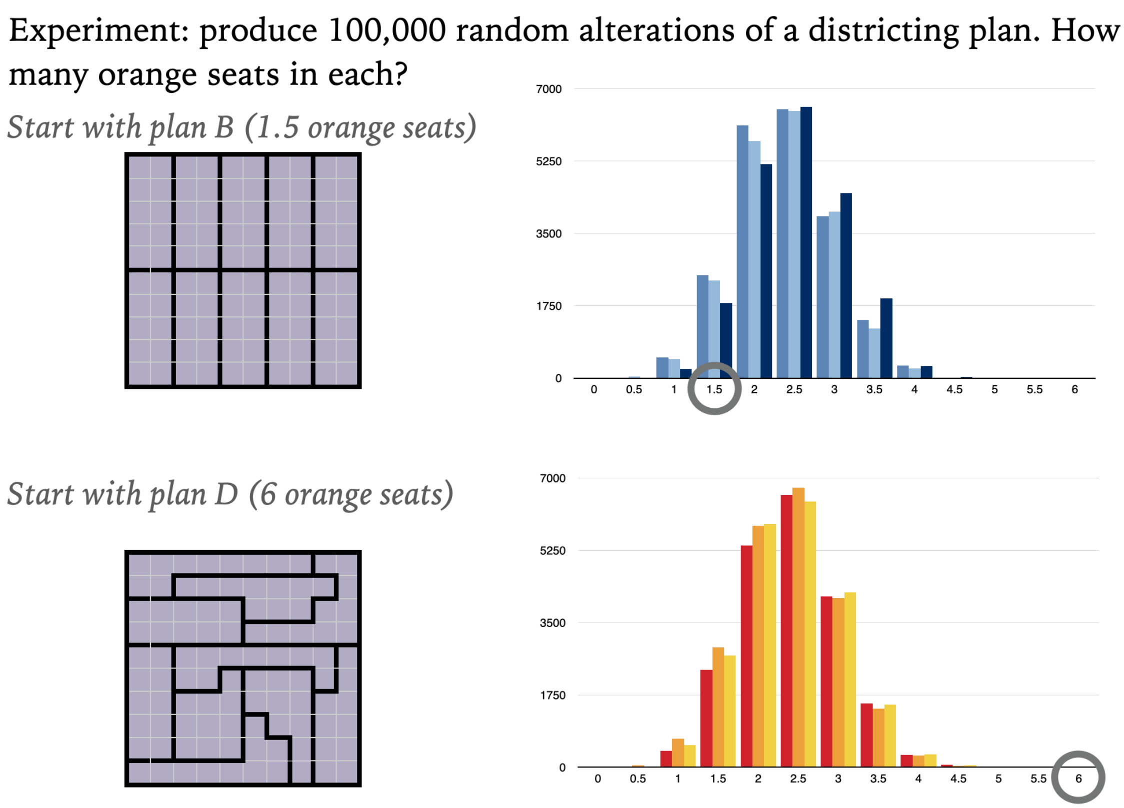
Gerrymandering with Math

We have seen geometric measures of shape (isoperimetric, Square Reock, ...) and measures of partisan asymmetry (mean-median, partisan bias). Measuring these things for just *one* plan gives us *one* data point. We can make up a threshold like "0.08" and say that a plan with a score above that is gerrymandered, but such thresholds are arbitrary, and therefore will not hold up in court. People may argue right back, "that's just how people live in this state: it's impossible to make a map that works out otherwise."

The new way to do it is to take a *random walk* on the *graph of districting plans*. The computer generates thousands of plans, and for each plan, you measure whatever is measure of choice: isoperimetric, mean-median, the number of districts that each party wins, etc. Then you make a *histogram* of how often you saw each value of the measure, and you see if the given plan is an *outlier*.

Below is a random walk on the space of districting plans for the 100-town, 10-district state with 60 pink towns and 40 orange towns that we saw on our first day.[4] The three different colors on each histogram represent the results from three different 100,000-step random walks.



3. Based on the graphics above, what can you say about plan B and plan D? Is either plan gerrymandered? Support your assertions with evidence.

4. The *graph of districting plans* for the 10×10 map is enormous. It is likely that plan B and plan D are far apart in the graph. Mathematicians worry whether the random walk is really exploring a representative sample of all districting plans. What do you think?

[4]This graphic, and the underlying calculation, are once again from Moon Duchin's talk.

Gerrymandering with Math

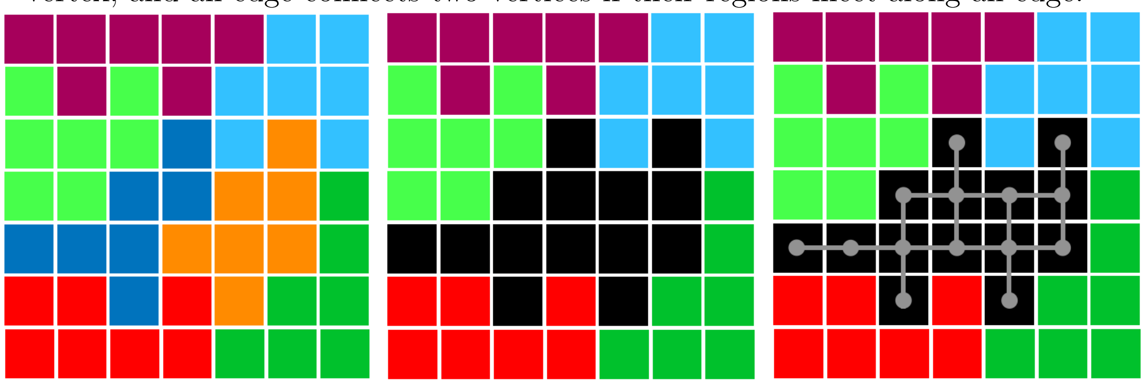
Homework for Thursday

- Write down some questions or topics you'd like to discuss with an expert on gerrymandering with math (Moon Duchin).
- Attend Moon's keynote talk on Wednesday afternoon at 5pm.
- Hang out with Moon after the talk and discuss the above!

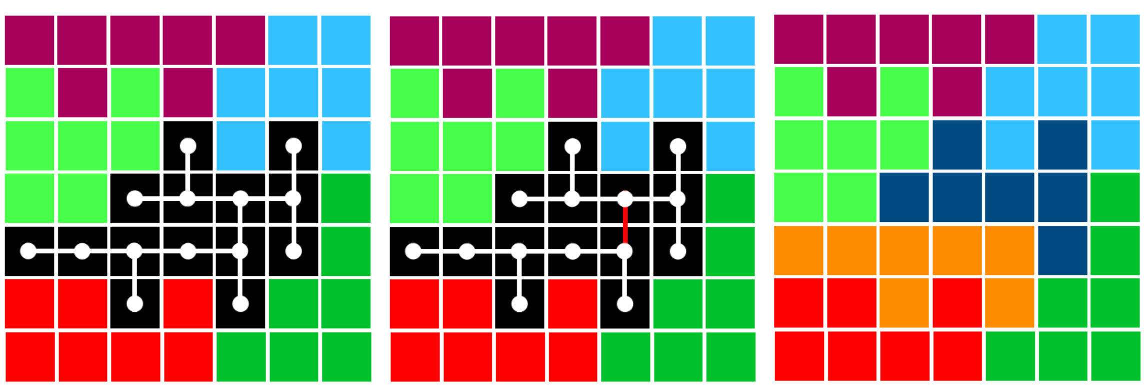
Gerrymandering with Math

When we made our graph of districting plans for our 9-town, 3-district state, we used the *flip* method, exchanging one square at each step. While flipping is fine for our tiny example, for a larger example like the 10×10 plan, or for a real state like Pennsylvania with over 400,000 Census blocks, flipping moves too slowly. Instead, we use *recombination*:

1. Start with a districting plan, here of a 7×7 grid broken into 7 districts.
2. Choose two adjacent regions, and merge them (here, the orange and blue).
3. Create the *adjacency graph* for all of the blocks in the merged region: each block is a vertex, and an edge connects two vertices if their regions meet along an edge.



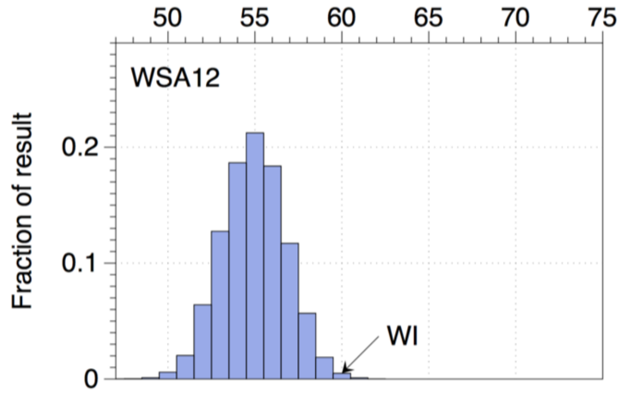
4. Find a *spanning tree* of the adjacency graph: a graph that includes every vertex of the adjacency graph, but has no cycles (loops).
5. Identify an edge of the spanning tree that you could cut to yield two equal-size sub-trees (shown here in red).
   - It is possible that there is no such edge in your spanning tree, so then you have to choose a different spanning tree.
6. Each of those subgraphs gives you one district, so now you have a new districting plan!



1. Do another step of the ReCom algorithm, starting with the ending map above.

2. Make a new graph of districting plans for the 3×3 plan, where the vertices are the same 10 plans as before, but now an edge connects two plans if they are related by one *ReCom move* (rather than one *flip* move as before). Is the graph planar?

3. The figure to the right, shows the result of a random walk on districting plans for Wisconsin's state assembly members.[5] The horizontal axis is the *number of seats out of 99 in the Wisconsin state assembly* that would have been won by Republicans, based on the geographic voting patterns in the 2012 election. The vertical axis shows how often that outcome occurred in the 19,184 districting plans visited in a random walk,.



(a) Identify the range of outcomes: what were all the different numbers of seats that Republicans won in these 19,184 plans?

(b) What was the most frequent outcome?

(c) The legislature's plan produced 60 seats for Republicans. Do you think this is an outlier? Explain why or why not. What outcomes would you accept as "reasonable" non-outliers?

(d) Explain why the *number of seats* measure doesn't work as well when the number of seats is small, for example for New Mexico's Congressional representatives that we looked at on Monday. In such cases, a measure like *mean-median* works better, because it is a continuous measure, with every number between -0.5 and +0.5 as a possible outcome, rather than a discrete measure that takes only a couple of possible values. Explain.

——— break ———

Sometimes outliers are good. Some people might say, "it's impossible to find a districting plan for Wisconsin where Republicans win 49 seats and Democrats win 50." The results of the random walk above show that this statement is false! The computer can help us to find outlier scenarios that we didn't think were possible. So can diligent people with Districtr.

Similarly, people might say, "it's impossible to make two majority-Black Congressional districts in Pennsylvania." Is it true, or not? Let's fire up Districtr and see for ourselves!

4. Go to `http://districtr.org`, click on Pennsylvania, and click on the left purple button, "17 Congressional districts built out of 2020 VTDs."

(a) On the "Data Layers" tab, click "Population by Race," click "Show population," and click "Black population" from the dropdown menu.

(b) Zoom in on the city of Philadelphia. Click the Population tab. Grab your paintbrush and paint your first district until you get to the Ideal line. Click "Lock already-drawn districts" and paint your second district.

(c) Click the Evaluation tab and see whether Black voters are over 50% in either of your districts. If not, modify them and try again until you get it!

Hint: this problem is hard. It will take a while.

[5]This figure, and the explanation of it, are from *Gerrymandering metrics: How to measure? What's the baseline?* by Moon Duchin, January 2018, available at `https://arxiv.org/abs/1801.02064`

Gerrymandering with Math

The social planner and the map-drawer. Get a partner. One of you is the Social Planner, a supporter of the Dot party, and one of you is the Map Drawer, a supporter of the Blank party. You are both interested in a 7×6 state where there are 21 Dot-supporting towns and 21 Blank-supporting towns.

1. Draw the 7×6 grid on the board.
2. The Social Planner draws dots in 21 of the squares, choosing where people live so as to maximize how many 7-town districts the Dot party will win.
3. The Map Drawer divides the state into six 7-town districts, so that the Blank party wins as many districts as possible.

After a few rounds:

- switch roles;
- determine how many Dot towns are needed, to ensure that Dot definitely wins three districts

Gerrymandering with Math

Here is another measure of gerrymandering.

- The *efficiency gap*. Let's say that a vote is *wasted* if it is a winning vote over 50% in a district, or if it is a losing vote. The *efficiency gap* is

$$\frac{\text{wasted Republican votes} - \text{wasted Democratic votes}}{\text{total votes}},$$

  which is the difference in wasted votes, as a proportion of the total votes in the election. The scholars who popularized the efficiency gap proposed that any election whose efficiency gap is greater than 0.08 in magnitude is probably gerrymandered.

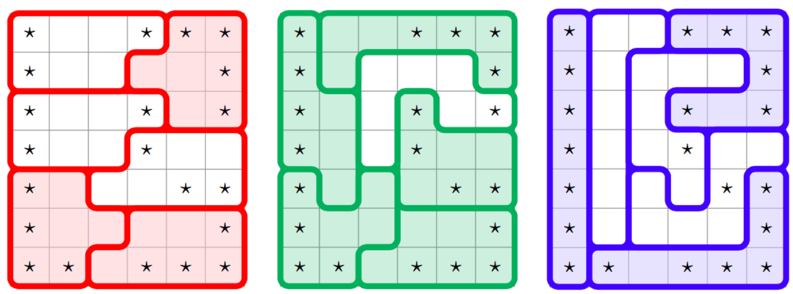
1. In New Mexico's 2012 Congressional election (from Monday), the Republican vote shares in the three districts were 0.349, 0.627, 0.376. Suppose each district had 1000 voters.

(a) Calculate the number of wasted votes for Republicans and Democrats in each district.

(b) Calculate the efficiency gap for New Mexico for this election. (Answer: 0.068.)

(c) Is New Mexico gerrymandered? In favor of which party?

2. A 42-town state has two parties, the *blank* party and the *star* party. The pictures to the right show the locations of the towns voting for each party, and the shading indicates who wins each district with each of the various plans.[6]



(a) For each districting plan shown above, compute the efficiency gap.

(b) For each plan, say which districts are *packed* and which are *cracked* in favor of each party, and discuss the relationship with the efficiency gap you computed for each district.

[6]The pictures are from *A formula goes to court: Partisan gerrymandering and the efficiency gap*, by Mira Bernstein and Moon Duchin, May 2017, available at `https://arxiv.org/abs/1705.10812`.

3. The efficiency gap seems to do a good job in measuring whether districts have been packed or cracked, but here are some undesirable properties that it has.[7] If you are having trouble calculating an efficiency gap, you may assume that the district has 100 voters.

(a) *Penalizes proportionality.* Suppose that the Democratic party wins 60% of the votes, and 60% of the seats. Compute the efficiency gap. It is nonzero and in fact greater than 0.08, so it suggests (based on a popularly suggested threshold) that this is an unacceptable gerrymander. In favor of which party?

(b) *Rewards certain landslides.* For the efficiency gap to be 0 in a district, what must the vote shares be for each party?

(c) *Volatile in competitive races.* For the first (red) districting plan in the previous problem, all of the districts are competitive. Suppose that at the last moment, some last-minute trend (such as the release of damaging information) pushes the voting slightly towards the "star" candidate, so that now there is an additional "star" town in each of the "blank"-winning districts in the picture. What does the efficiency gap say about the districting plan now?

4. *What the efficiency gap actually measures.* Suppose that a state has n districts, each with P voters. Suppose that Republicans win r of the n districts. Let the overall Republican vote share in the state be V, and let the Republican seat share be $S = r/n$.

(a) Show that the total number of Republican (R) votes in the state is VPn, and find the total number of Democratic (D) votes.

(b) Show that there are $\frac{1}{2}Pr$ non-wasted R votes, and thus $P\left(Vn - \frac{1}{2}r\right)$ wasted R votes.

(c) Calculate the number of wasted D votes. Check your answer by adding it to the wasted R votes and making sure that half of the total votes in the election are wasted (!).

(d) Show that the efficiency gap is $2V - r/n - 1/2$, or in other words $2V - S - 1/2$.

(e) One way of understanding this is that efficiency gap "expects" a party to get a *winner's bonus* of twice as much seat share over 50% as they have vote share over 50%. Explain.

[7] This list of properties is from the same source as above.