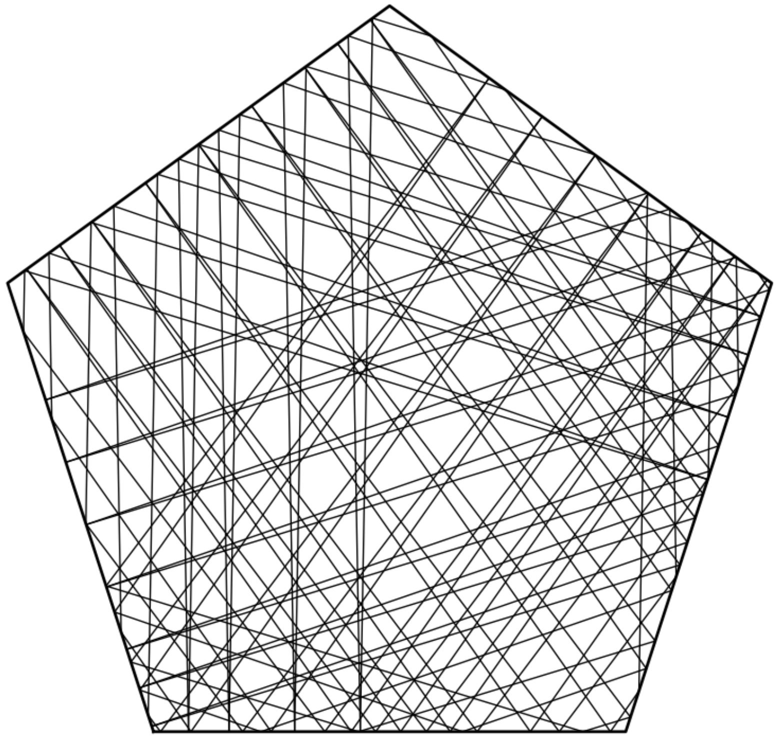
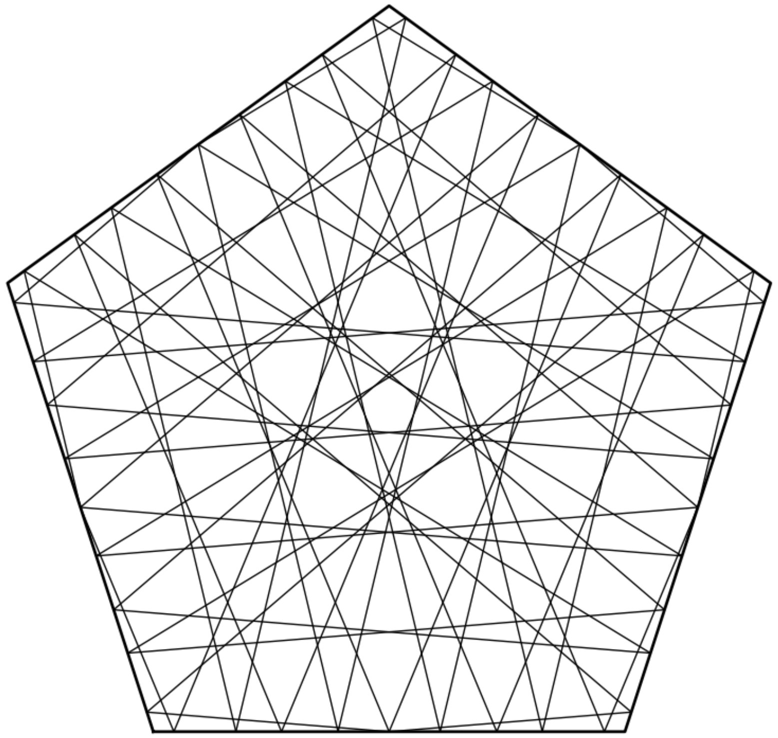
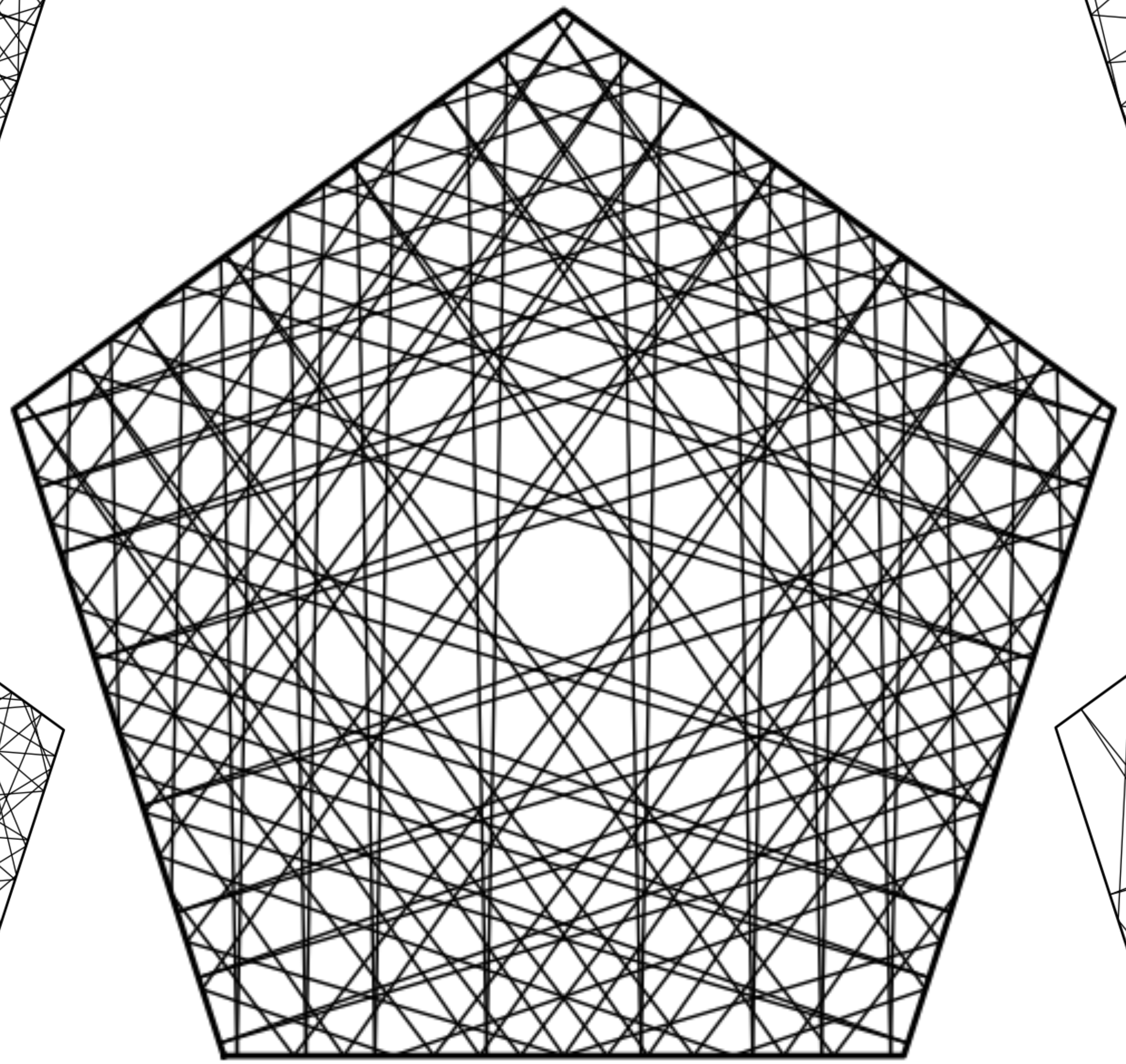
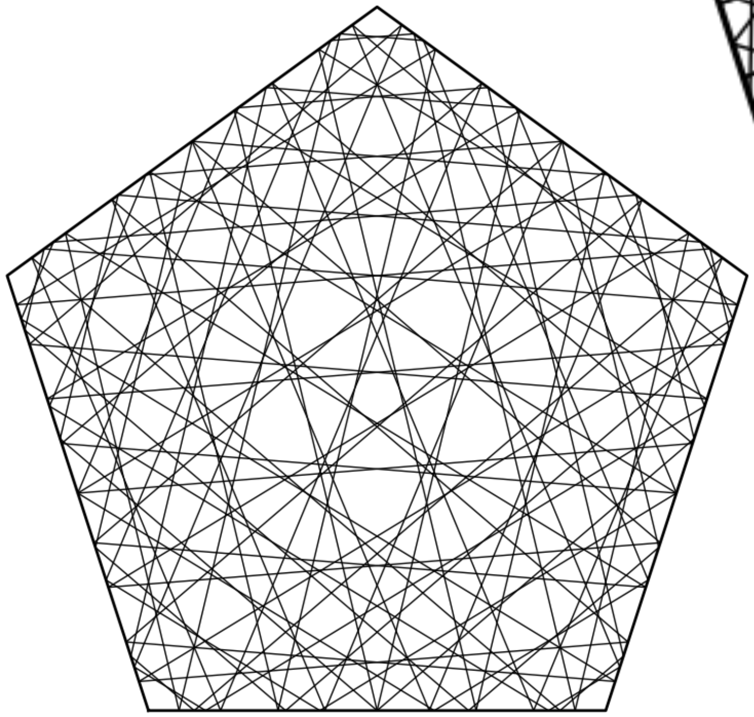
Periodic Paths on the Pentagon

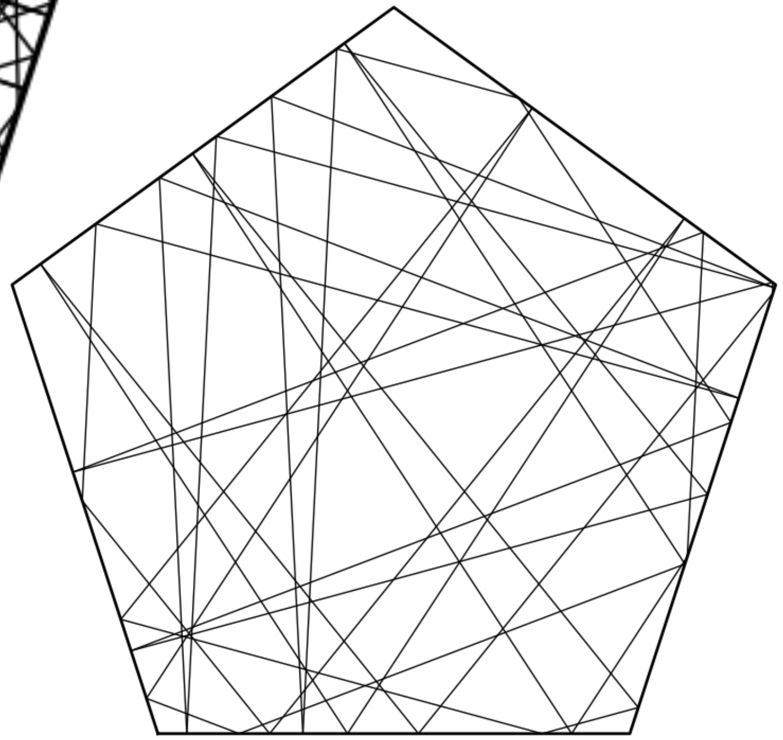


Diana Davis
Boston University
January 28, 2019









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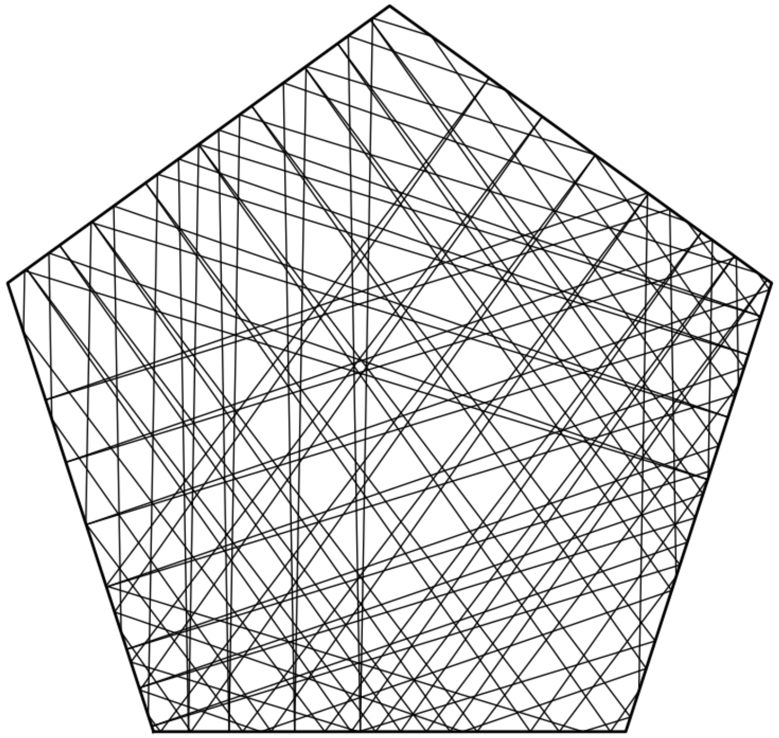
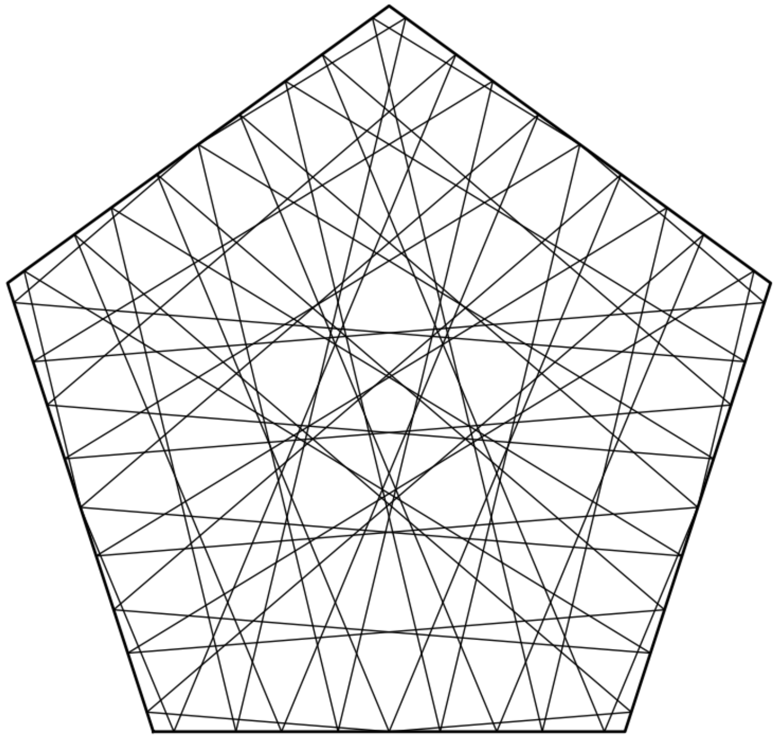
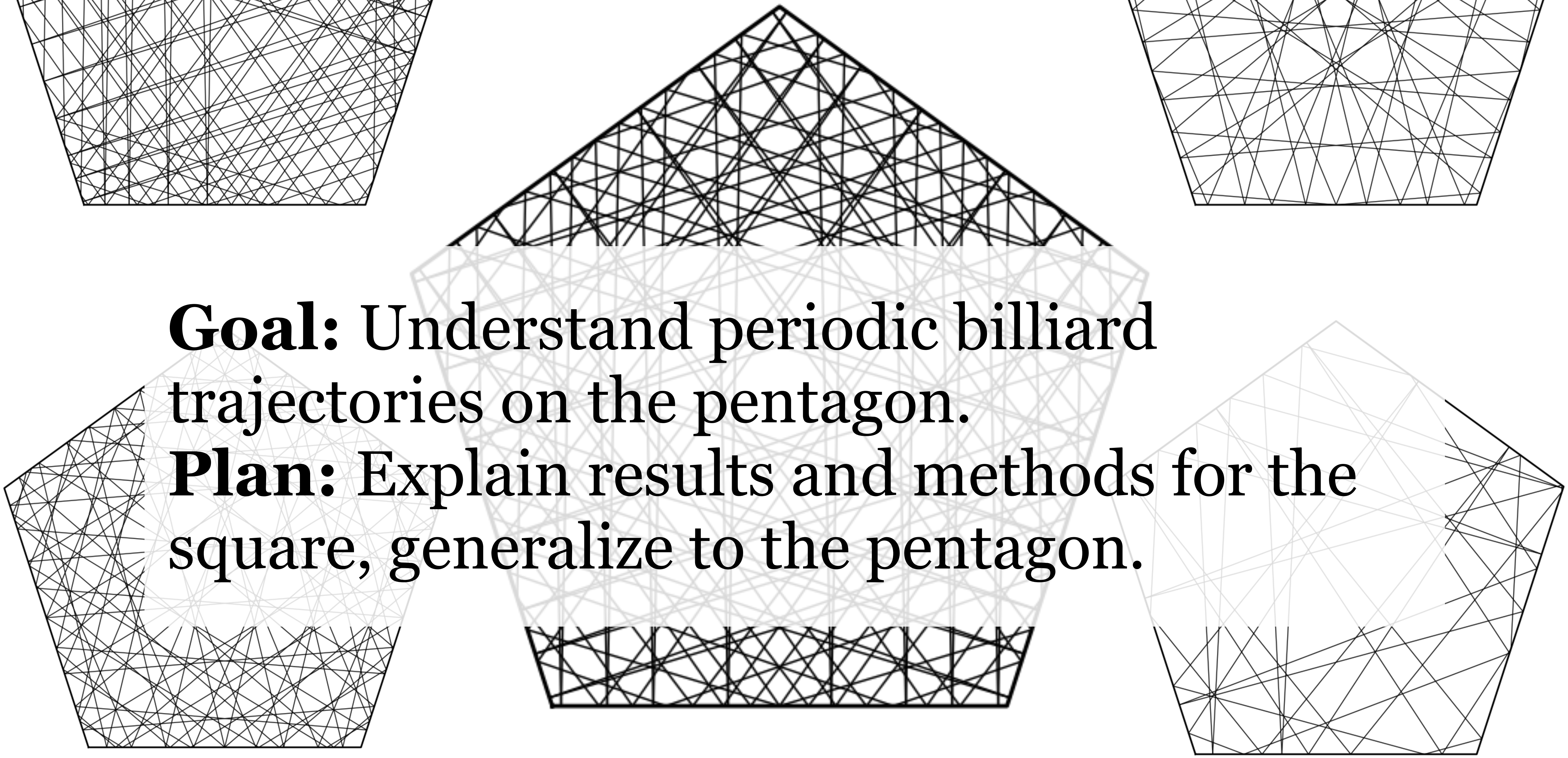


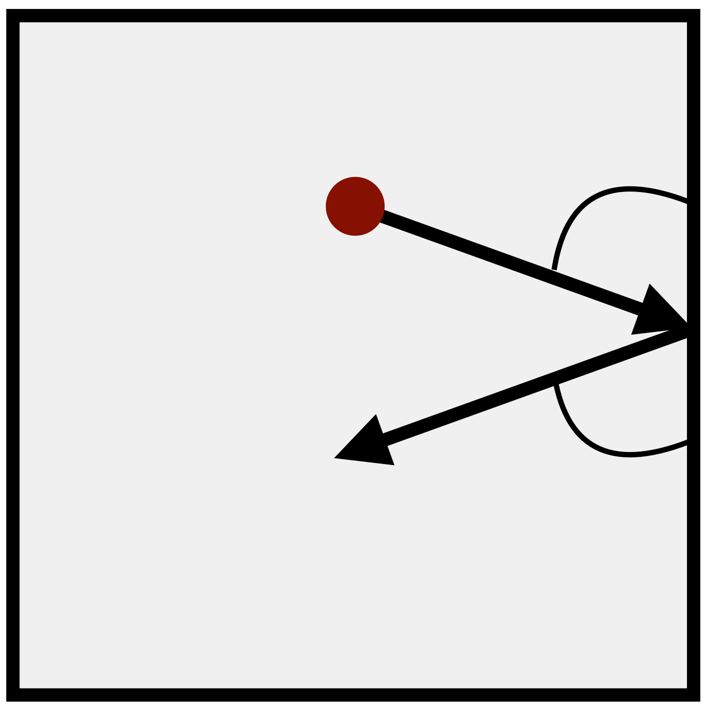


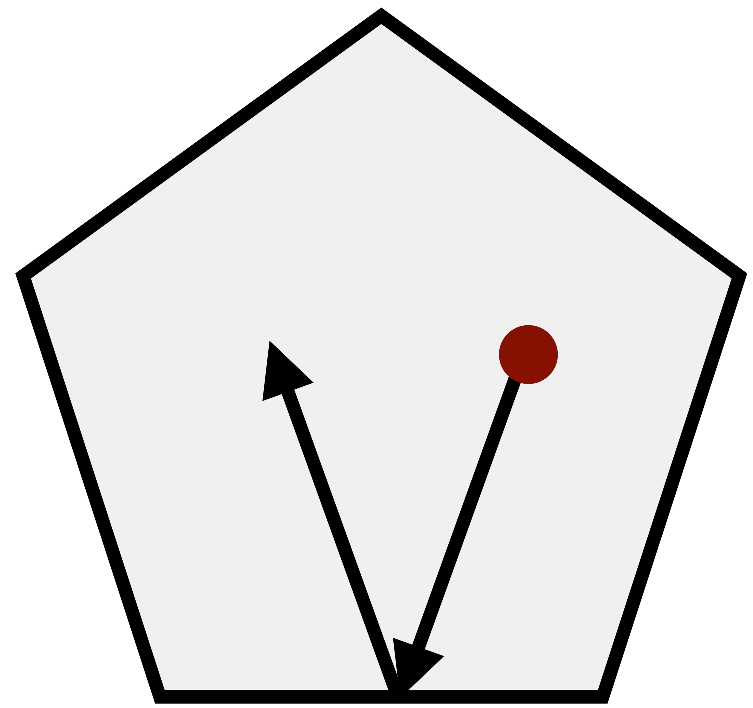


Goal: Understand periodic billiard trajectories on the pentagon.
Plan: Explain results and methods for the square, generalize to the pentagon.

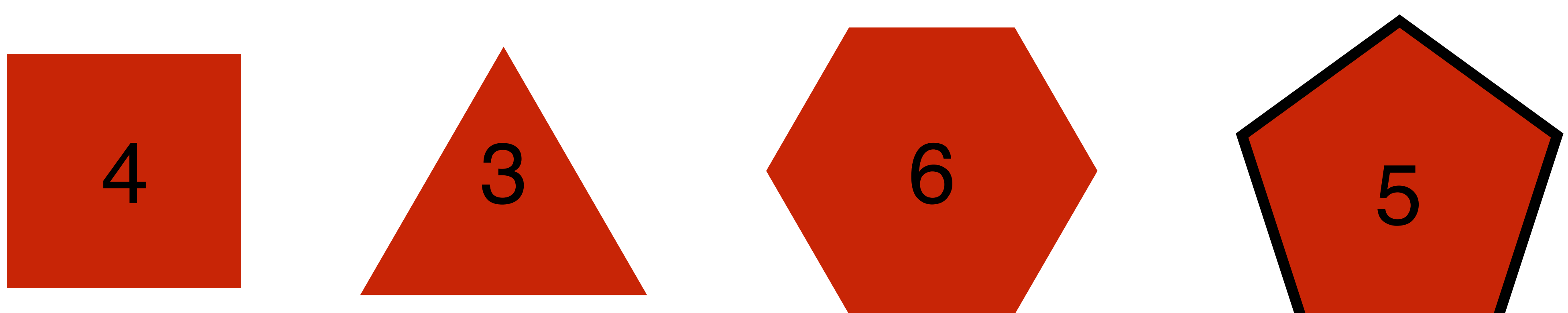
BILLIARDS:

- A particle bouncing around inside a polygon
- The angle of incidence equals the angle of reflection





We understand periodic billiard paths on:



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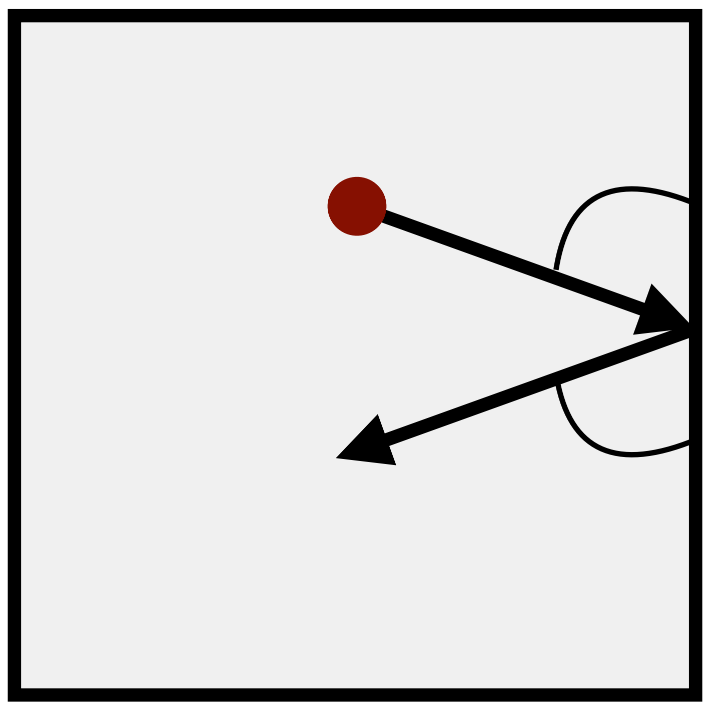
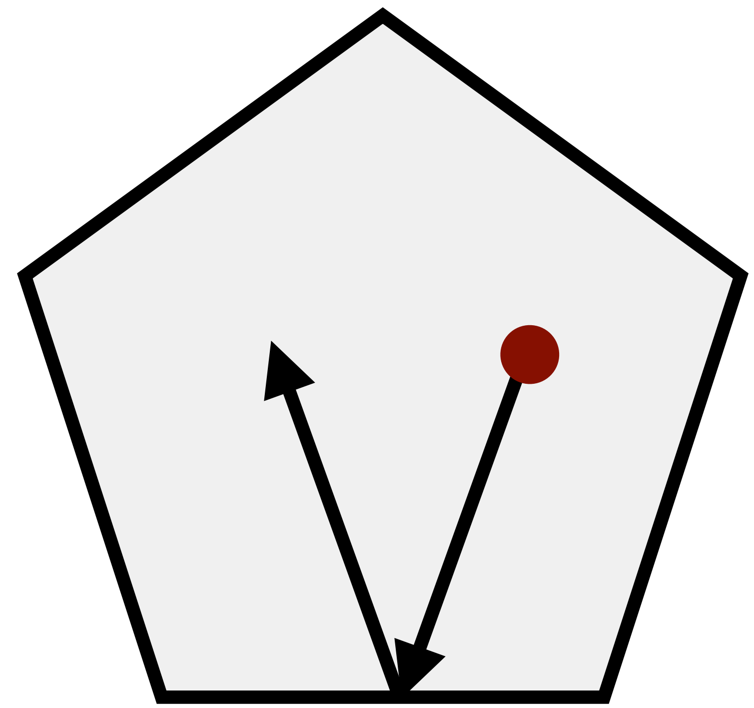




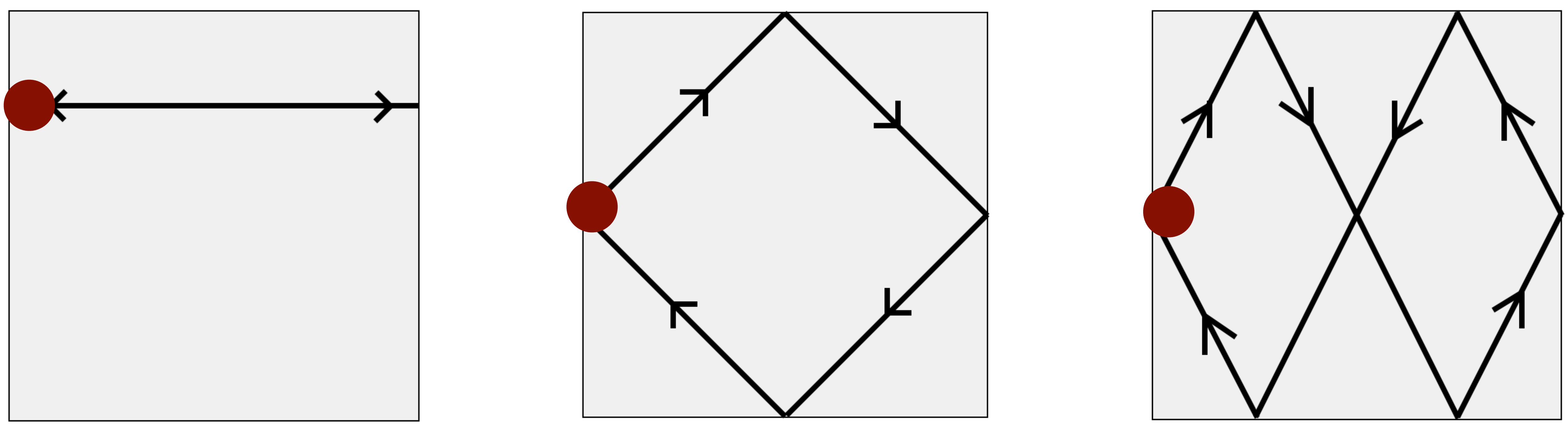
Billiards is a vibrant and fast-growing field:

- Two of the 2014 Fields Medals were for work in this area (Maryam Mirzakhani and Artur Avila).

BILLIARDS:

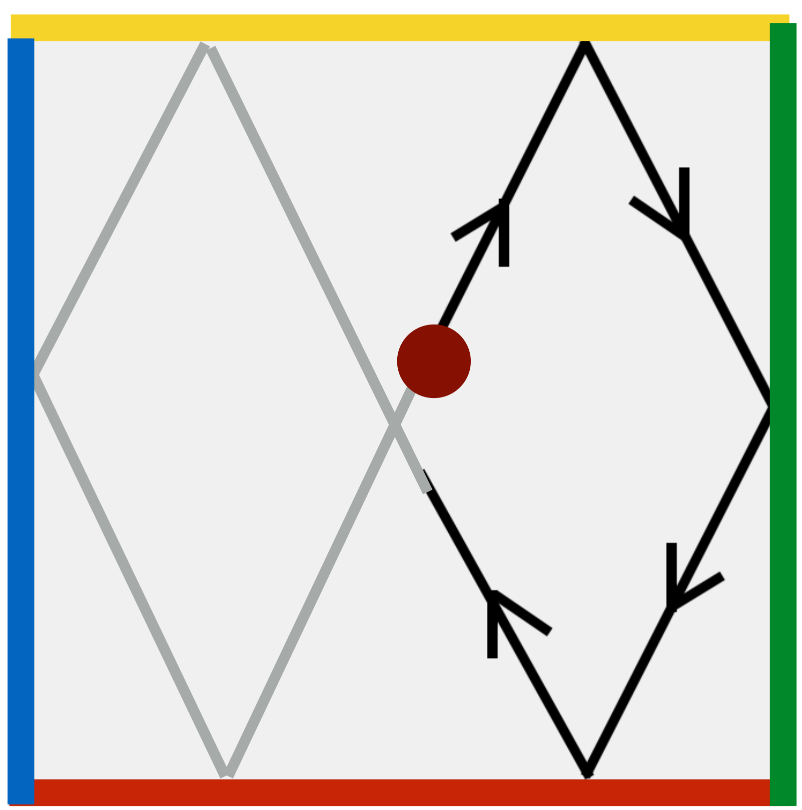
- A particle bouncing around inside a polygon
- The angle of incidence equals the angle of reflection

Some periodic paths in the square:



Which directions are periodic?

Which directions are periodic?
Idea: "unfold" the path across each edge.



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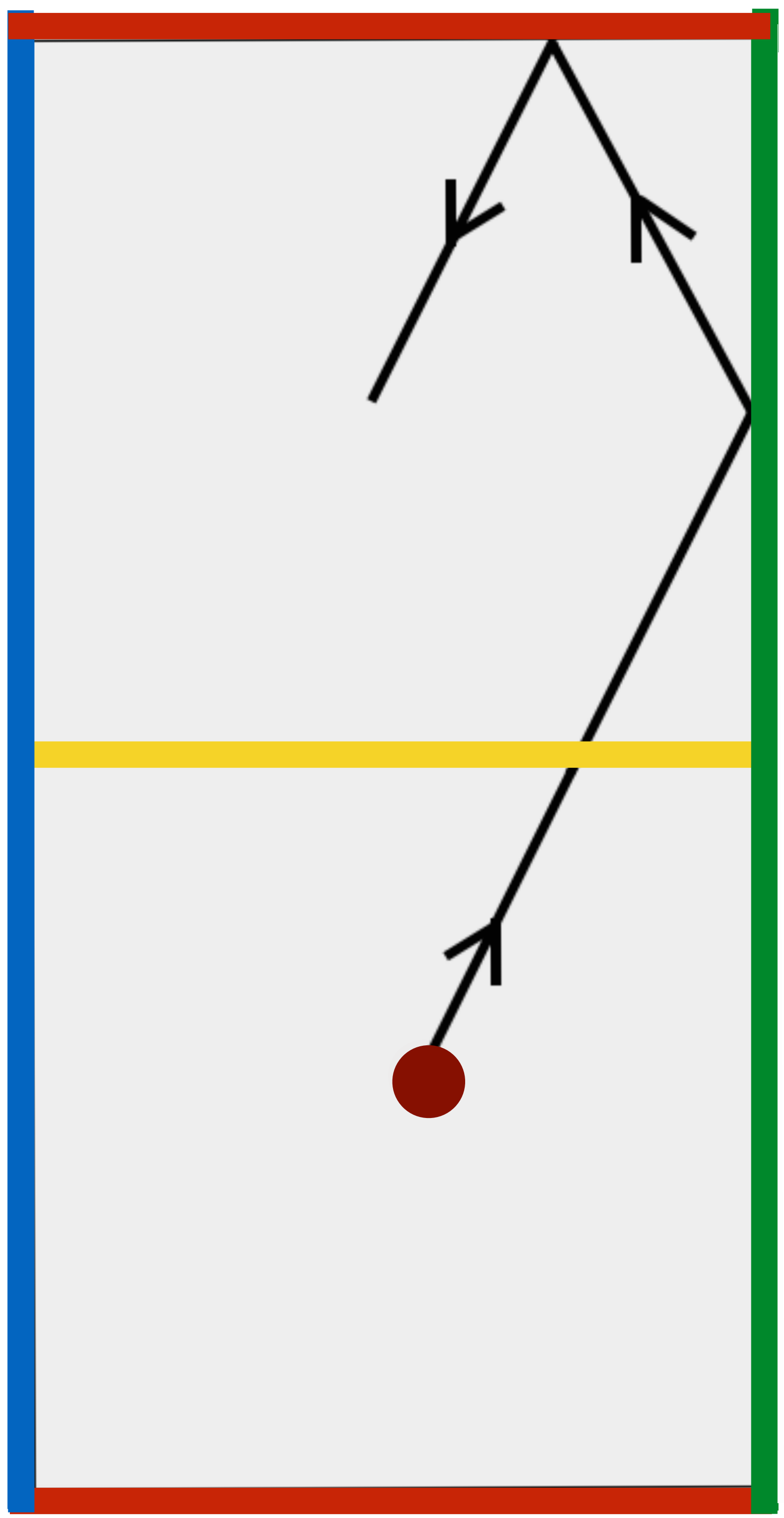

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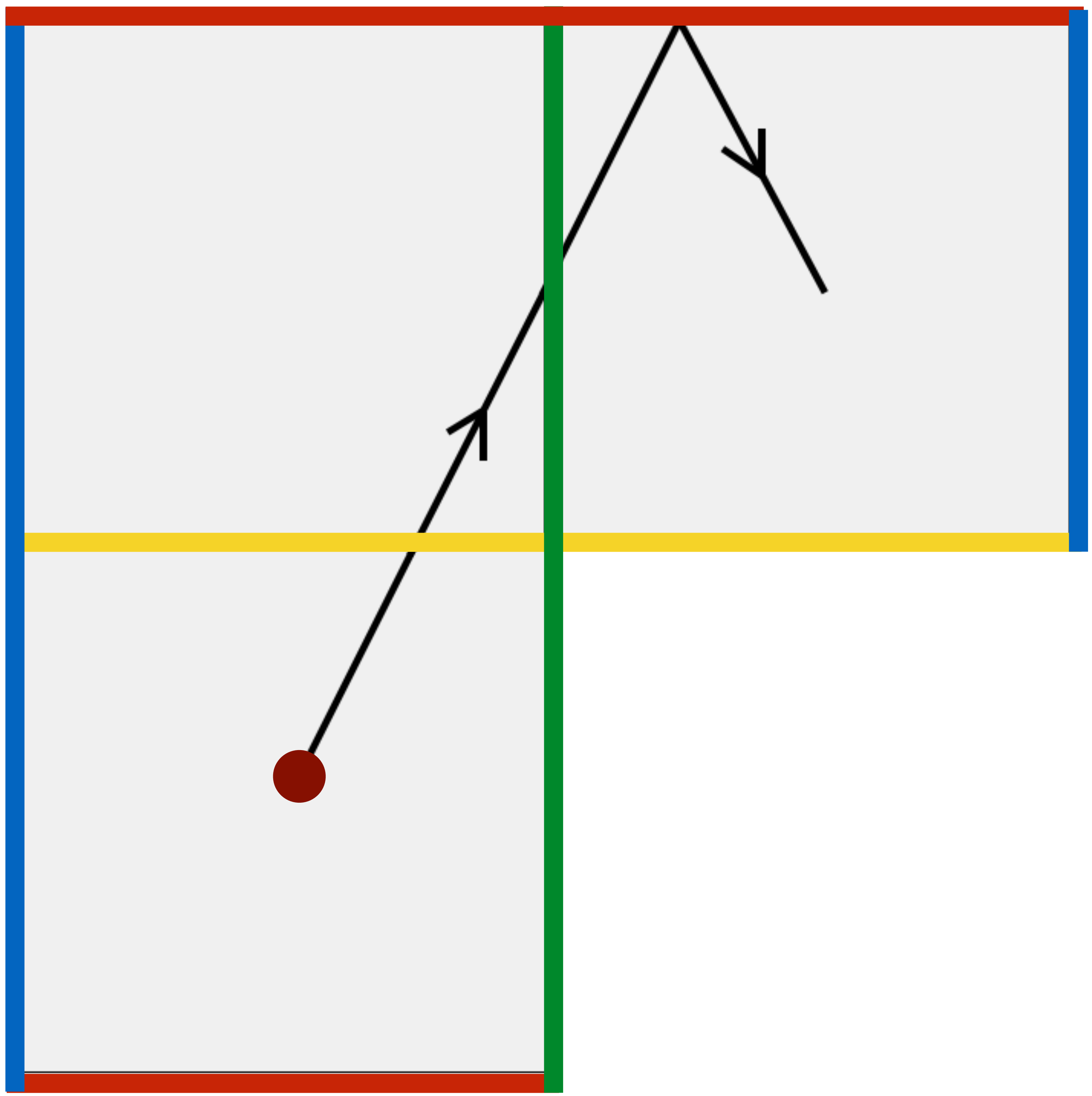

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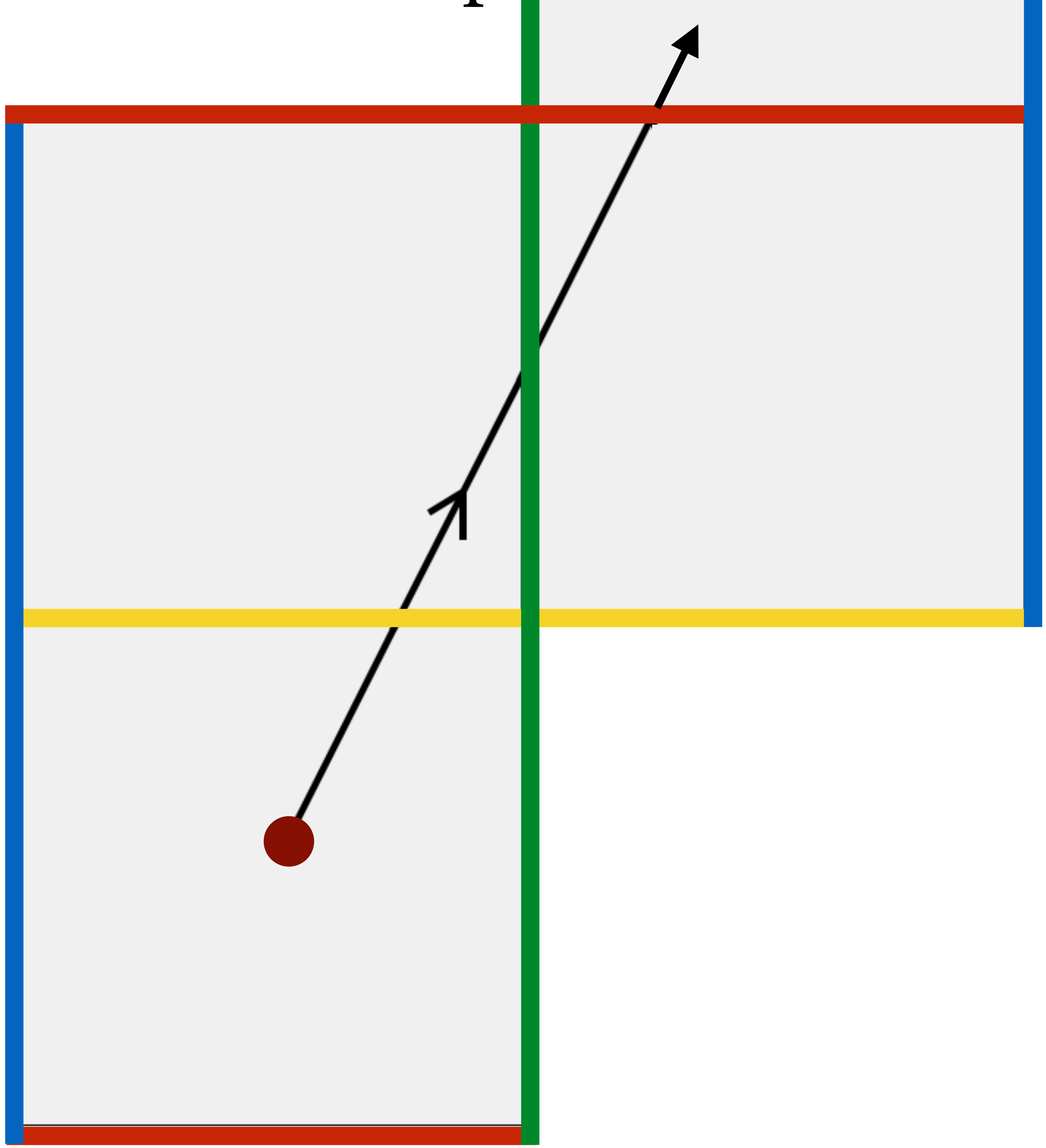

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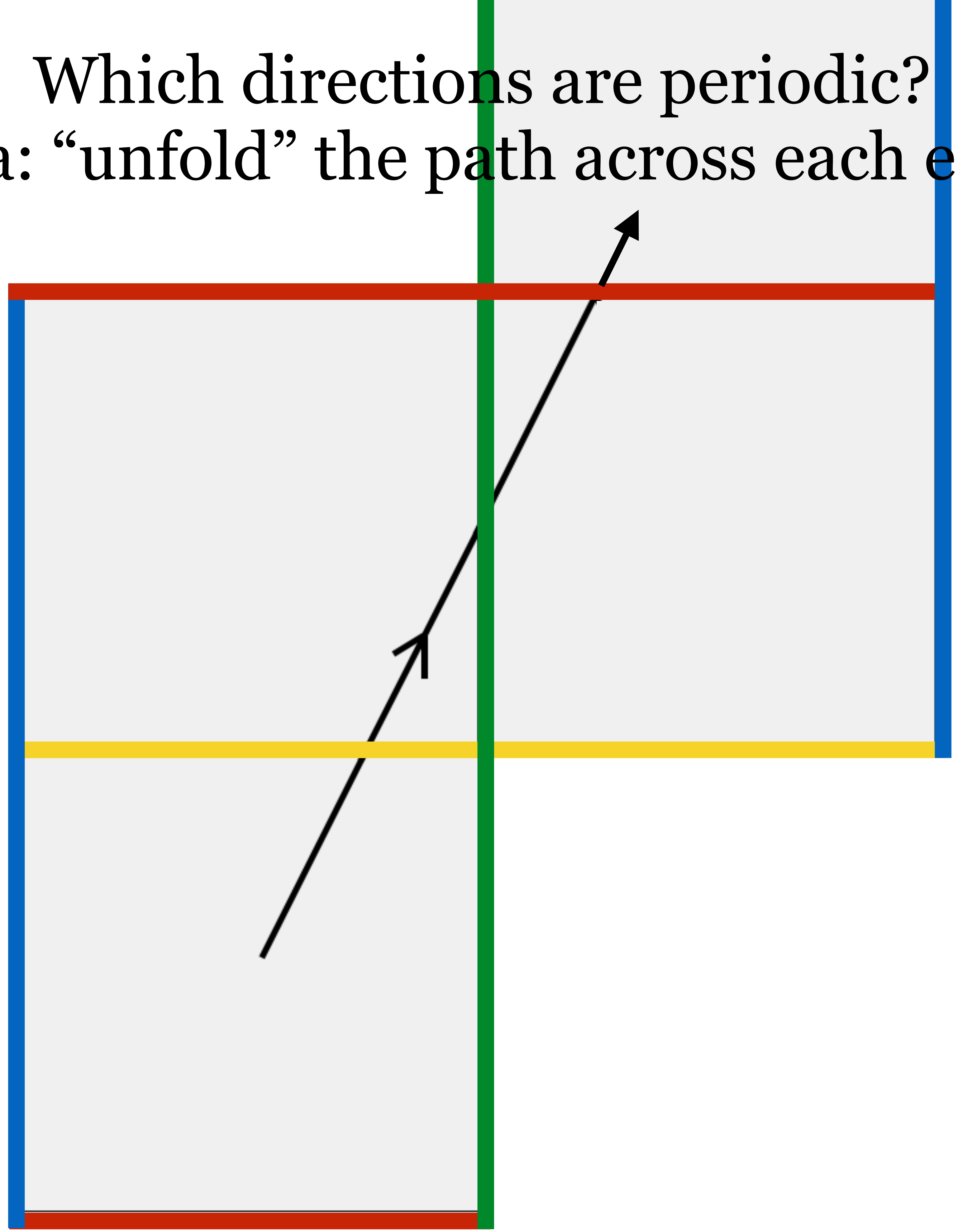

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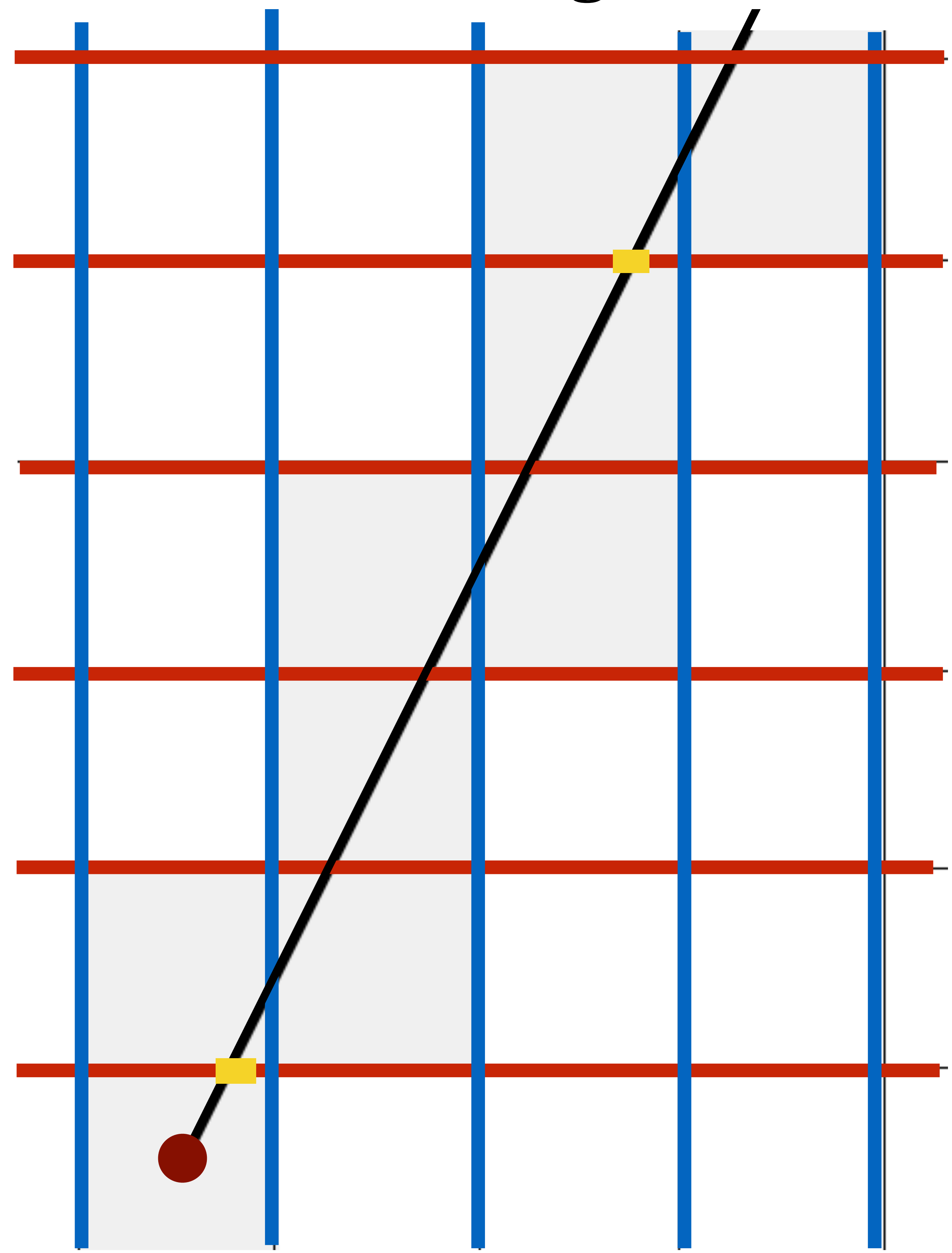
Idea: "unfold" the path across each edge.

A billiard path on the square corresponds to a line on a piece of graph paper.

Theorem:
A trajectory is periodic when its slope is rational.

Proof: For rational slopes, the trajectory eventually hits a corresponding point, and repeats.



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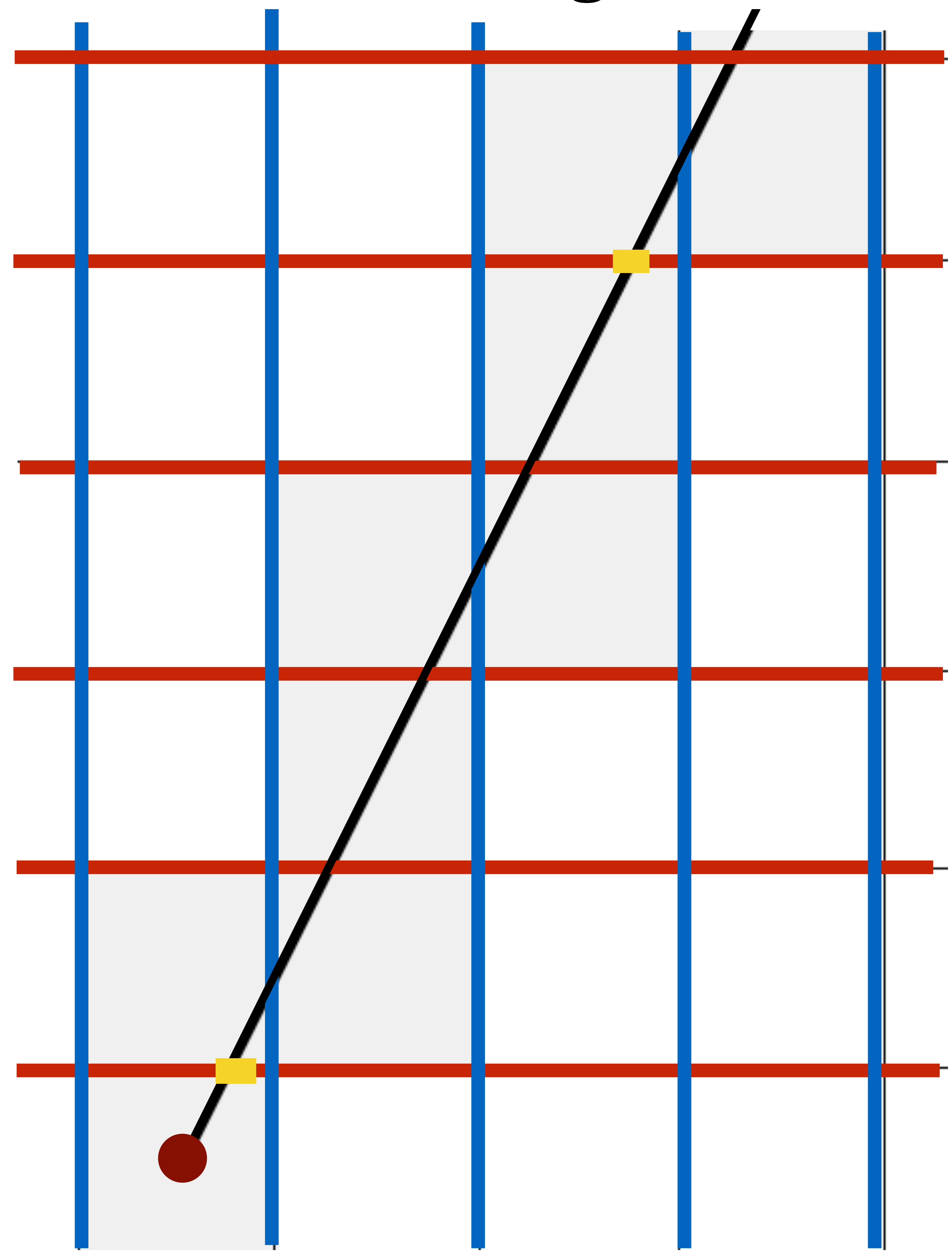
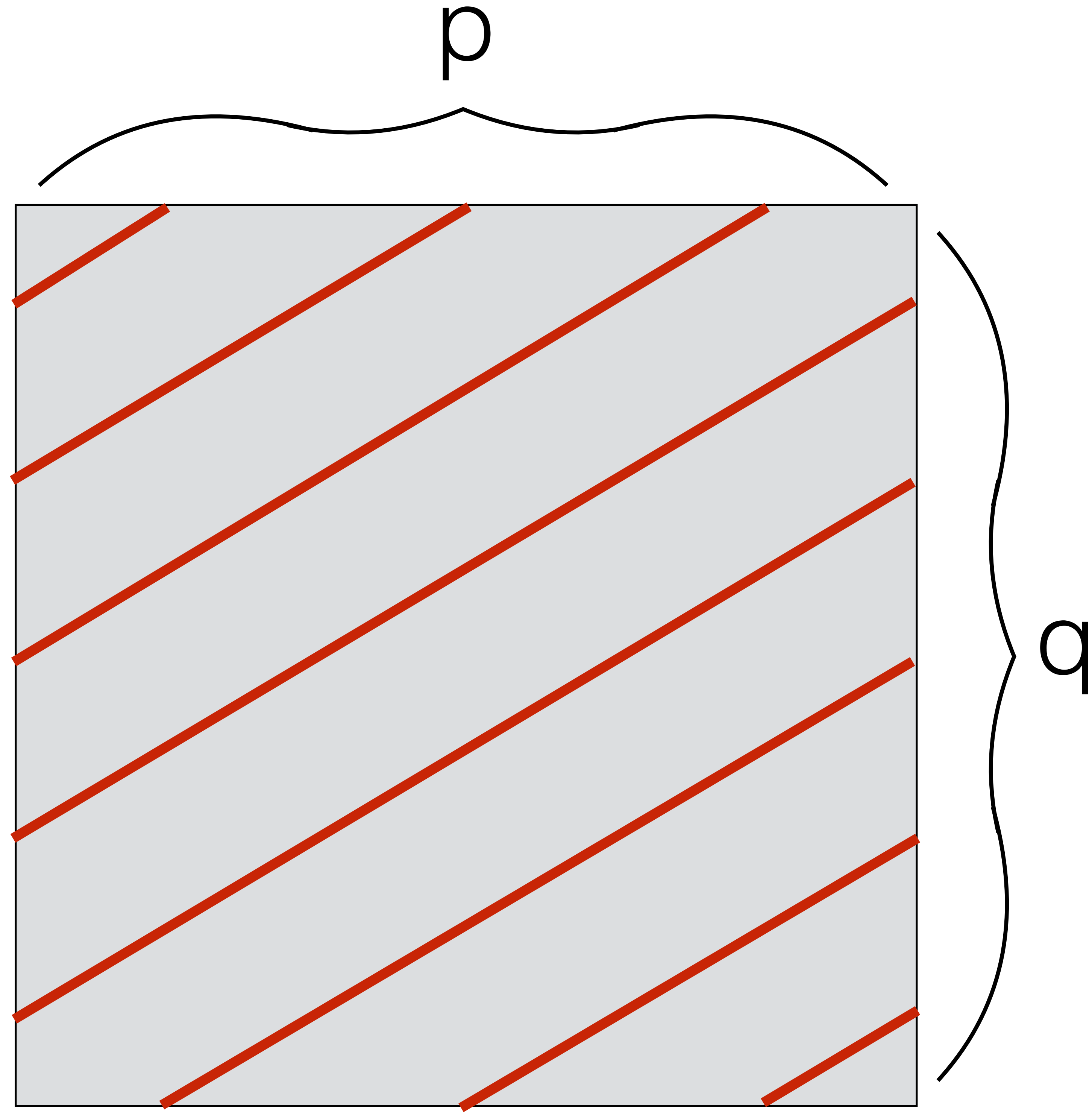
Periodic directions on the square: Vectors [*p*,*q*] for integers *p* and *q*.



Periodic directions on the square torus and billiard table are those with rational slope p/q.

Trajectory with slope p/q on square **torus** has period p+q



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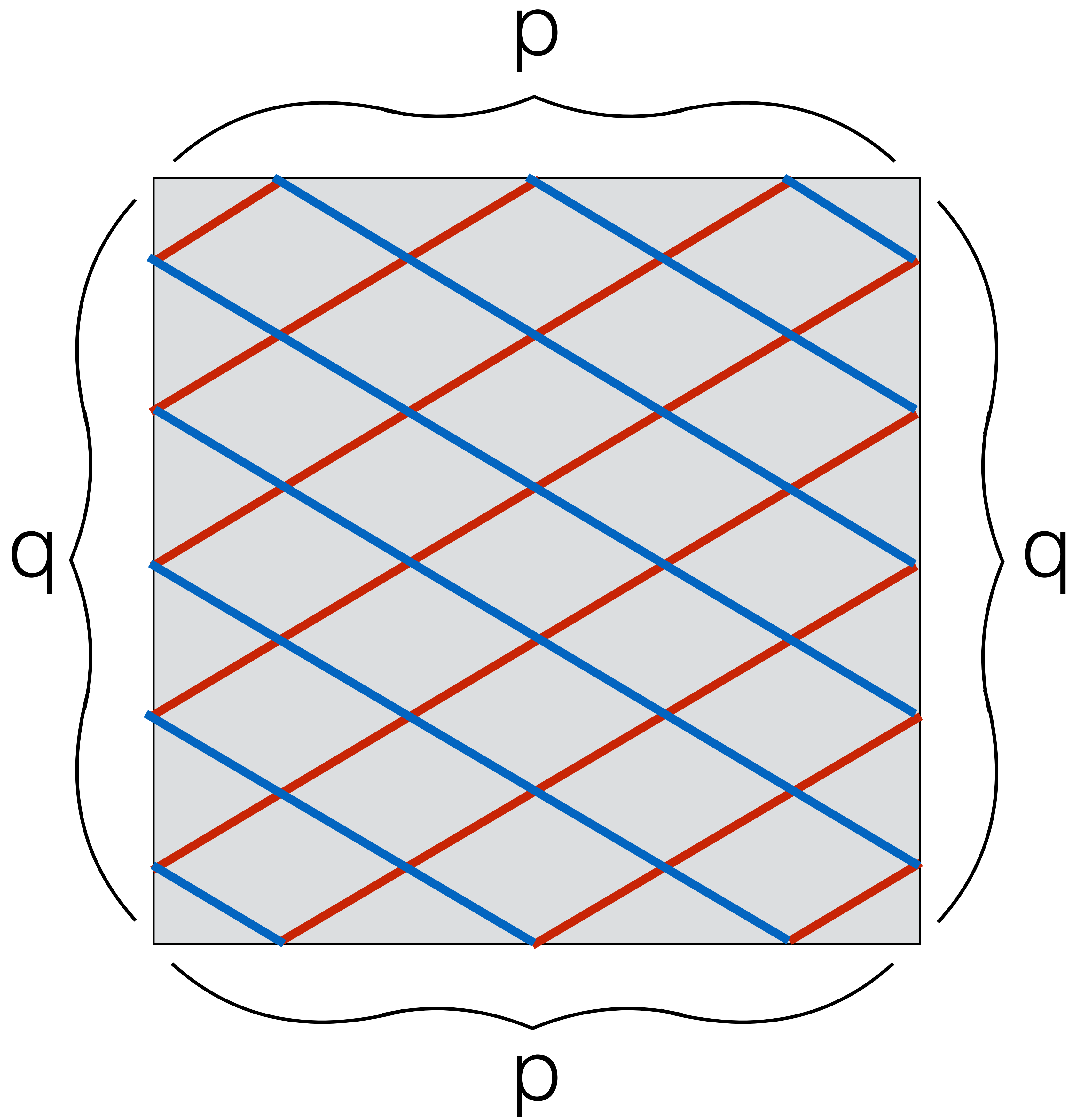
Trajectory with slope p/q on square **billiard table** has period 2(p+q)



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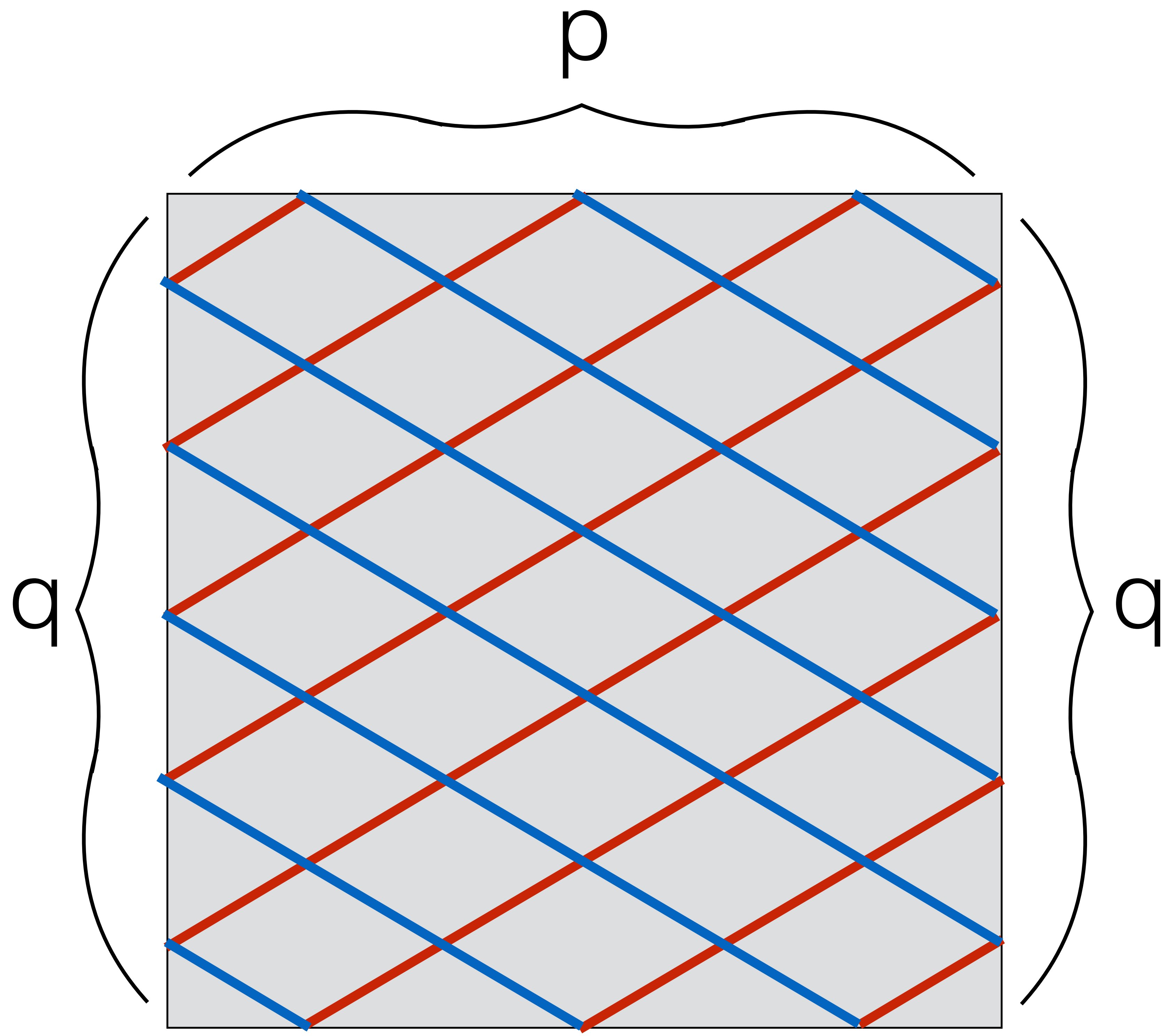
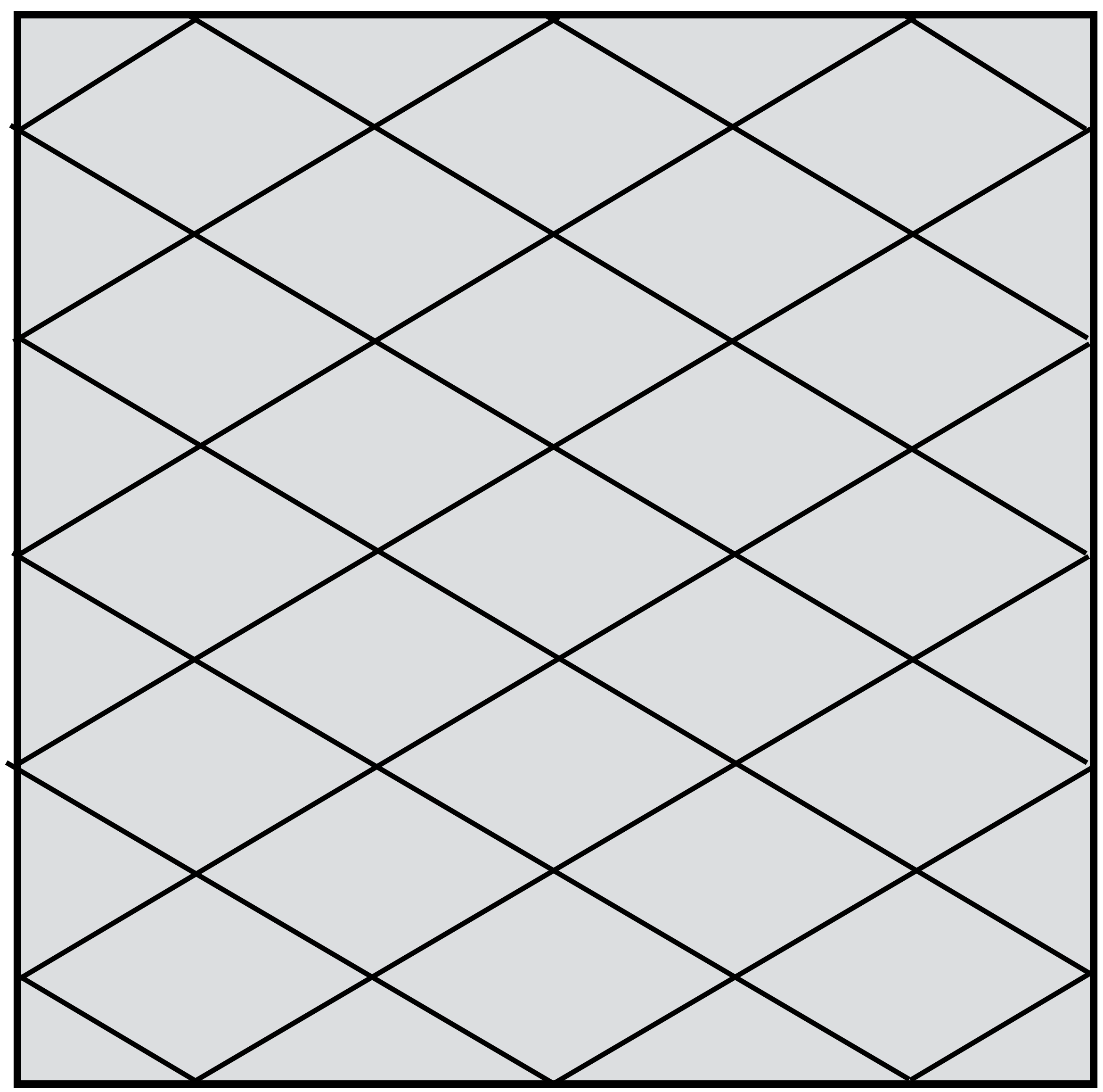


Burning question: What is the period of a trajectory in a given periodic direction on the pentagon?

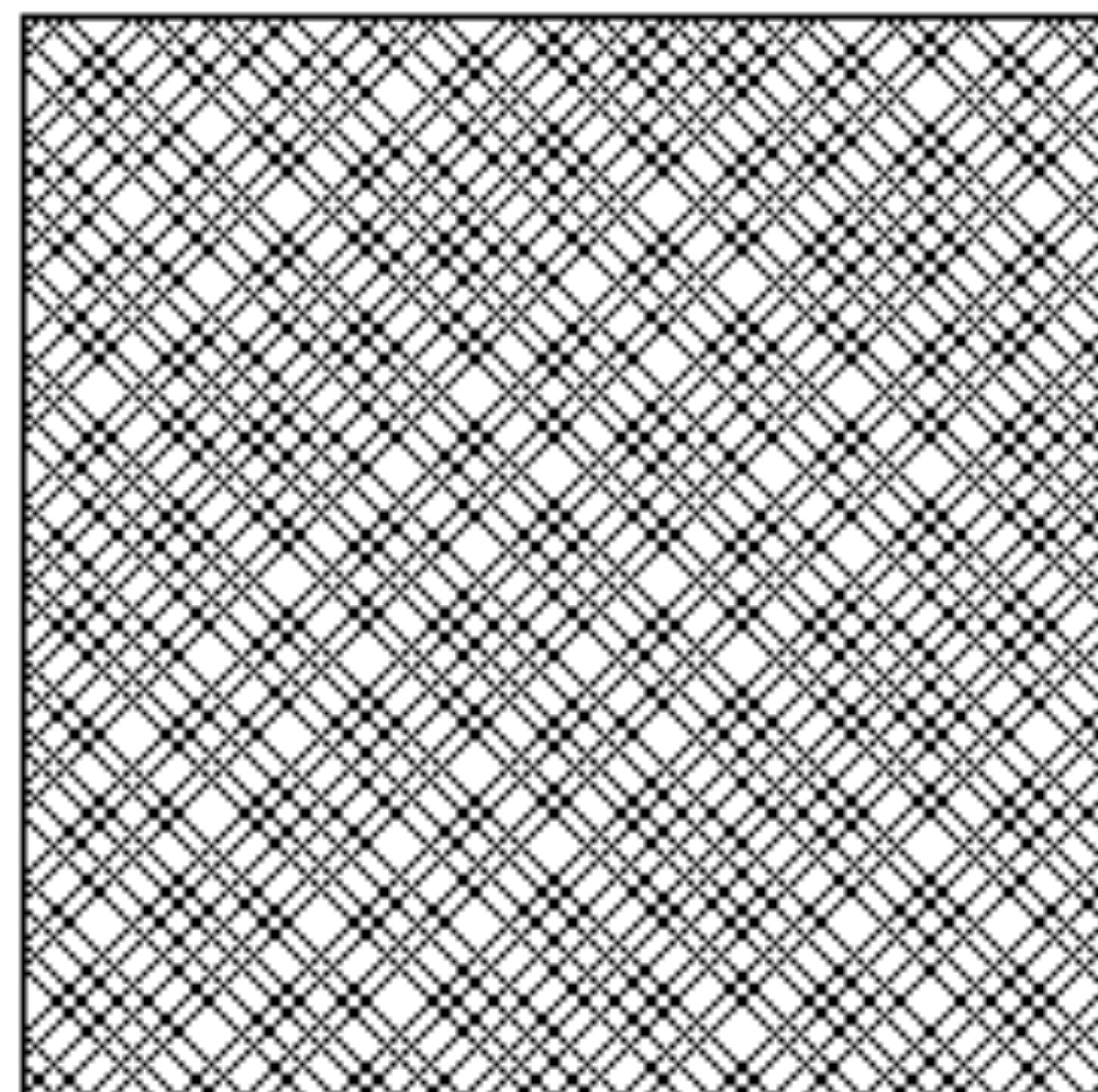
In the square billiard table, a trajectory with:

rational slope p/q is periodic, with period $2(p+q)$.

irrational slope is dense, filling in the entire table.



Picture by Curtis T. McMullen



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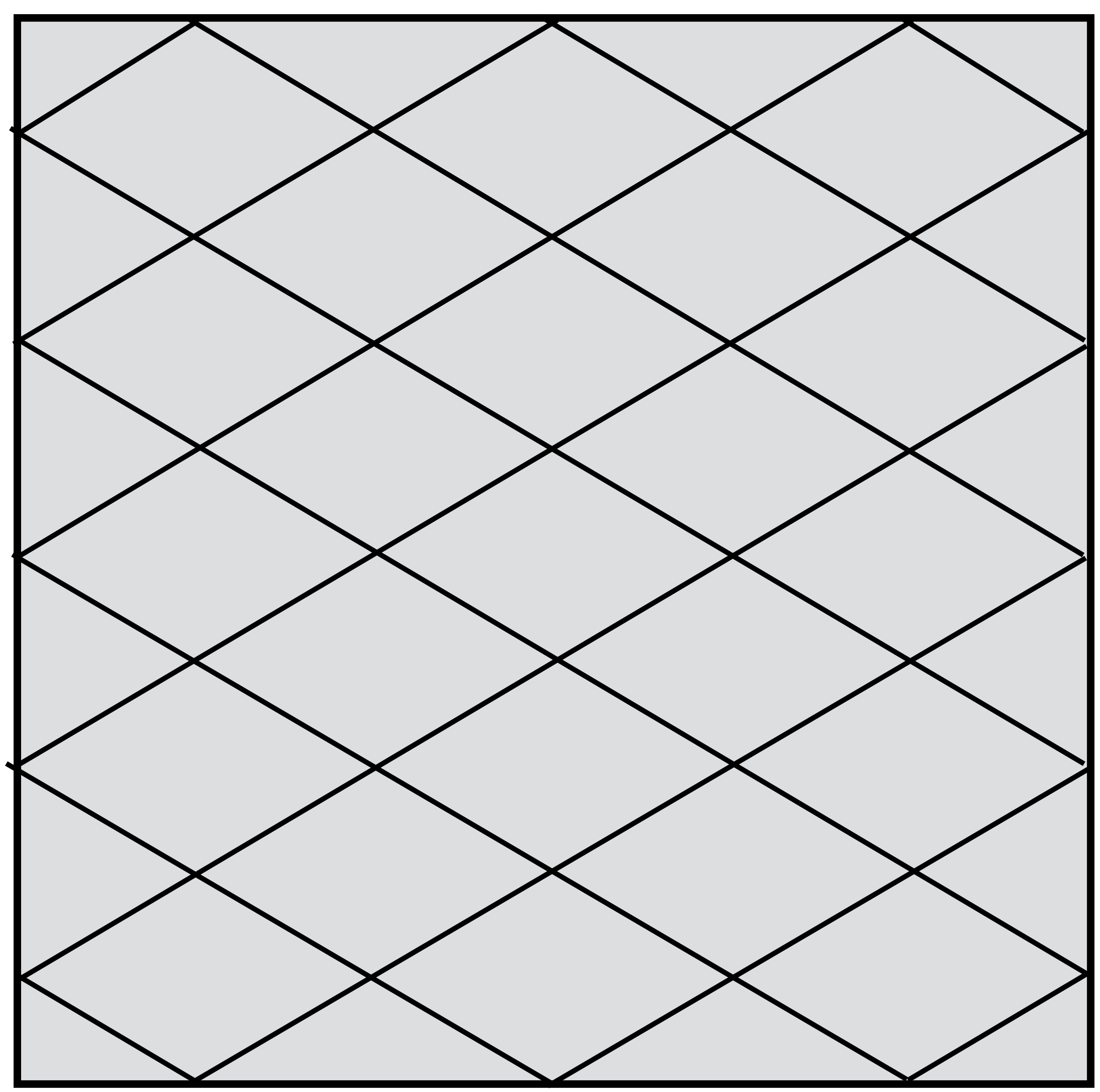

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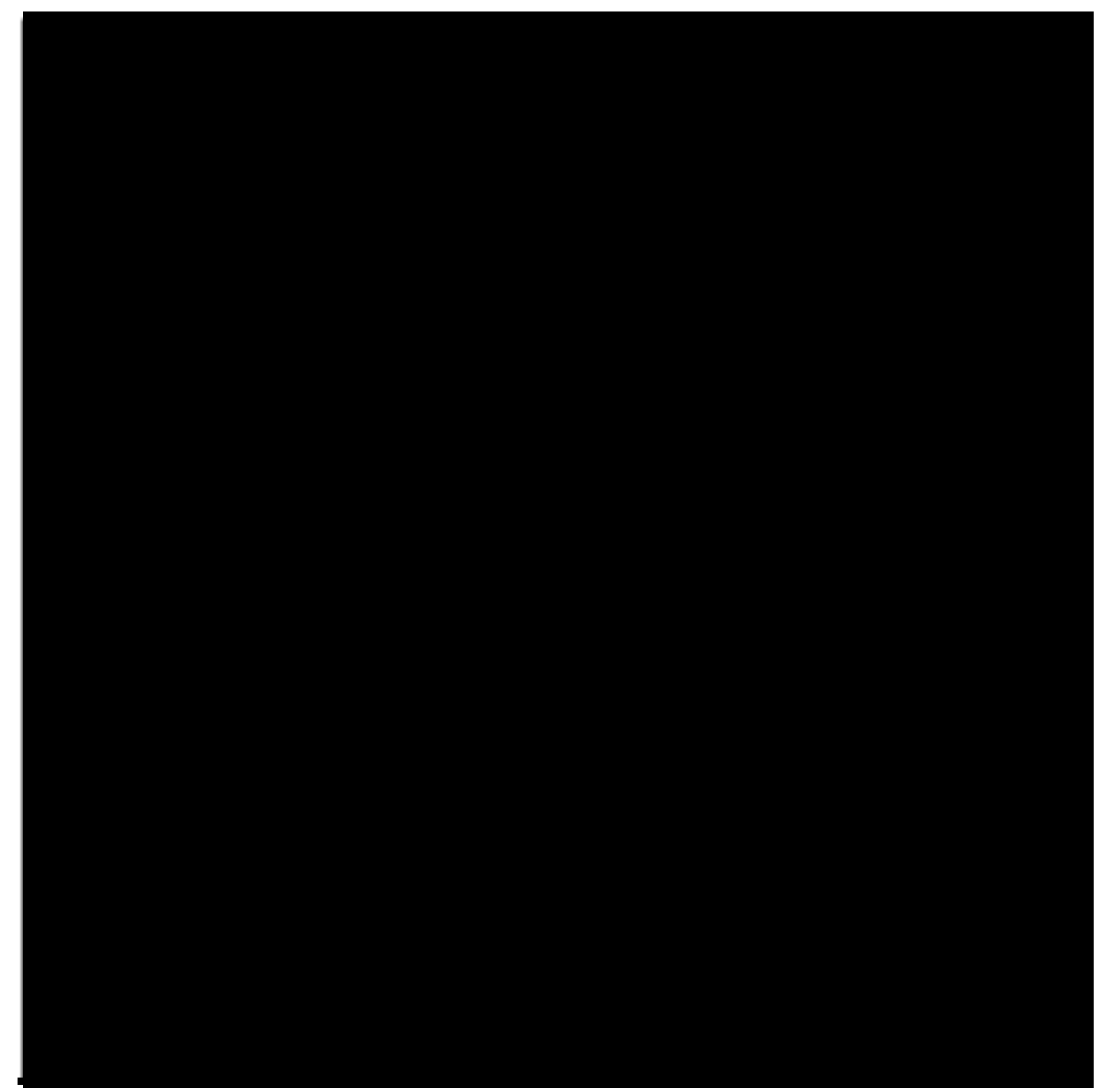

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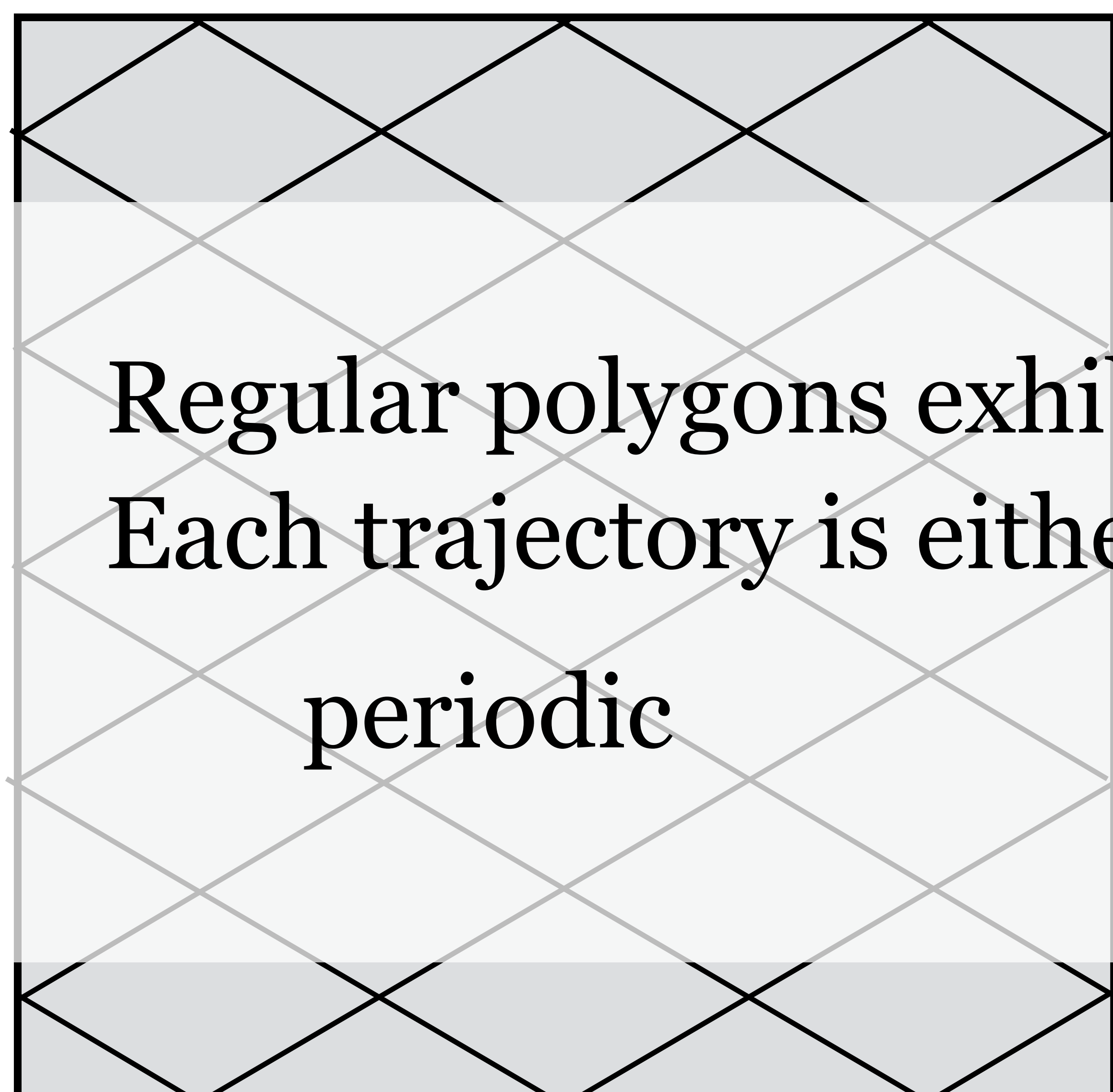

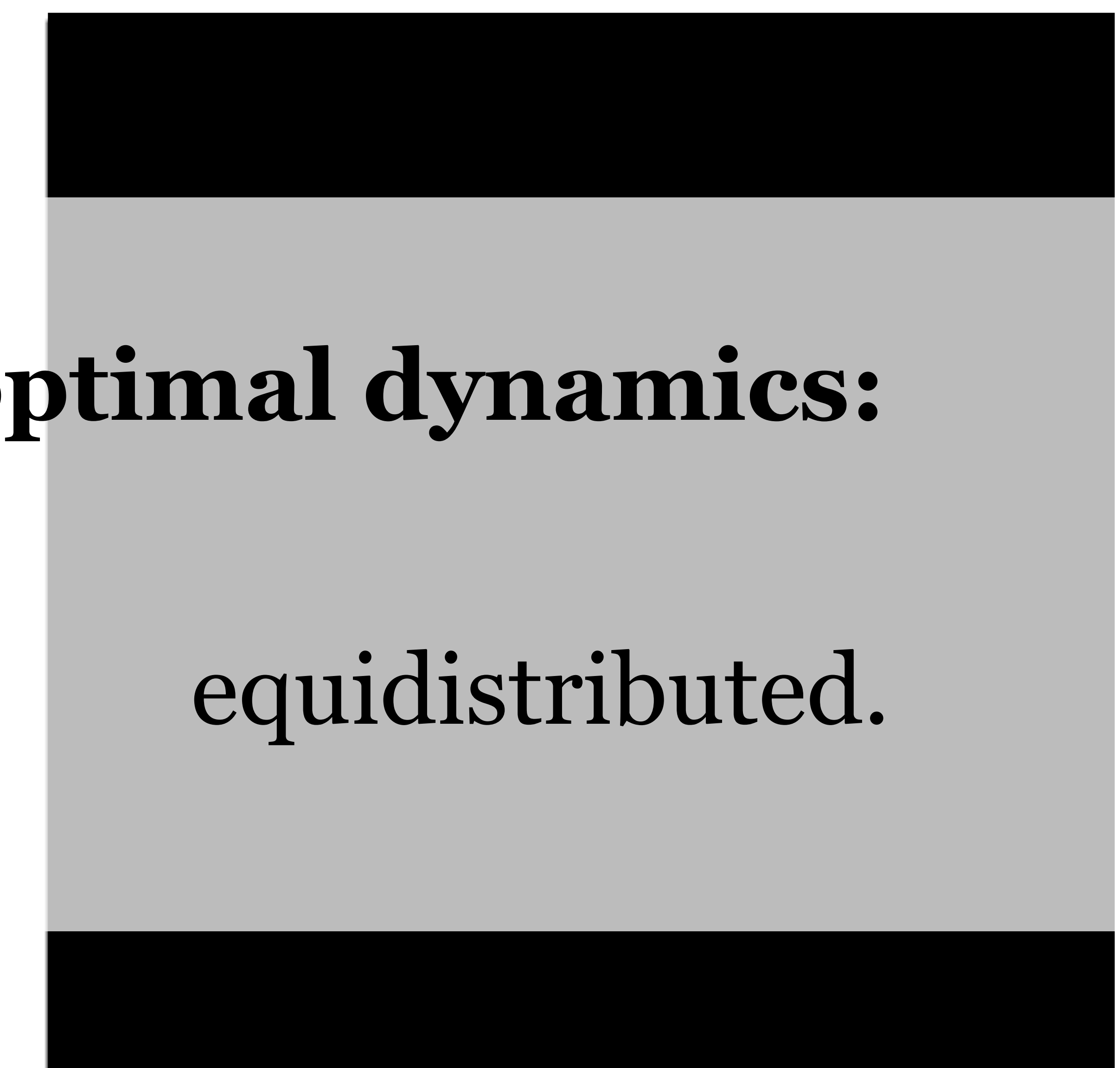

Regular polygons exhibit **optimal dynamics:**
Each trajectory is either

periodic or equidistributed.

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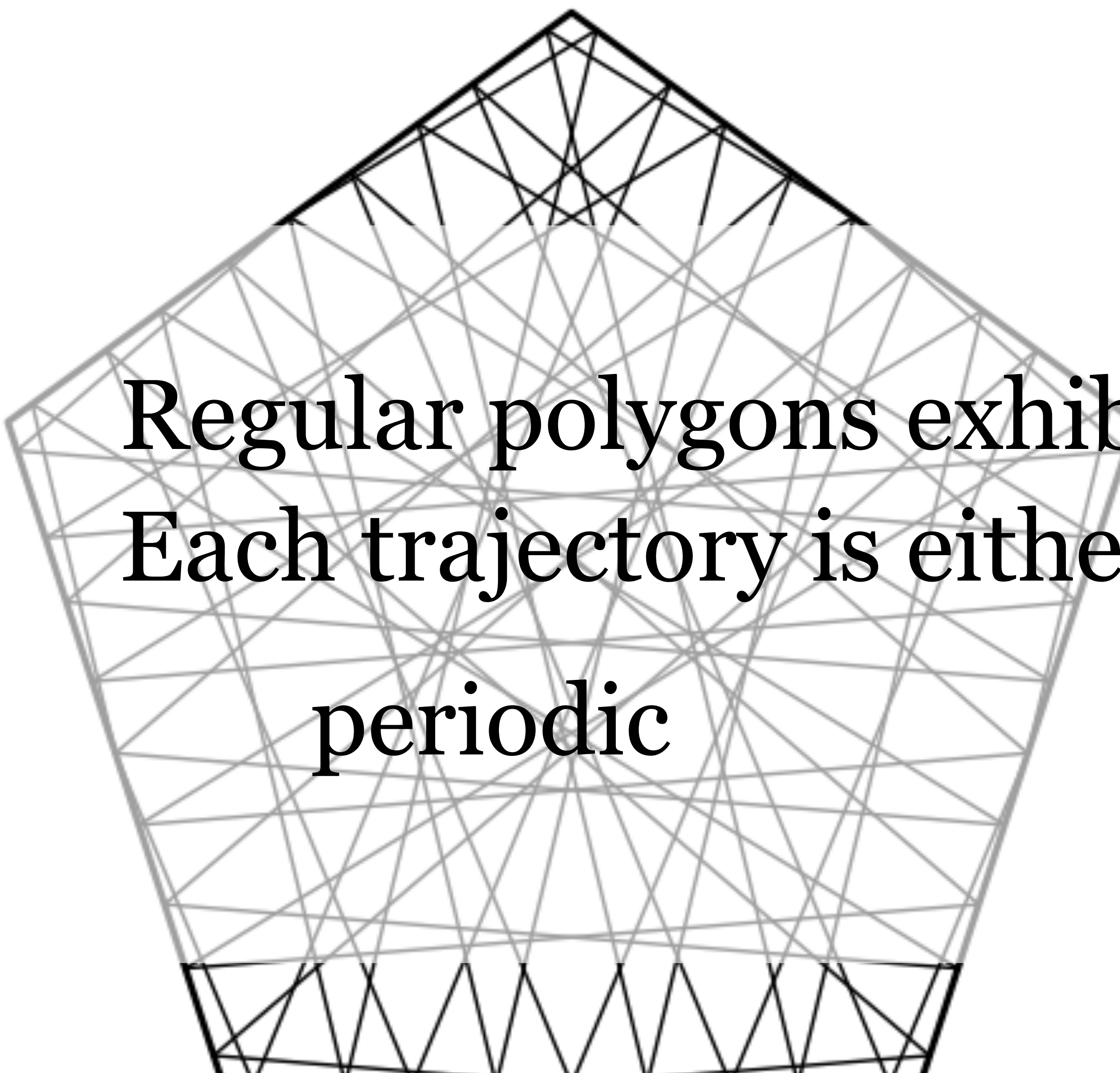

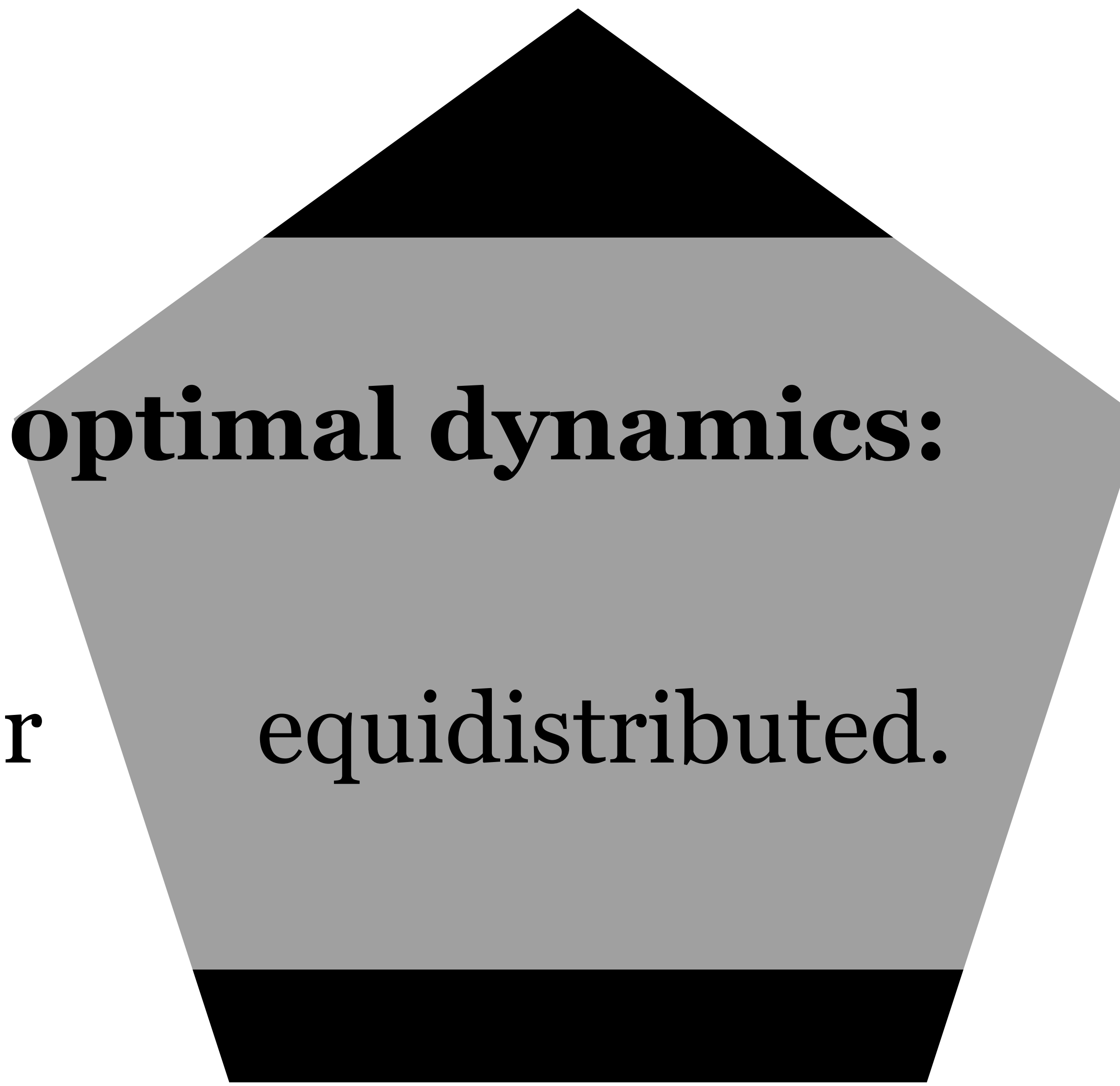

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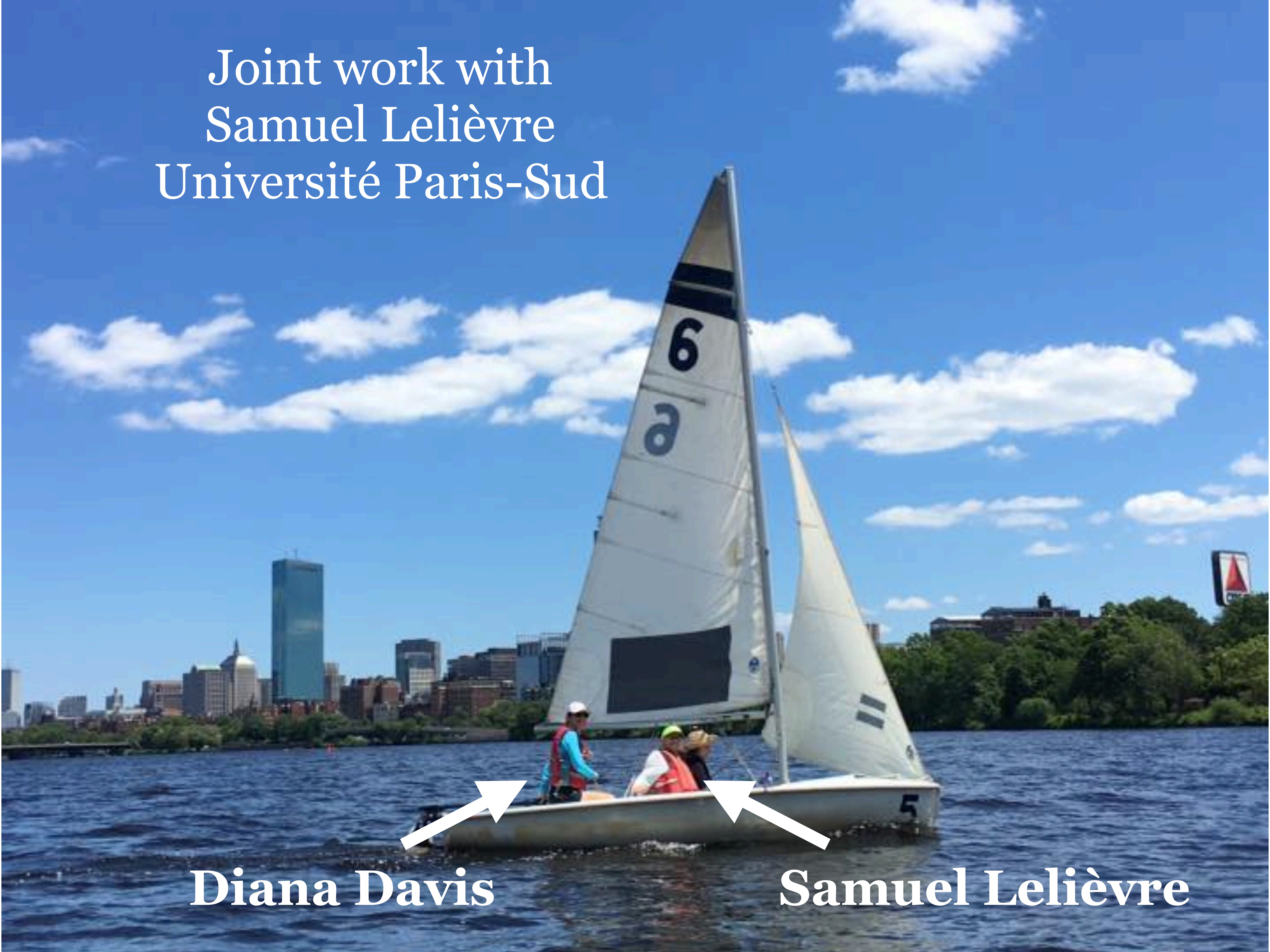
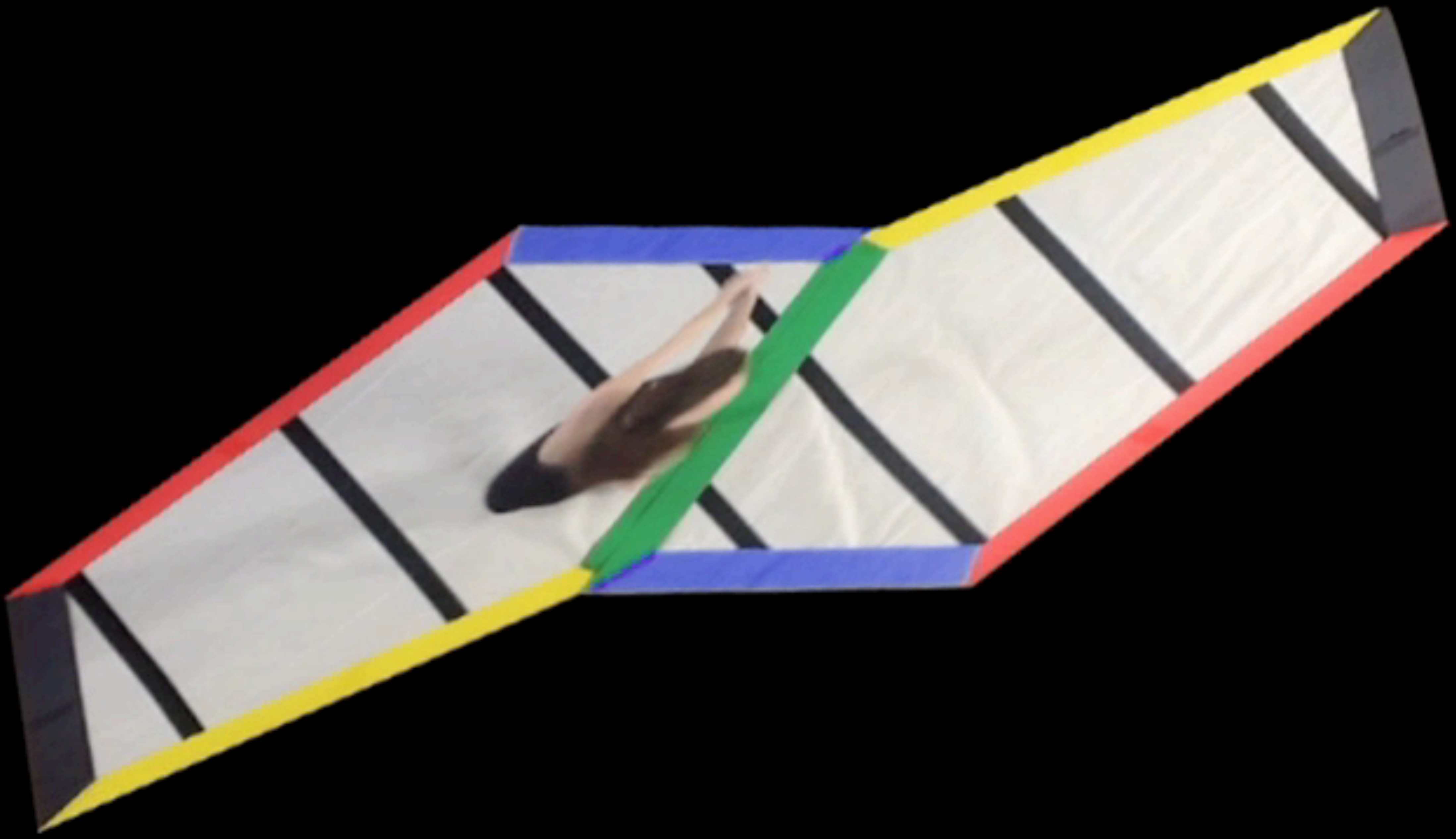

Joint work with
Samuel Lelièvre
Université Paris-Sud
Diana Davis
Samuel Lelièvre

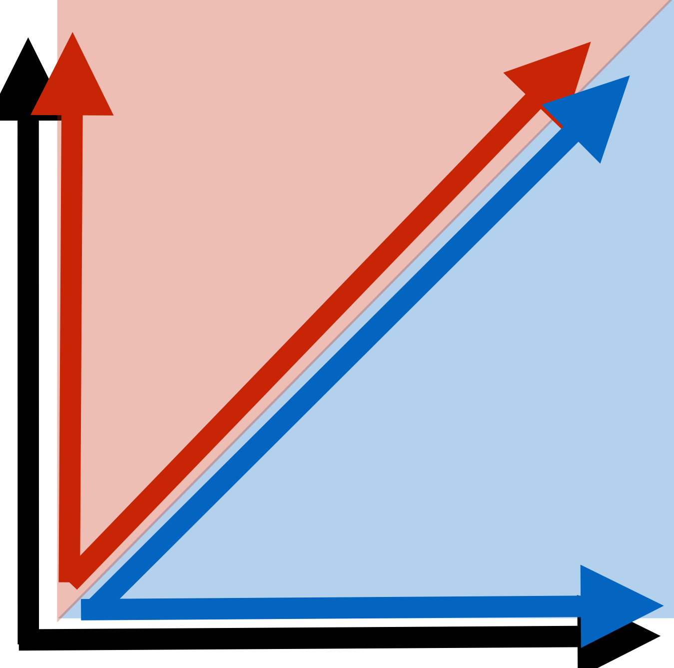


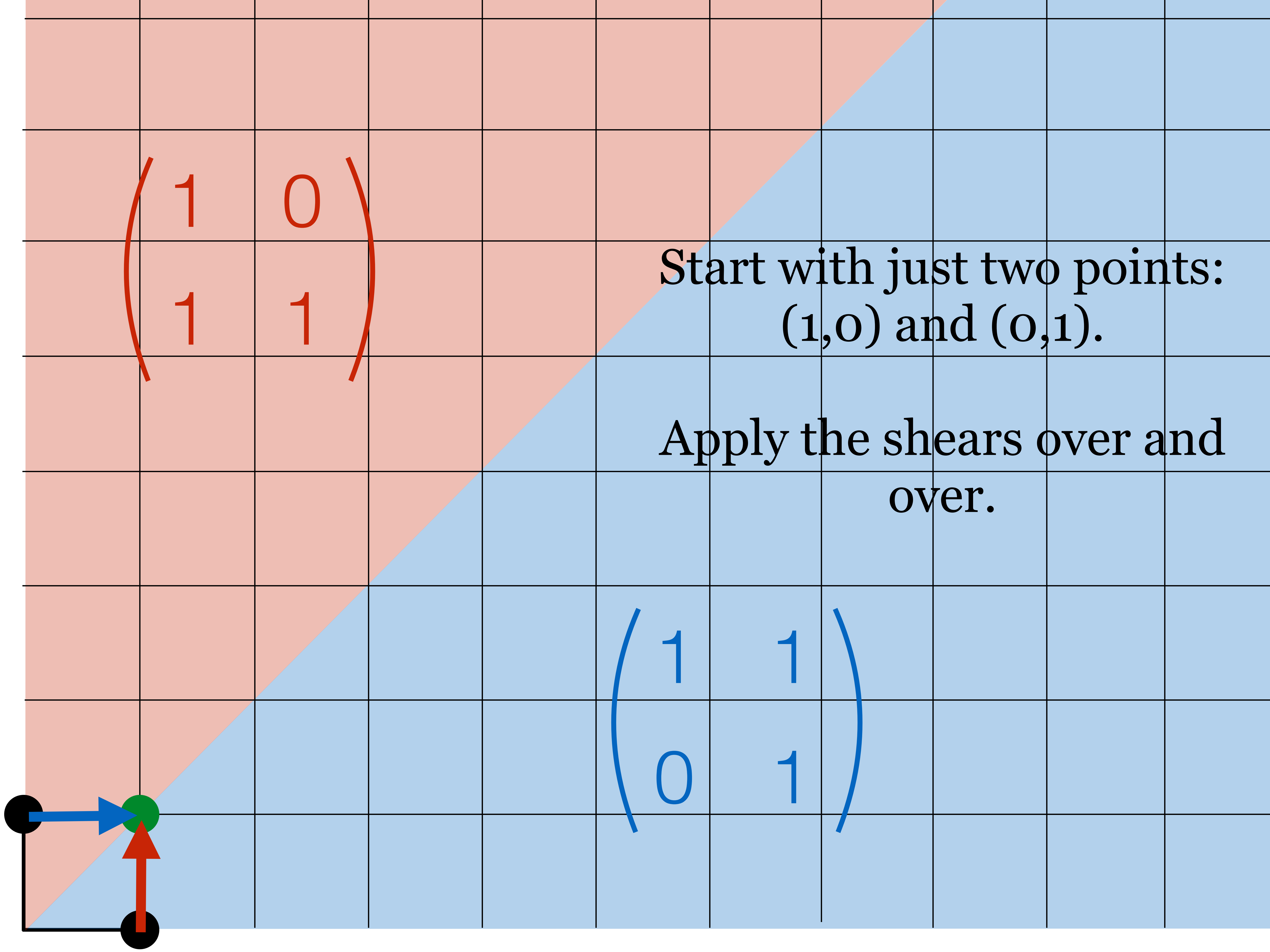
useful tool: shear

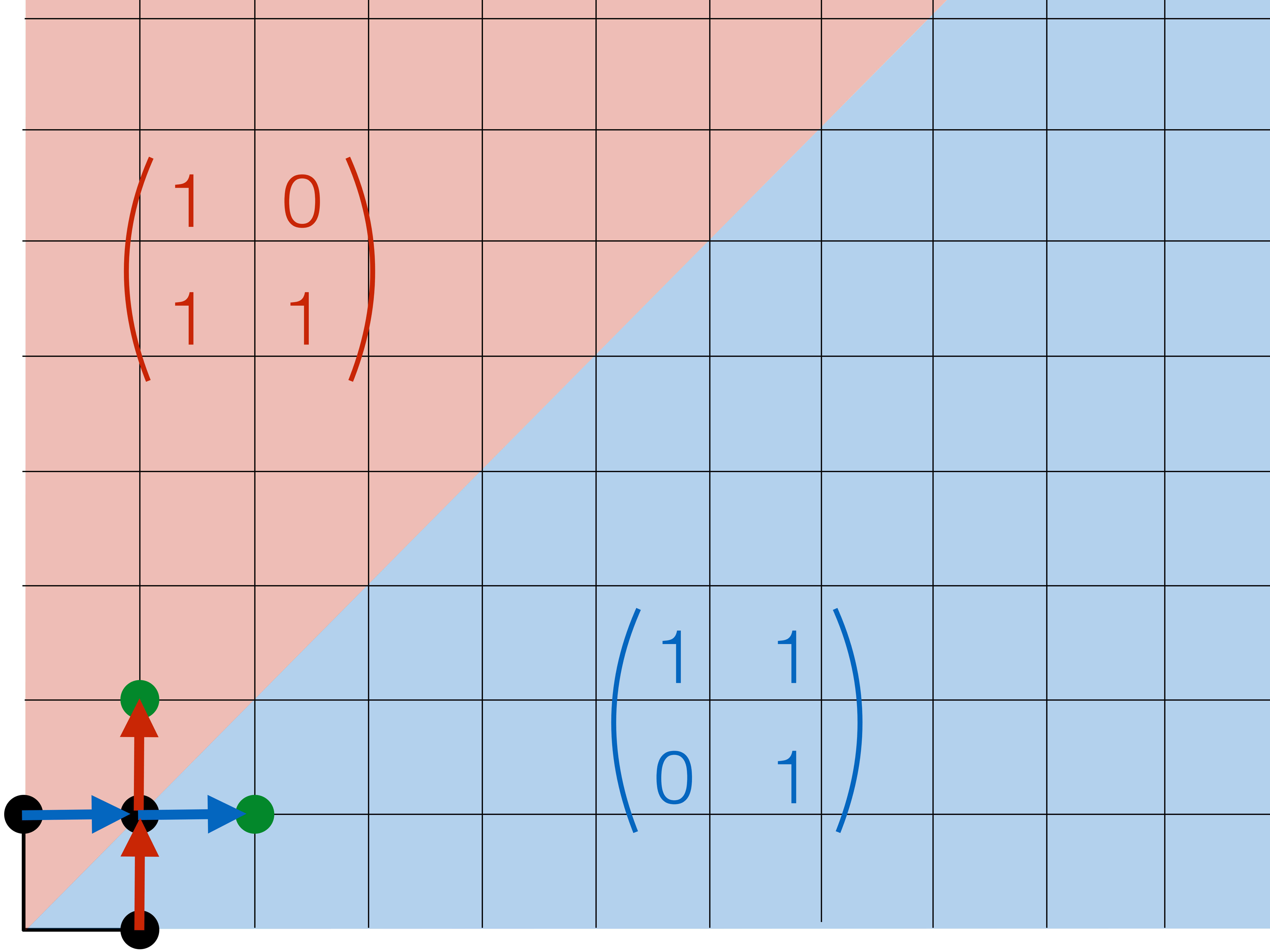
$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Use shears to generate every "primitive" (visible) lattice point, and put a tree structure on those points.

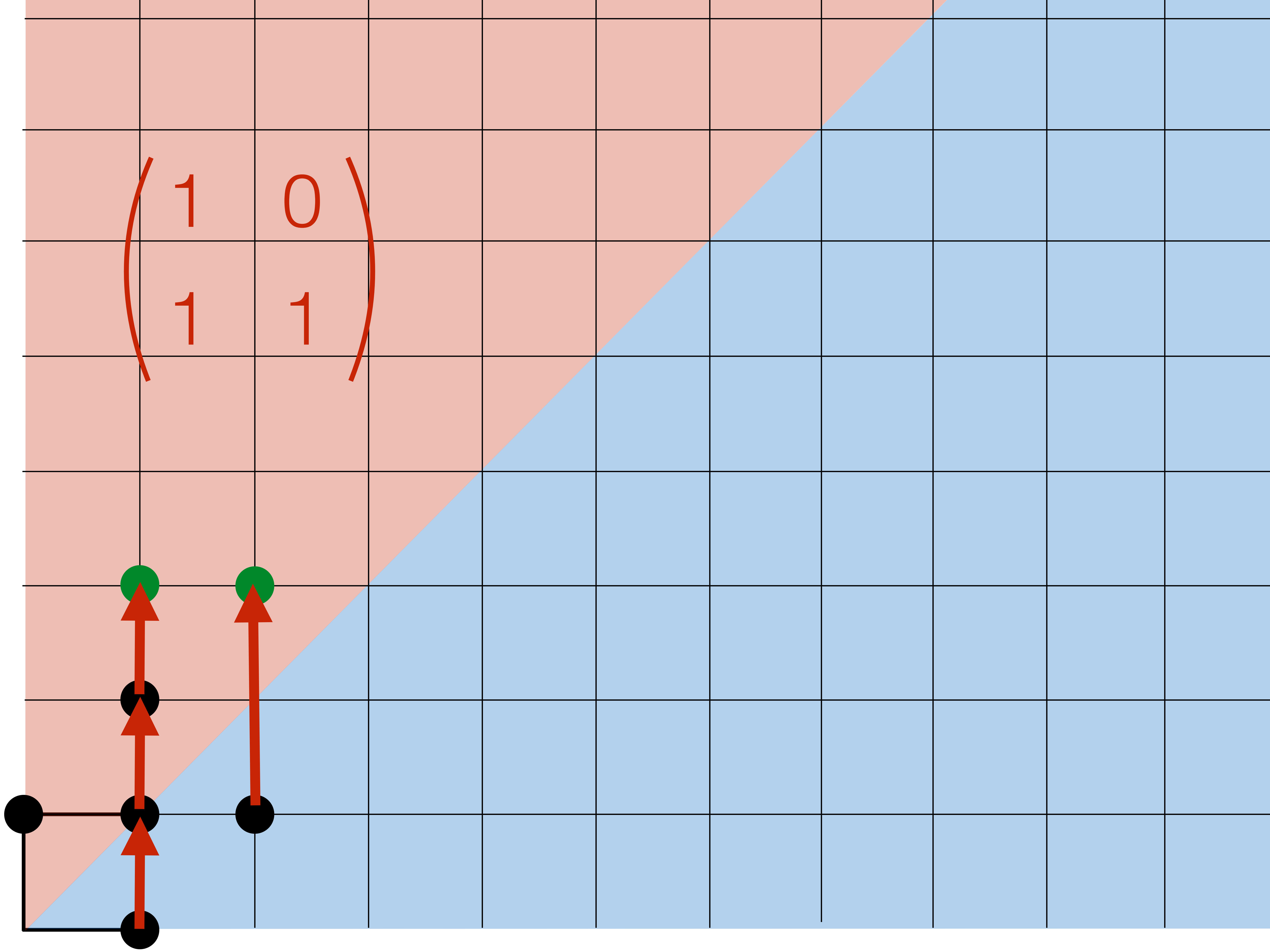
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

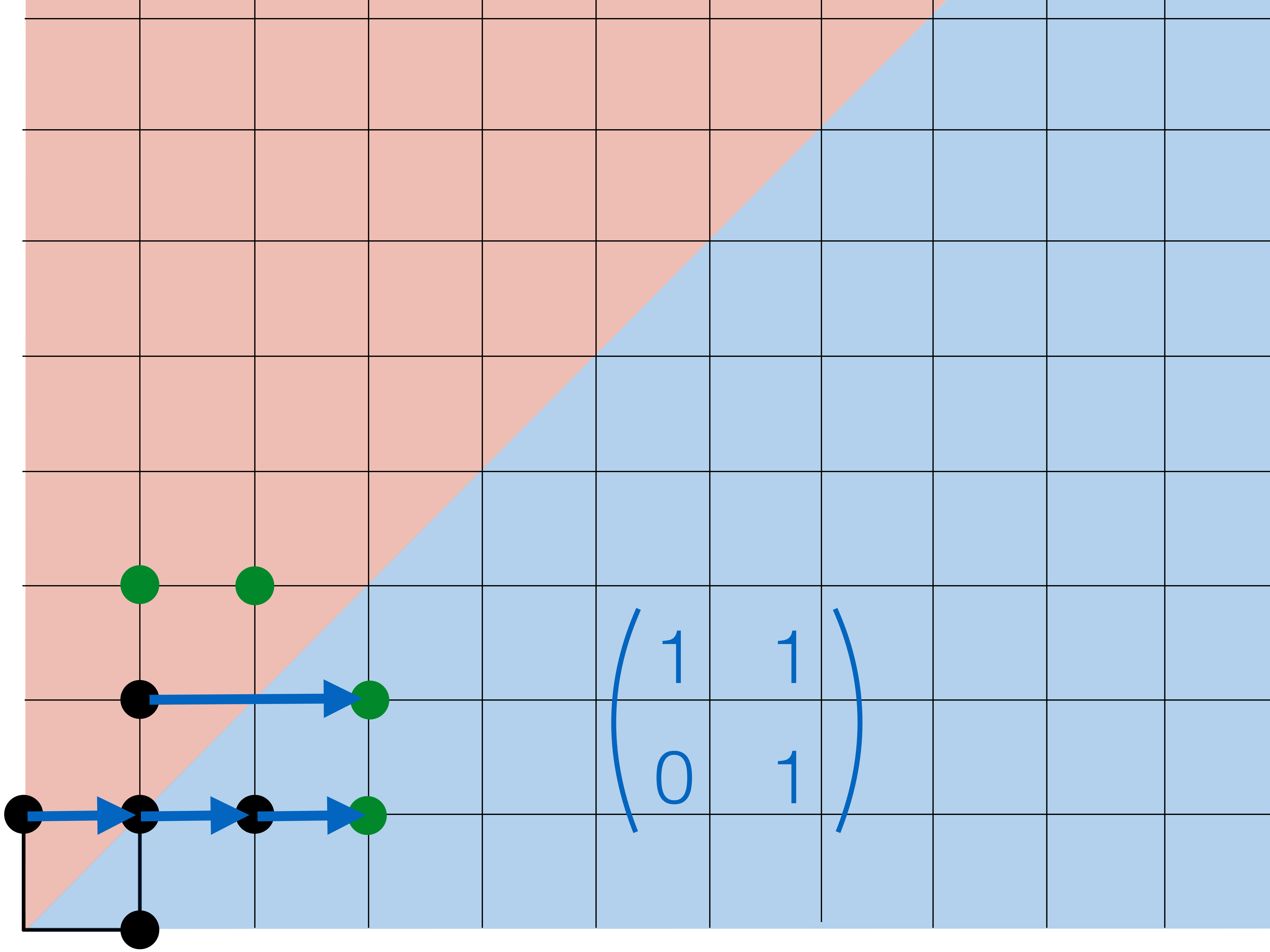



$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$
Start with just two points: (1,0) and (0,1).
Apply the shears over and over.
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$



$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$


1 0
1 1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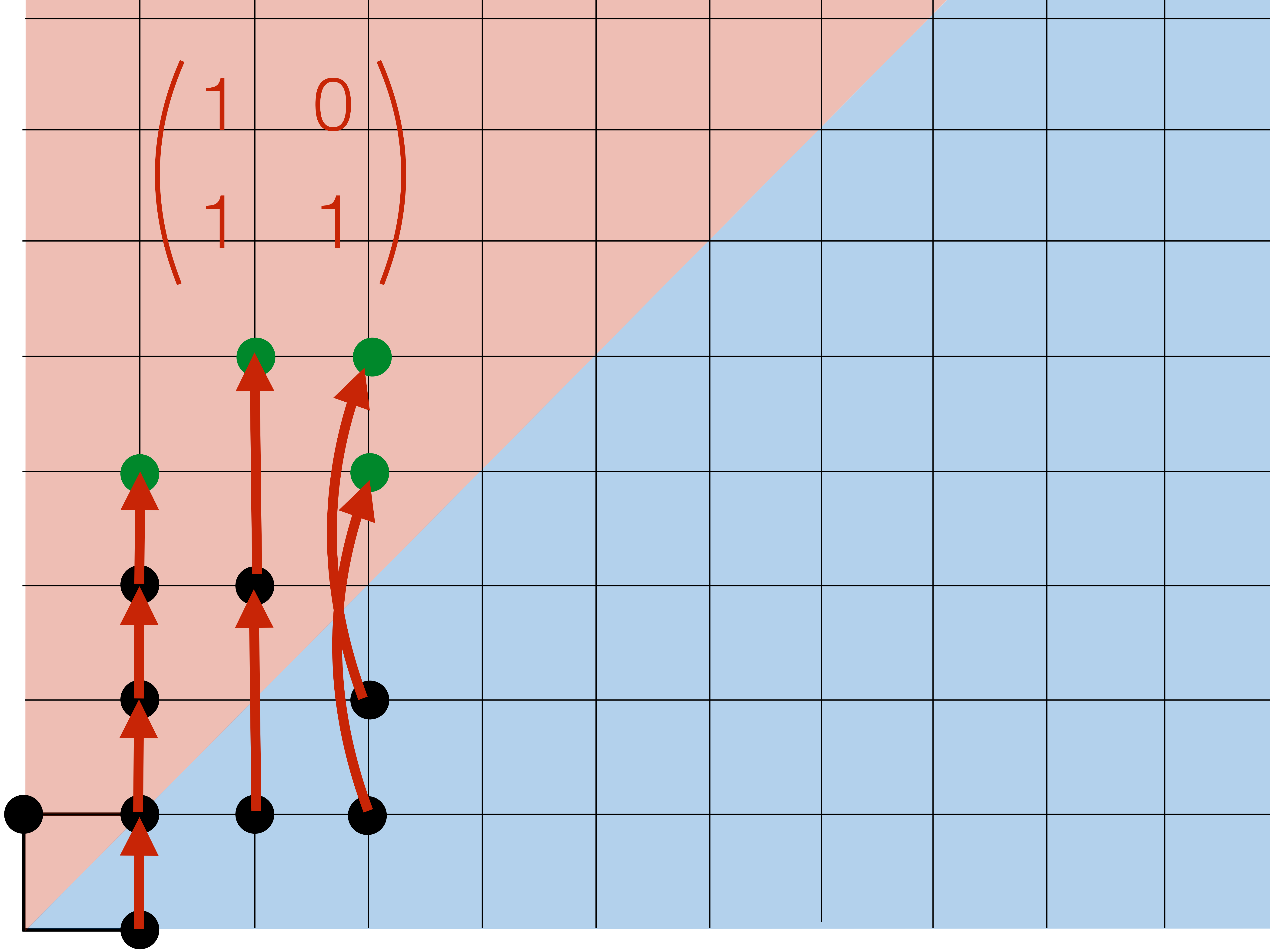

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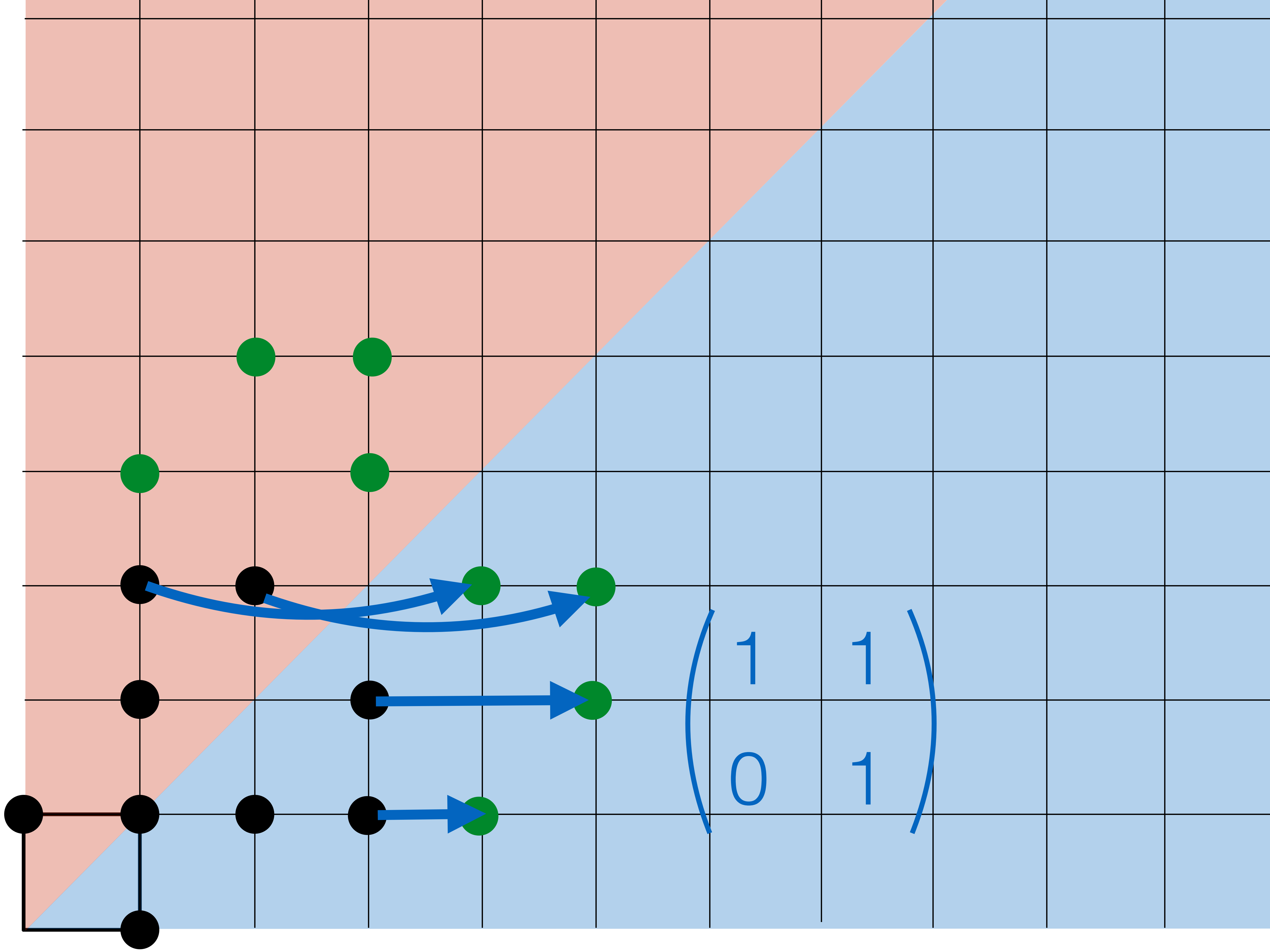

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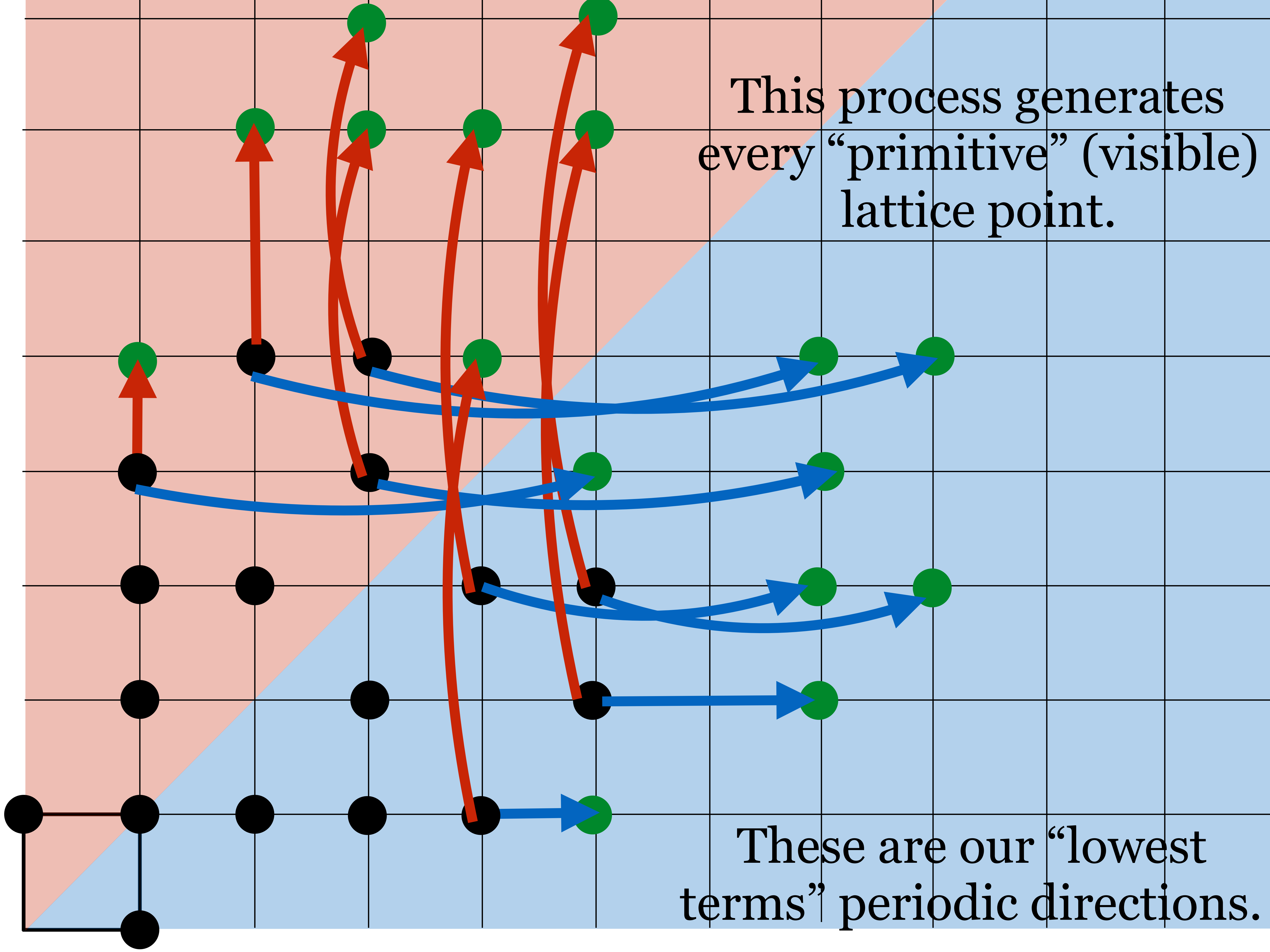
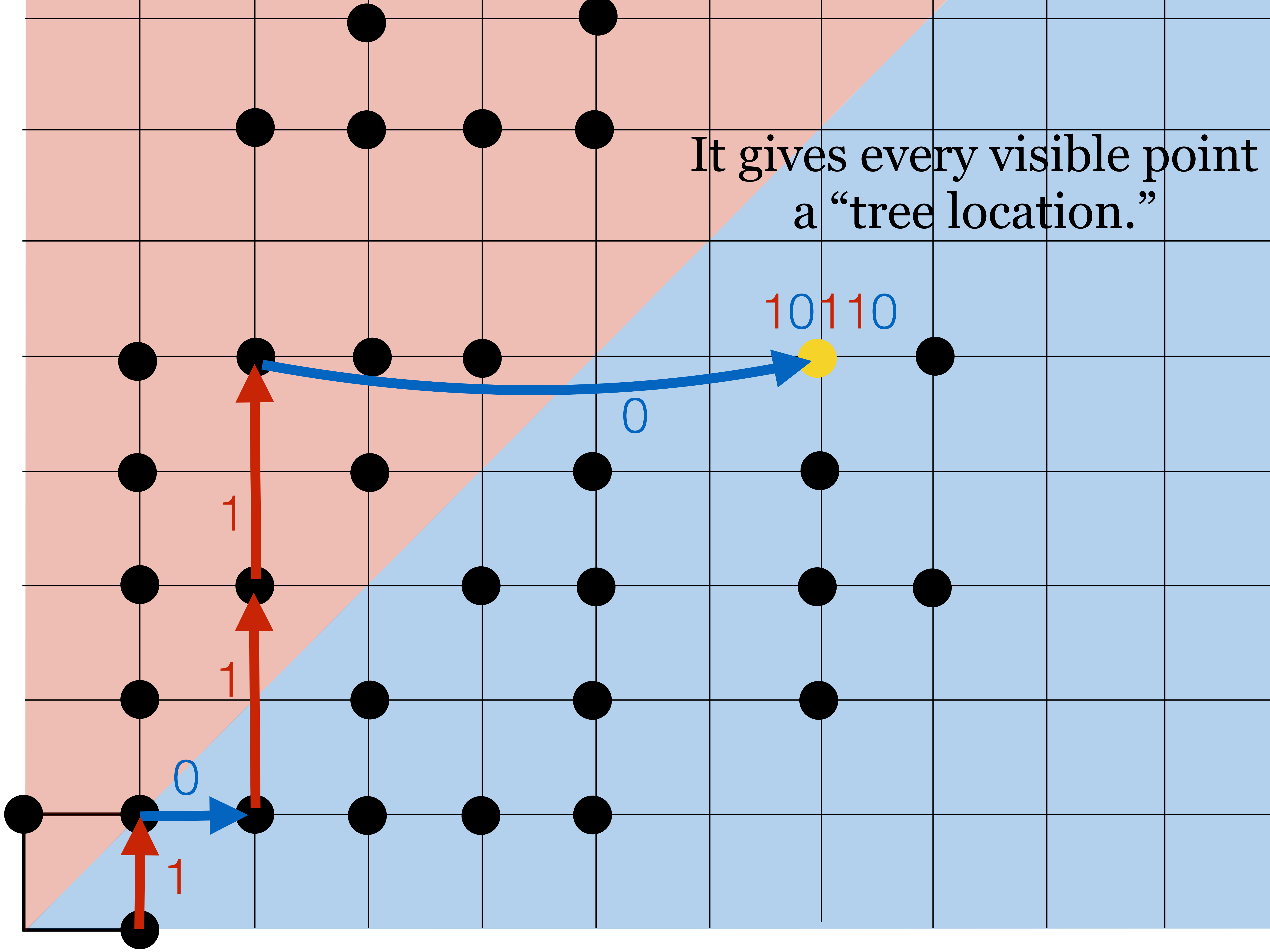
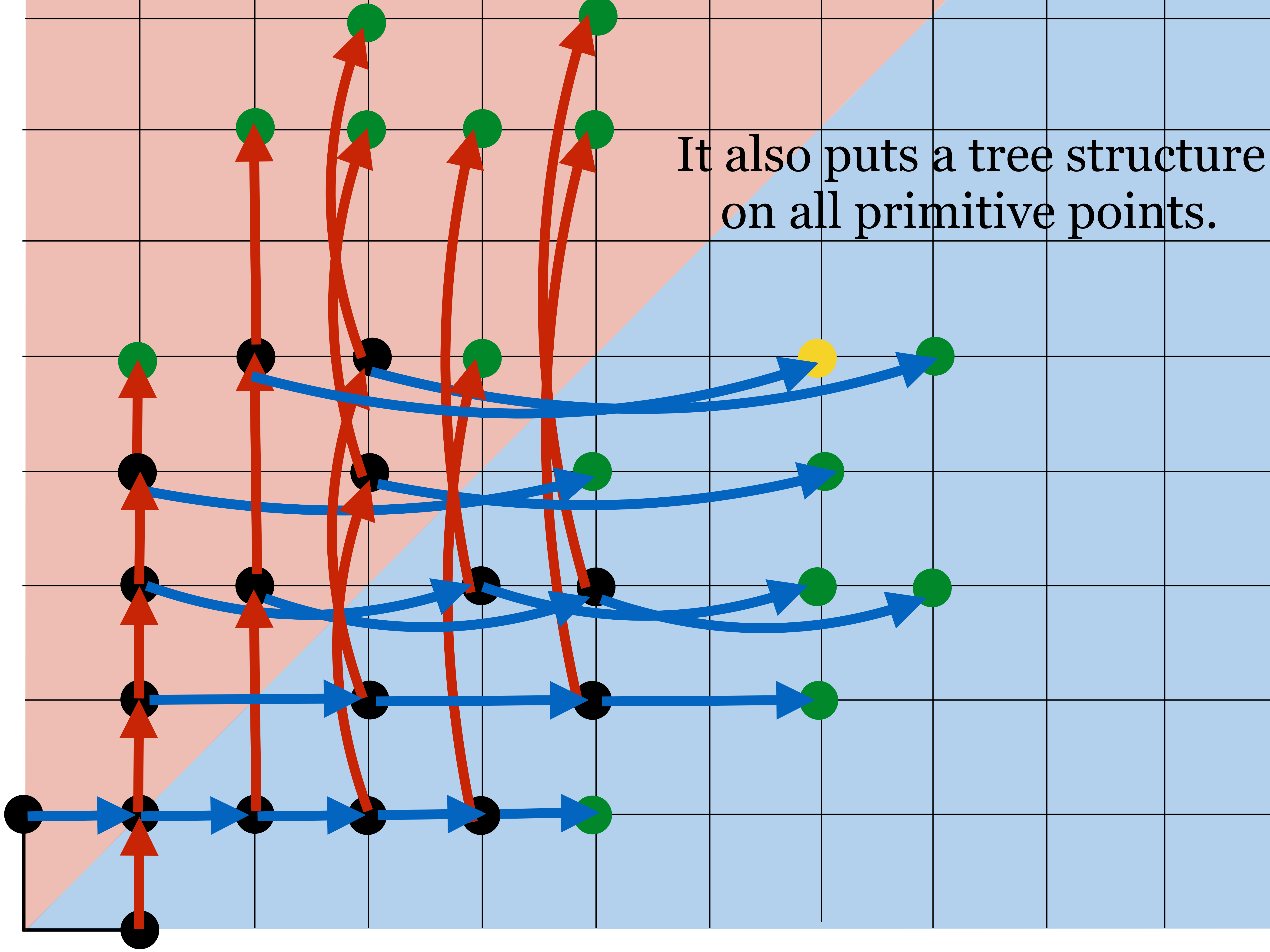
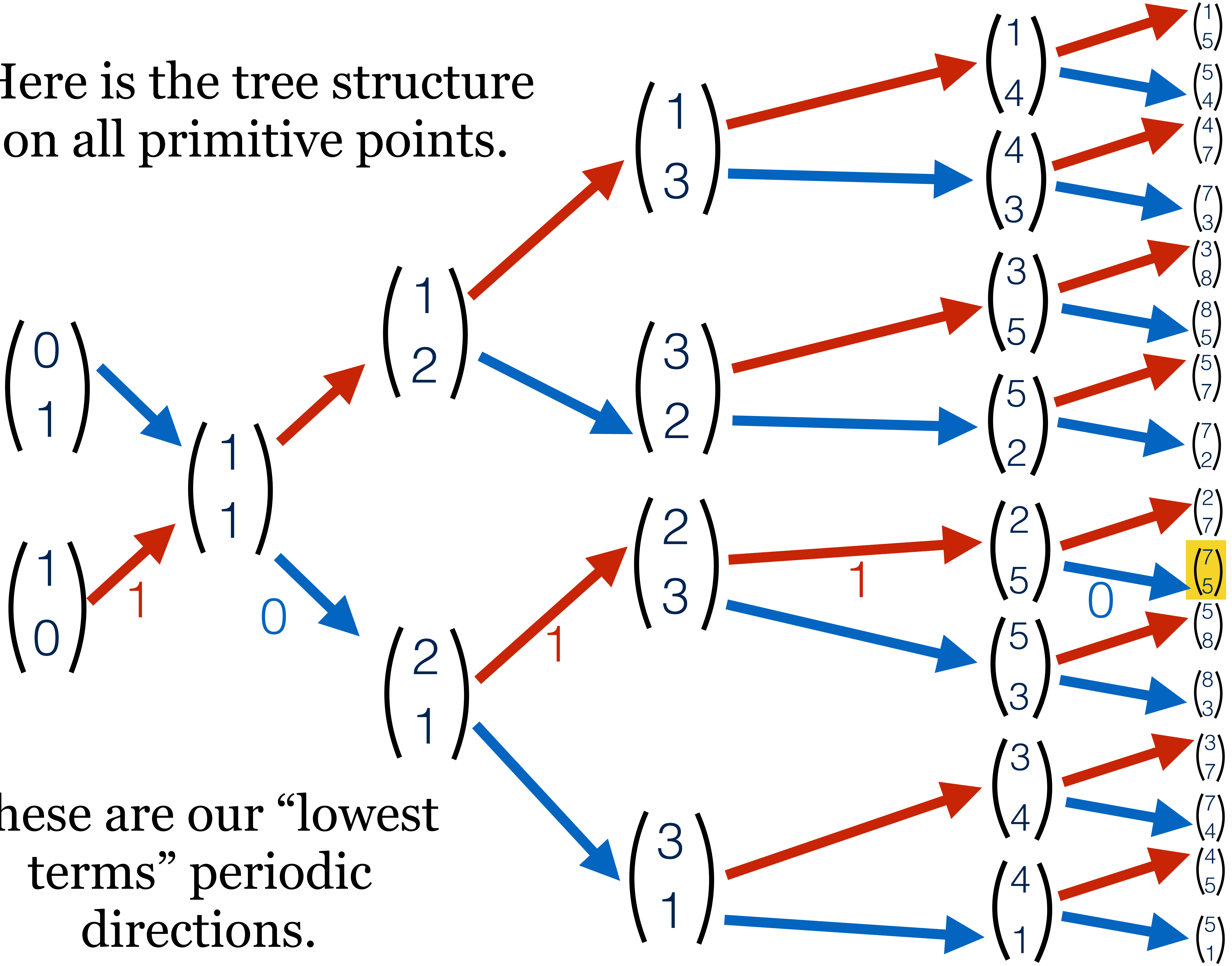

This process generates every "primitive" (visible) lattice point.
These are our "lowest terms" periodic directions.


It gives every visible point
a “tree location.”
10110
0
1
1
0
1



It also puts a tree structure on all primitive points.

Here is the tree structure on all primitive points.



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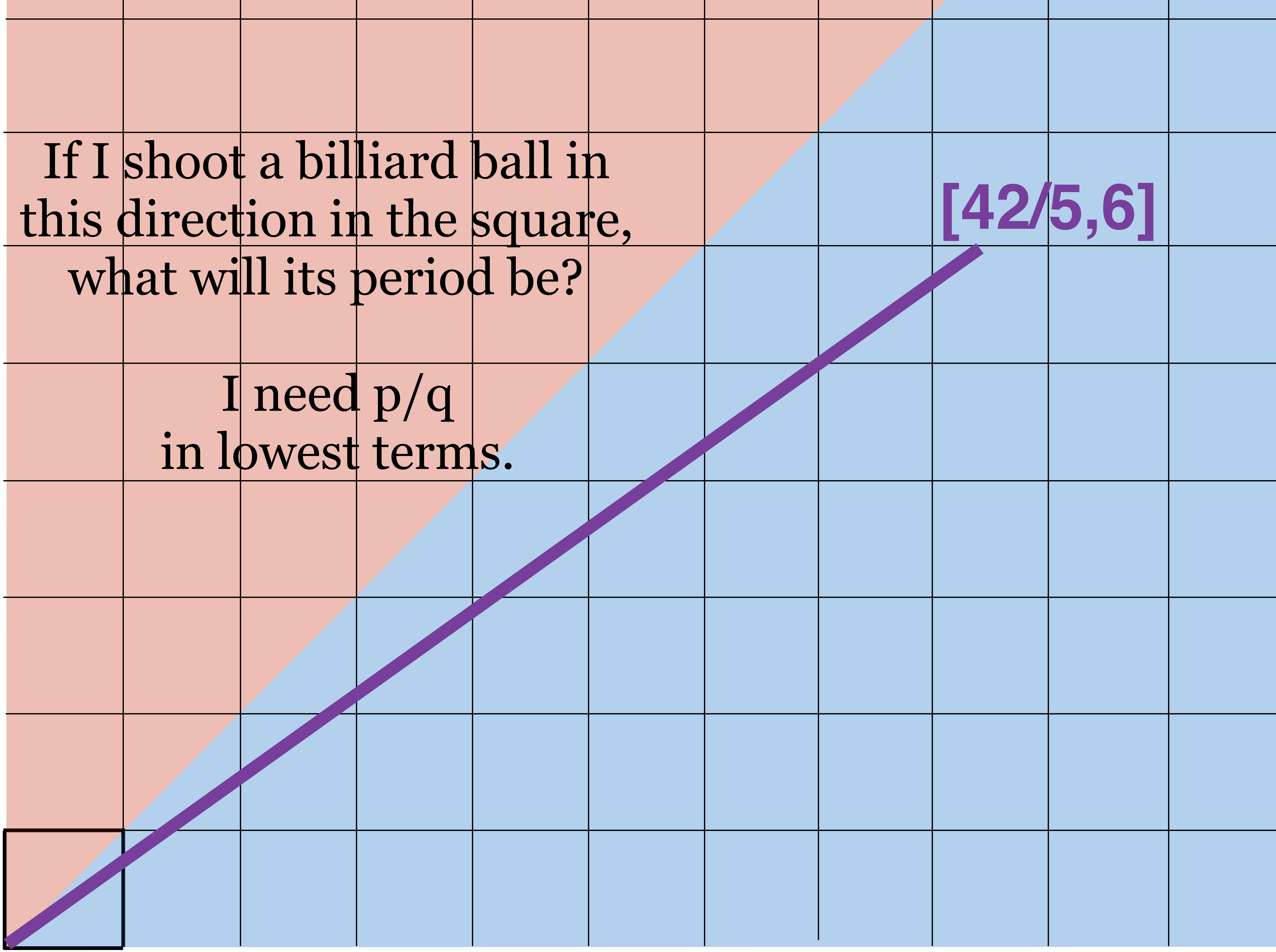
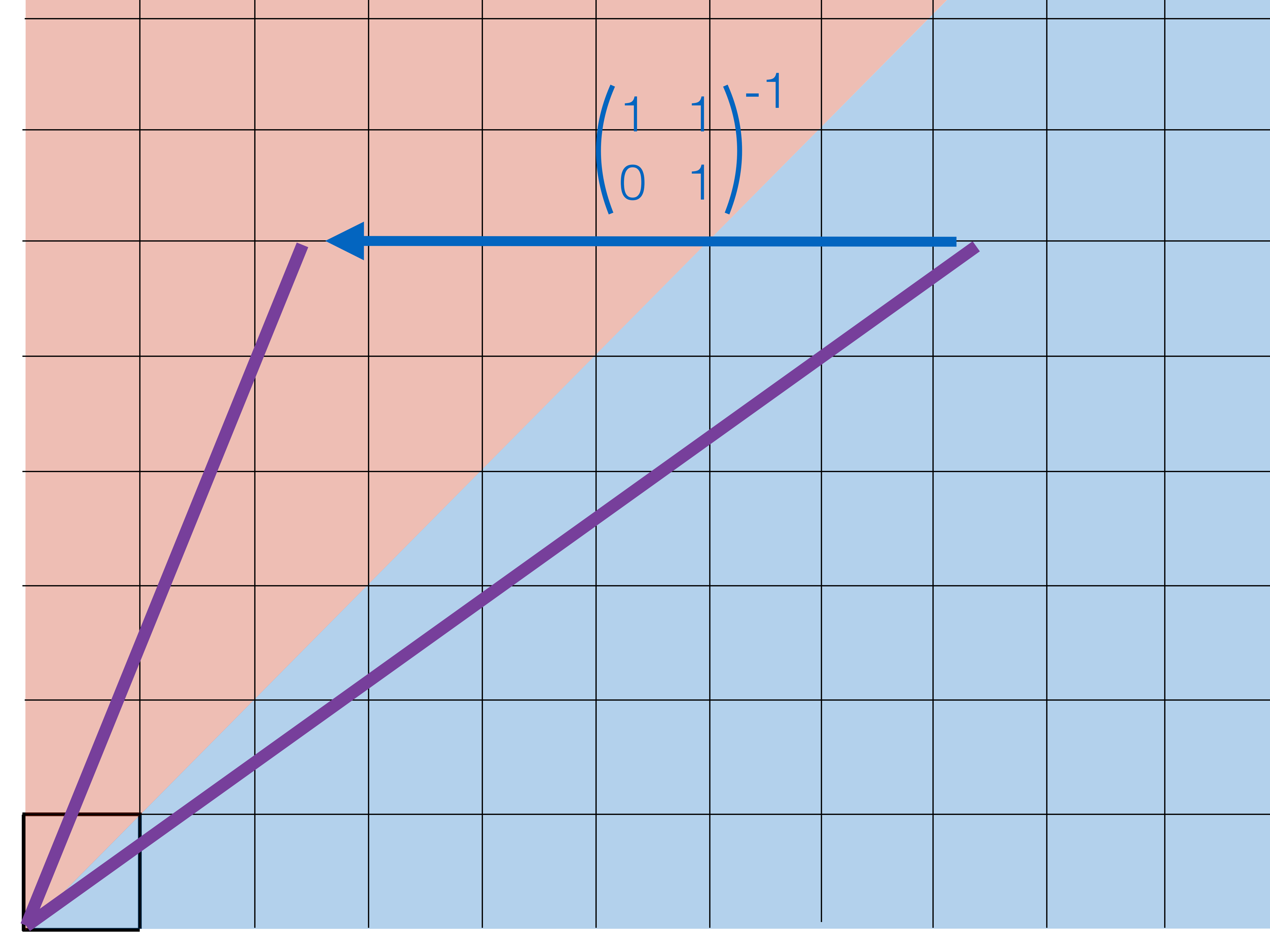
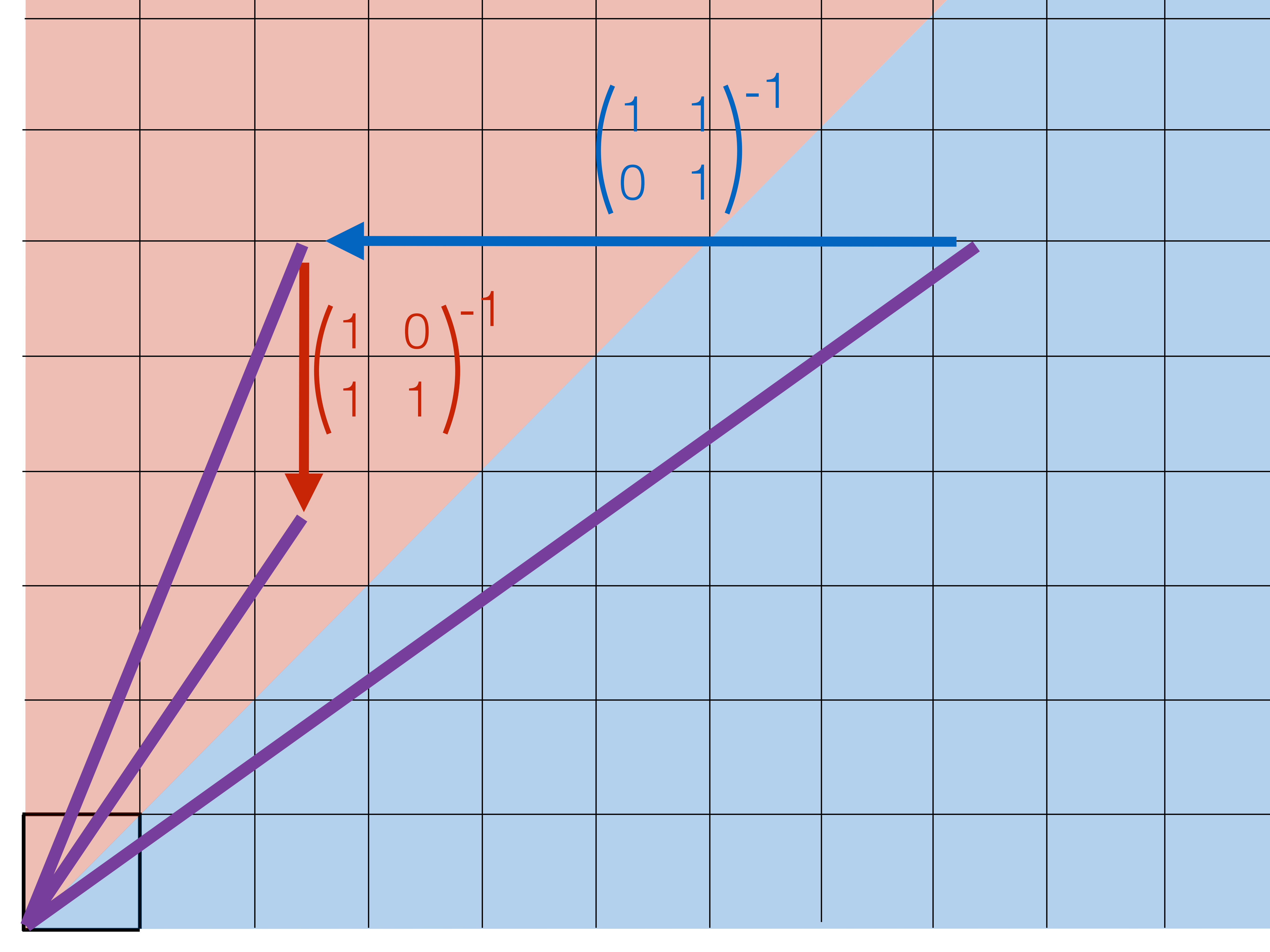

If I shoot a billiard ball in
this direction in the square,
what will its period be?
I need p/q
in lowest terms.
[42/5,6]


$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$


$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$
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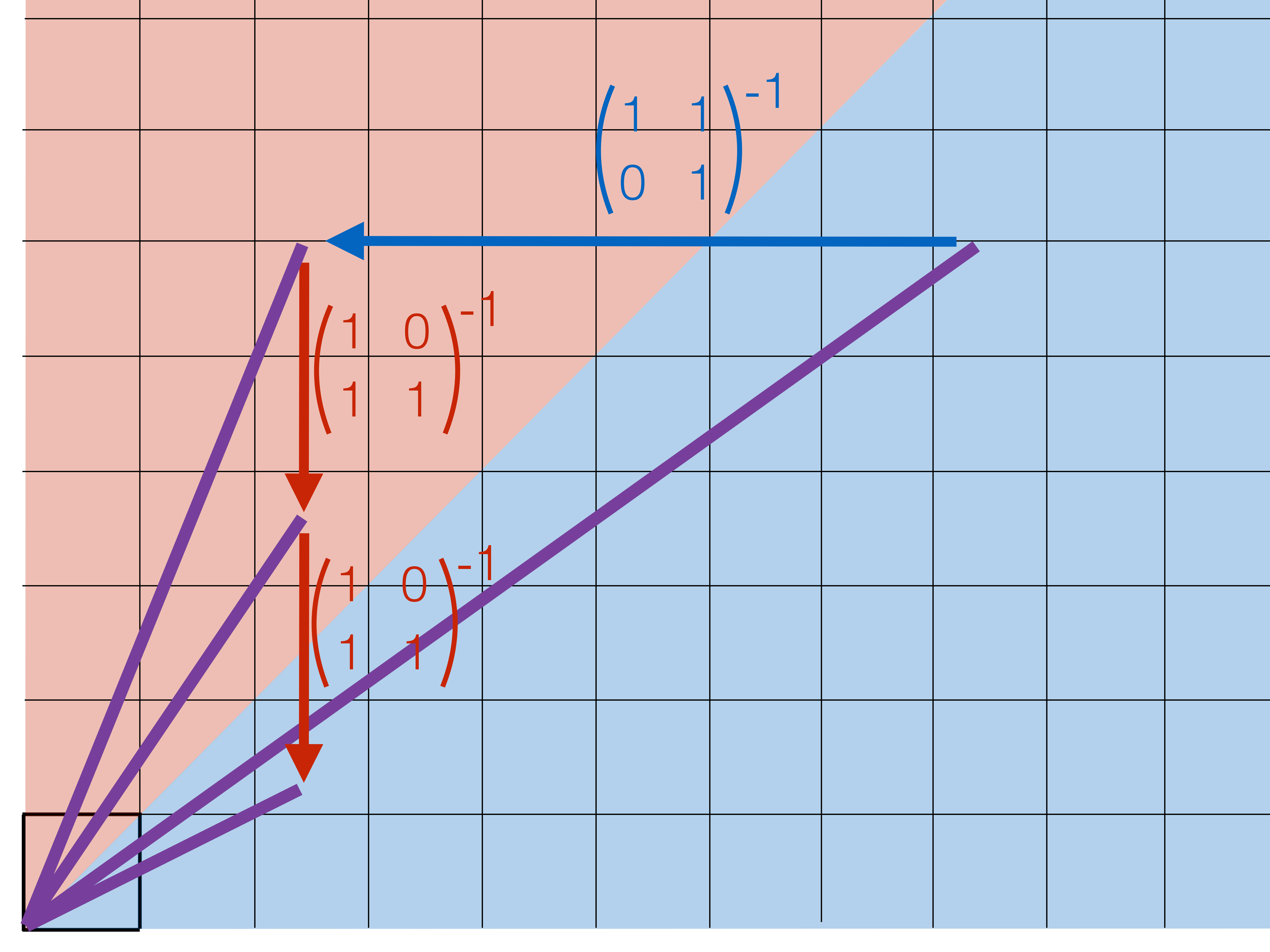

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$
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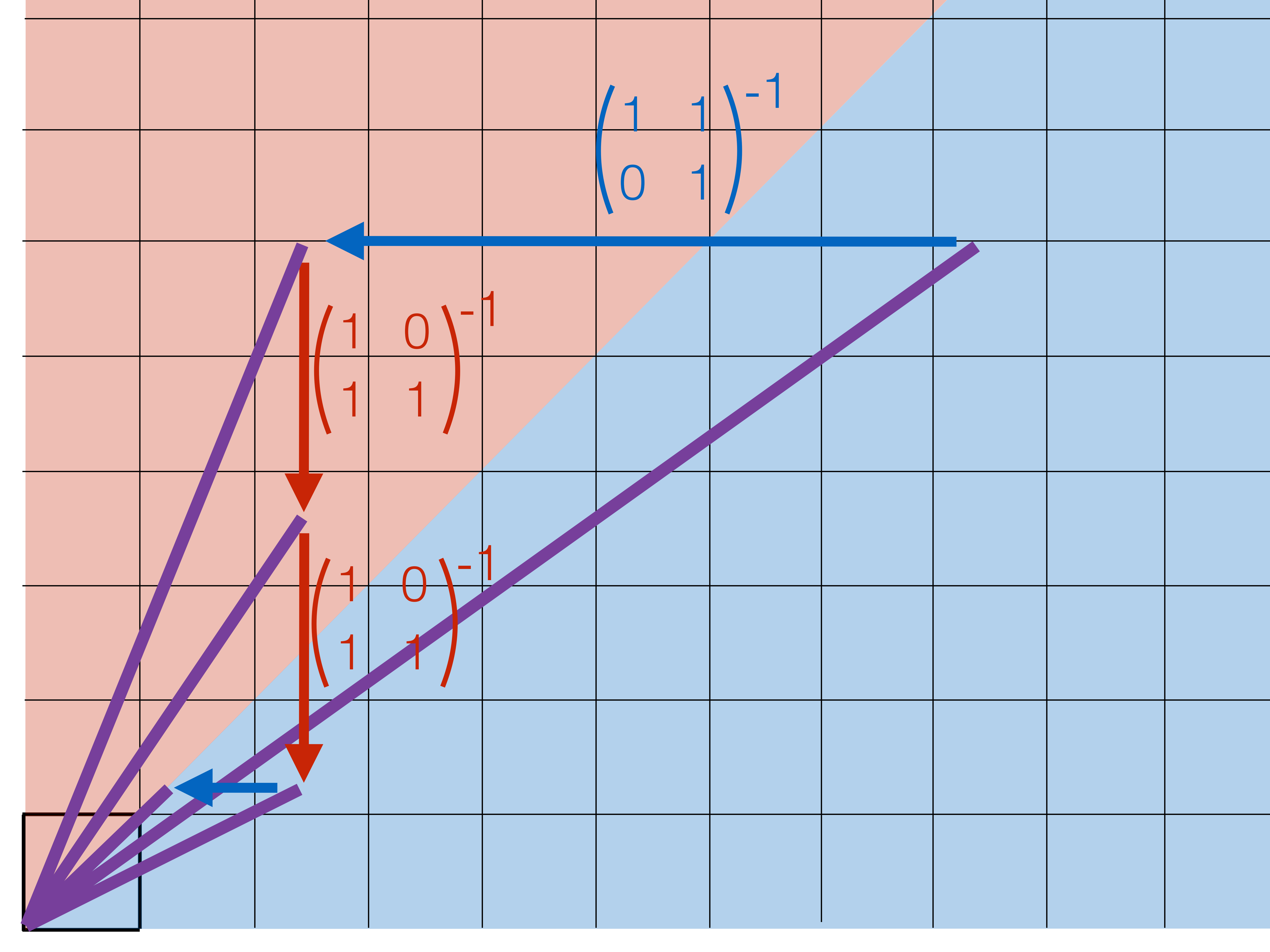

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$
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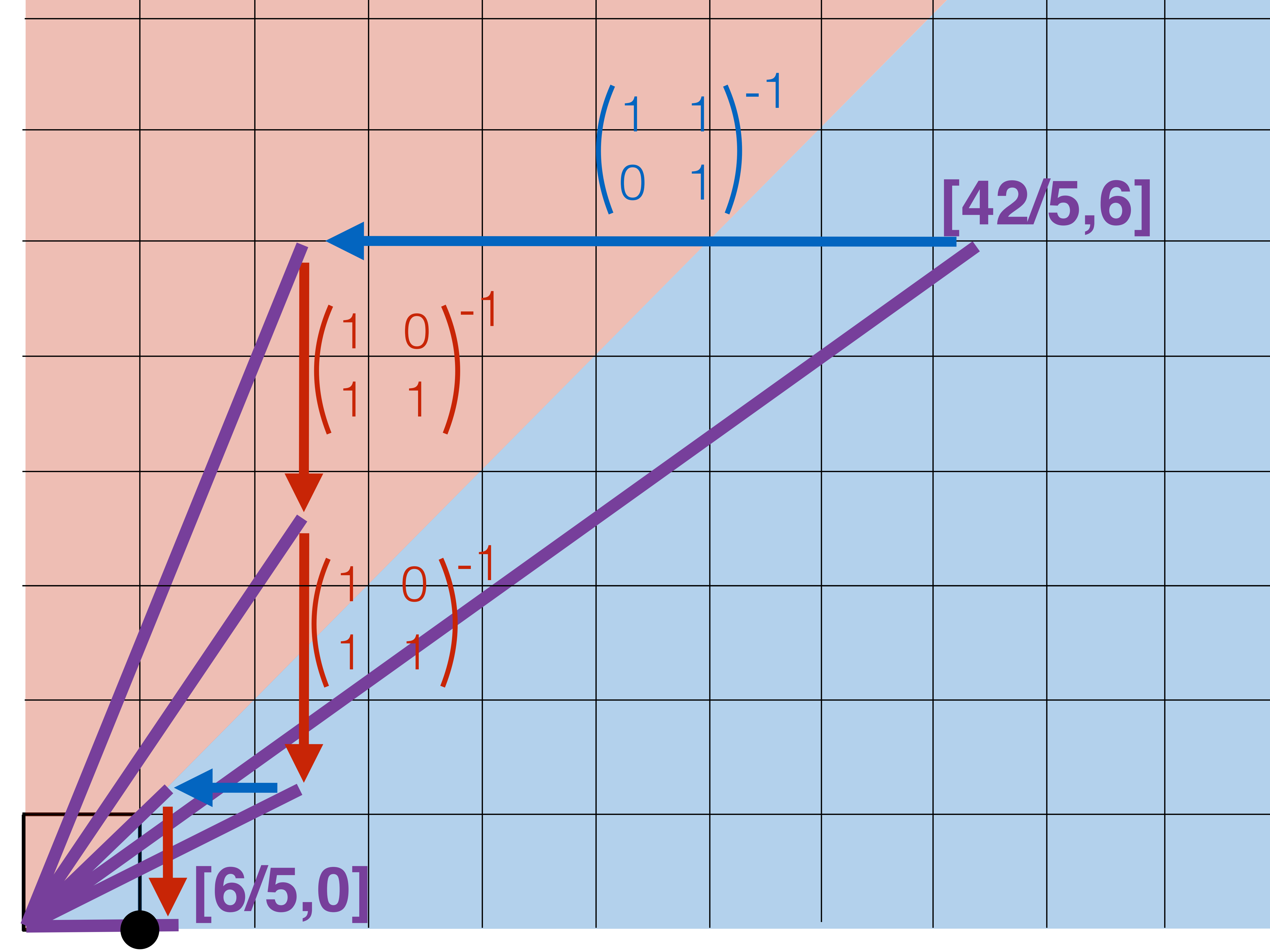
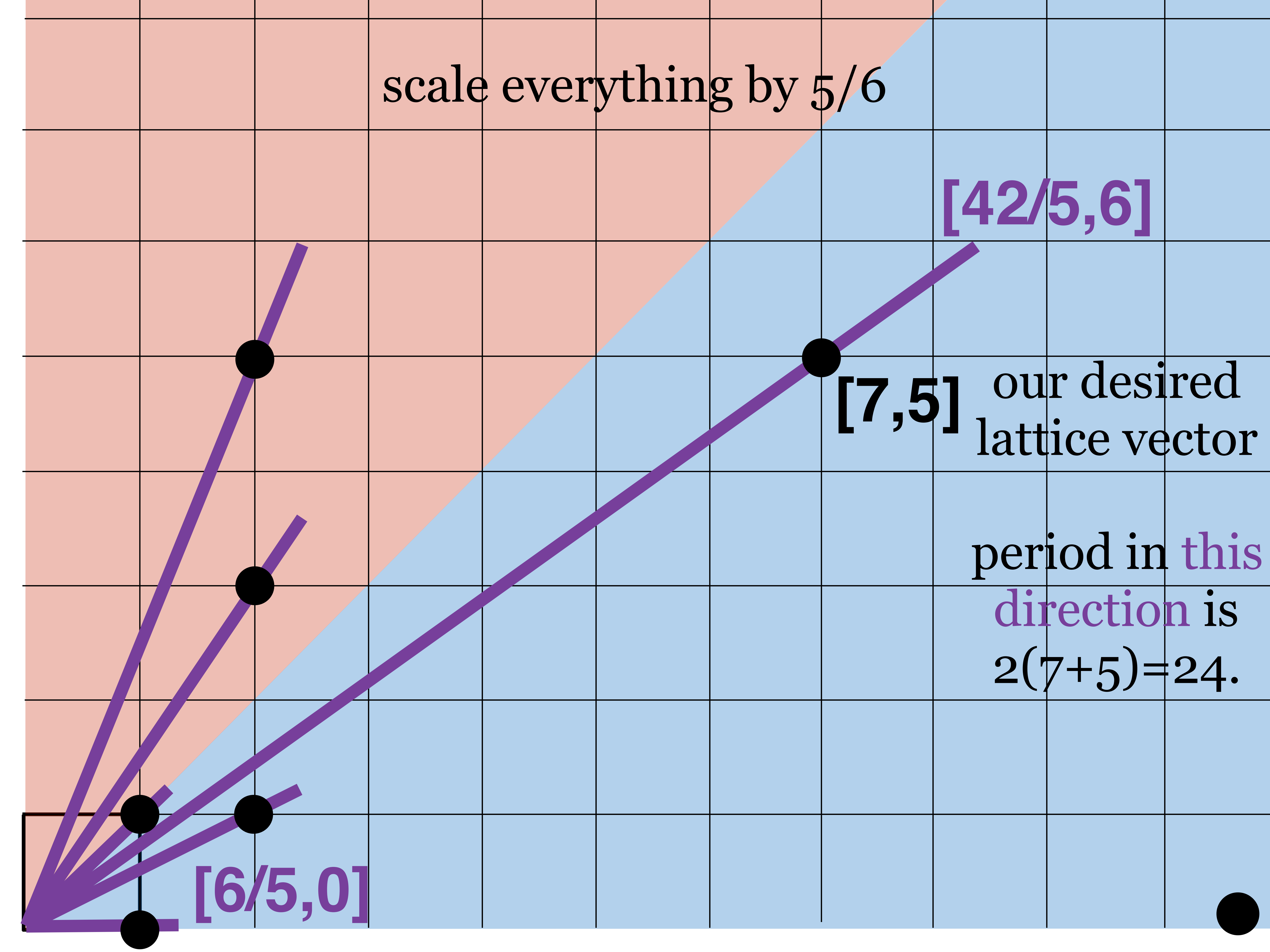
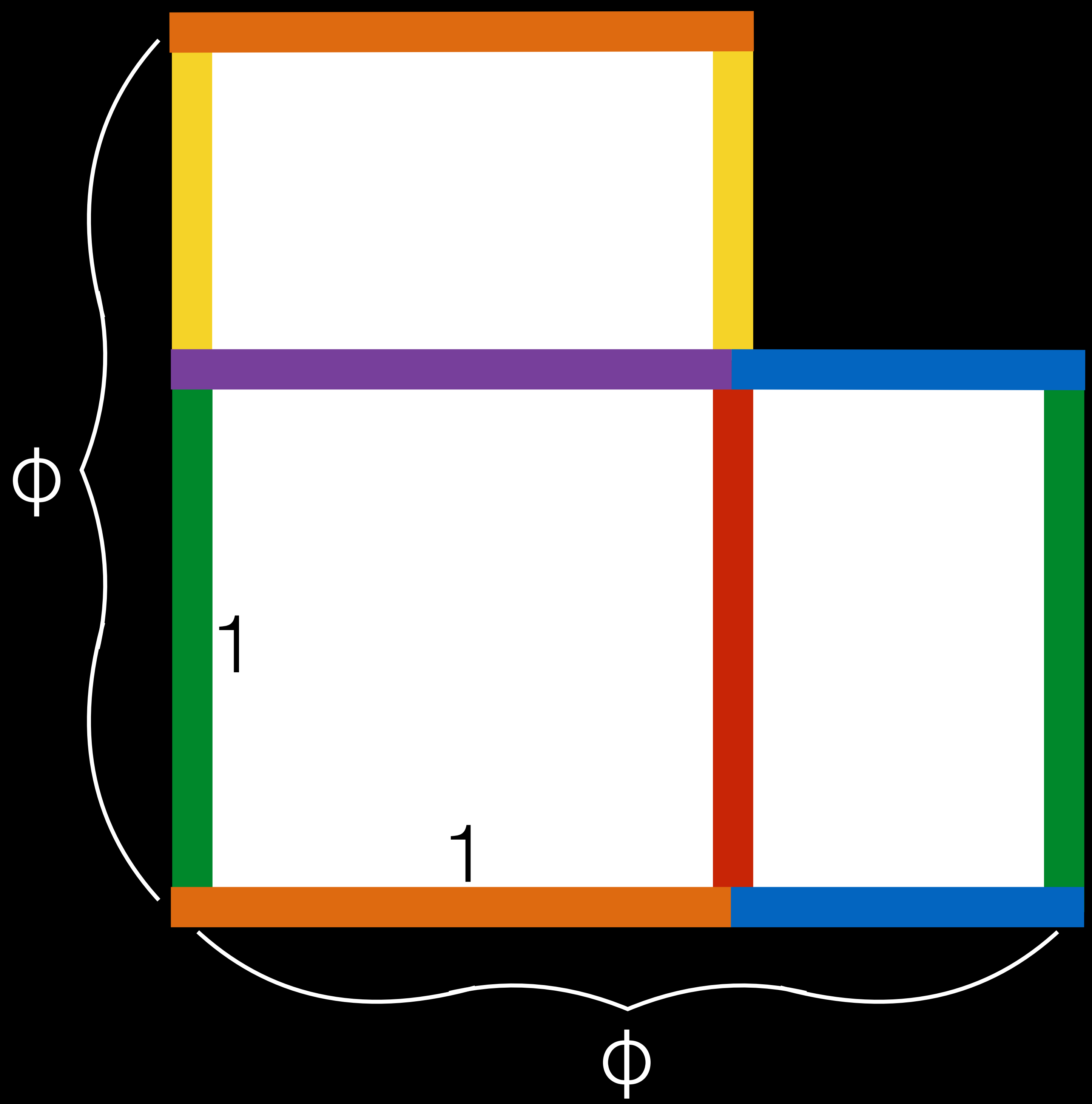


$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$
[42/5,6]
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1}$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1}$
[6/5,0]


scale everything by 5/6
[42/5,6]
[7,5]
our desired
lattice vector
period in this
direction is
2(7+5)=24.
[6/5,0]

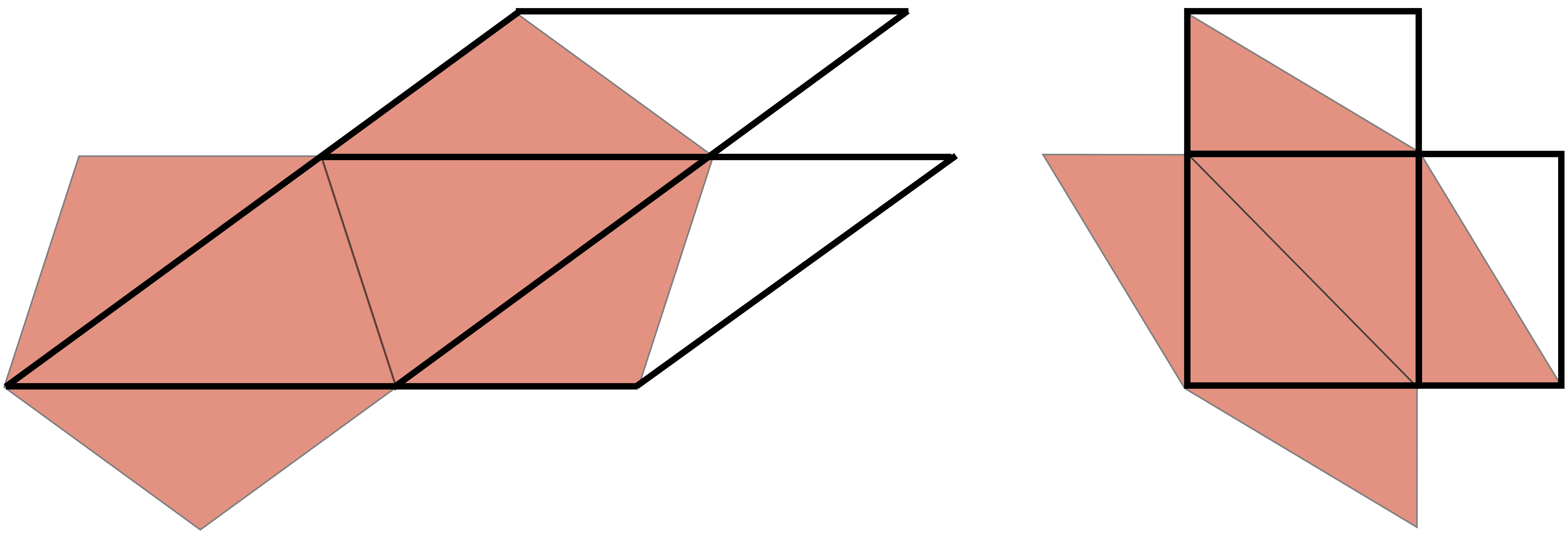
The double pentagon

The golden L



Fact: The double pentagon and the golden L are the same surface.

Proof. Shear; cut and paste.



The golden L is easier to use.

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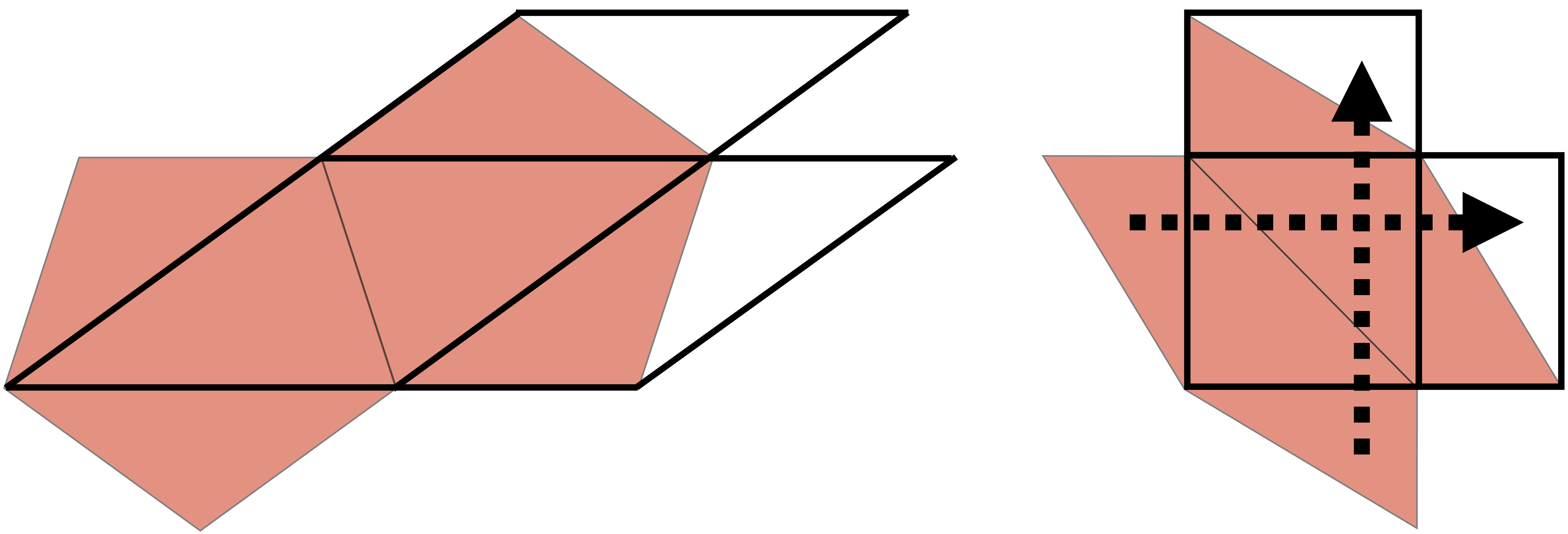
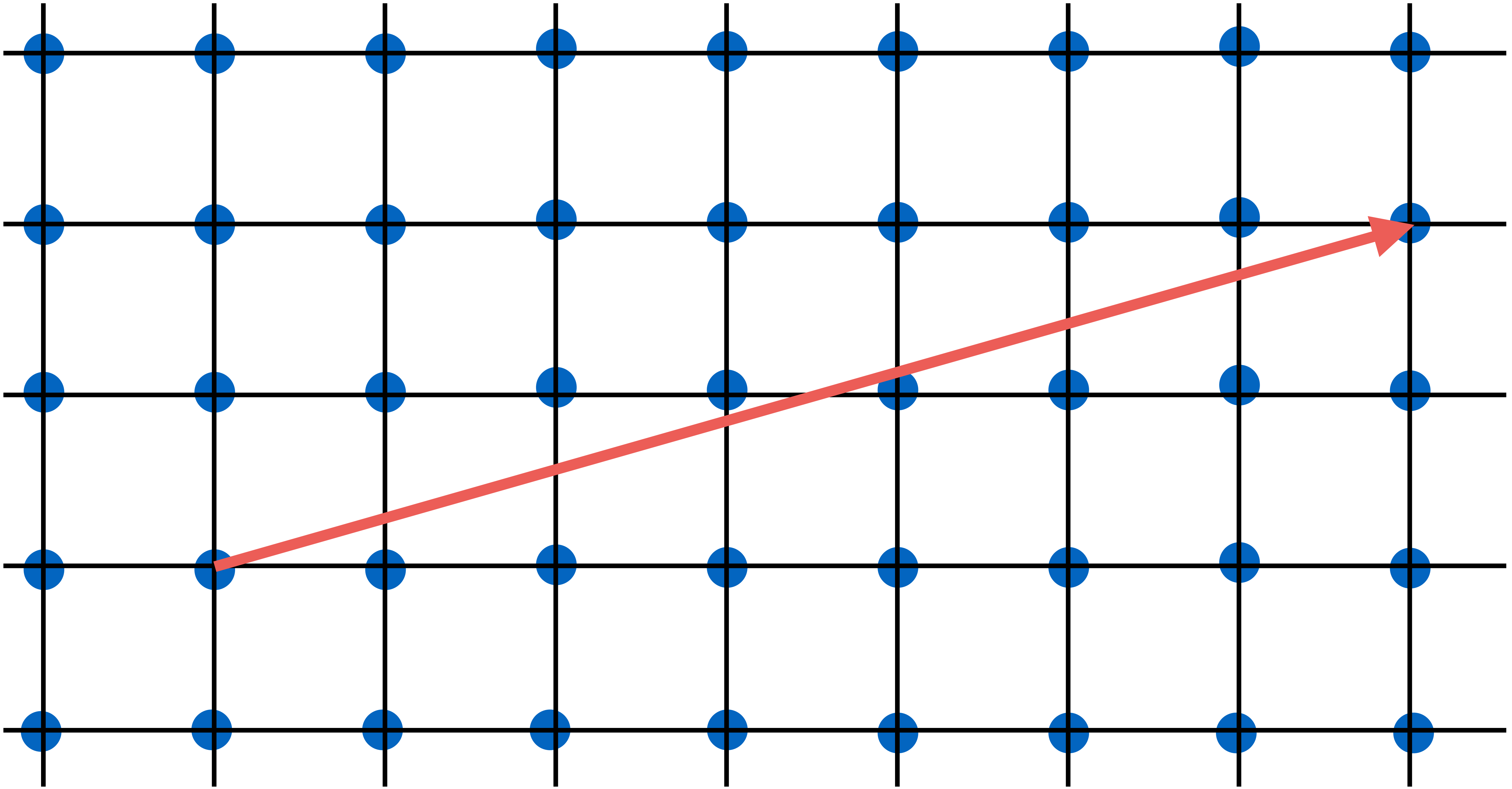


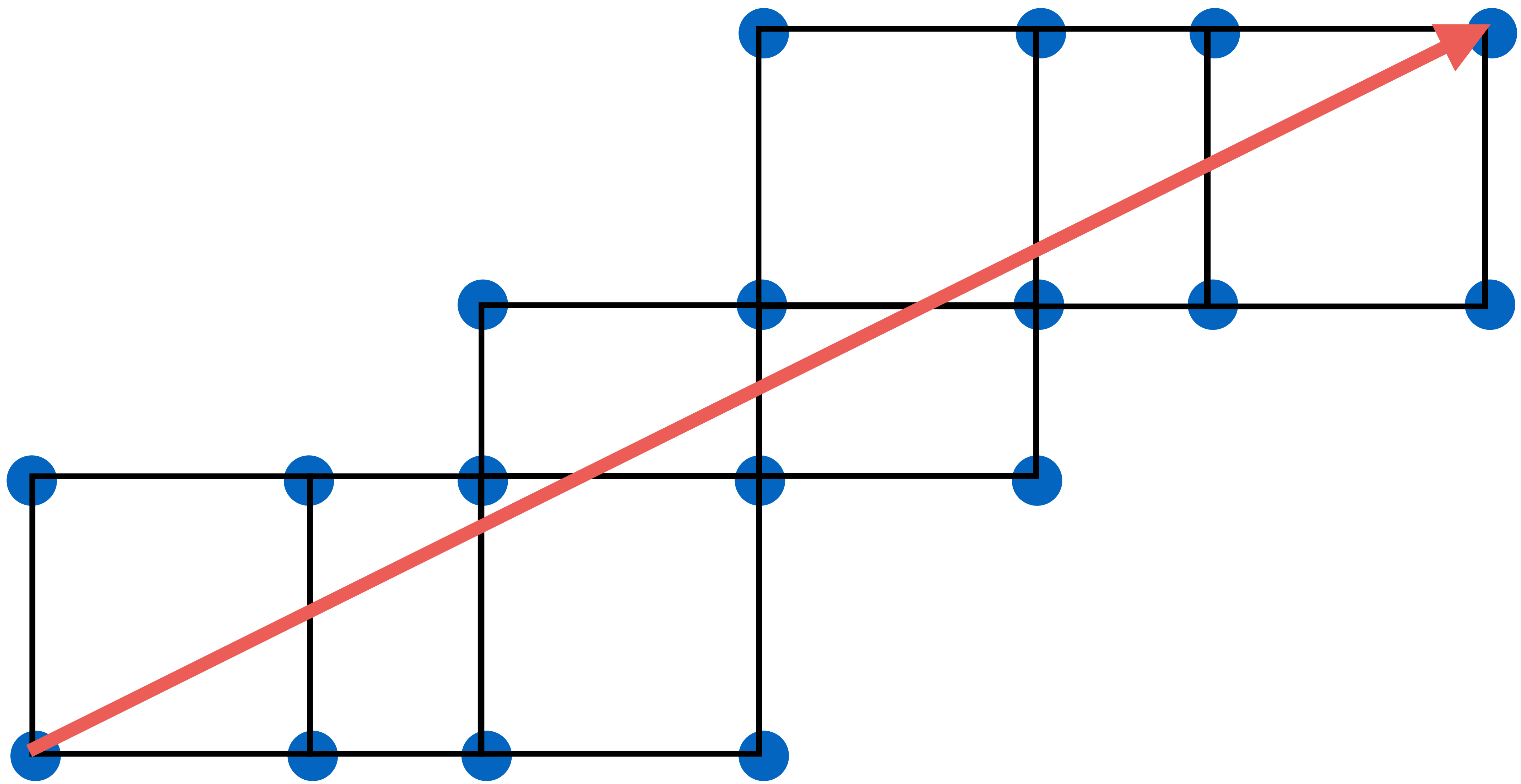
The golden L is easier to use.


Diana:
double pentagon expert.
Samuel:
golden L expert.

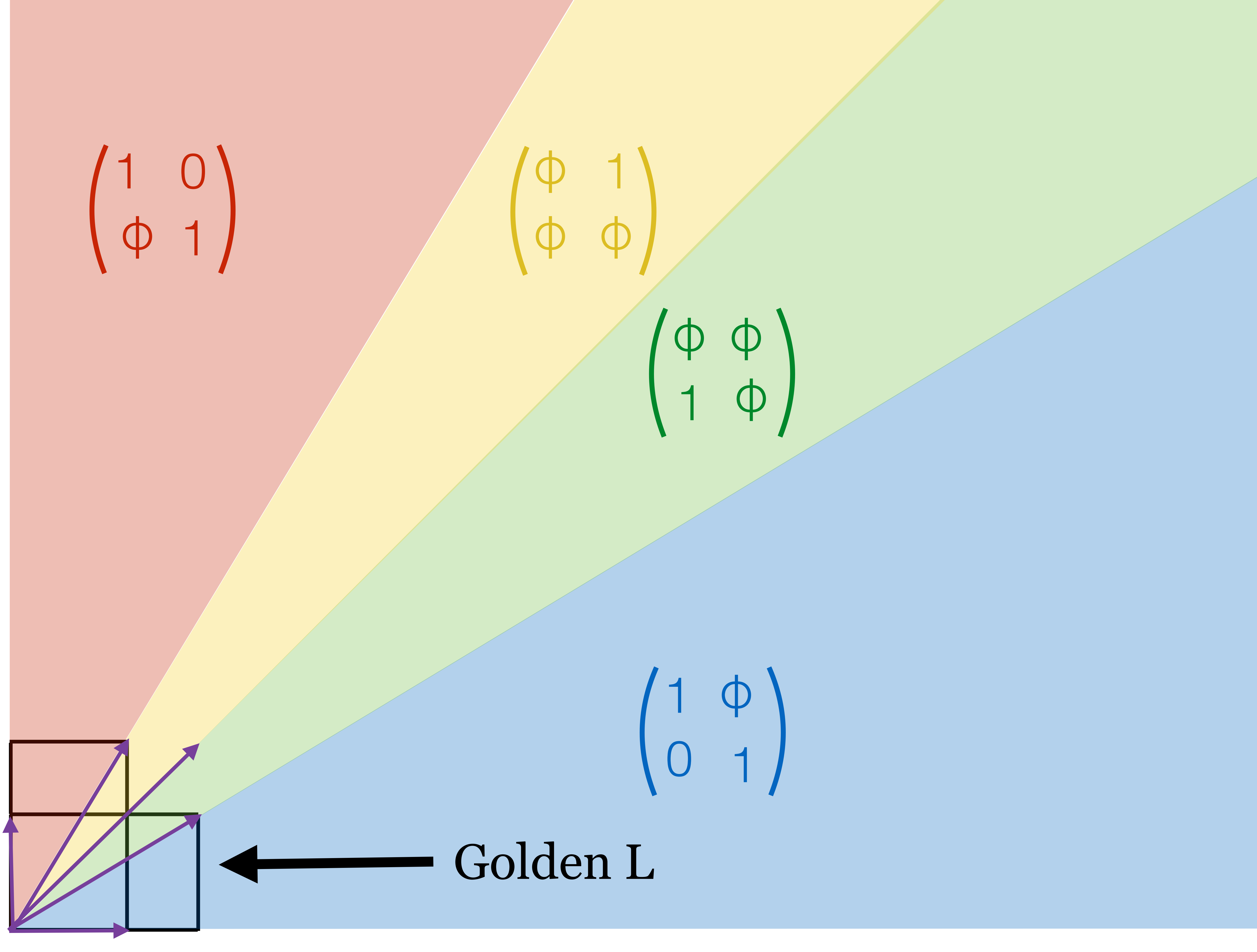
Periodic directions on the **square** are those with vectors connecting **corners of squares**.



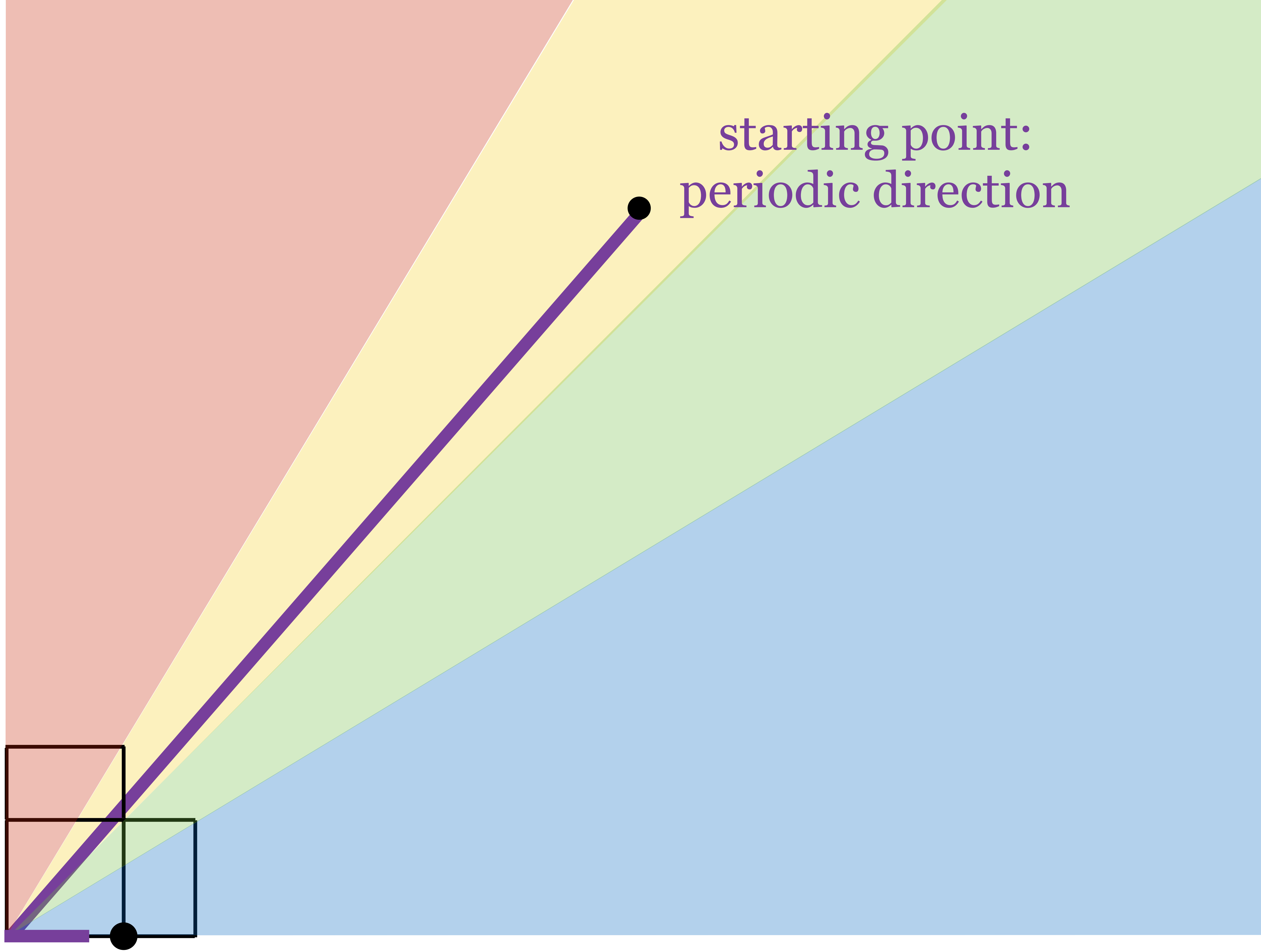
Periodic directions on the **golden L** are those with vectors connecting **corners of unfolded golden Ls.**

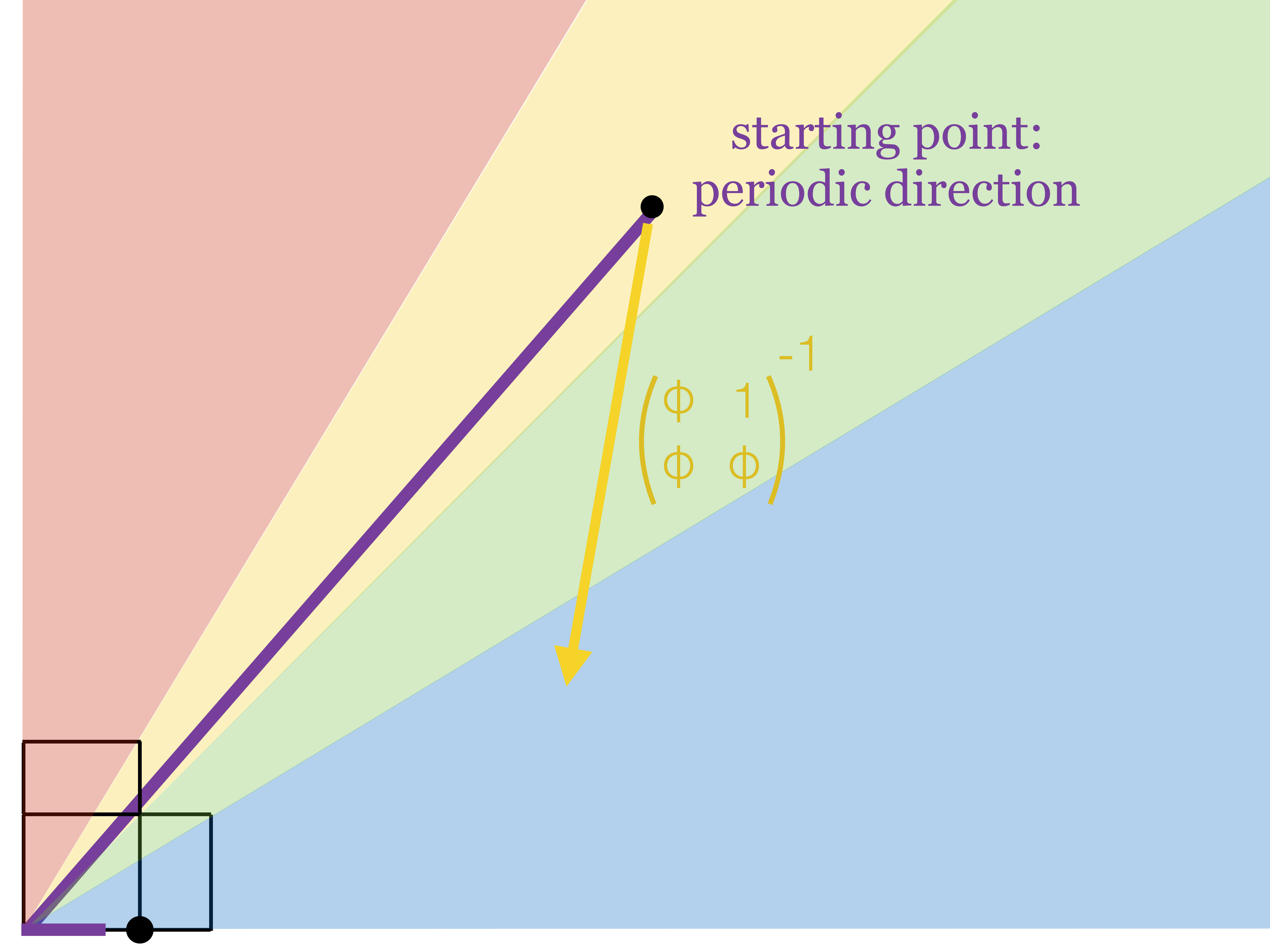


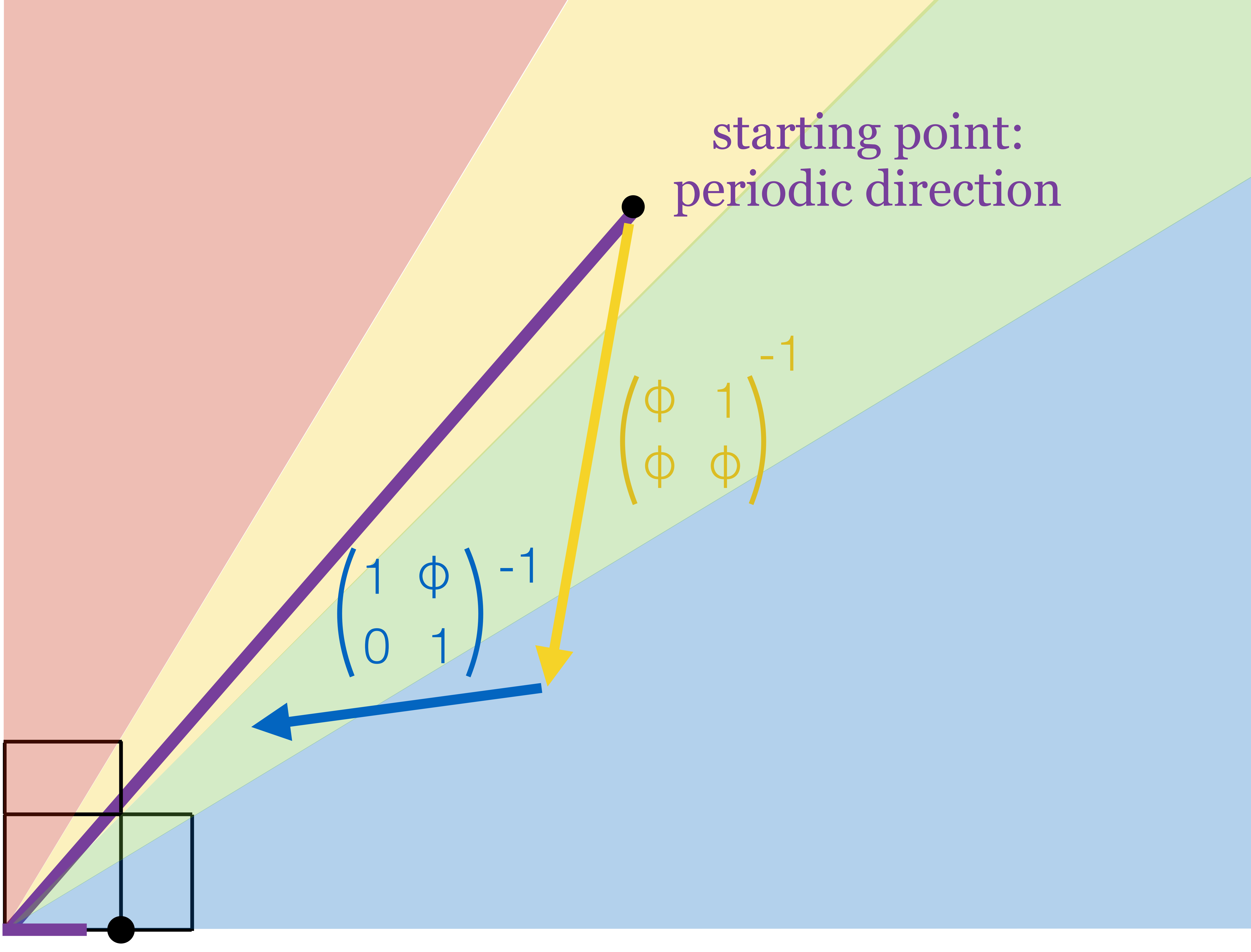
Burning question: What is the period of a trajectory in a given periodic direction on the pentagon?

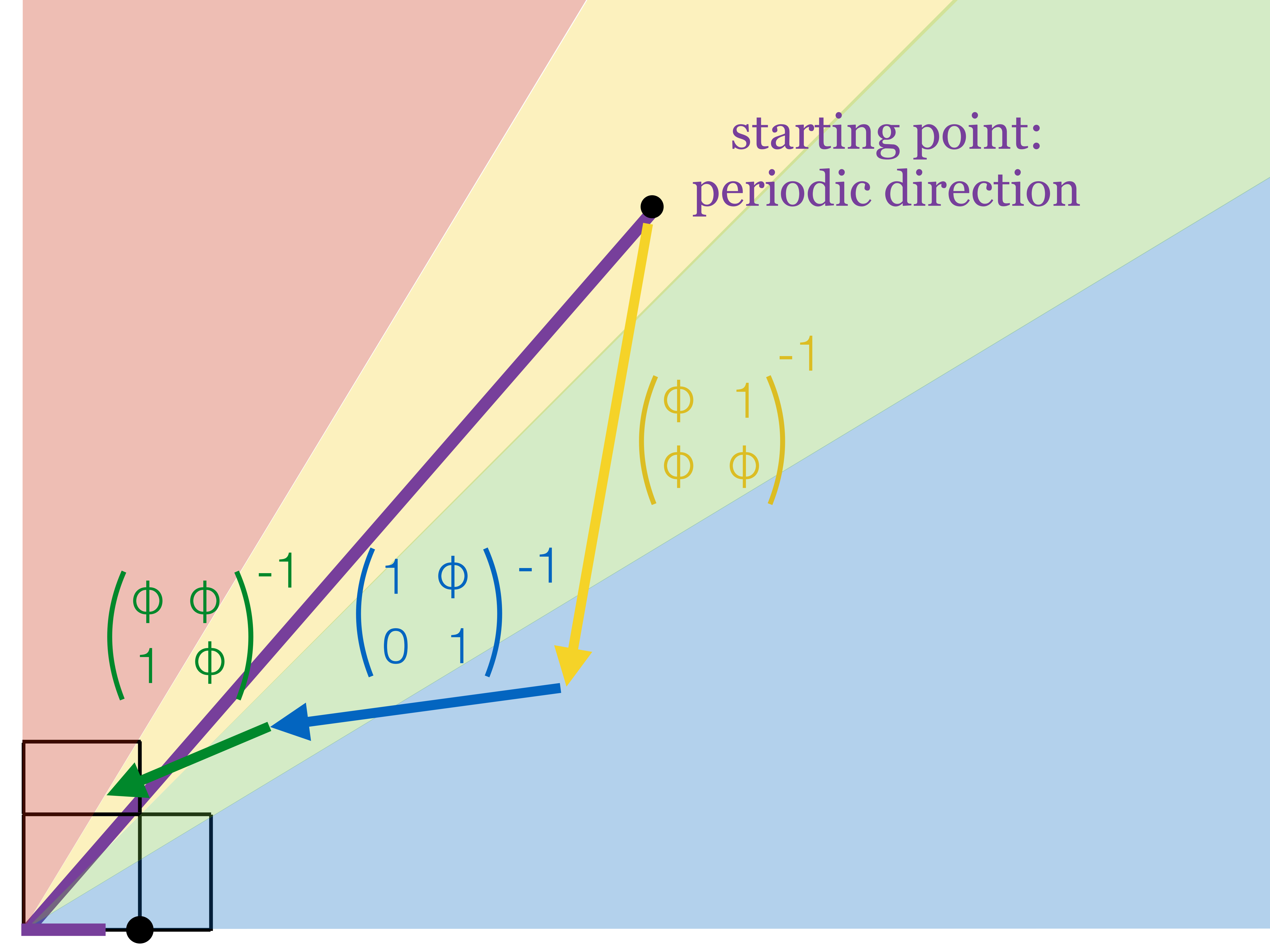


$\begin{pmatrix} 1 & 0 \\ \phi & 1 \end{pmatrix}$
$\begin{pmatrix} \phi & 1 \\ \phi & \phi \end{pmatrix}$
$\begin{pmatrix} \phi & \phi \\ 1 & \phi \end{pmatrix}$
$\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}$
Golden L

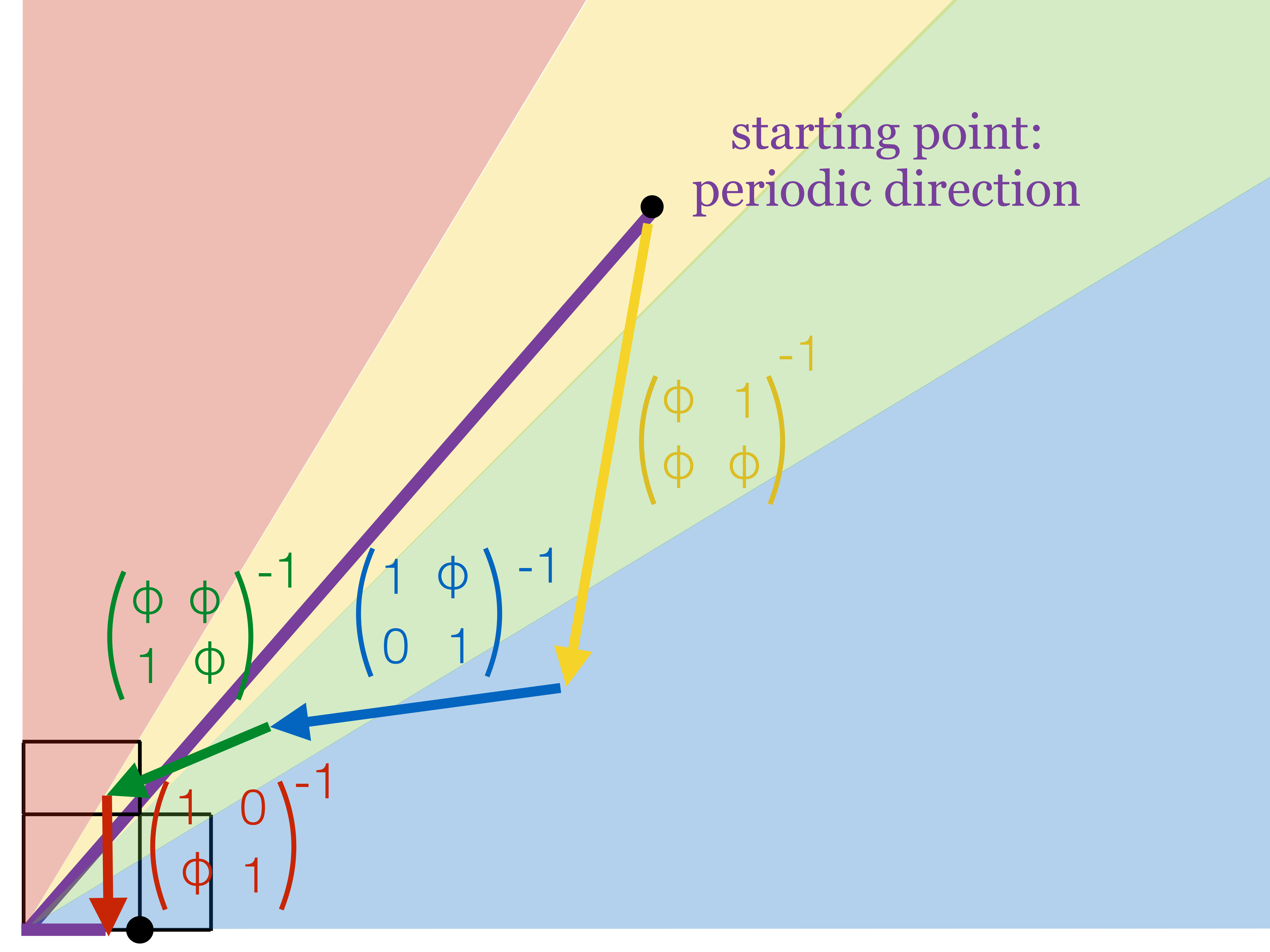

starting point:
periodic direction


starting point:
periodic direction
$\begin{pmatrix} \phi & 1 \\ \phi & \phi \end{pmatrix}^{-1}$

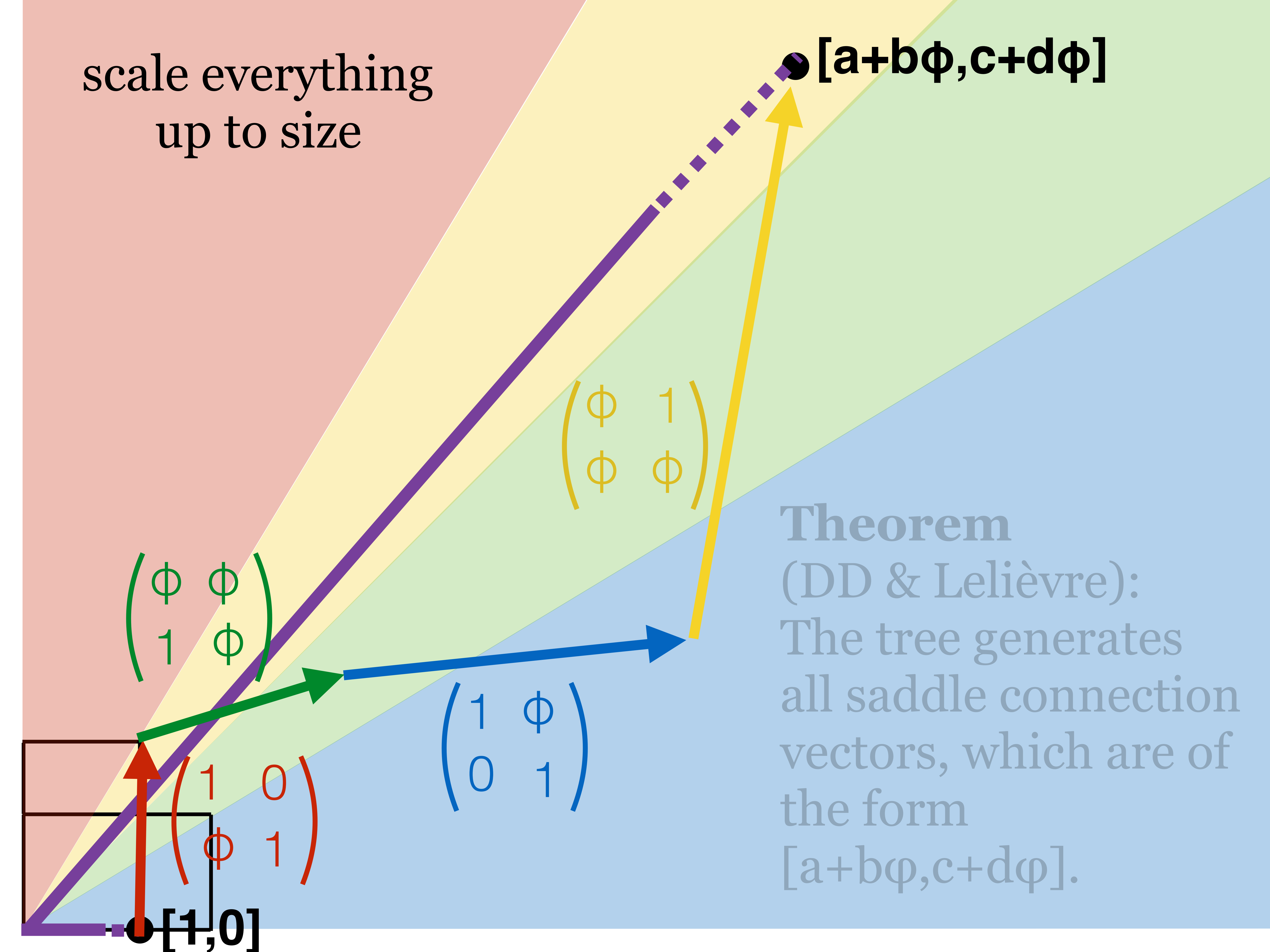

starting point:
periodic direction
$\begin{pmatrix} \phi & 1 \\ \phi & \phi \end{pmatrix}^{-1}$
$\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}^{-1}$

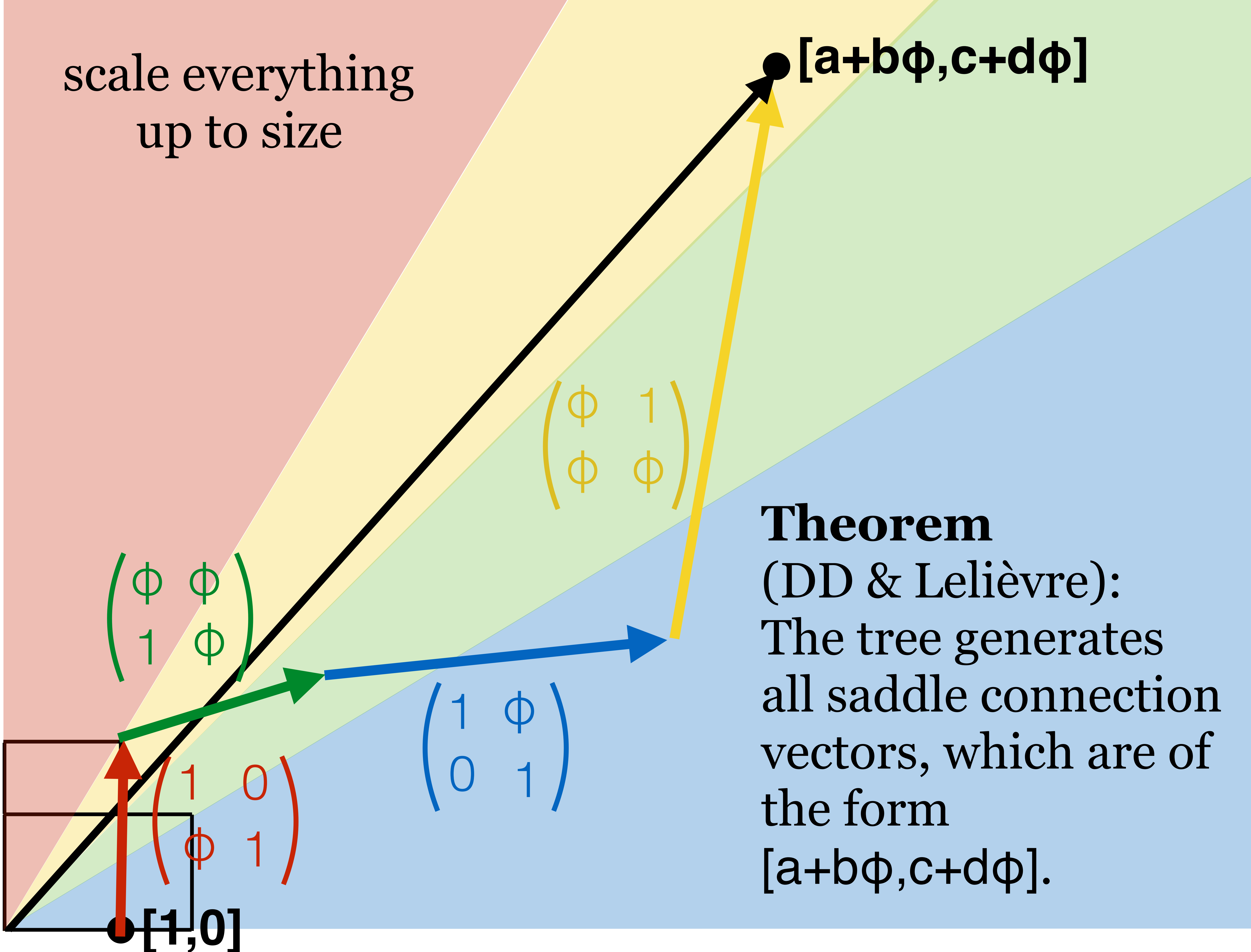


starting point:
periodic direction
$\begin{pmatrix} \phi & 1 \\ \phi & \phi \end{pmatrix}^{-1}$
$\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}^{-1}$
$\begin{pmatrix} \phi & \phi \\ 1 & \phi \end{pmatrix}^{-1}$



starting point:
periodic direction
$\begin{pmatrix} \phi & 1 \\ \phi & \phi \end{pmatrix}^{-1}$
$\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}^{-1}$
$\begin{pmatrix} \phi & \phi \\ 1 & \phi \end{pmatrix}^{-1}$
$\begin{pmatrix} 1 & 0 \\ \phi & 1 \end{pmatrix}^{-1}$


scale everything
up to size
[a+bϕ,c+dϕ]
[1,0]
Theorem
(DD & Lelièvre):
The tree generates
all saddle connection
vectors, which are of
the form
[a+bφ,c+dφ].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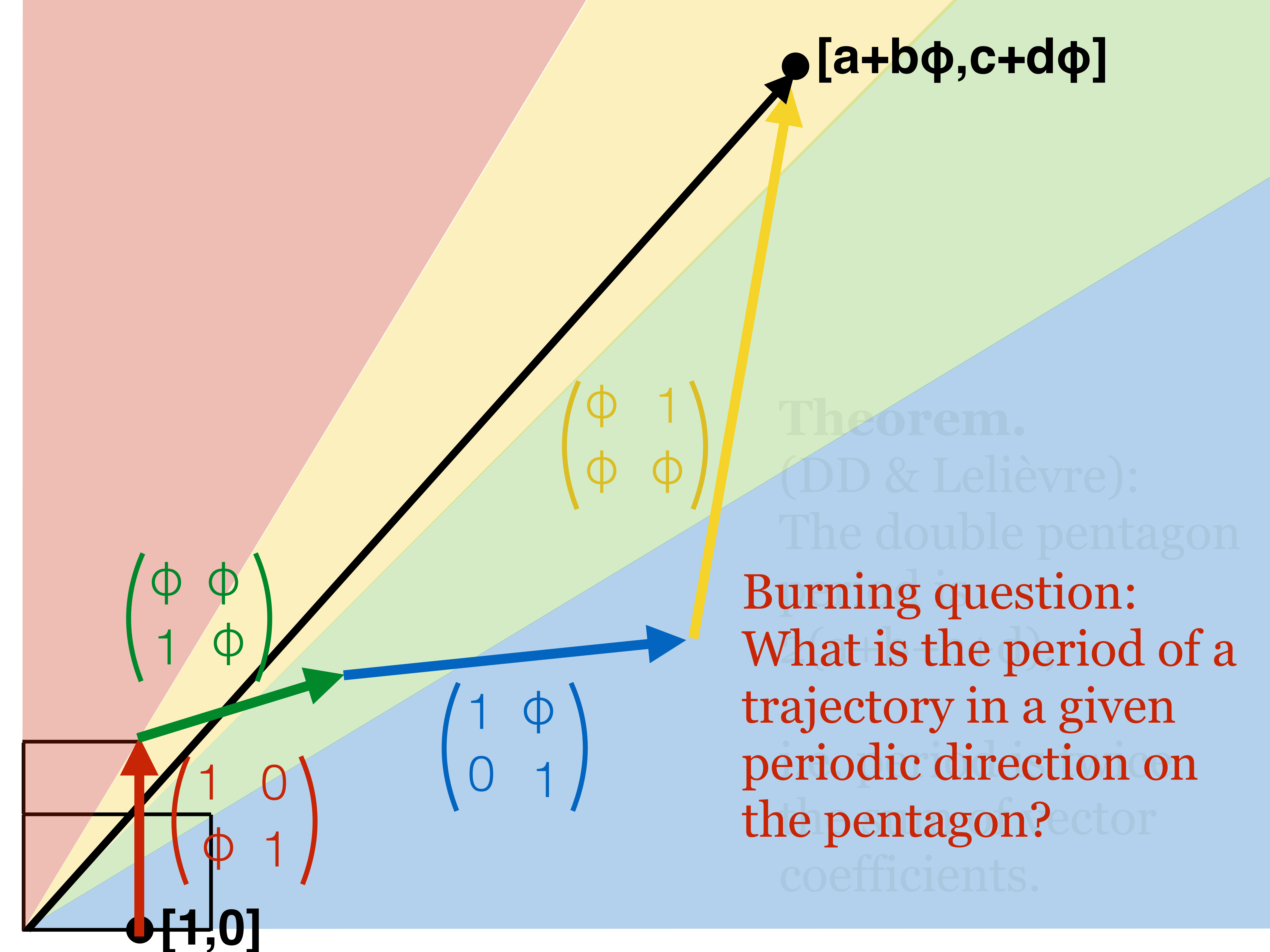
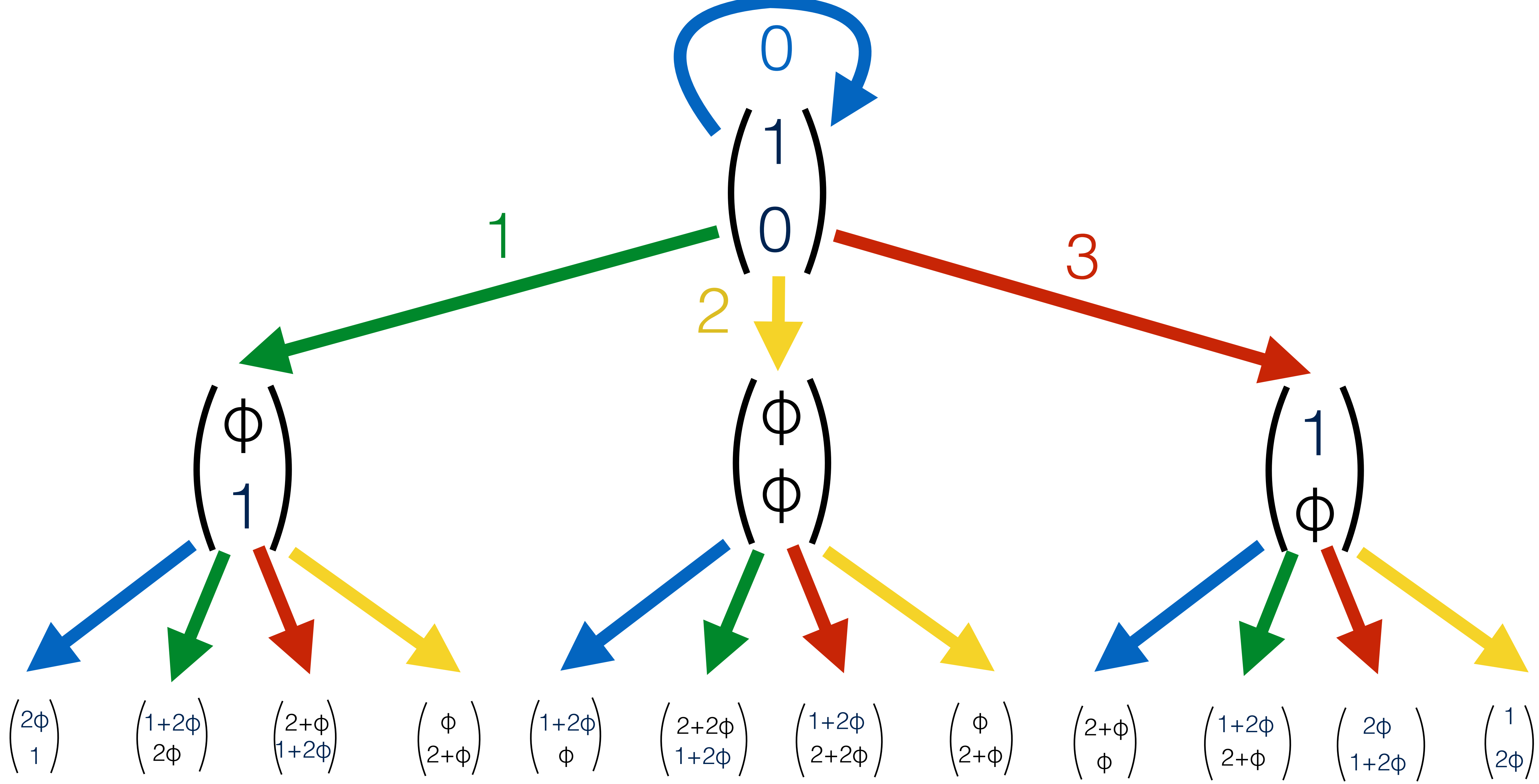
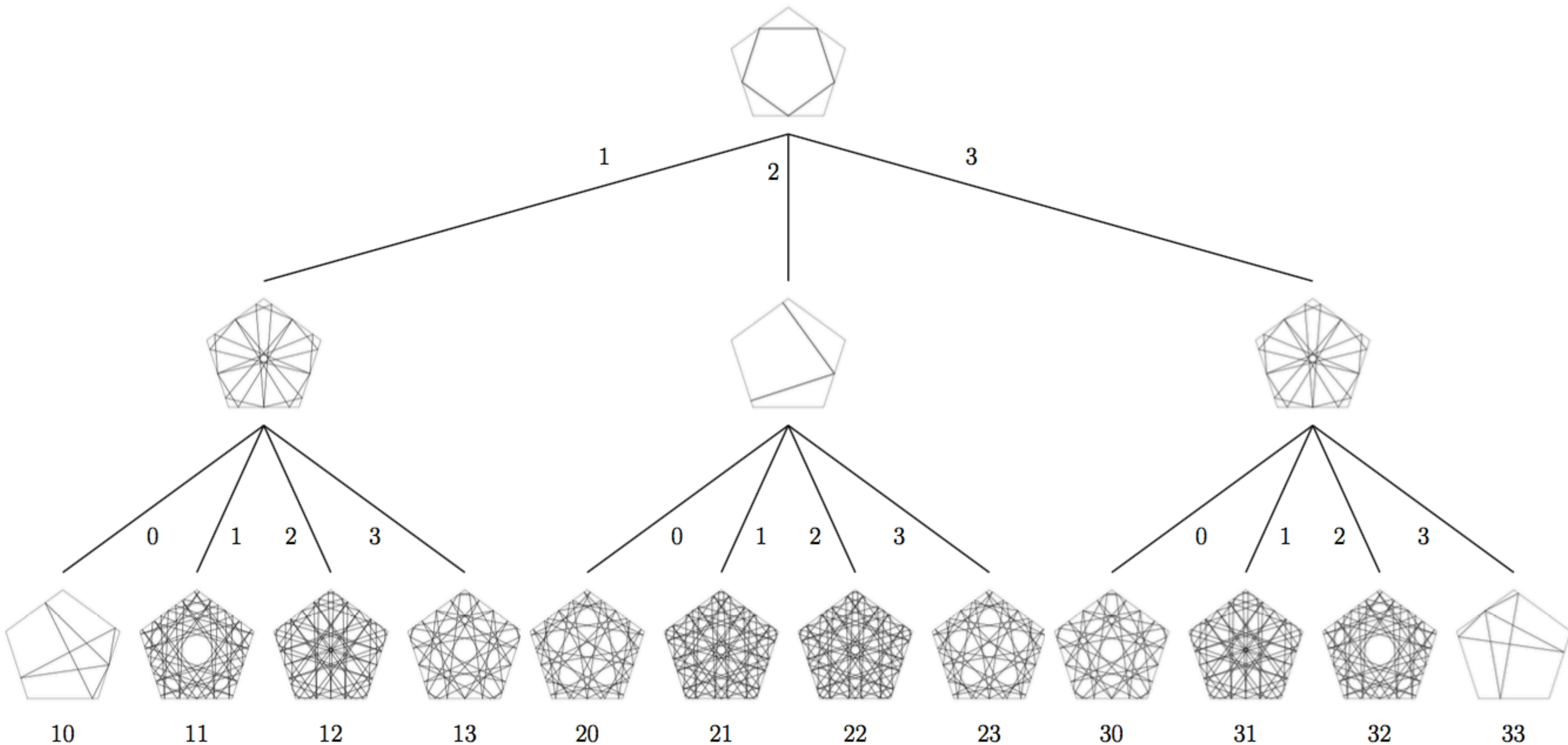
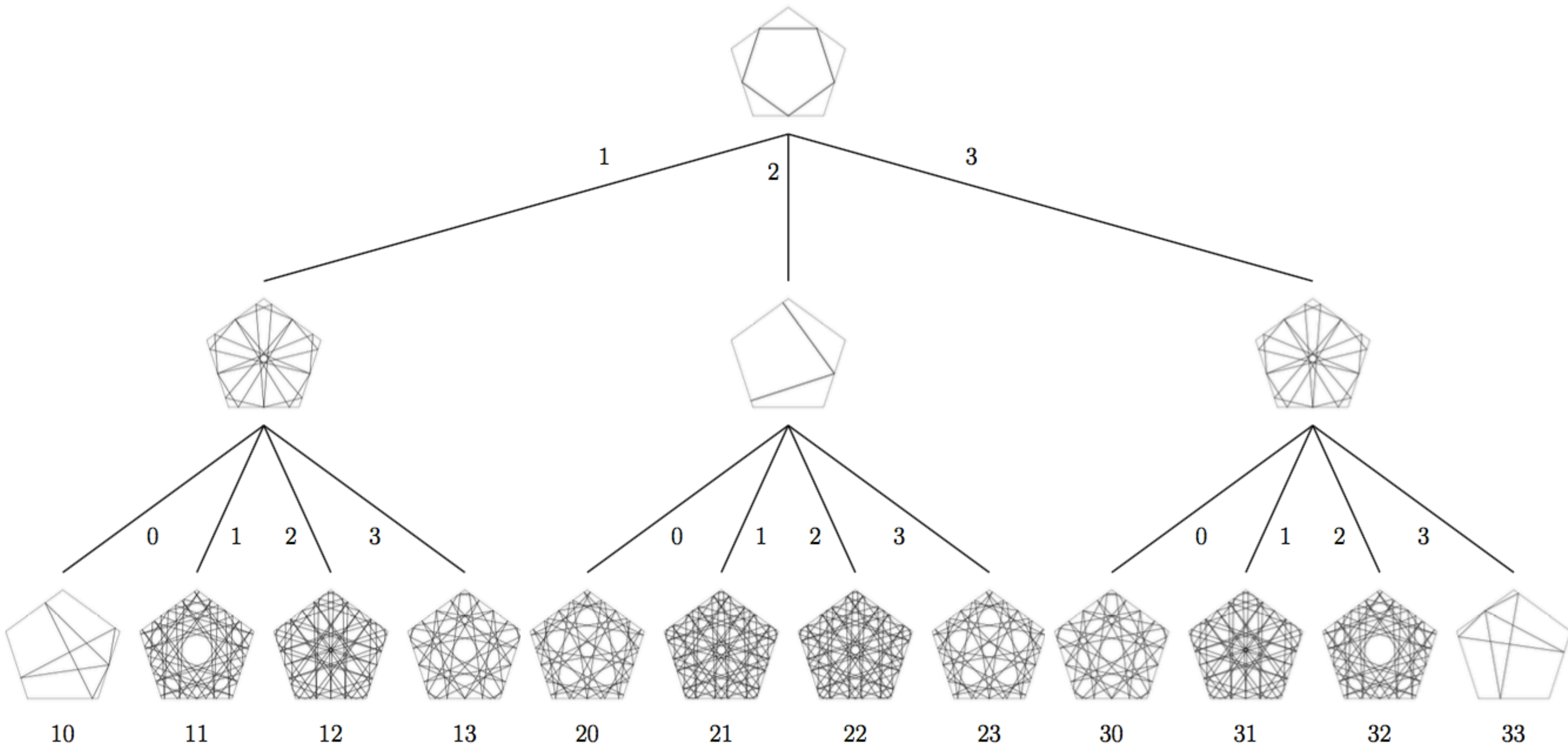

[a+bφ,c+dφ]
[1,0]
Theorem.
(DD & Lelièvre):
The double pentagon
coefficients.
Burning question:
What is the period of a
trajectory in a given
periodic direction on
the pentagon?



Here is the tree structure in direction vectors (short direction vector shown).



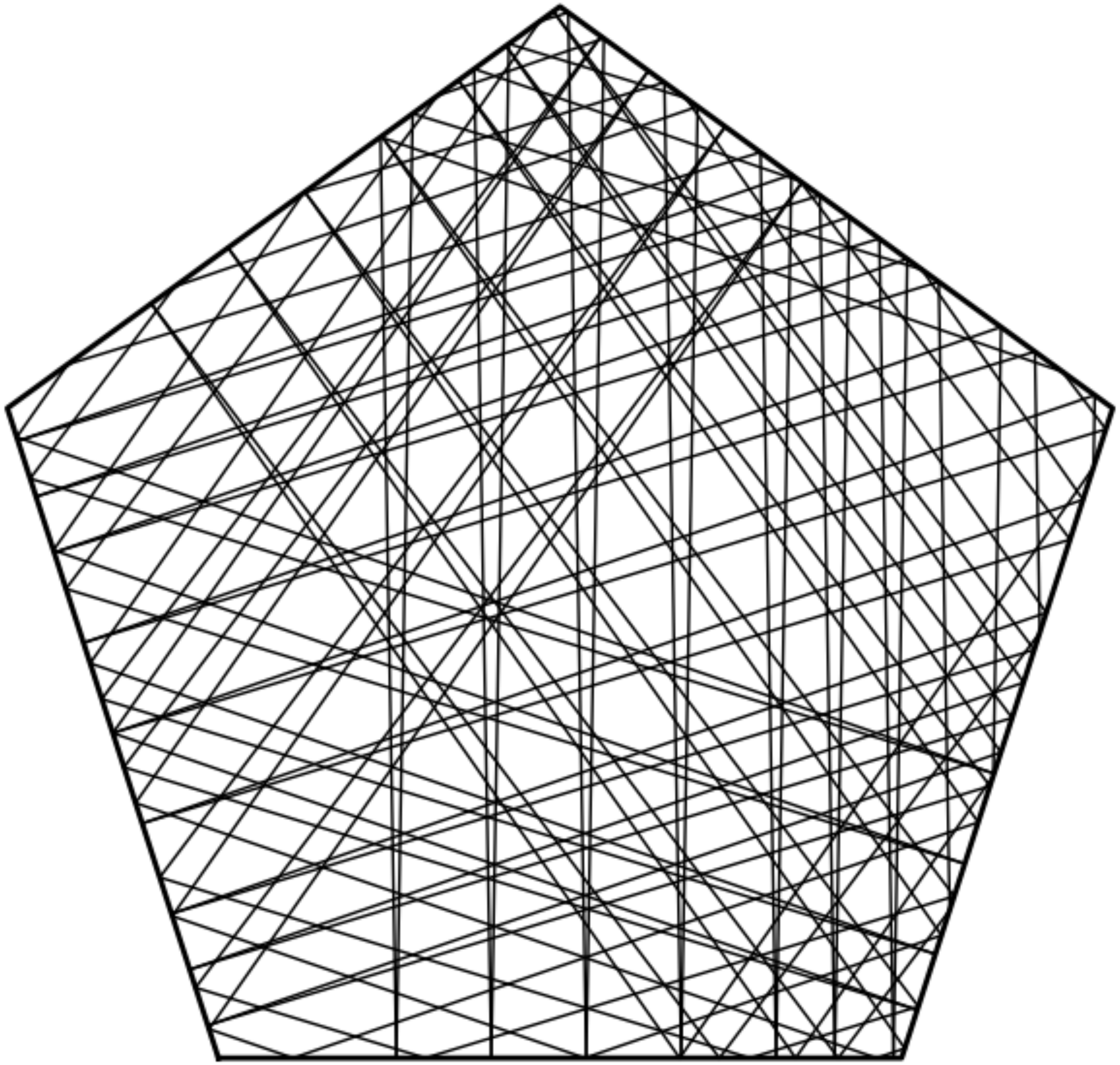
Here is the tree structure in trajectory pictures.
(short trajectory shown).



Here is the tree structure in trajectory pictures.
(short trajectory shown).

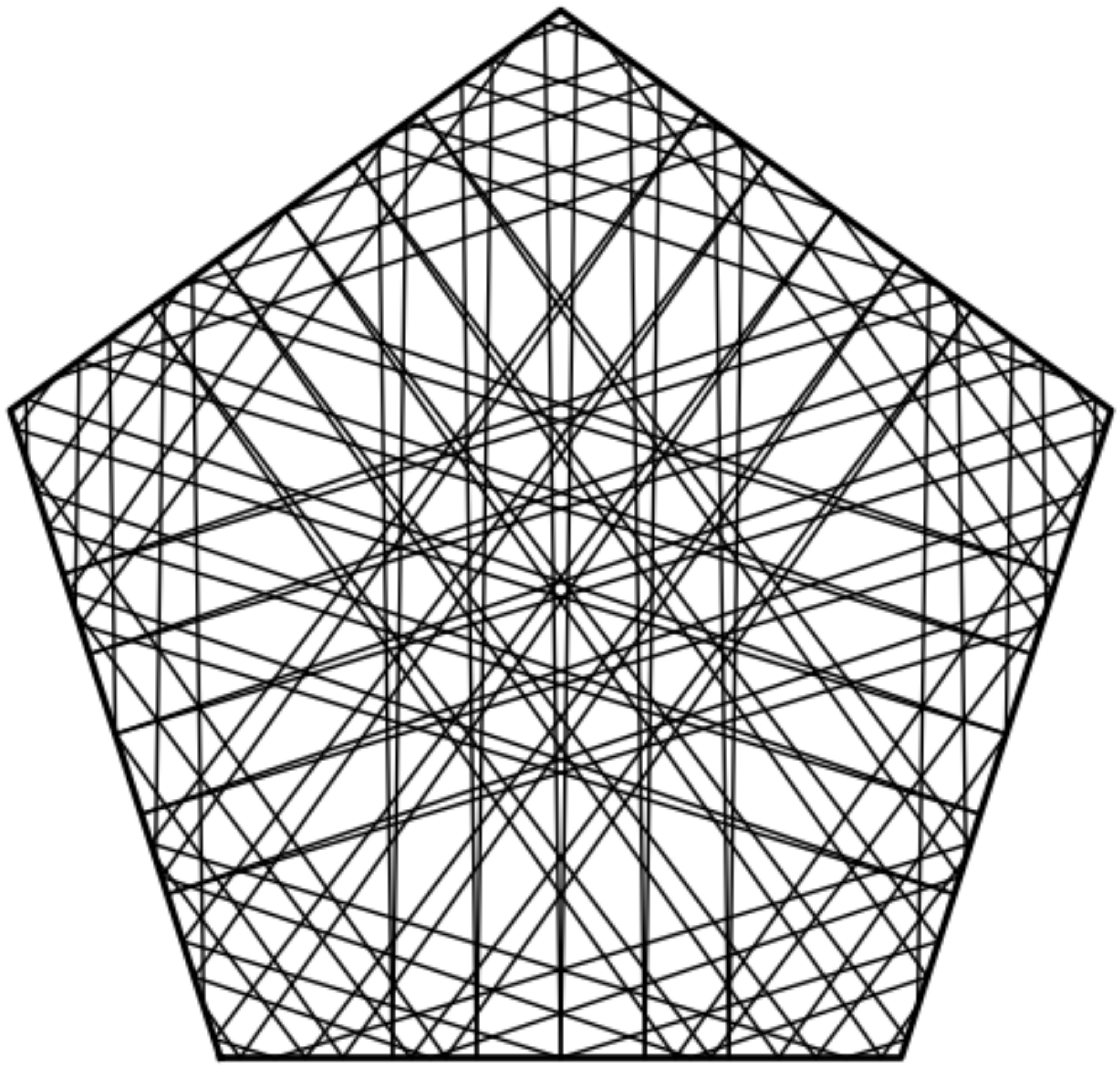
Theorem. (DD & Lelièvre, 2017)
The tree is symmetric.

Here is our friend 3102



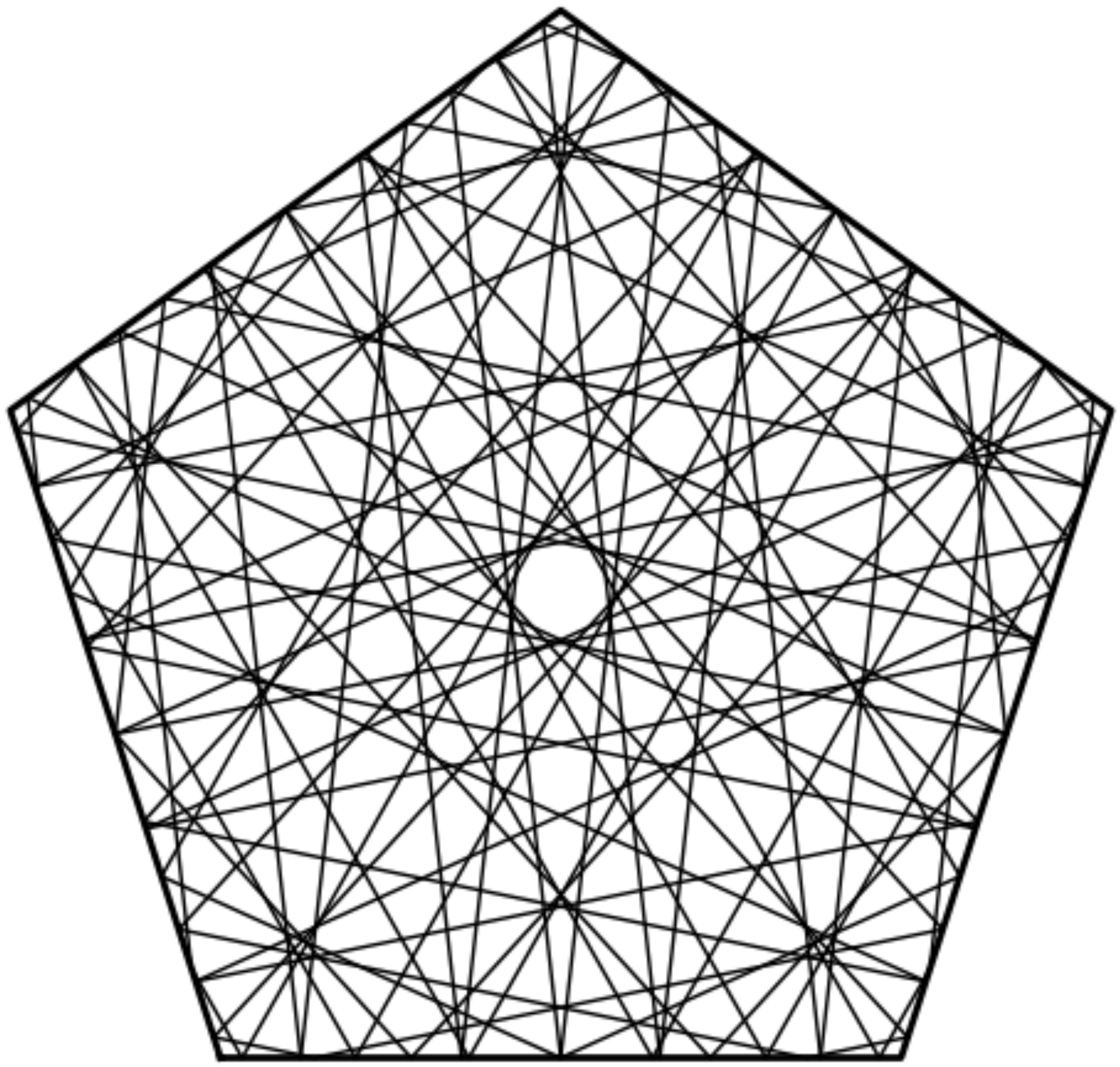
pictures by Diana Davis and Samuel Lelièvre

Here is our friend 102



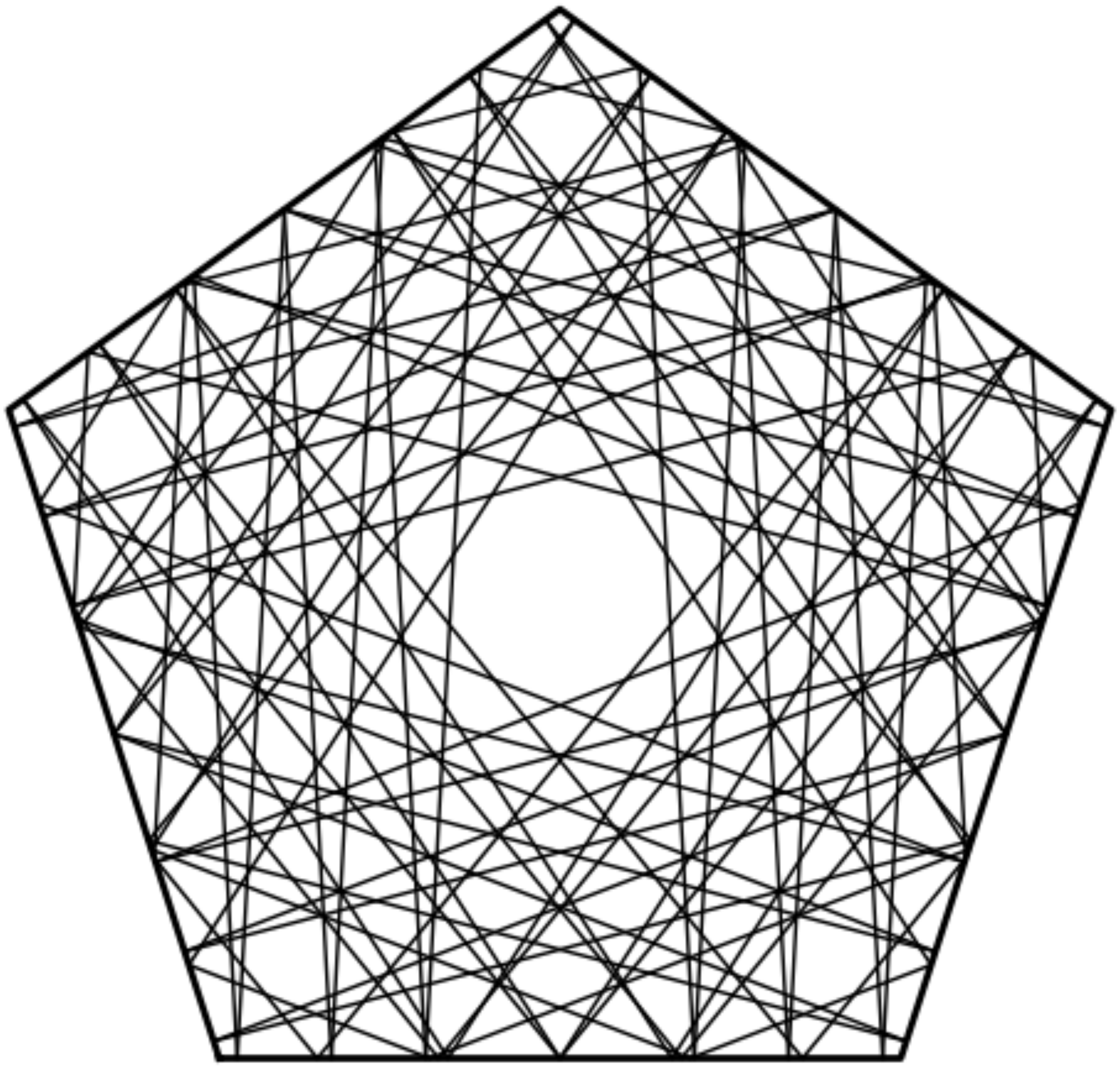
pictures by Diana Davis and Samuel Lelièvre

Here is our friend 130



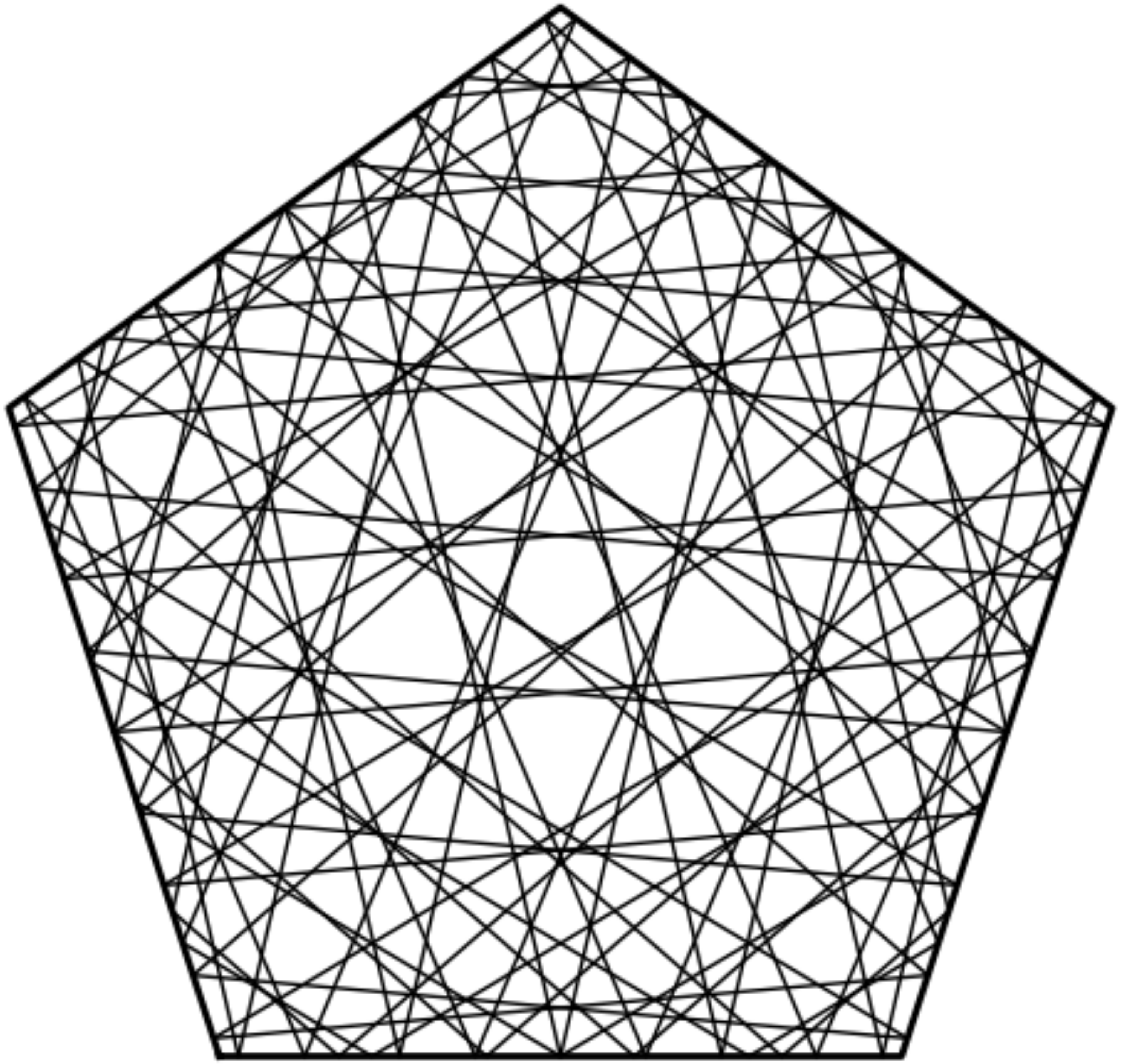
pictures by Diana Davis and Samuel Lelièvre

Here is our friend 101

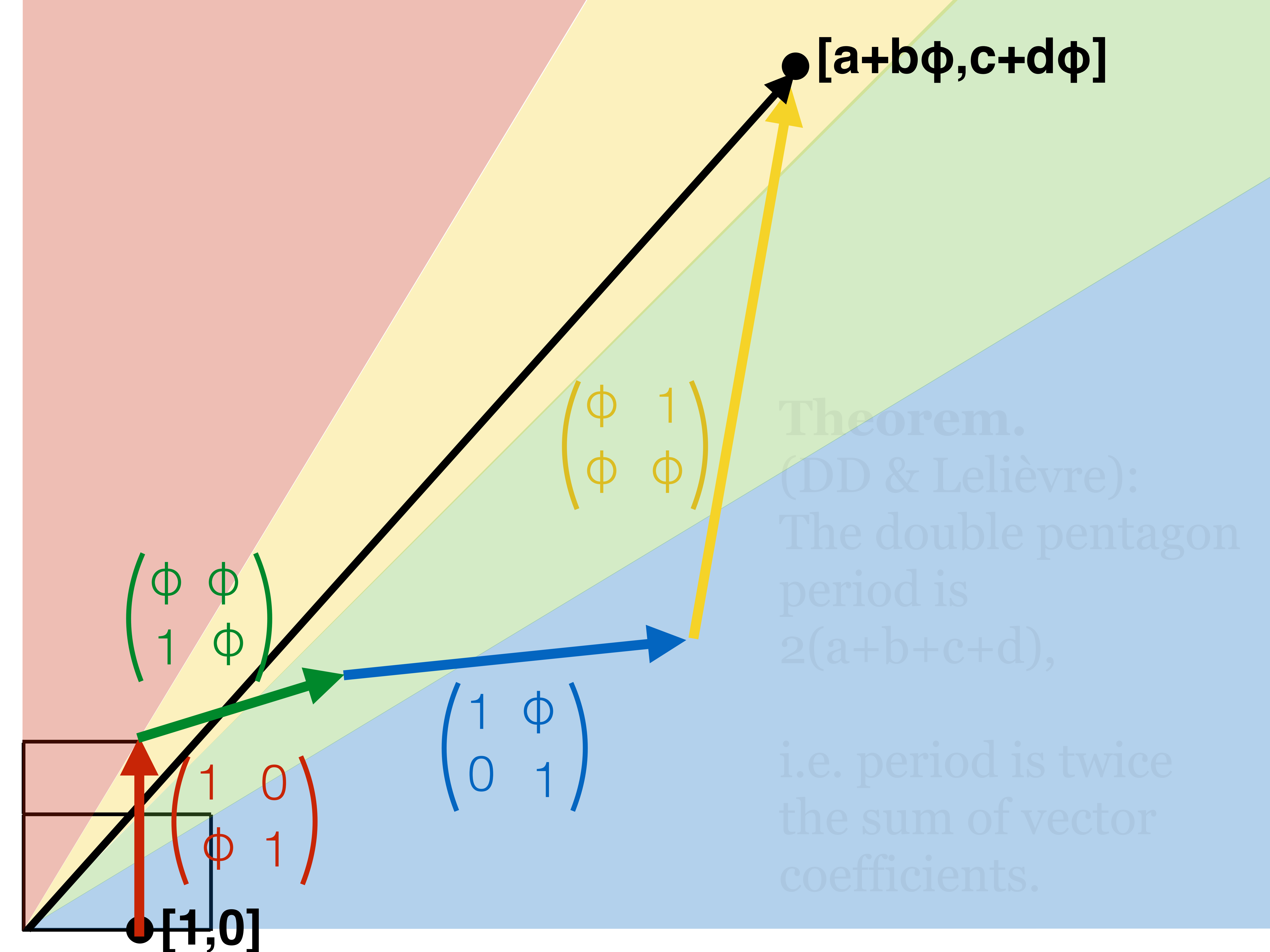


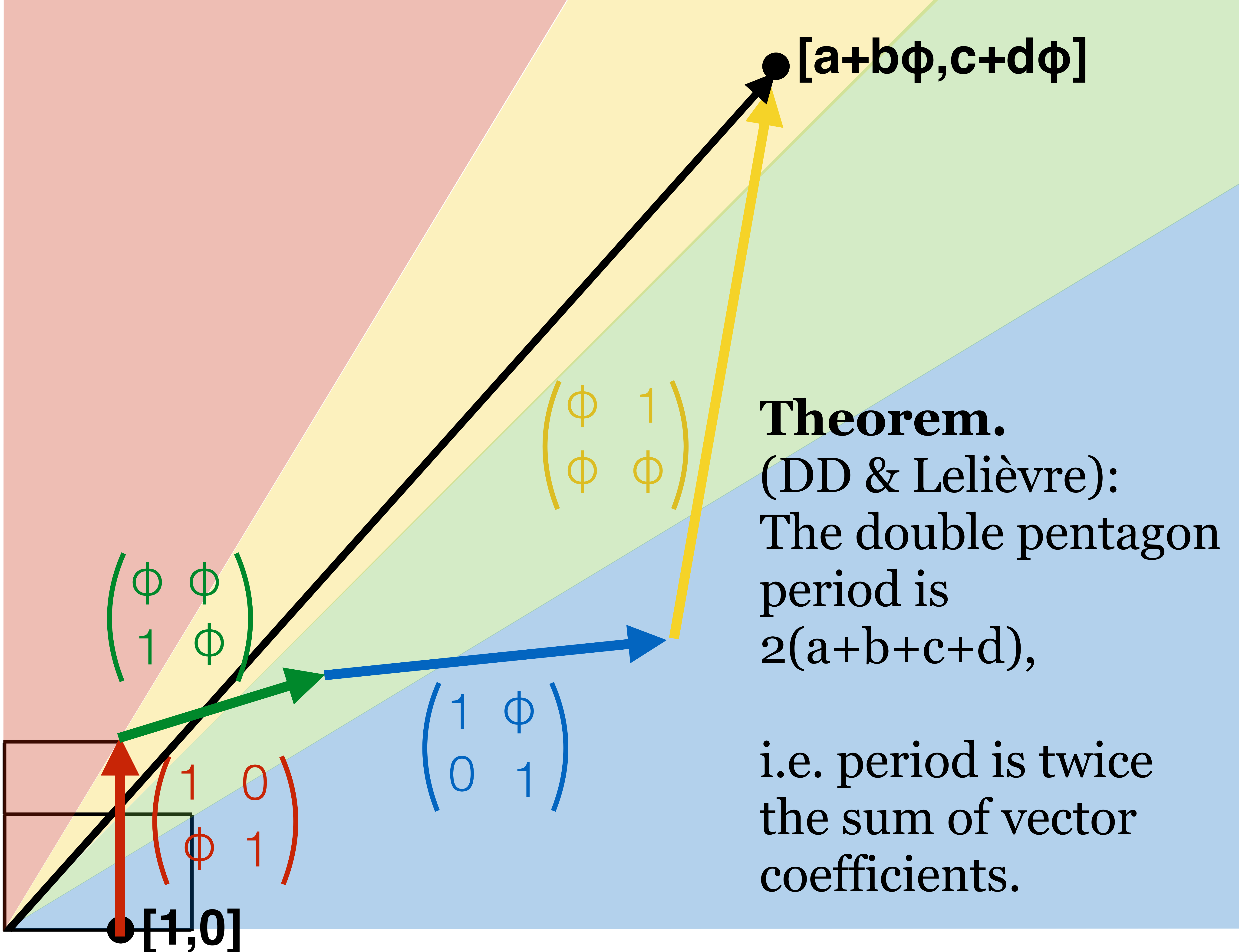
pictures by Diana Davis and Samuel Lelièvre

Here is our friend 2000



Burning question: What is the period of a trajectory in a given periodic direction on the pentagon?


[a+bϕ,c+dϕ]
[1,0]
$\begin{pmatrix} 1 & 0 \\ \phi & 1 \end{pmatrix}$
$\begin{pmatrix} \phi & \phi \\ 1 & \phi \end{pmatrix}$
$\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} \phi & 1 \\ \phi & \phi \end{pmatrix}$
Theorem.
(DD & Lelièvre):
The double pentagon period is 2(a+b+c+d),
i.e. period is twice the sum of vector coefficients.


[a+bϕ,c+dϕ]
ϕ 1
ϕ ϕ
ϕ ϕ
1 ϕ
1 ϕ
0 1
1 0
ϕ 1
[1,0]
Theorem.
(DD & Lelièvre):
The double pentagon
period is
2(a+b+c+d),

i.e. period is twice
the sum of vector
coefficients.

Theorem: The period of the double pentagon trajectory is twice the sum of the vector coefficients.

Idea of proof:

(1) Base case (horizontal trajectory, below) satisfies this property.

(2) Applying the group actions preserves the doubling.

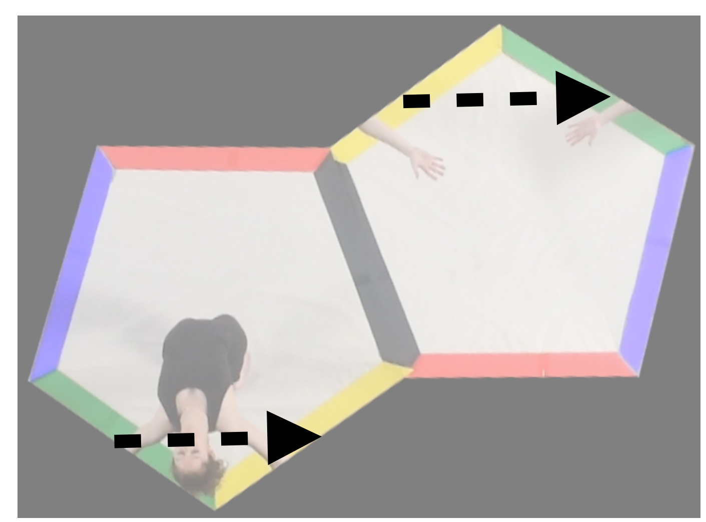
Base case:

VECTOR PART

[1,0]

sum of coefficients = 1

SEQUENCE PART



$1^2 2^1$ period = 2

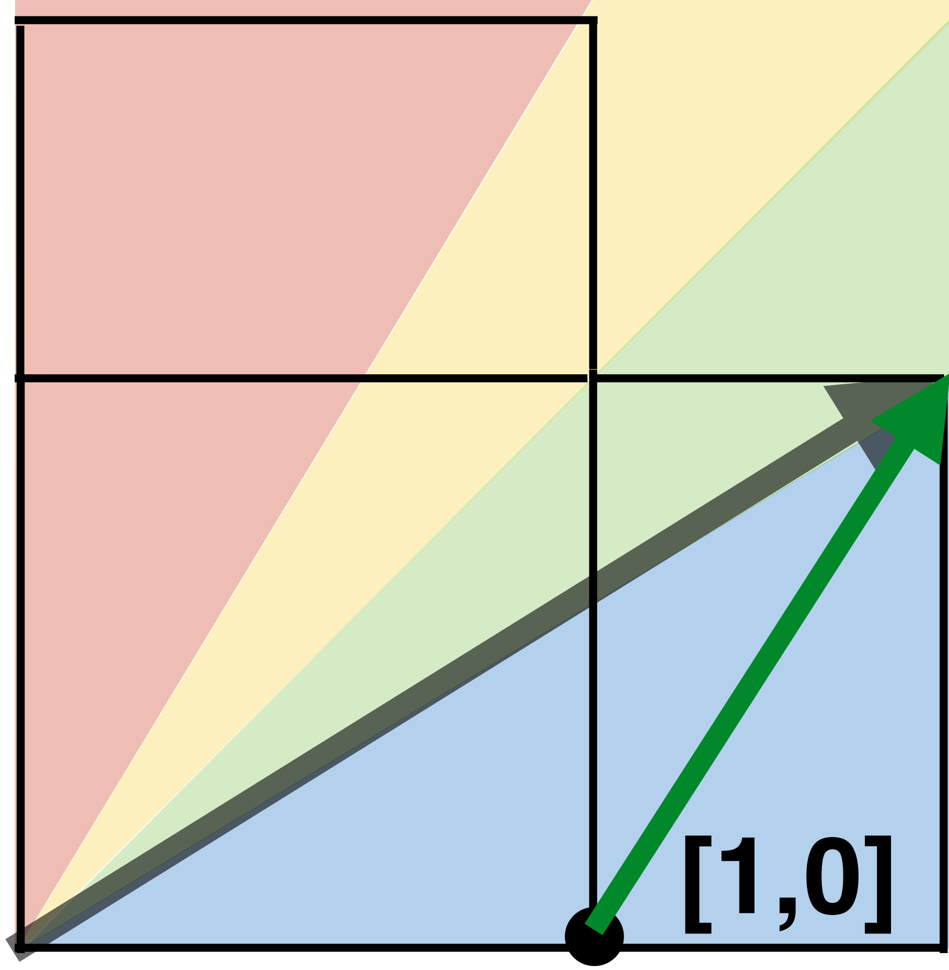
Theorem: The period of the double pentagon trajectory is twice the sum of the vector coefficients.

VECTOR PART

- start with [1,0]
- apply one of the 4 matrices
- get a vector of the form

$[a+b\phi, c+d\phi]$

$$\begin{pmatrix}\phi & \phi\\ 1 & \phi\end{pmatrix}[1,0] = [\phi,1]$$



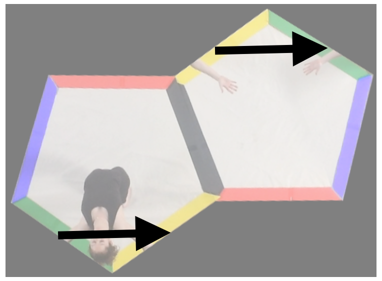
$$\begin{pmatrix}\phi & \phi\\ 1 & \phi\end{pmatrix}$$

Theorem: The period of the double pentagon trajectory is twice the sum of the vector coefficients.

SEQUENCE PART

$$r_1 = \begin{cases} 1^2 \to 3^4 4^5 \\ 2^1 \to 5^4 4^3 \\ 2^3 \to 5^4 4^3 3^2 2^1 \\ 3^2 \to 1^2 2^3 3^4 4^5 \\ 3^4 \to 1^2 2^3 3^4 \\ 4^3 \to 4^3 3^2 2^1 \\ 4^5 \to 4^3 3^2 \\ 5^4 \to 2^3 3^4 \end{cases}$$

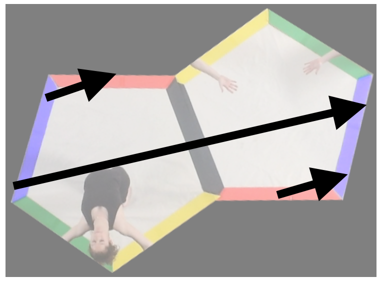
- start with sequence



$1^2 2^1$

- apply corresponding combinatorial operation
- get a new sequence

$3^4 4^5 5^4 4^3$



Theorem: The period of the double pentagon trajectory is twice the sum of the vector coefficients.

VECTOR PART

[1,0] sum of coefficients = 1

Apply matrix $\begin{pmatrix} \phi & \phi \\ 1 & \phi \end{pmatrix}$, etc.

[ϕ,1] sum of coefficients = 2

SEQUENCE PART

$1^{2}2^{1}$ period = 2

Apply combinatorial operation to edge sequence

$3^{4}4^{5}5^{4}4^{3}$ period = 4

Applying the group actions preserves the doubling.

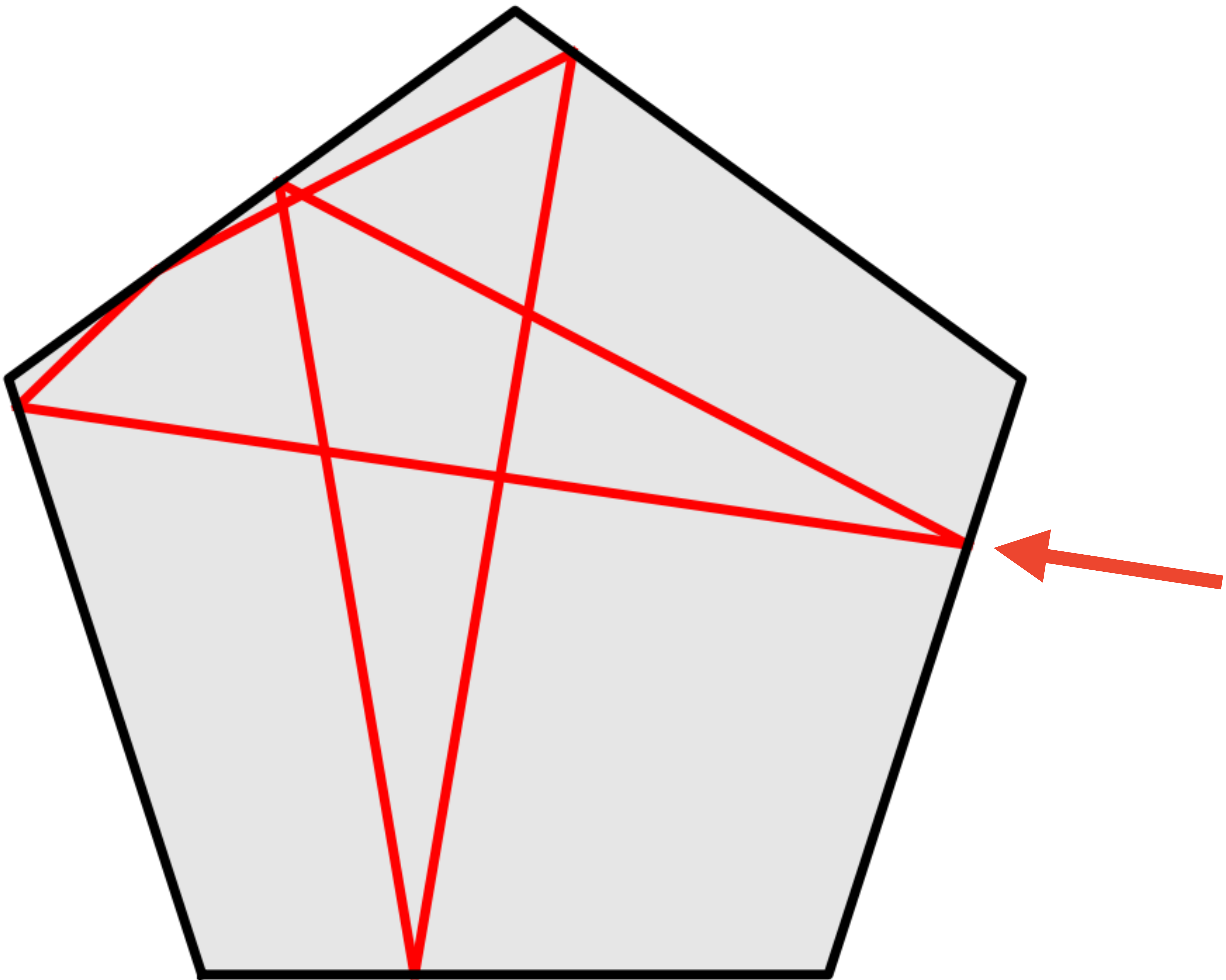
Goal: Understand periodic billiard trajectories on the pentagon.
Plan: Explain results and methods for the square, generalize to the pentagon.
Done!

Plan for rest of talk: Explore and understand things that are different in the pentagon than on the square:

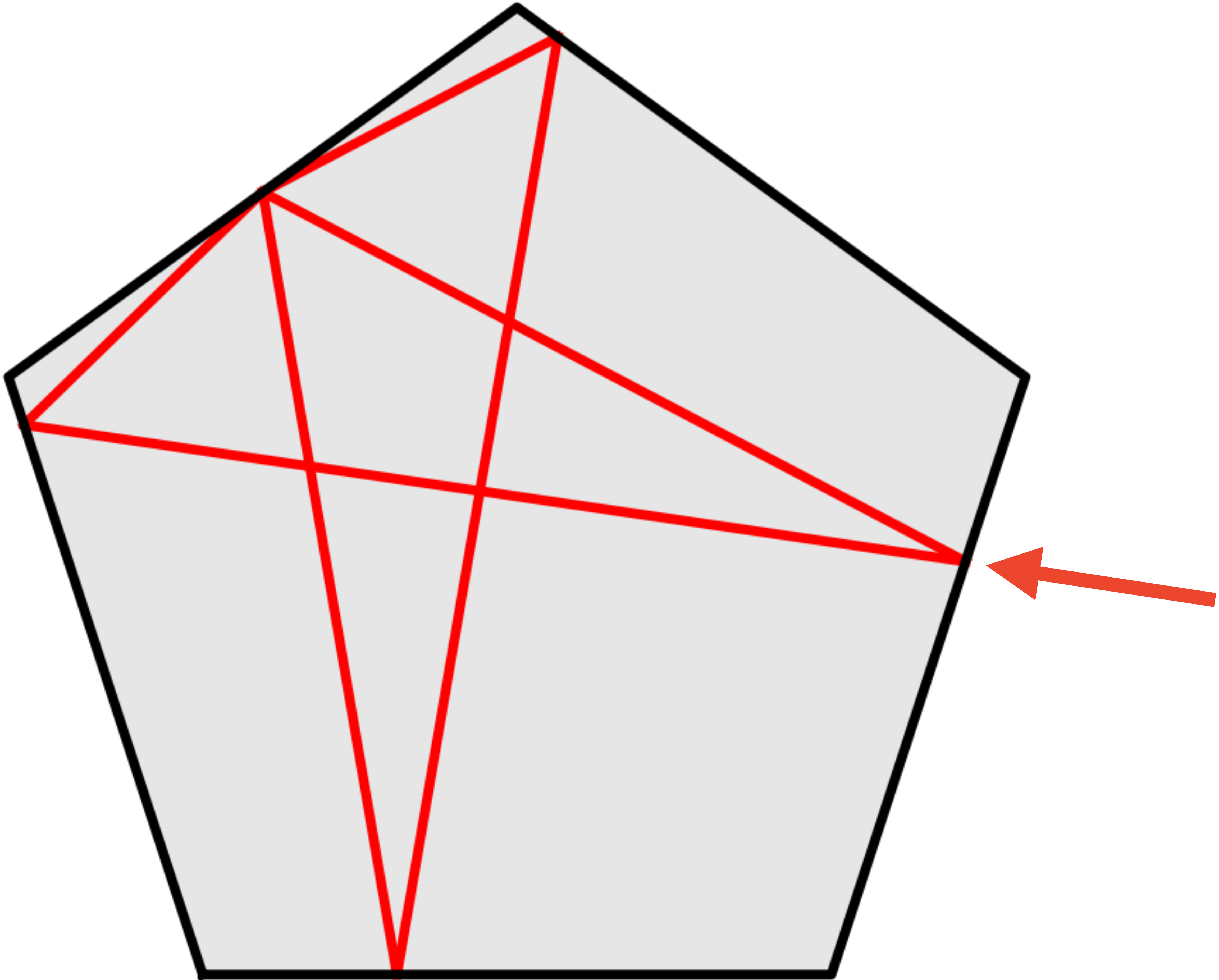
- Buddies
- Symmetry
- Families

BUDDIES

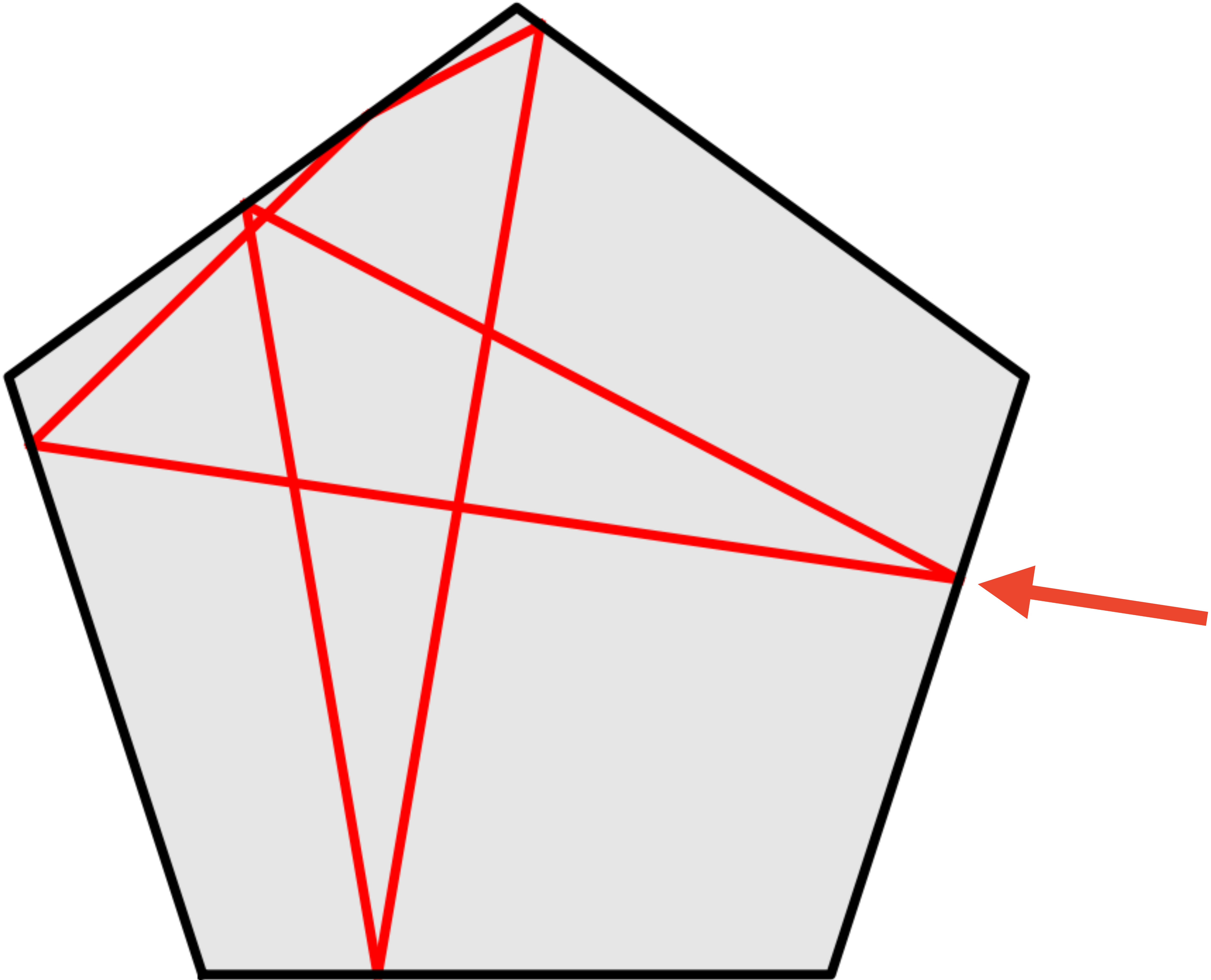
BUDDIES: the parallel trajectories in a given direction



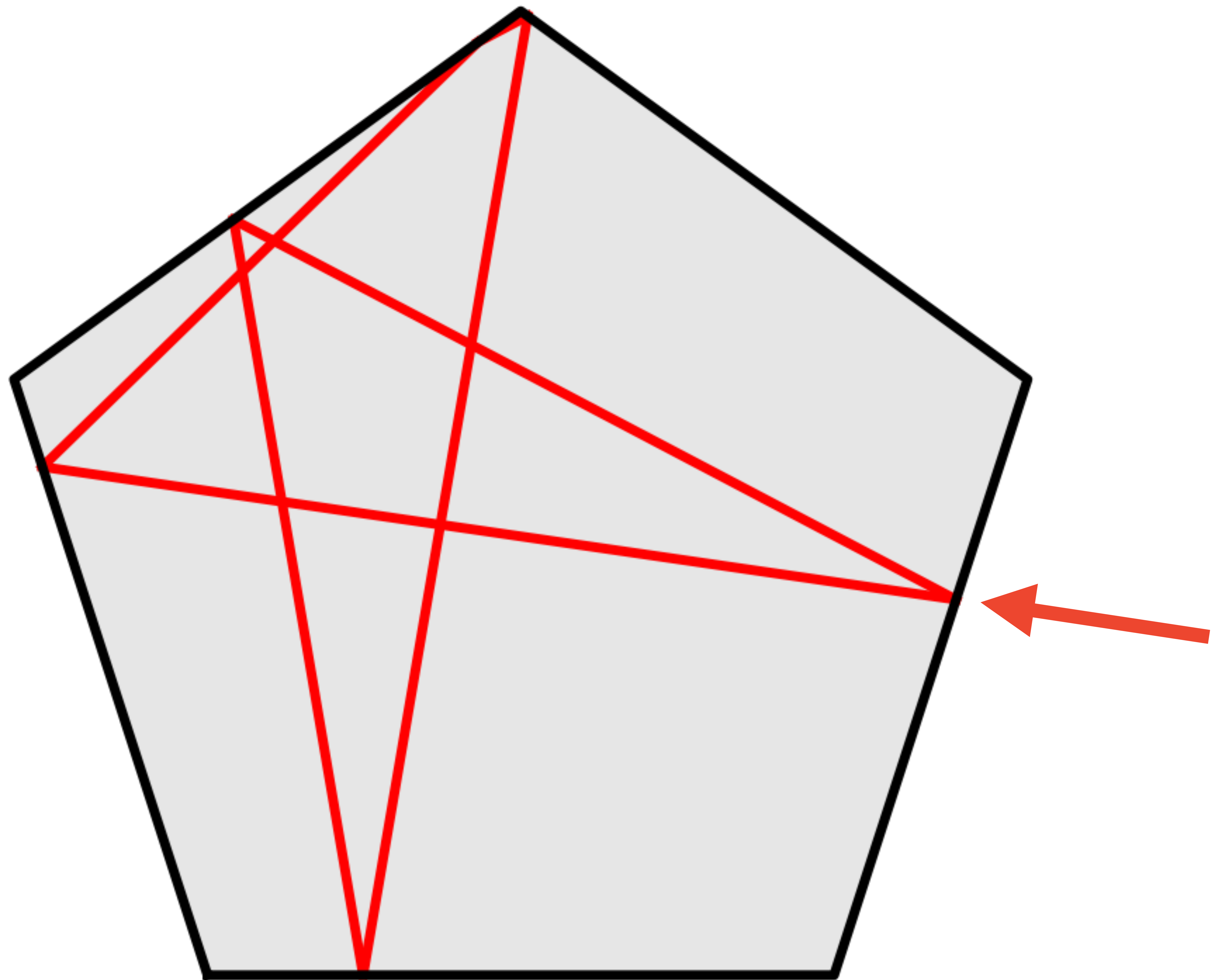
BUDDIES: the parallel trajectories in a given direction



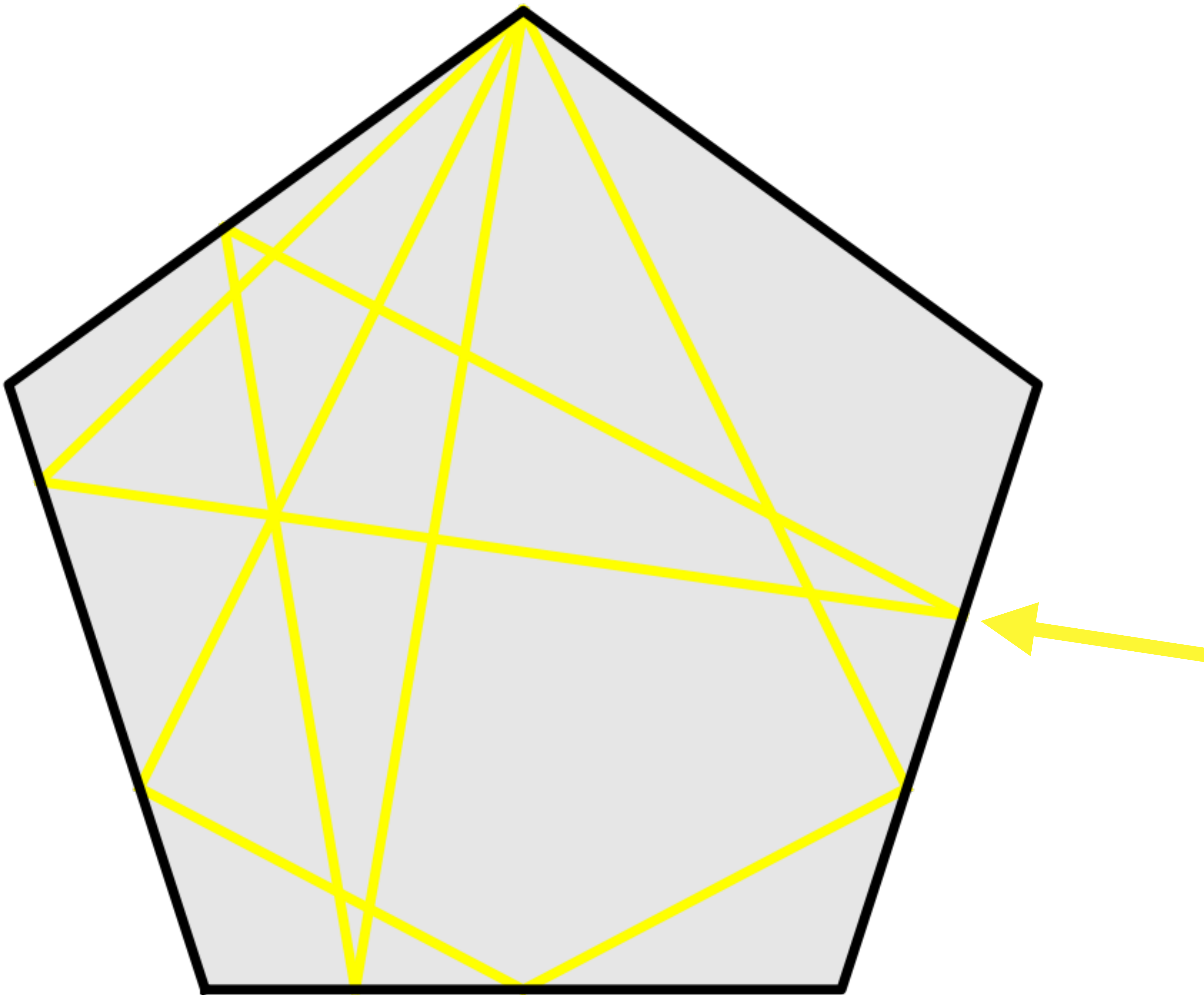
BUDDIES: the parallel trajectories in a given direction



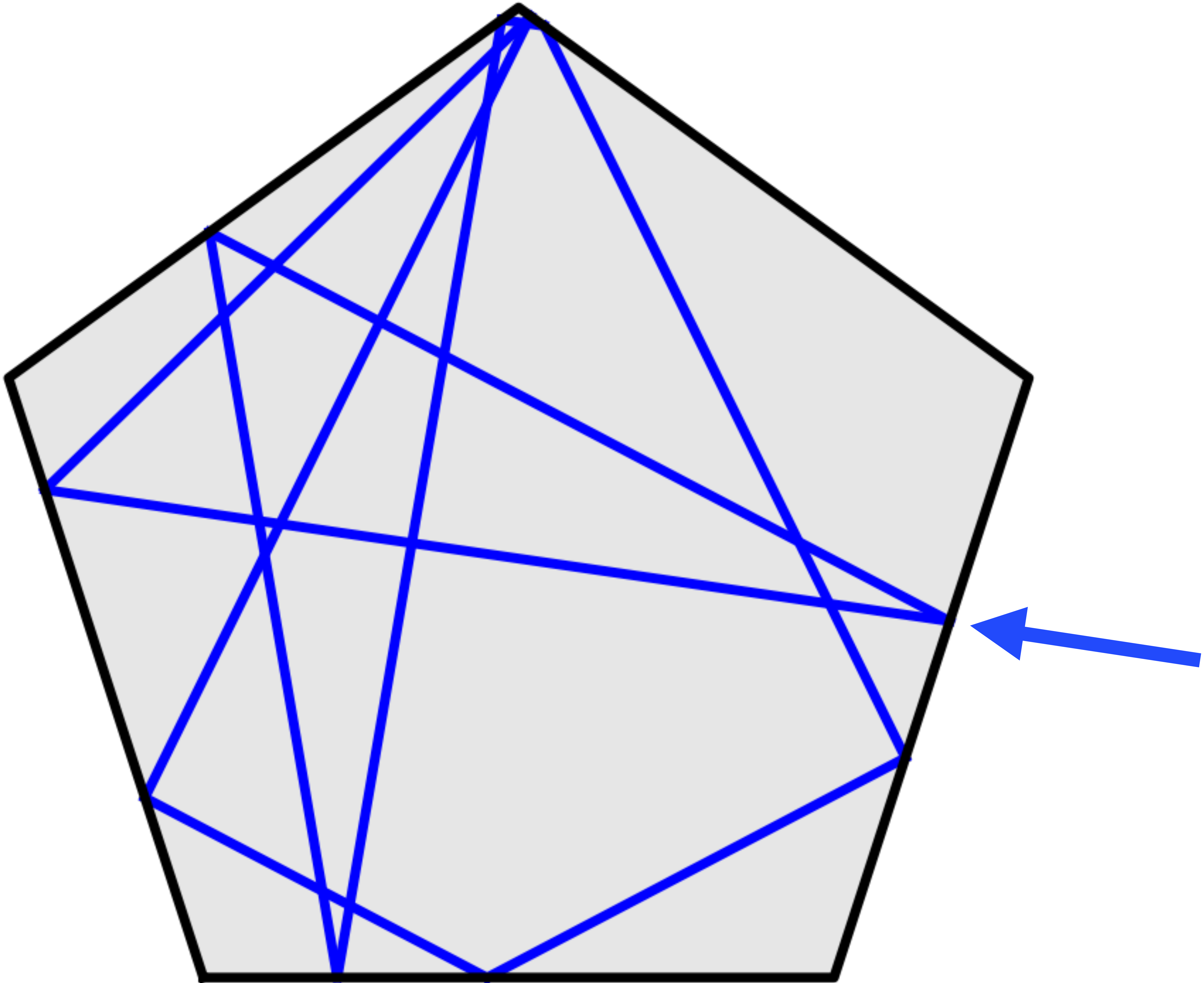
BUDDIES: the parallel trajectories in a given direction



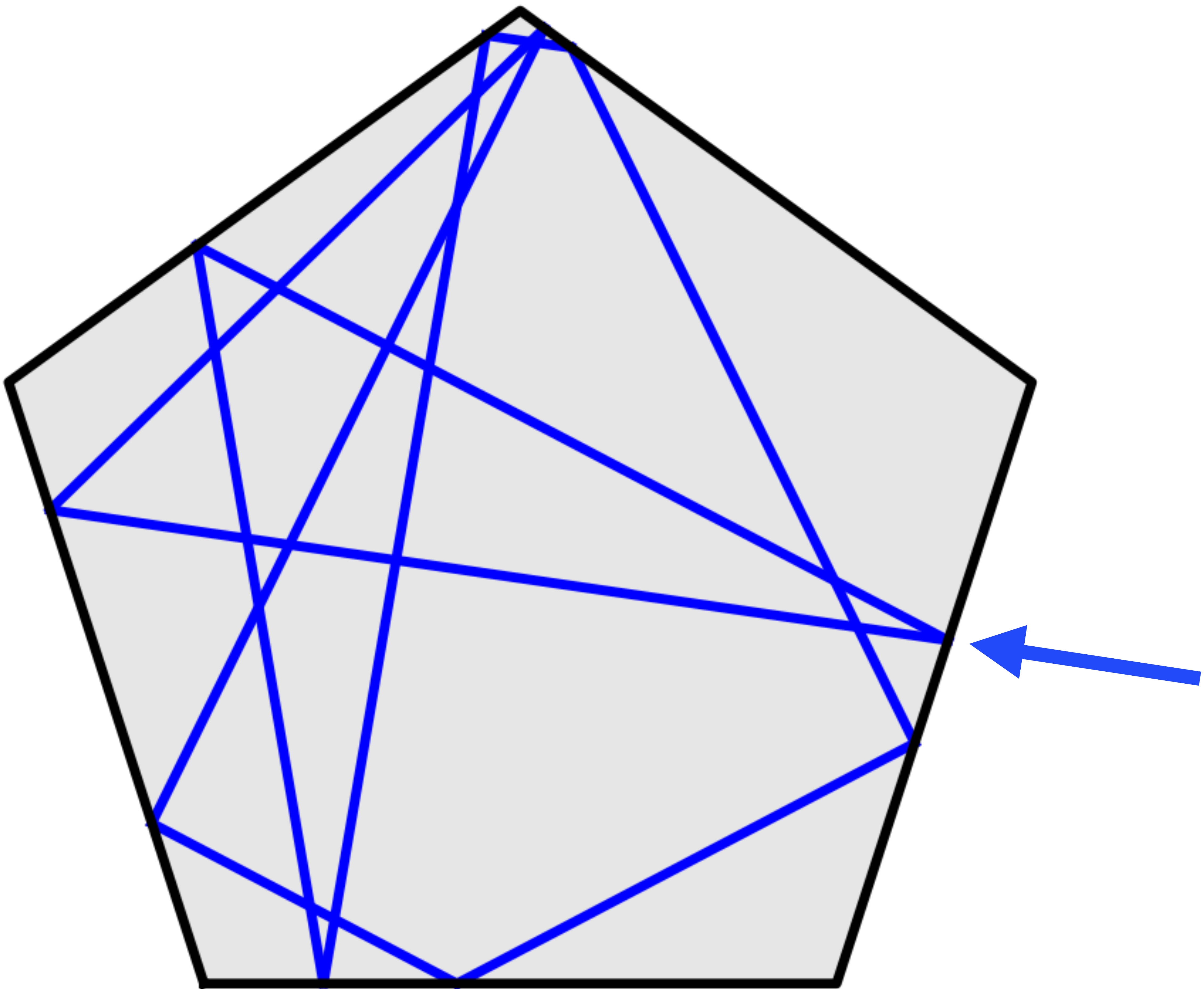
BUDDIES: the parallel trajectories in a given direction



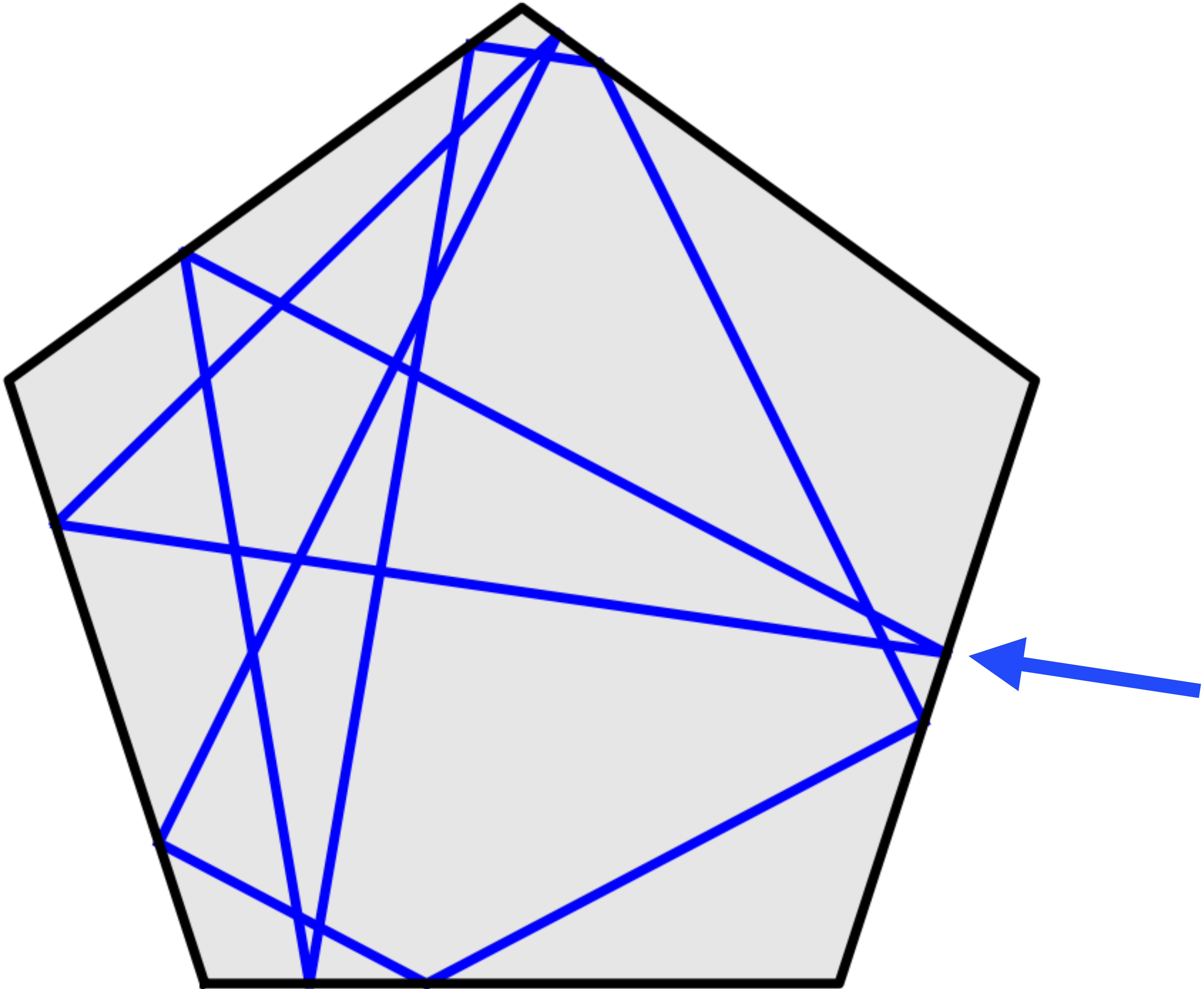
BUDDIES: the parallel trajectories in a given direction



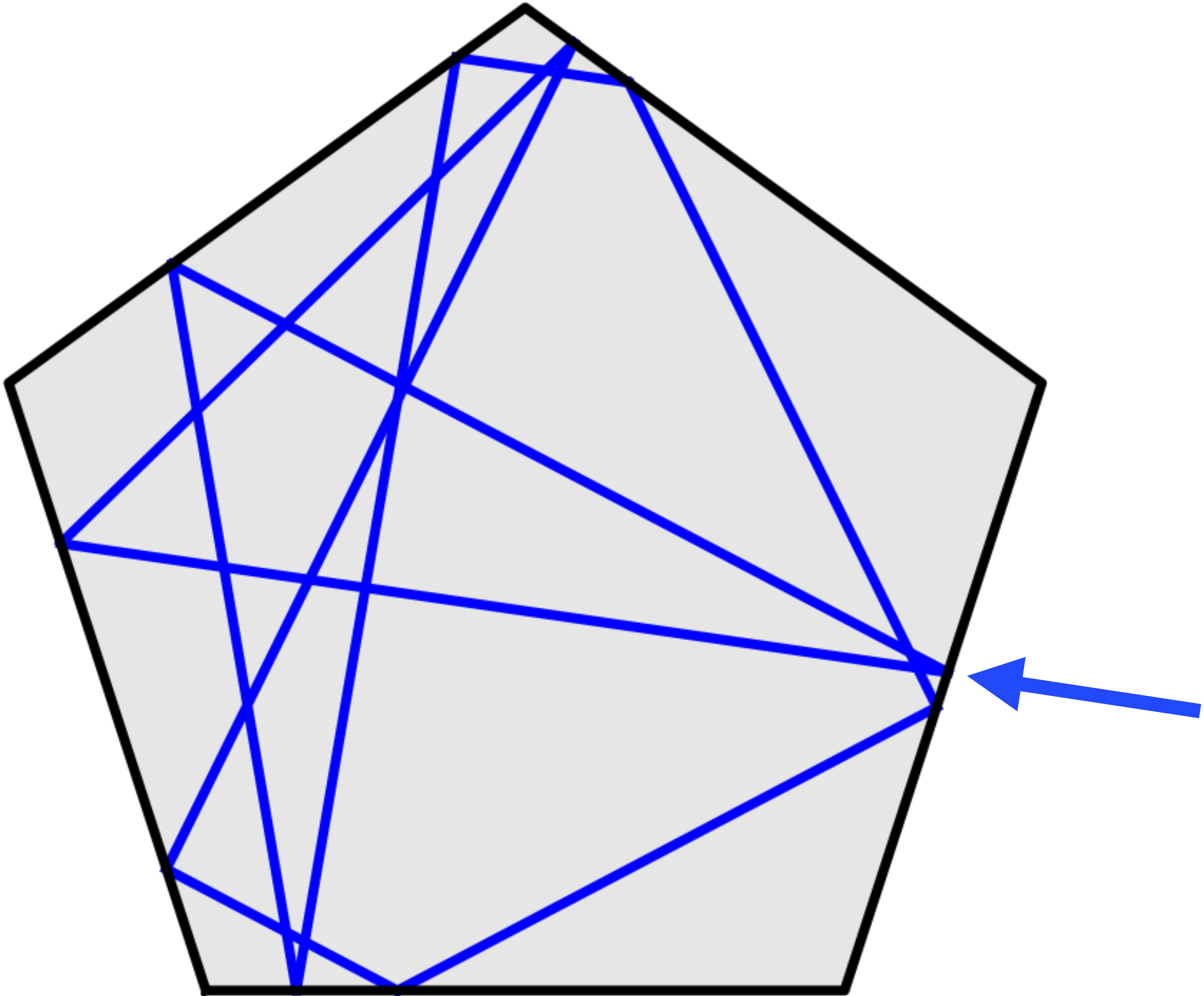
BUDDIES: the parallel trajectories in a given direction



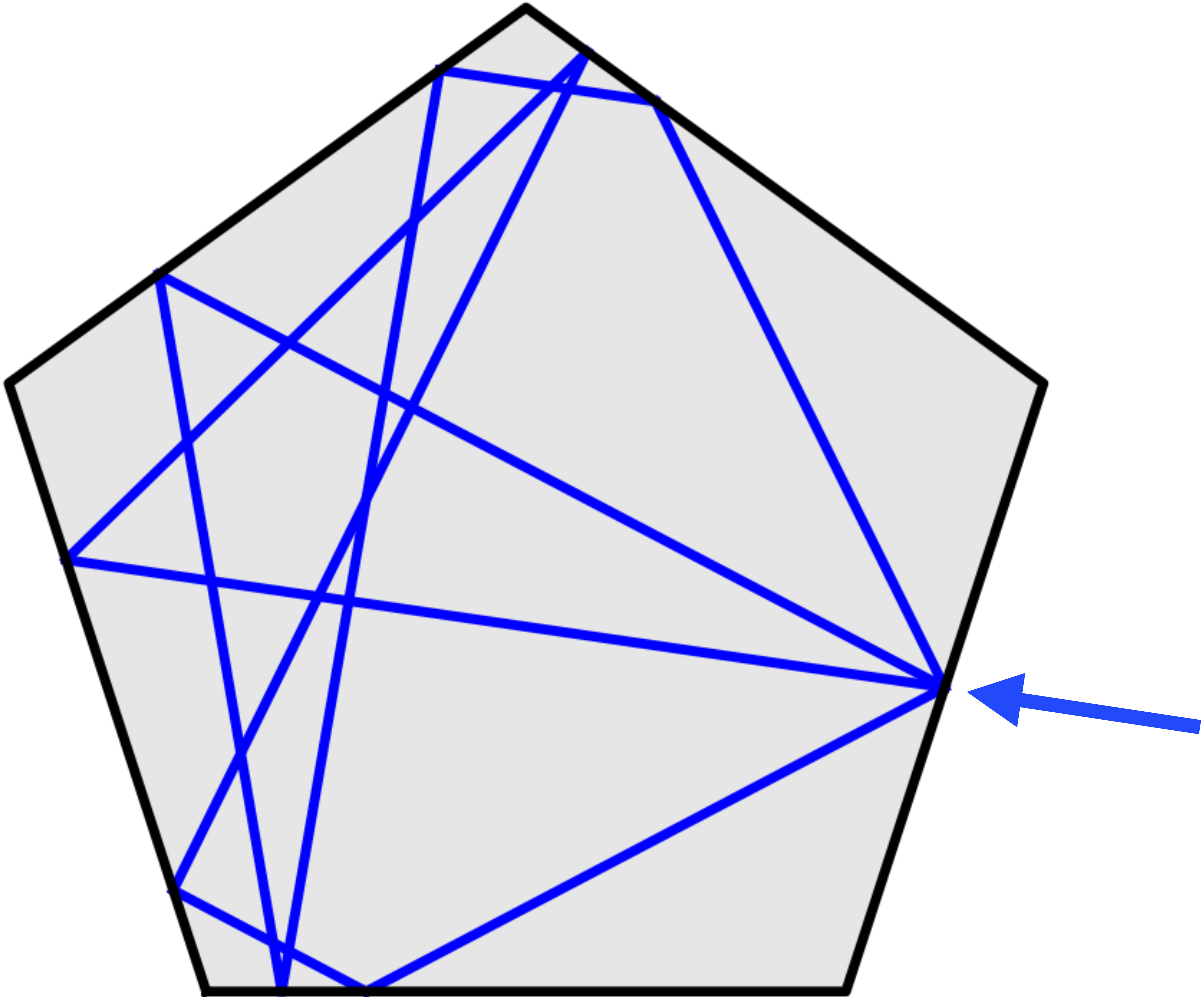
BUDDIES: the parallel trajectories in a given direction



BUDDIES: the parallel trajectories in a given direction

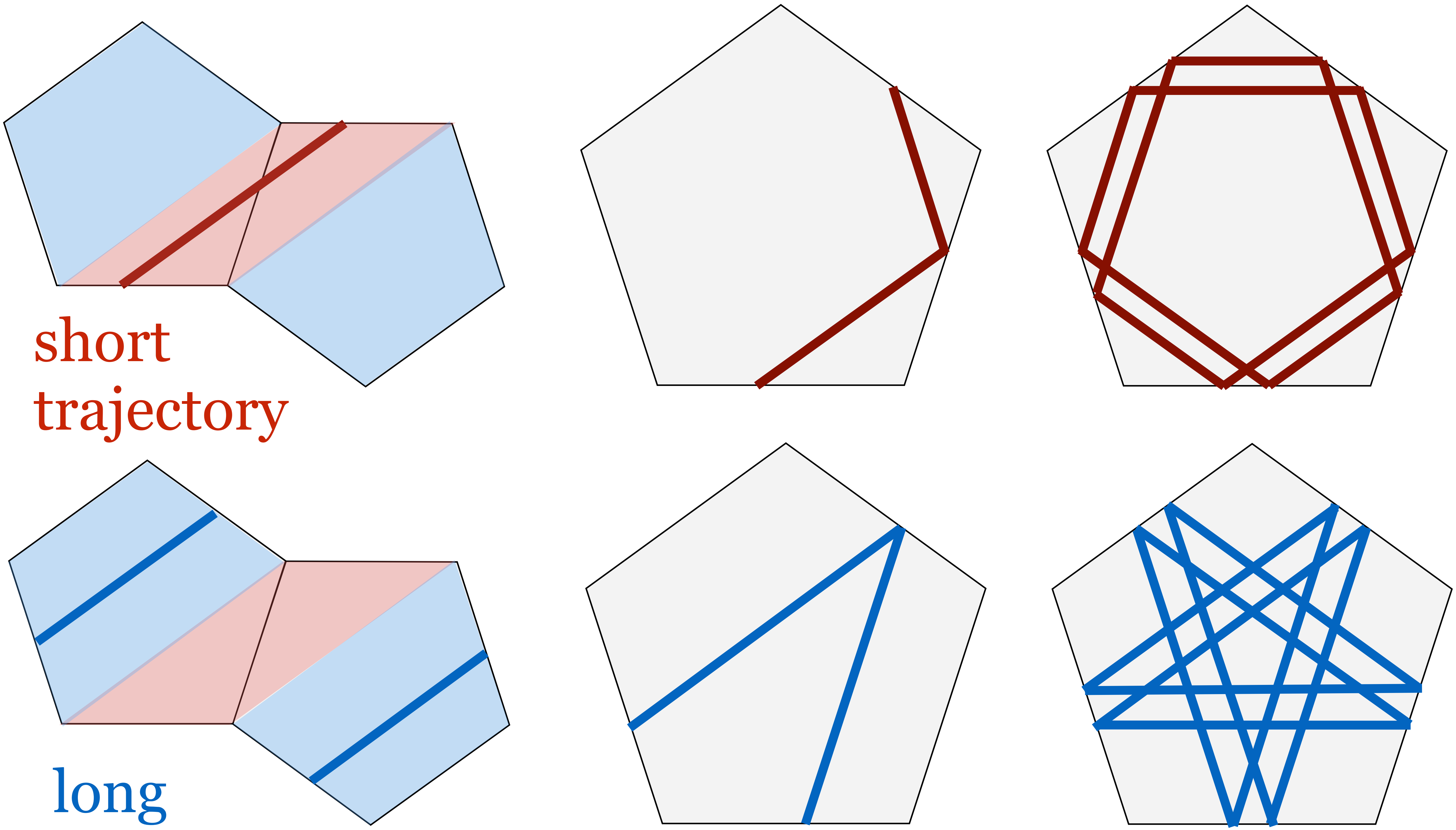


BUDDIES: the parallel trajectories in a given direction

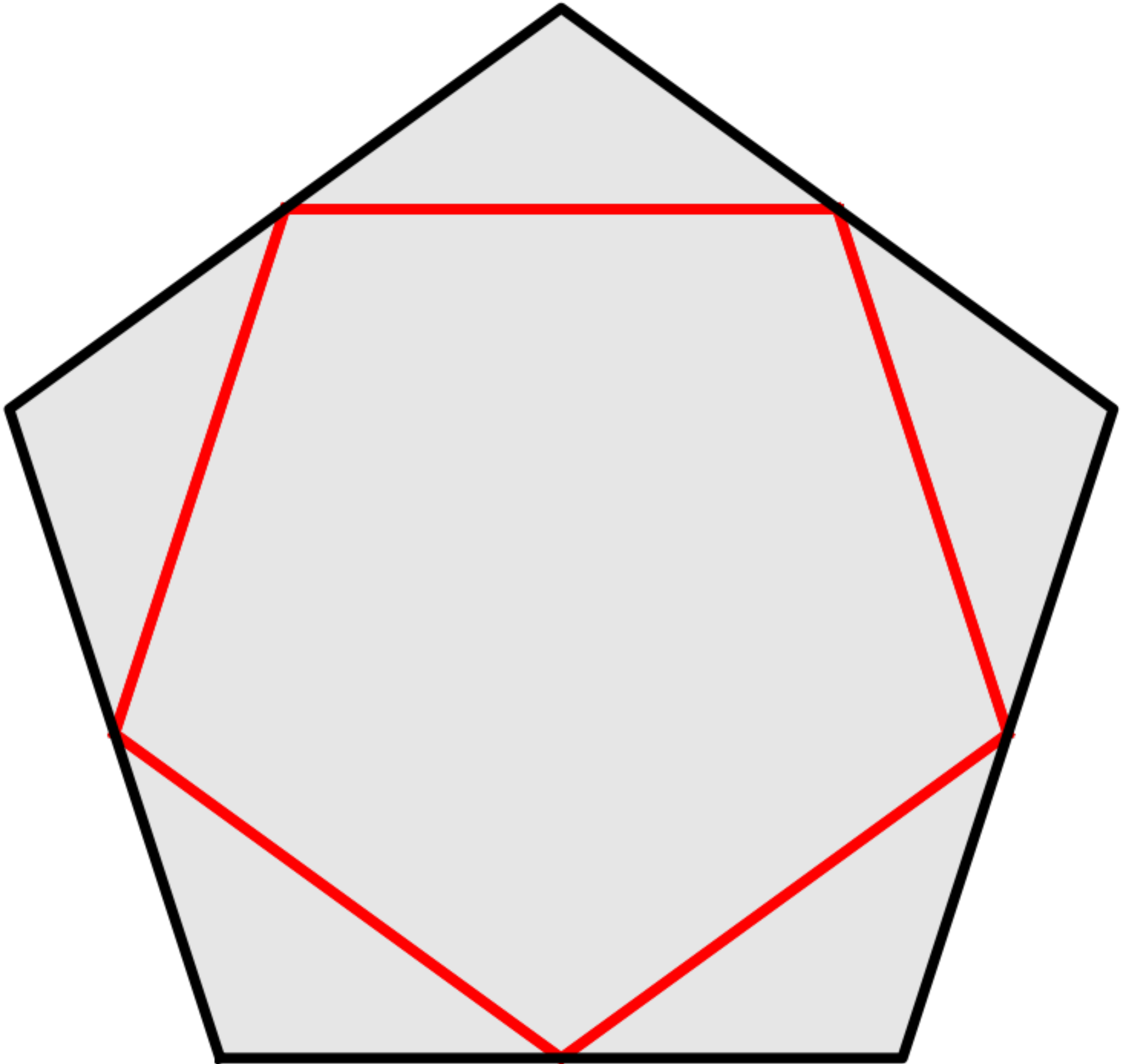


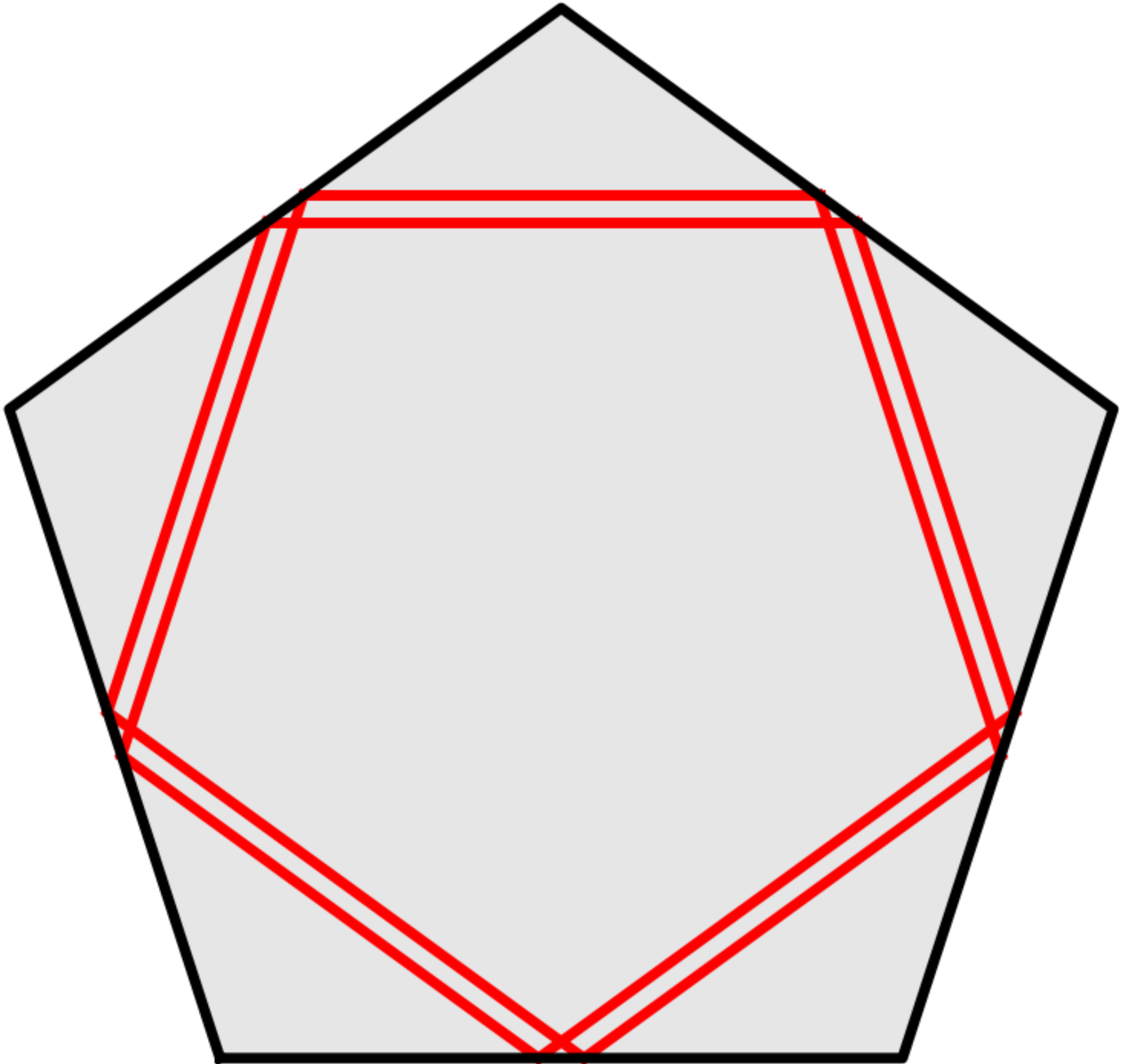
Two trajectories in a given direction:

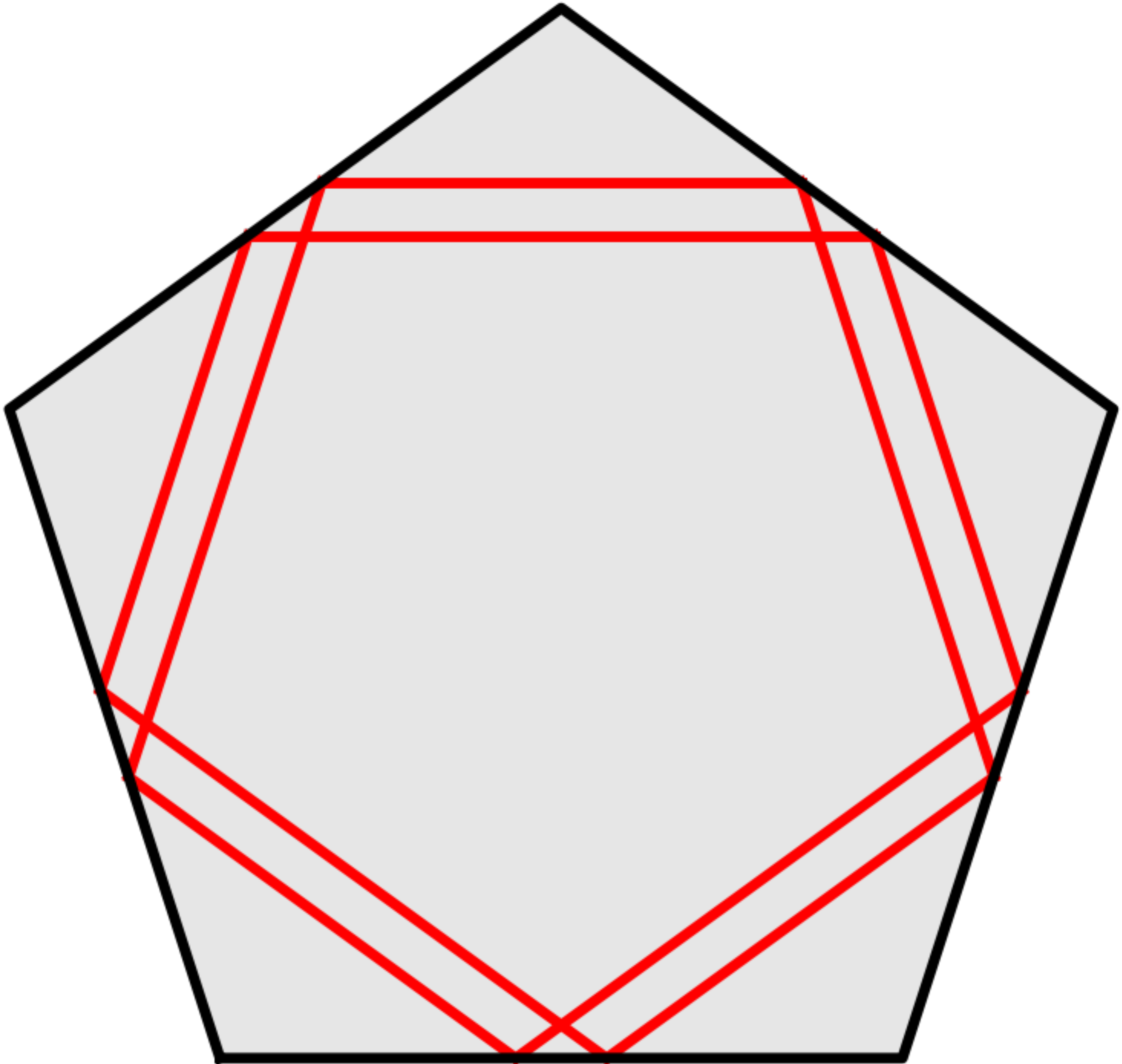
- Short trajectory (core of short cylinder)
- Long trajectory (core of long cylinder)

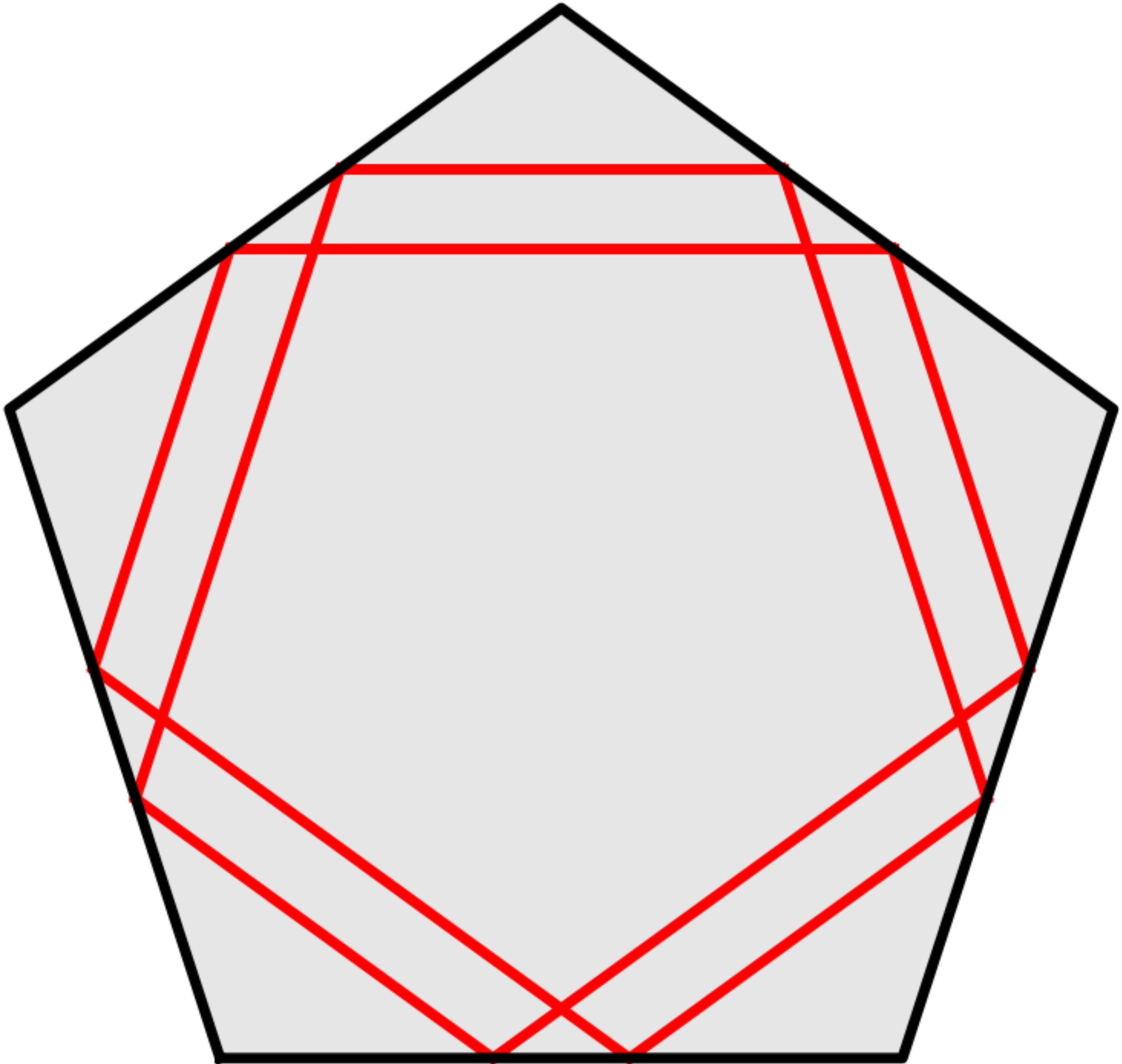


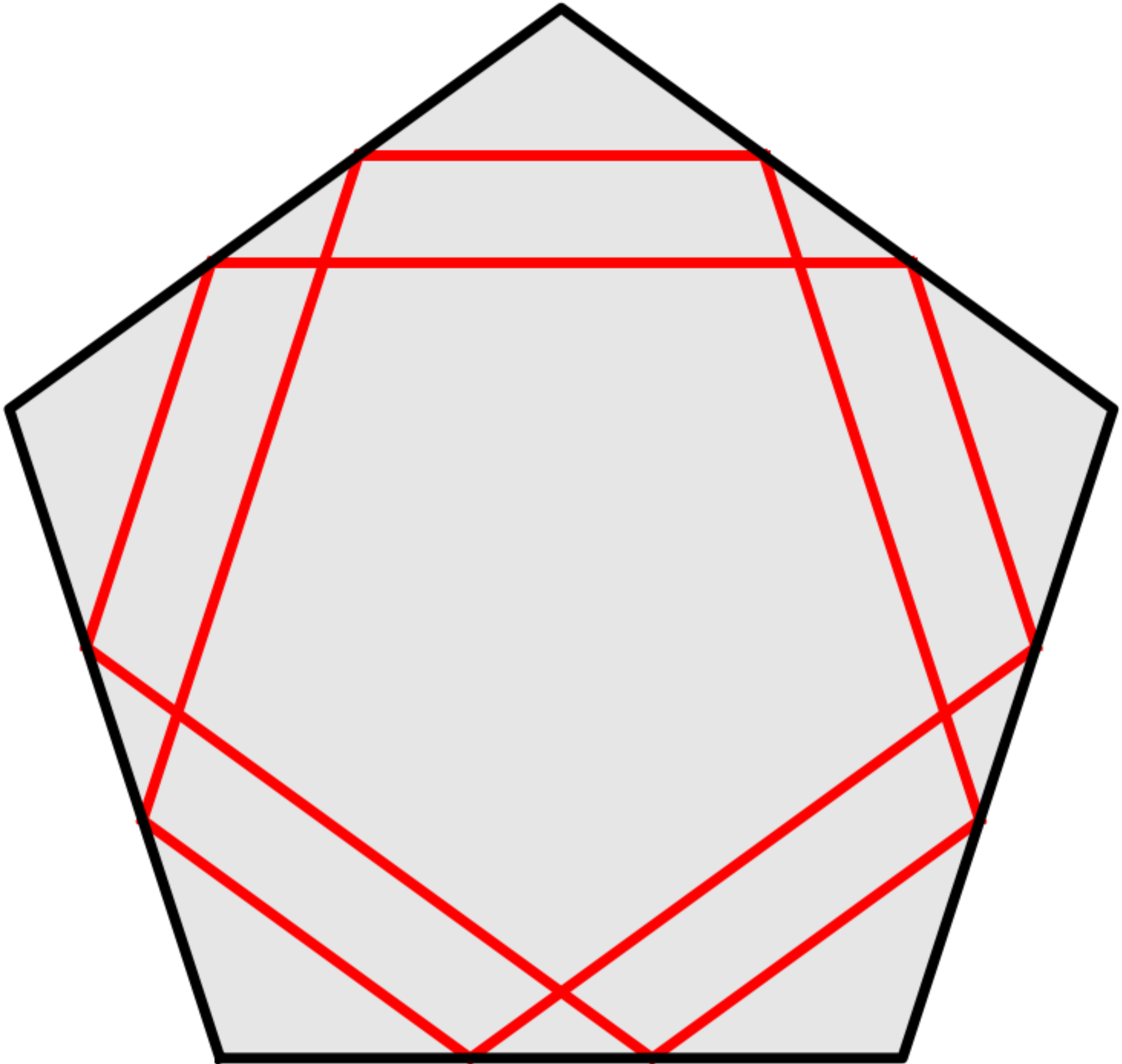
long
trajectory in same direction, ϕ times as long

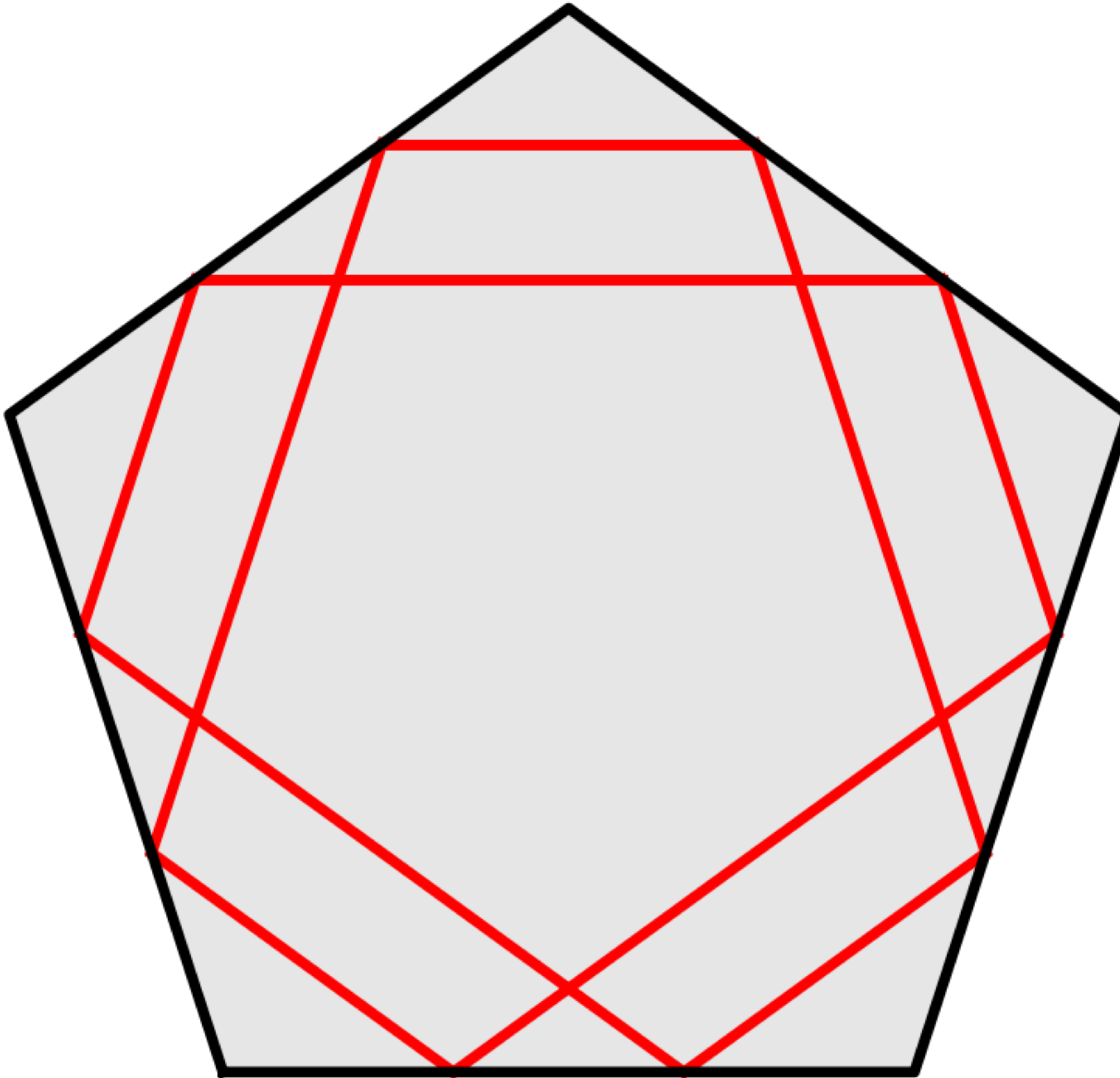


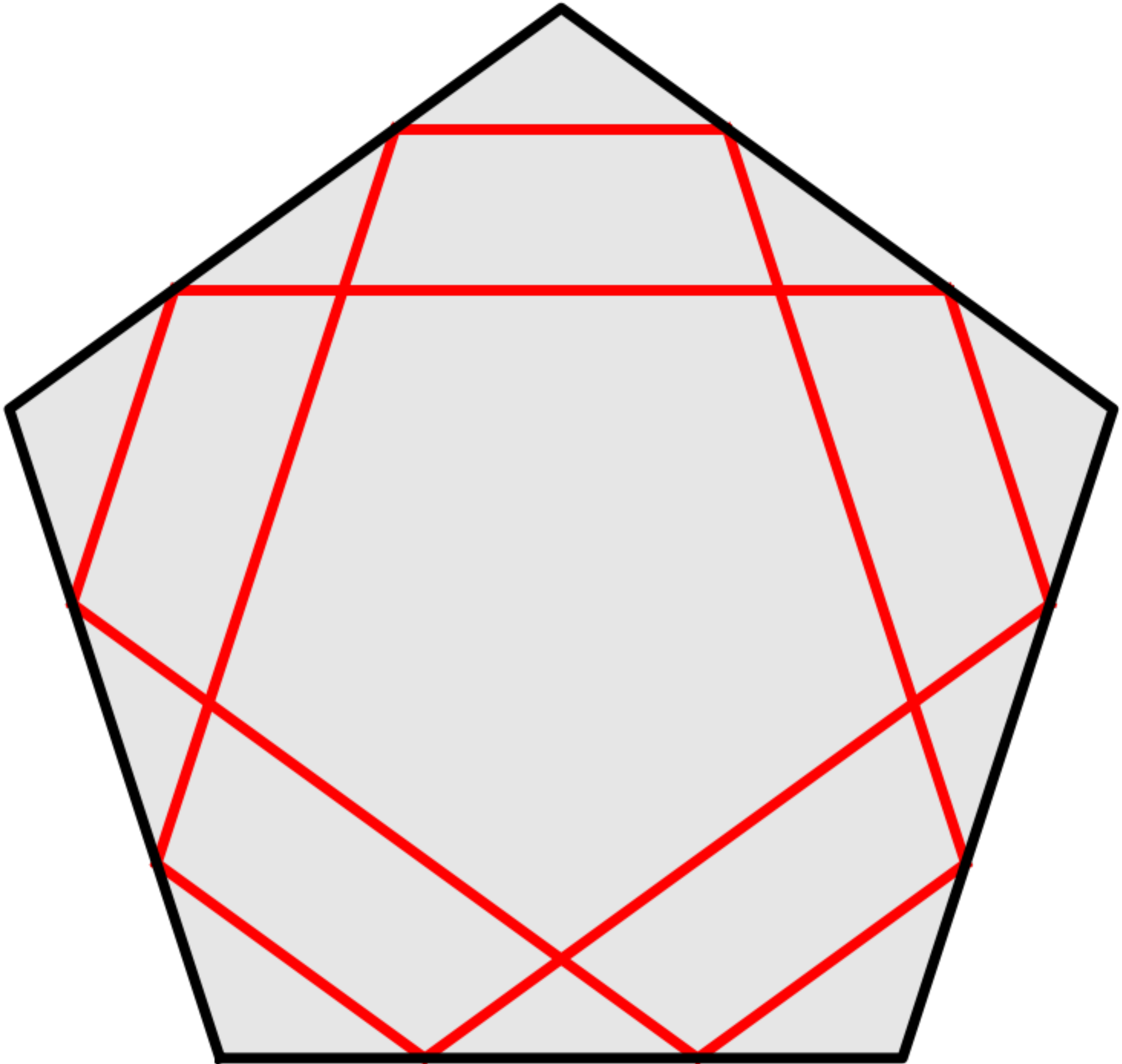


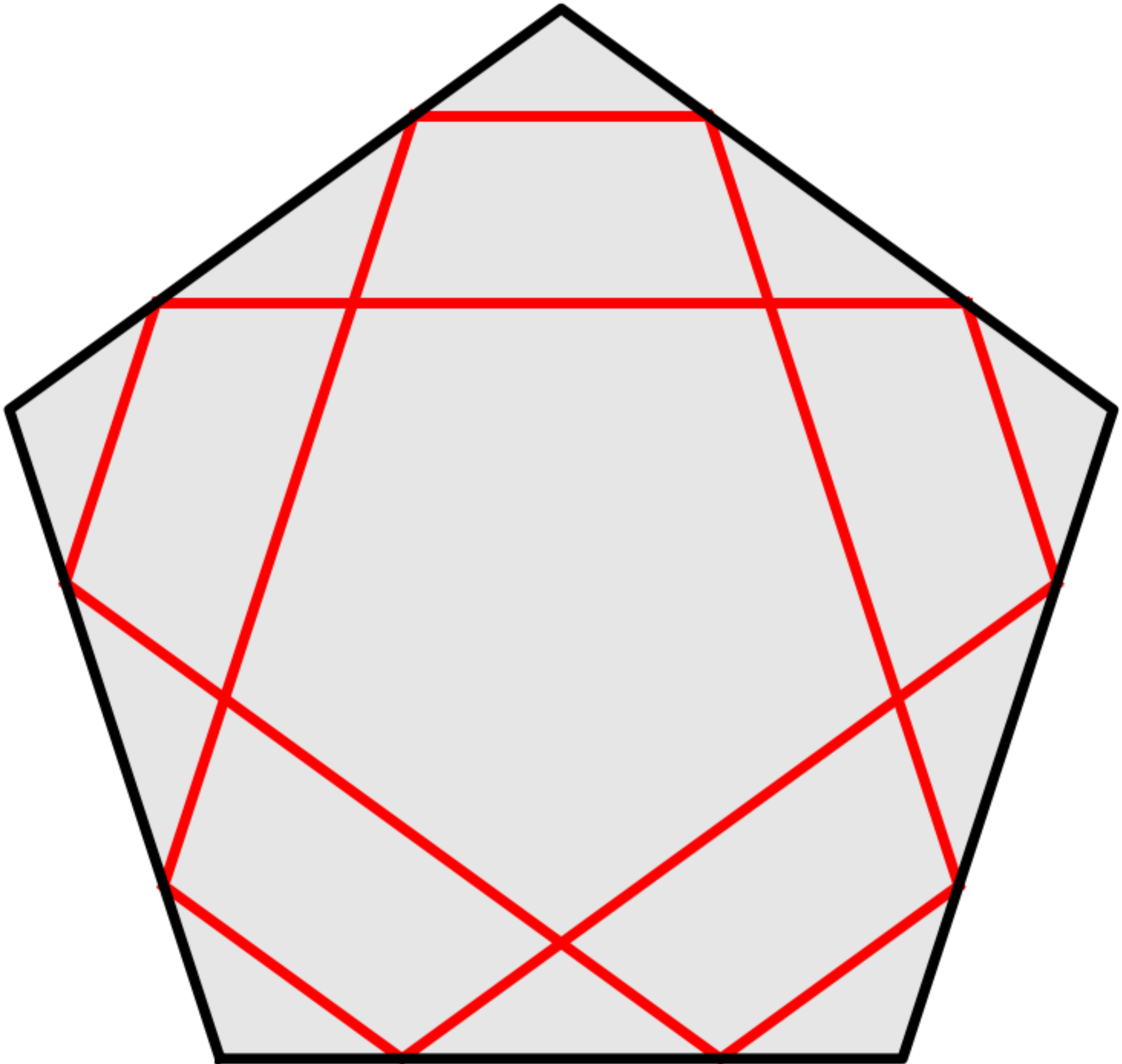


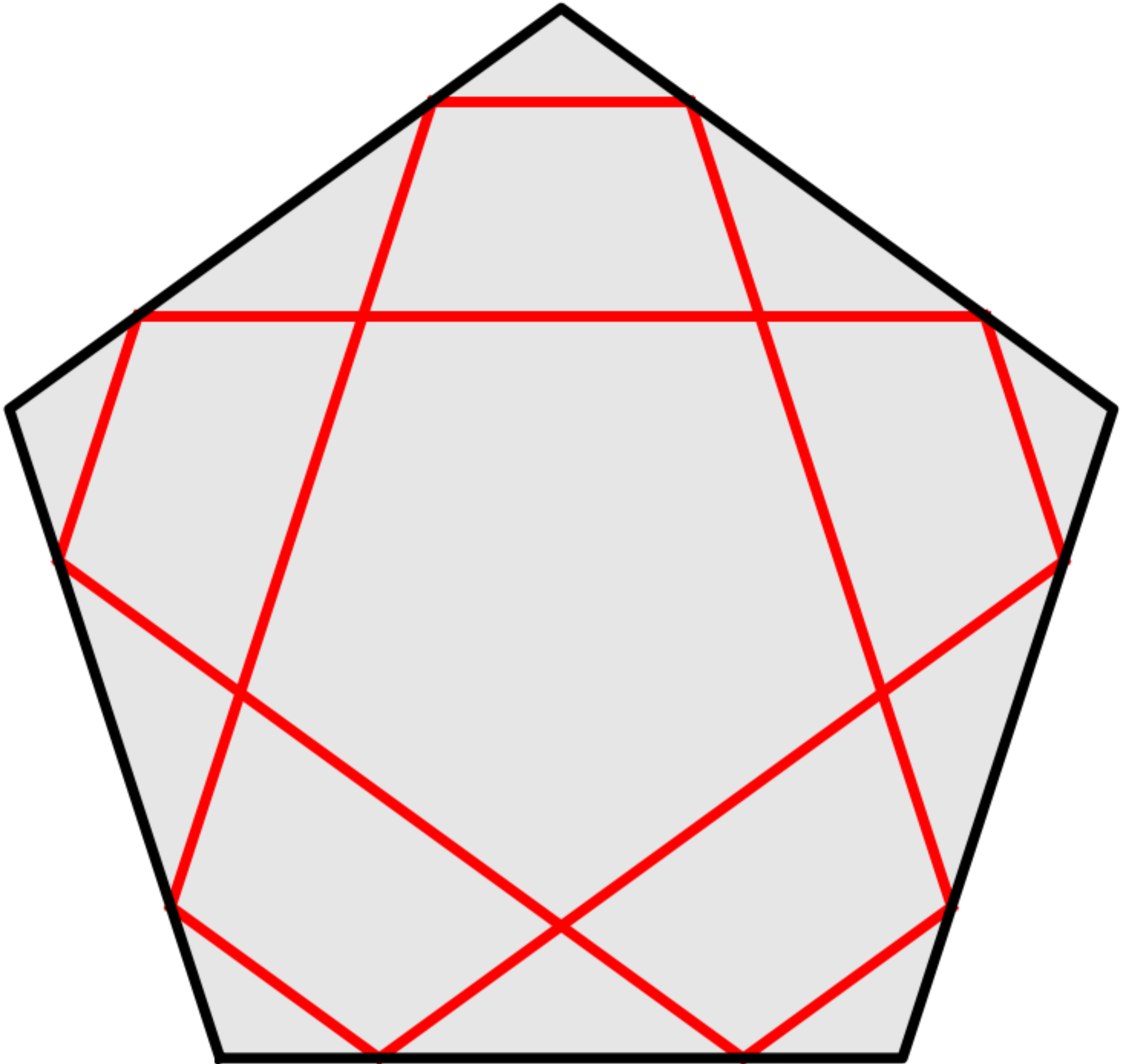


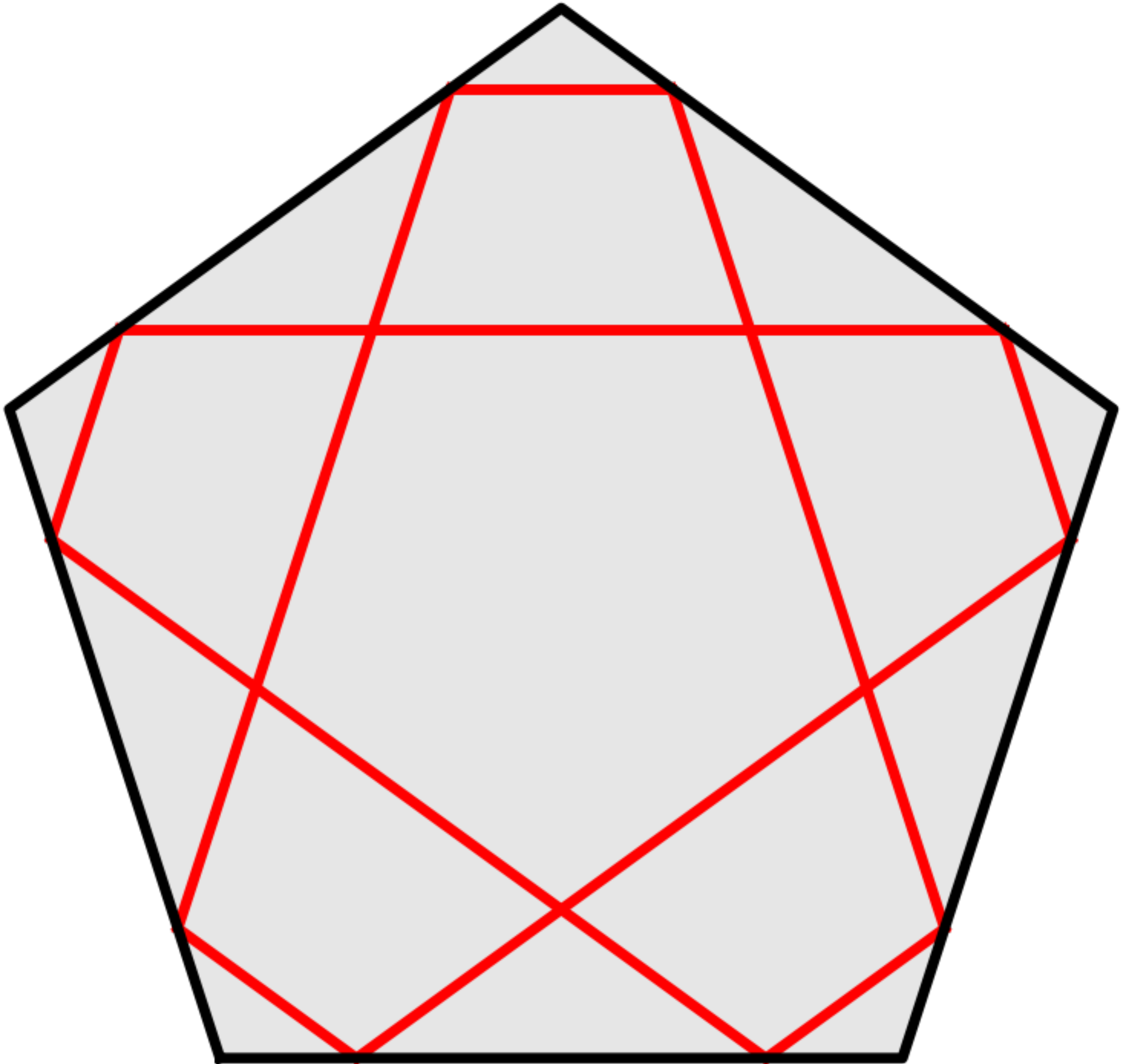


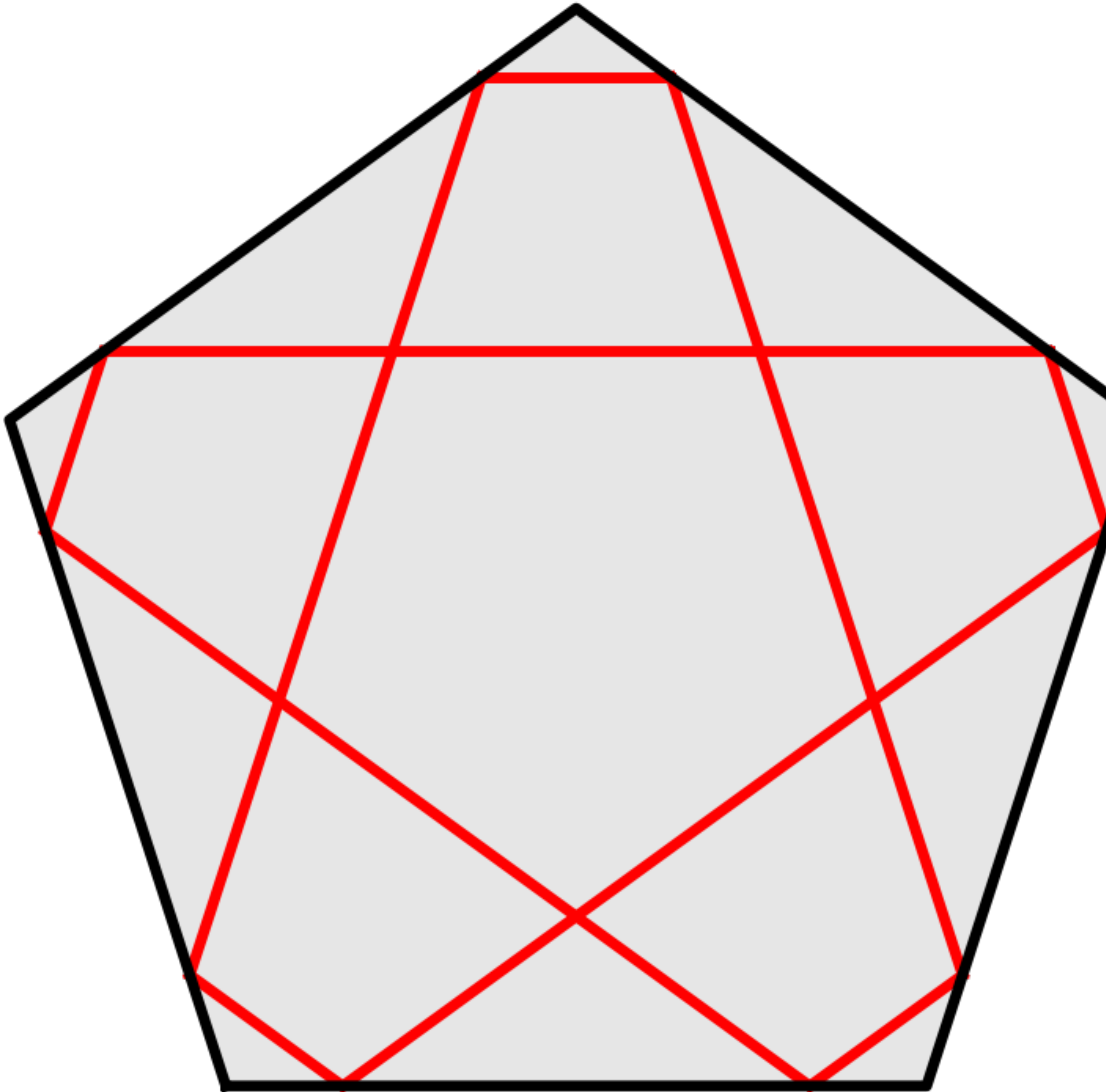


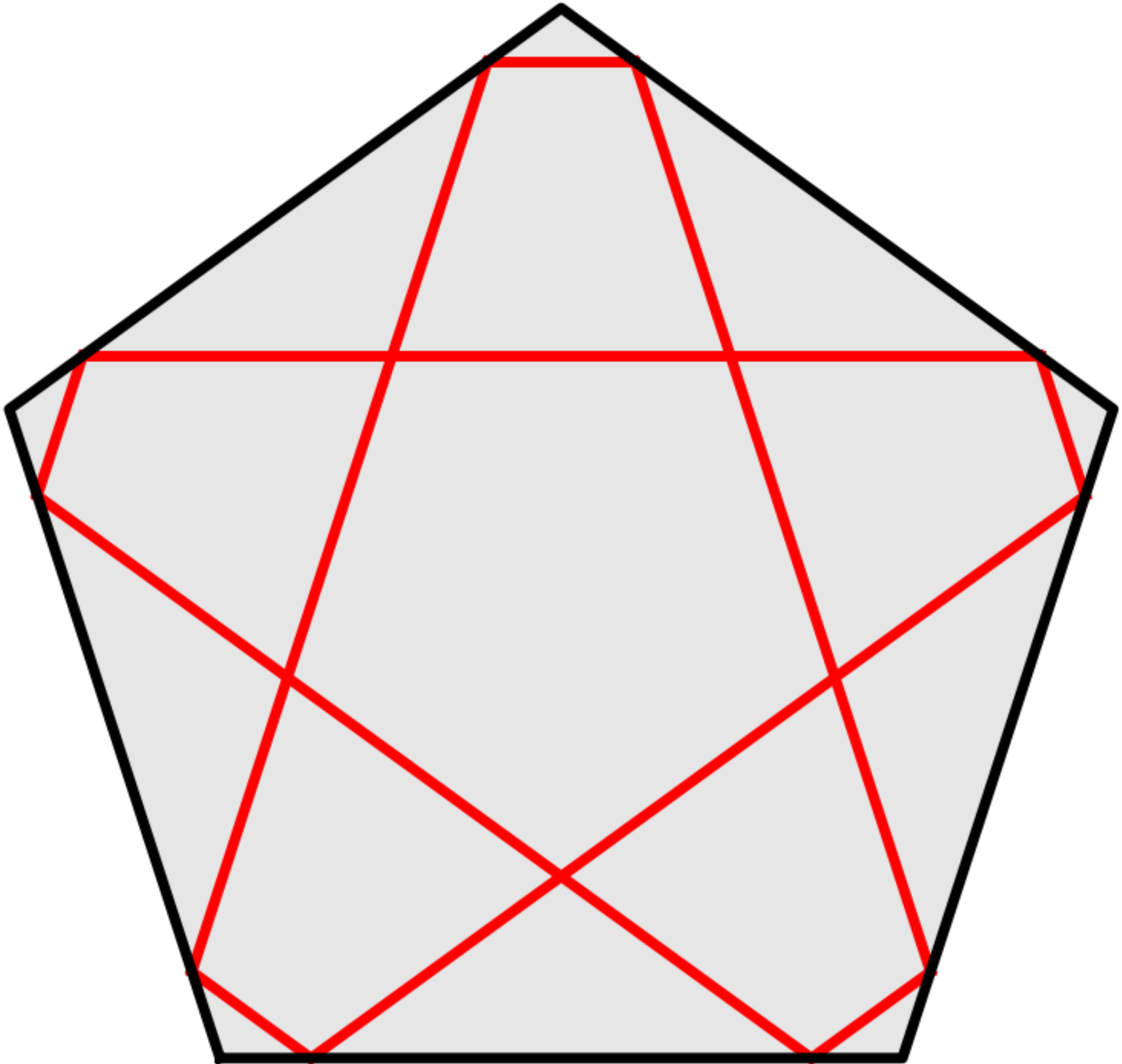


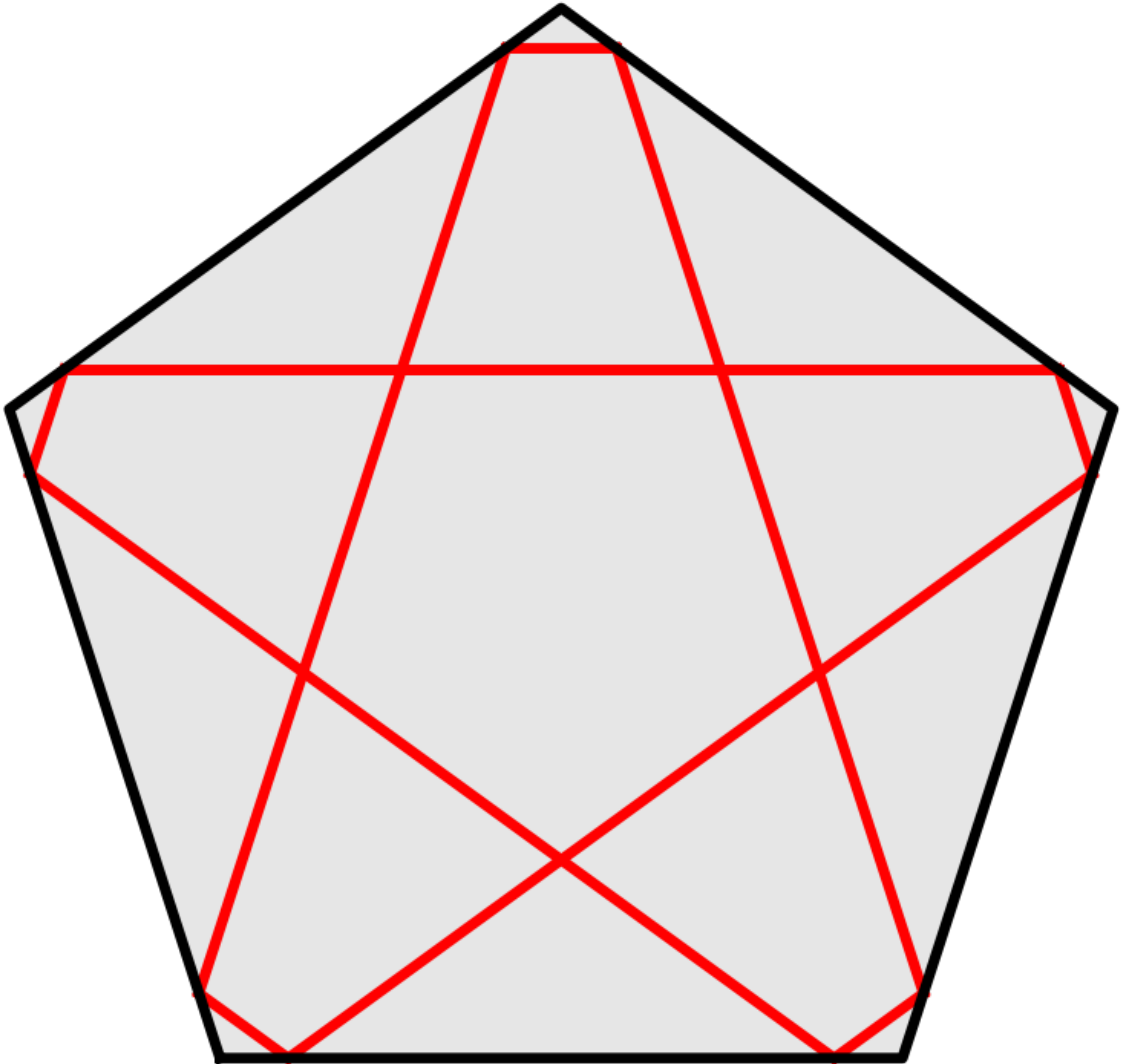


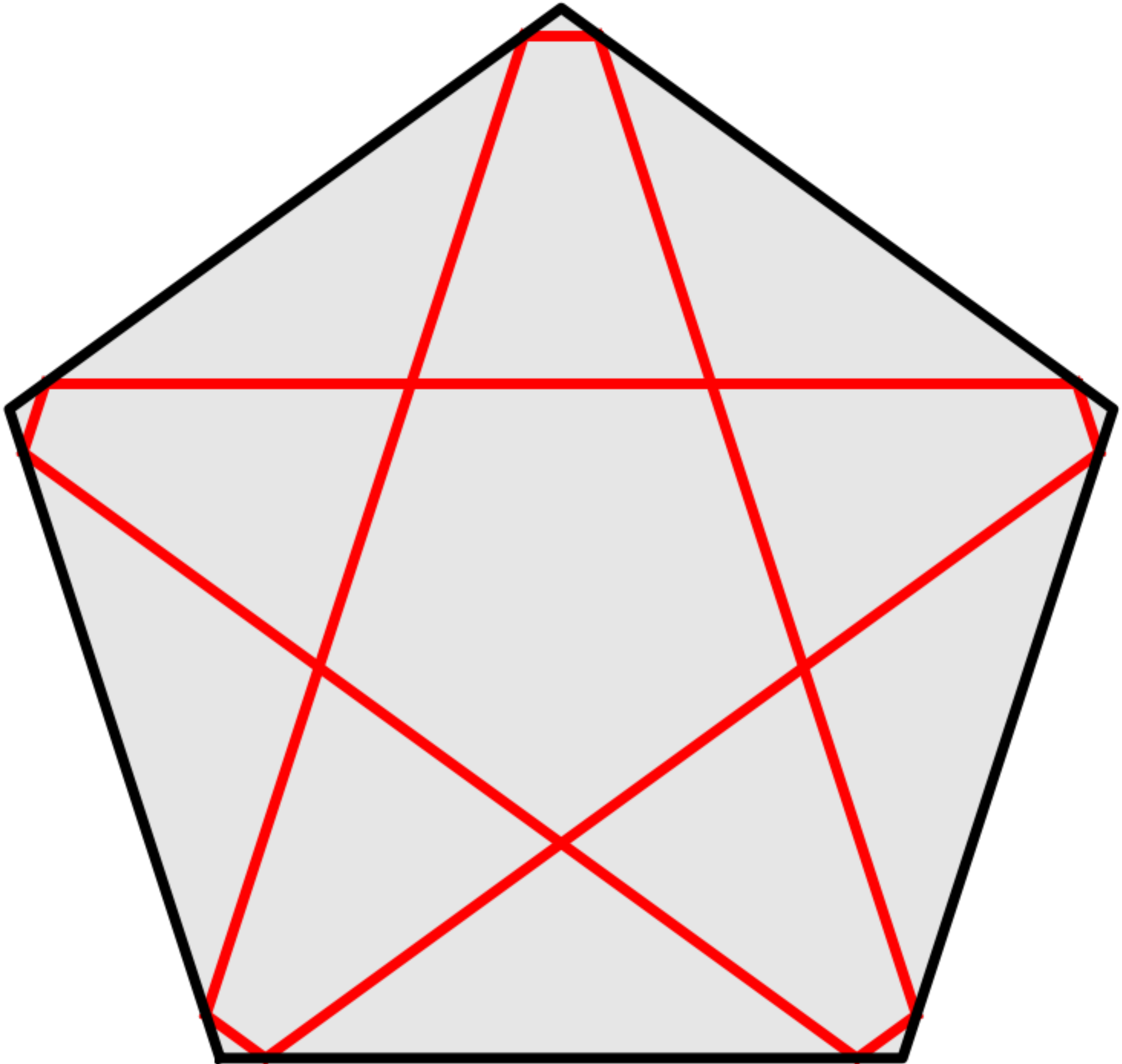


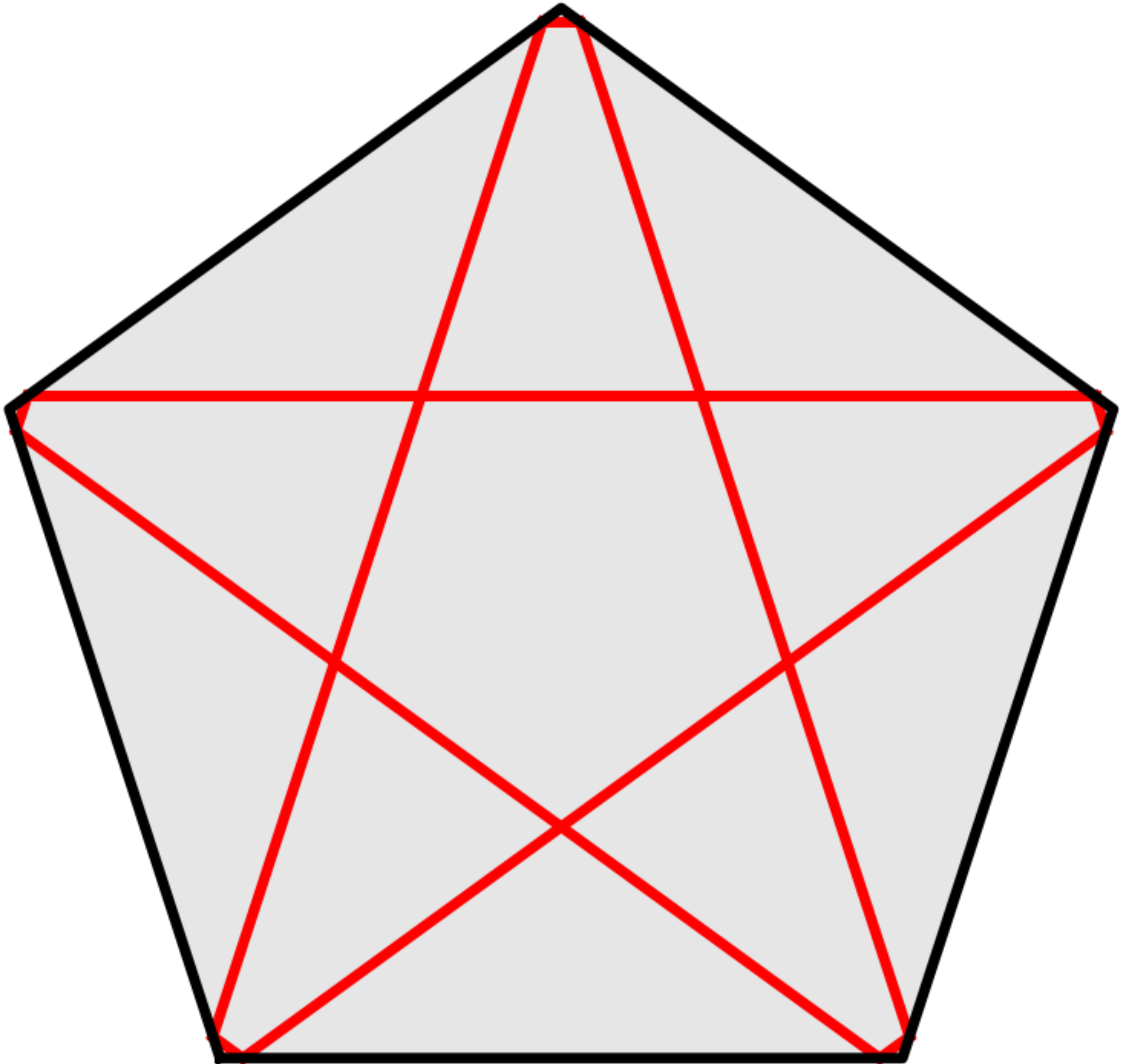


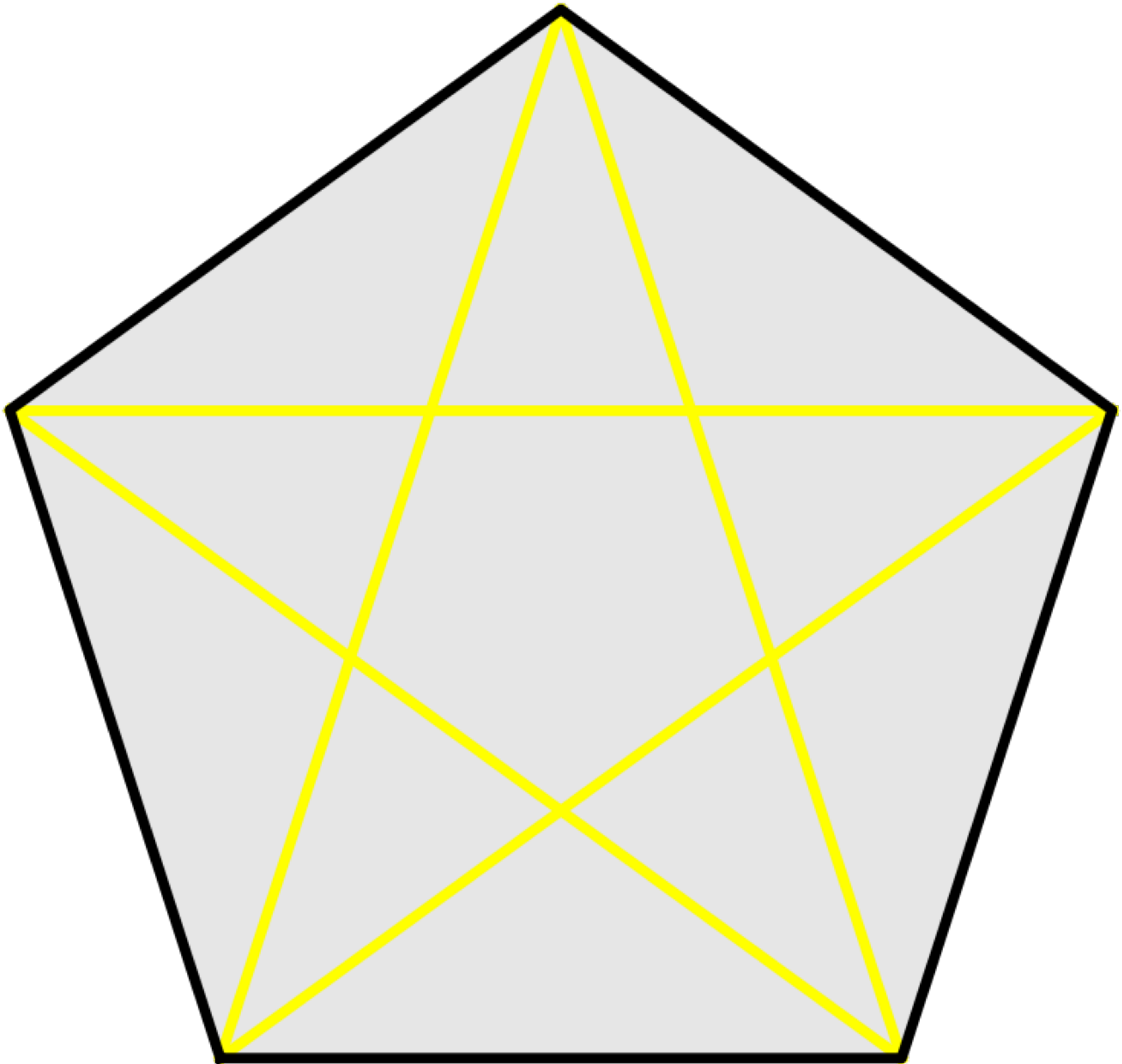


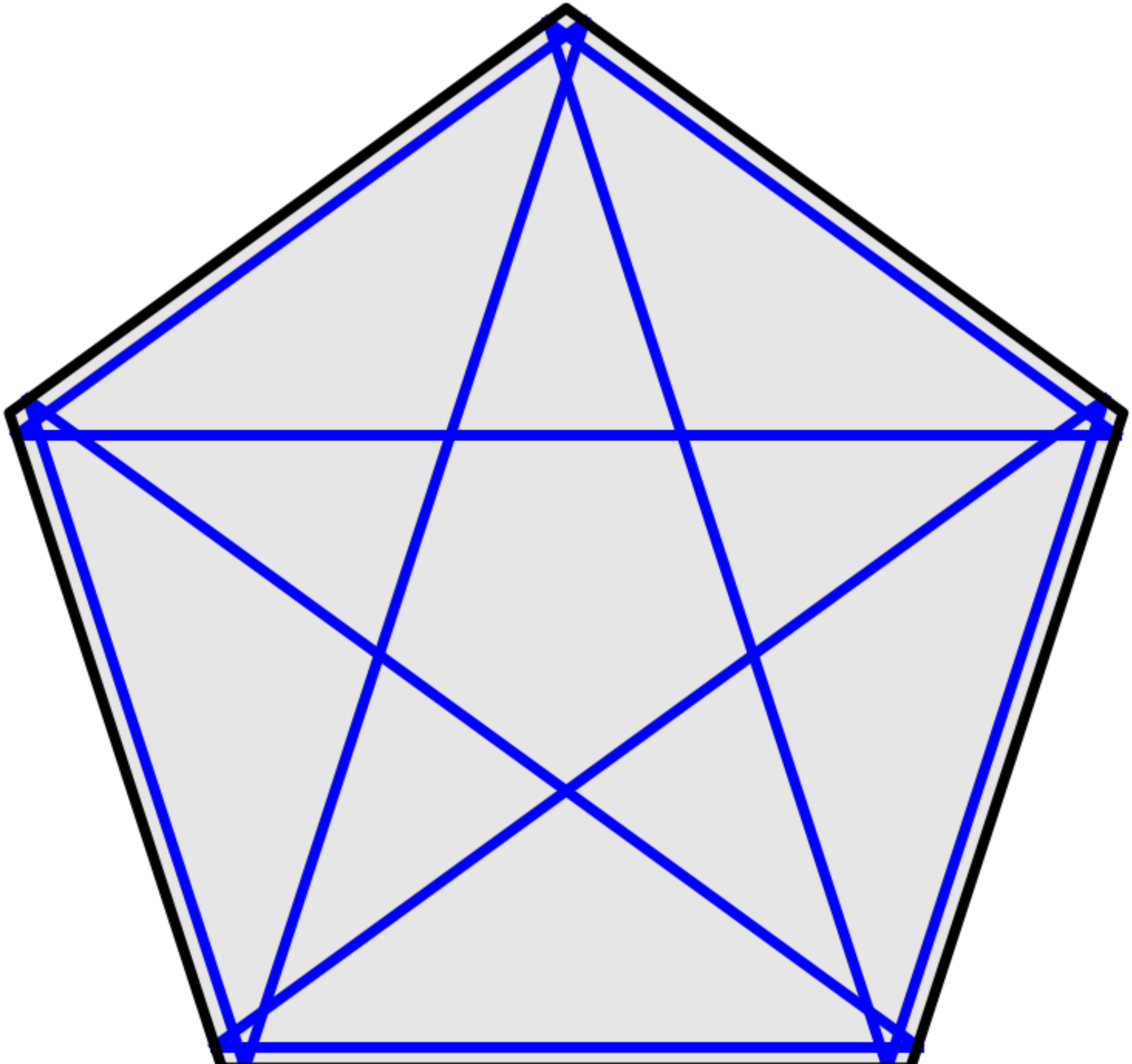


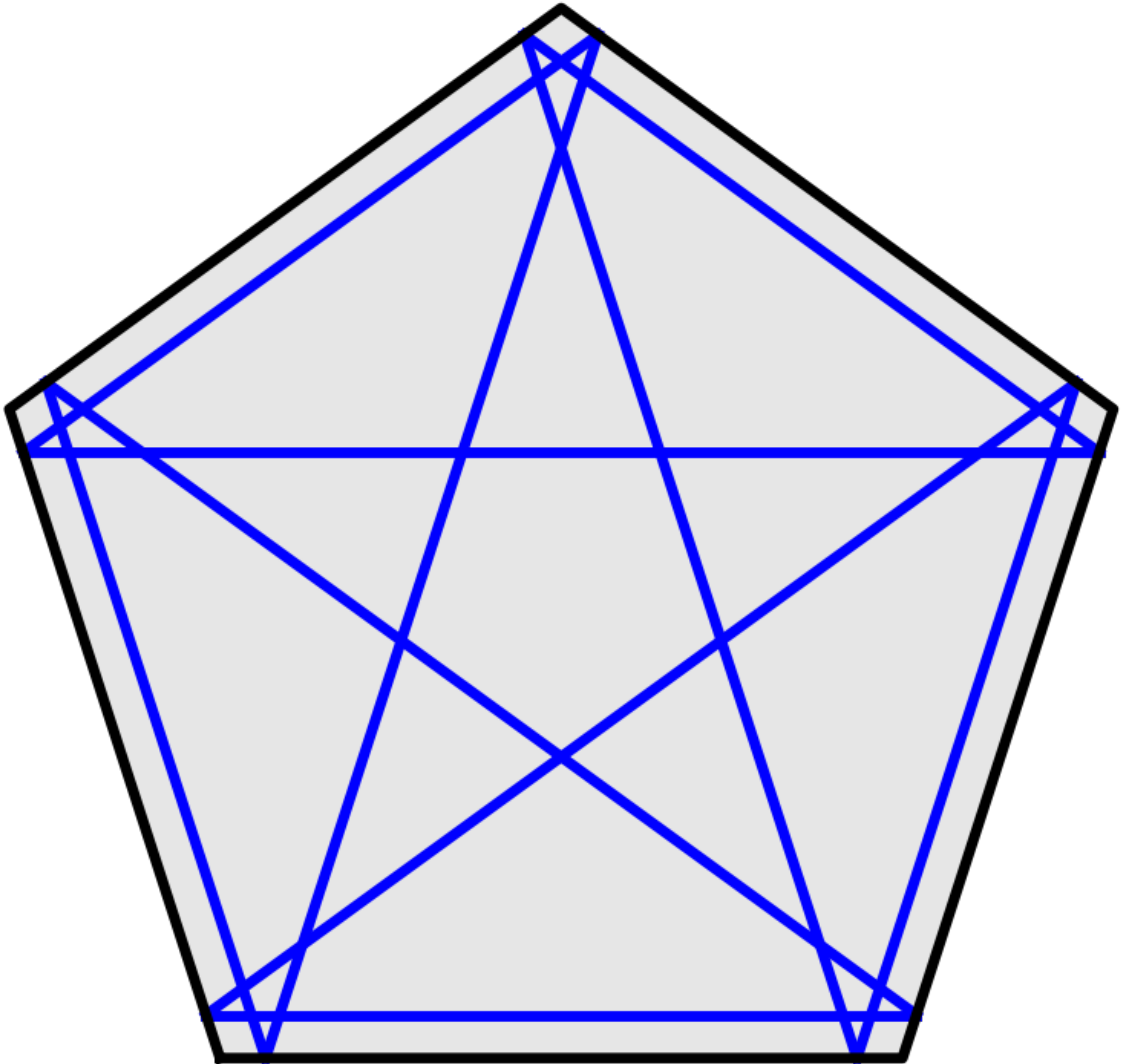


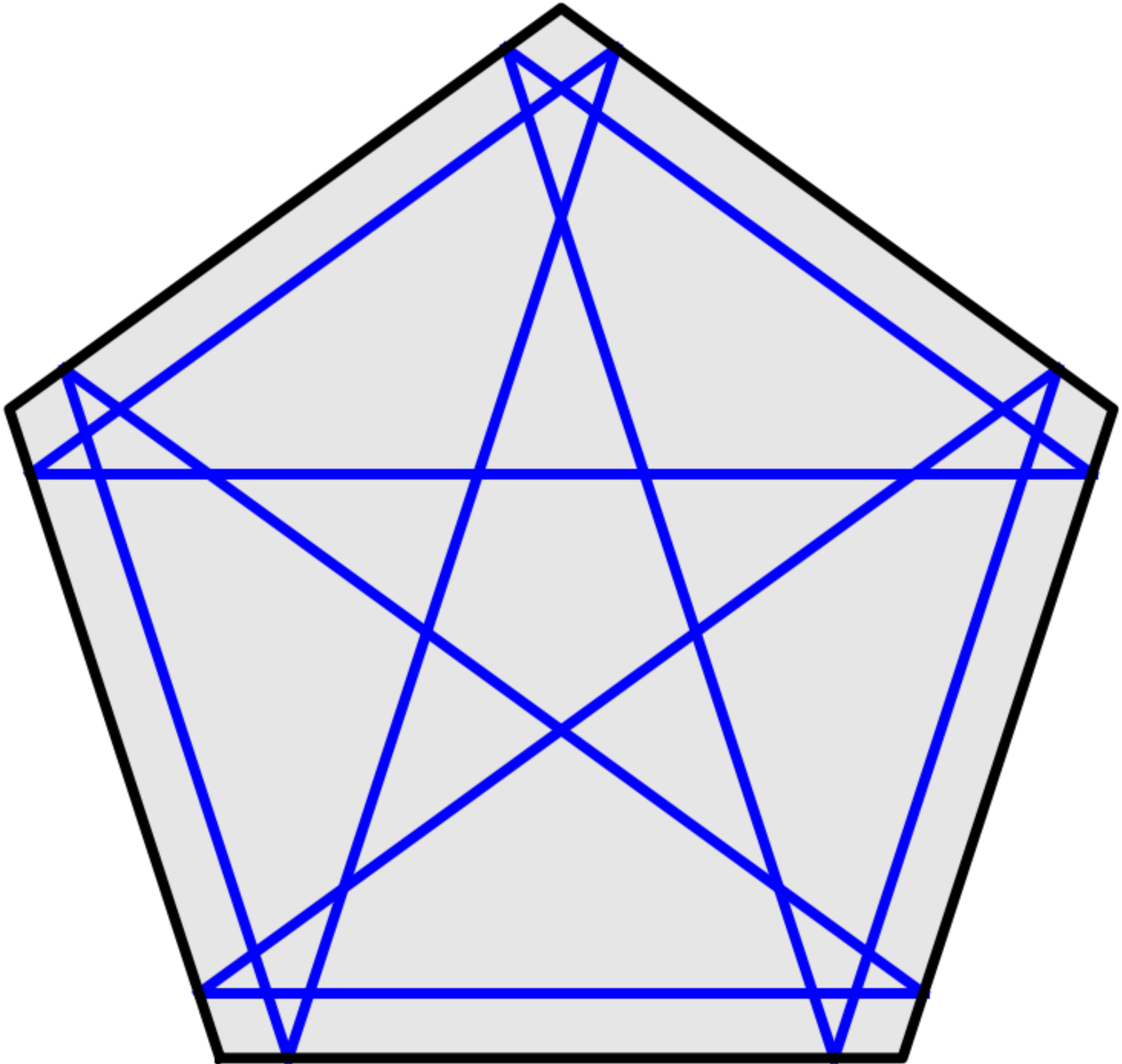


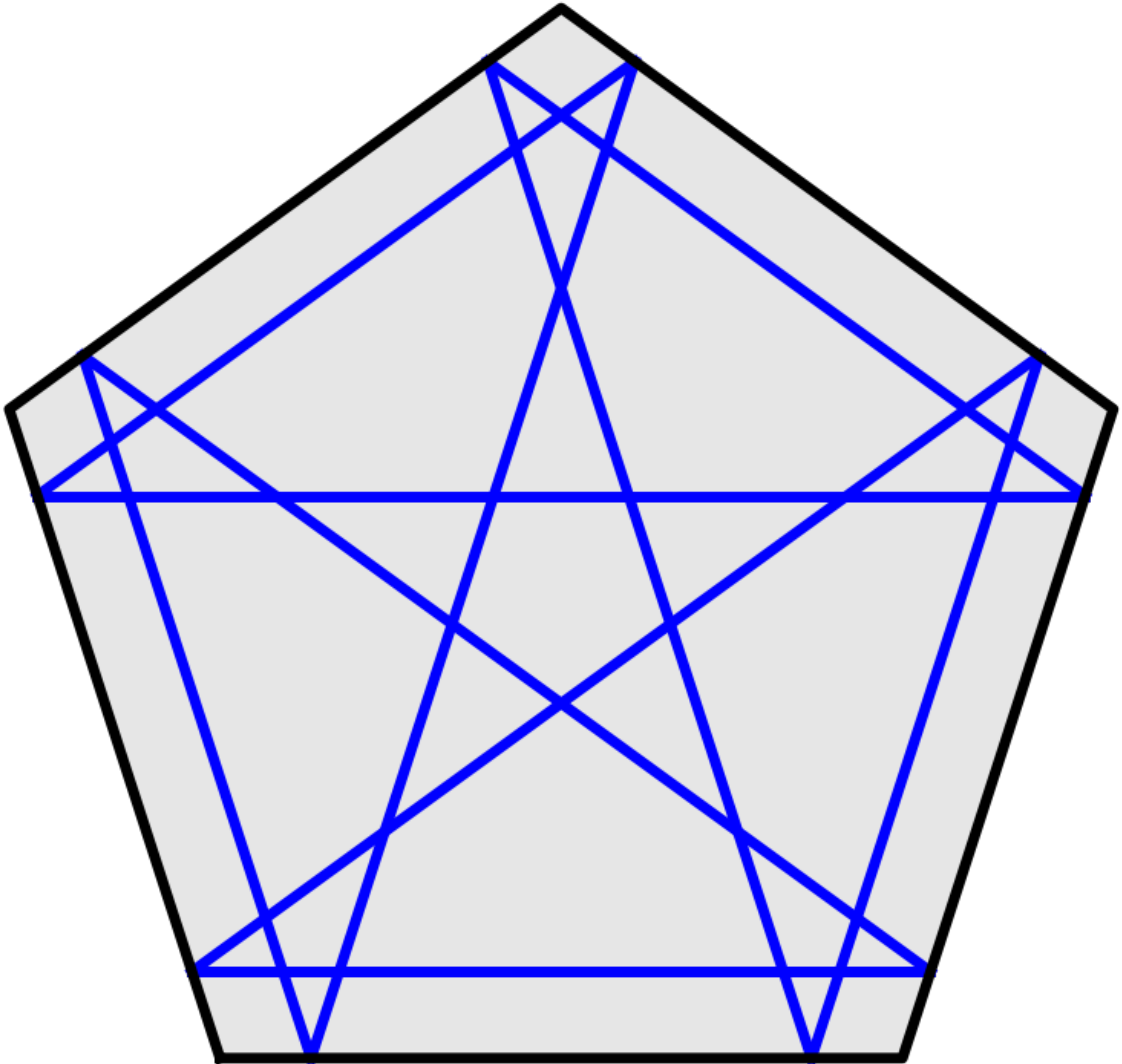


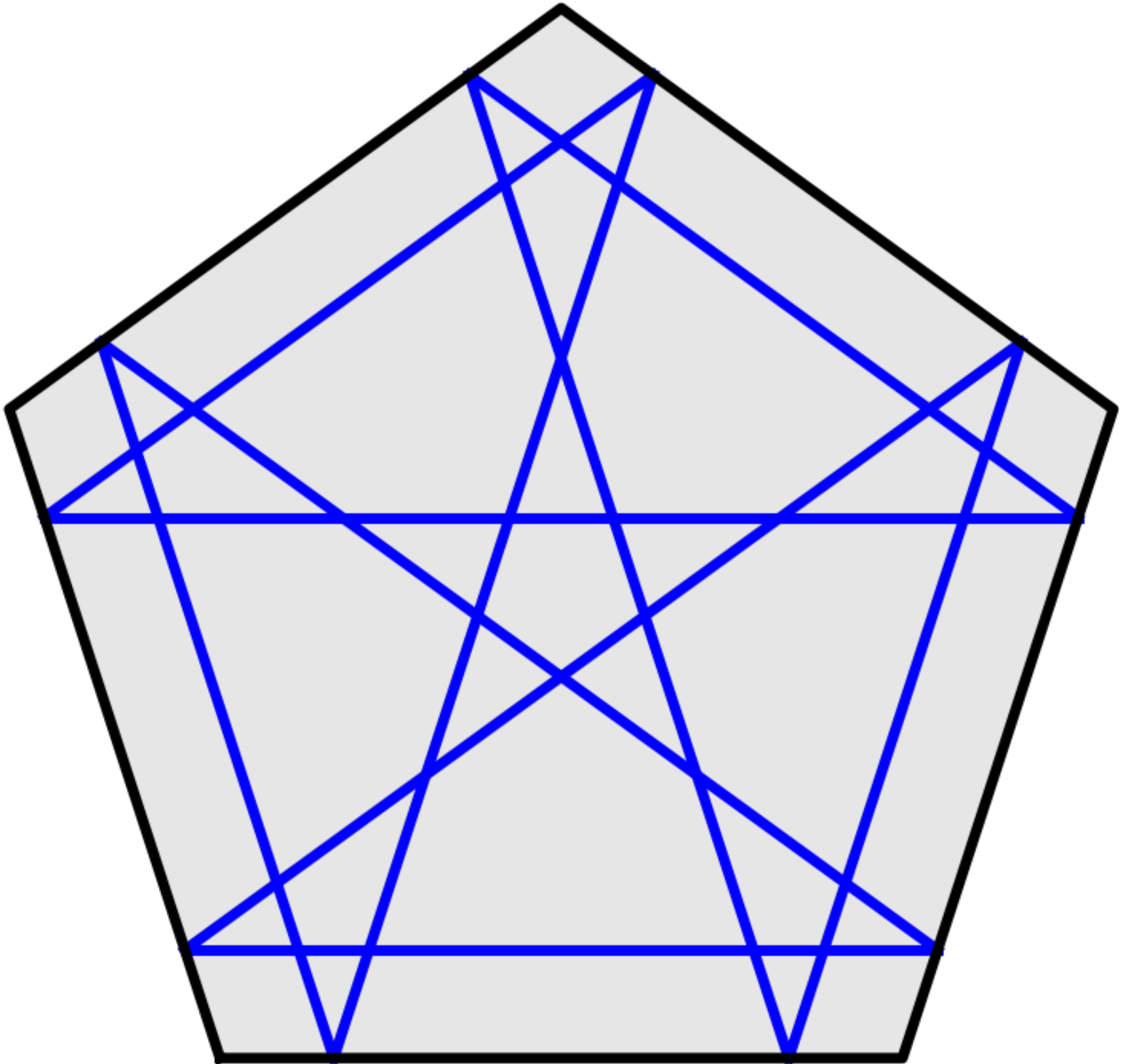


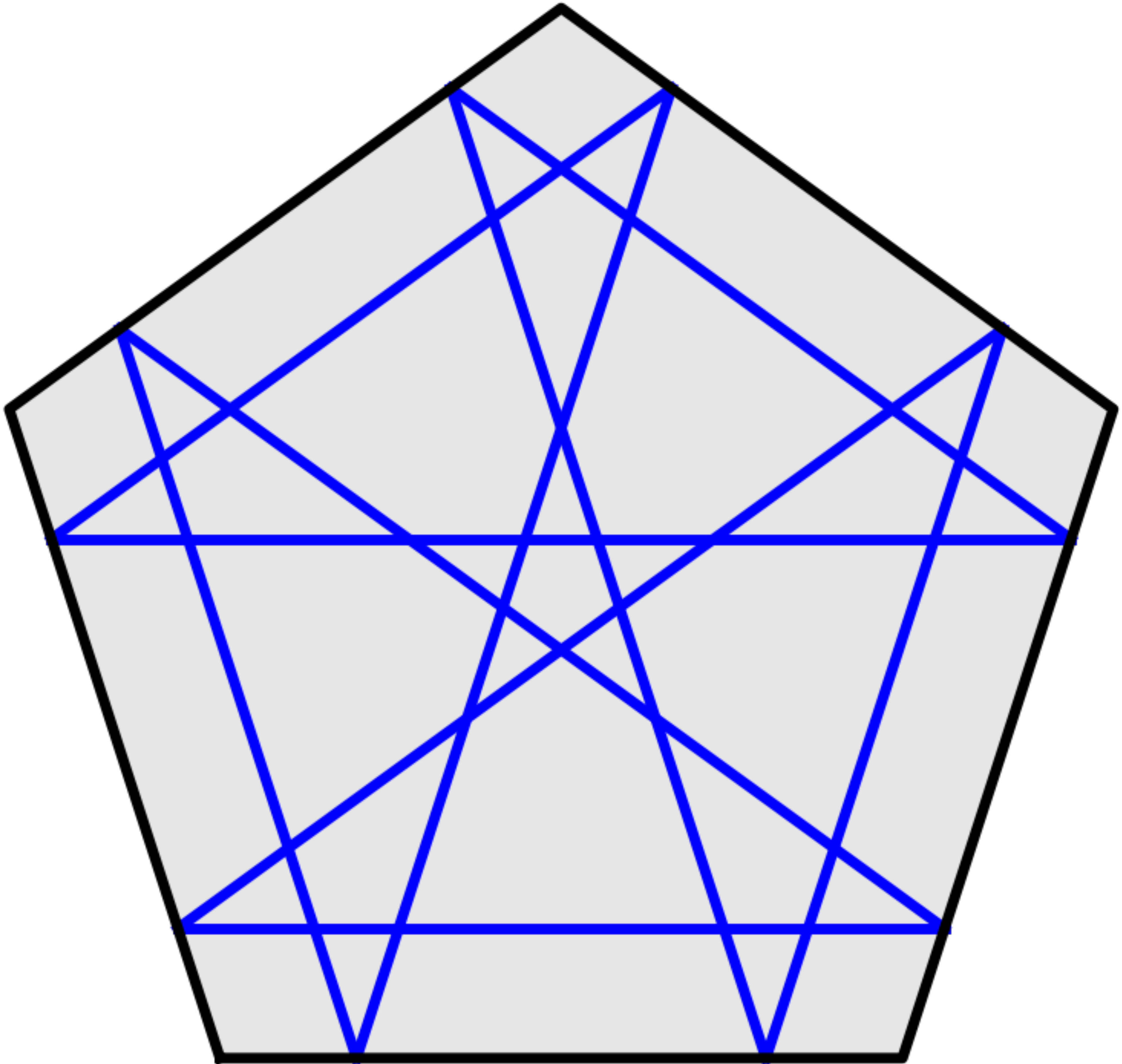


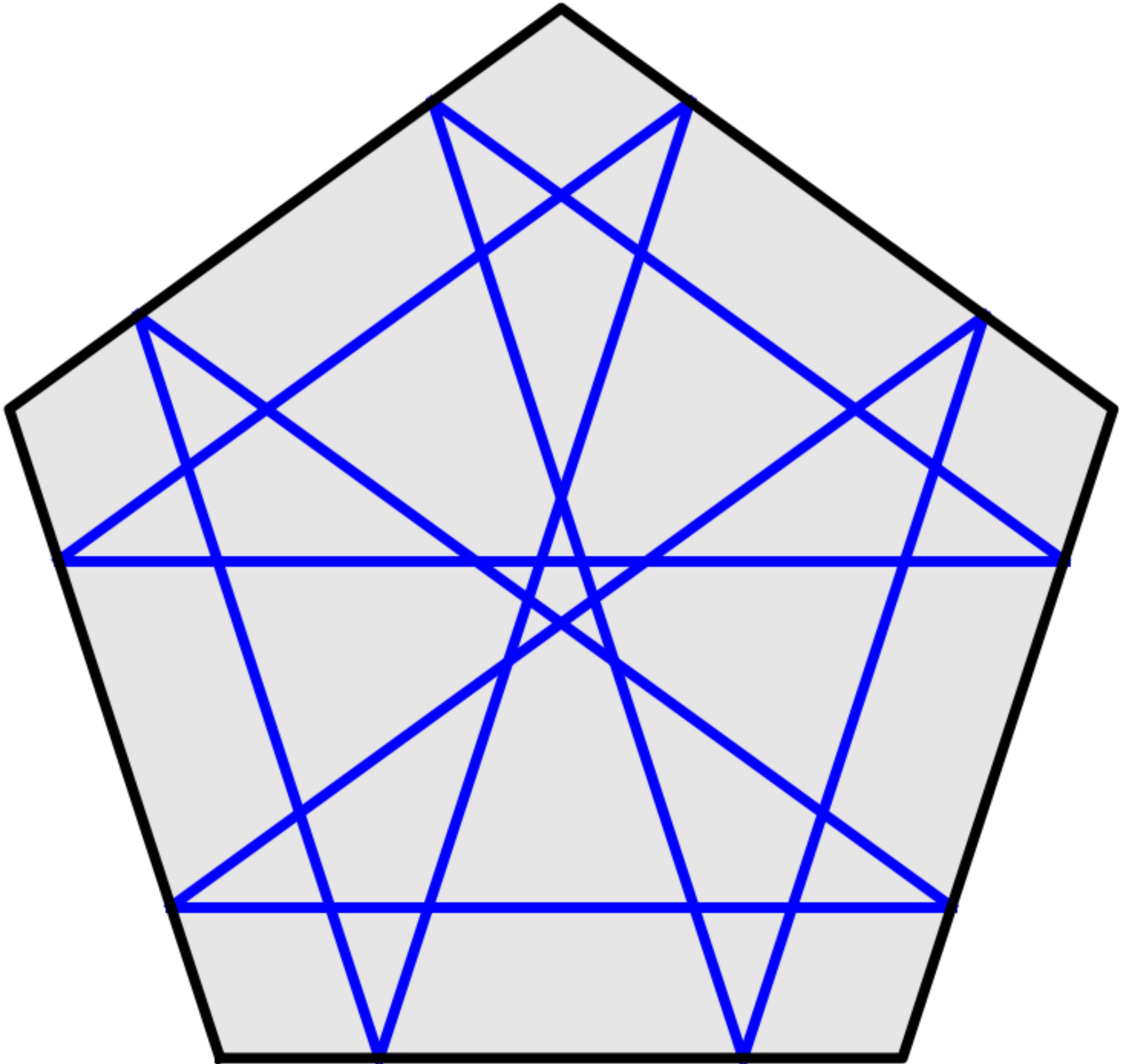


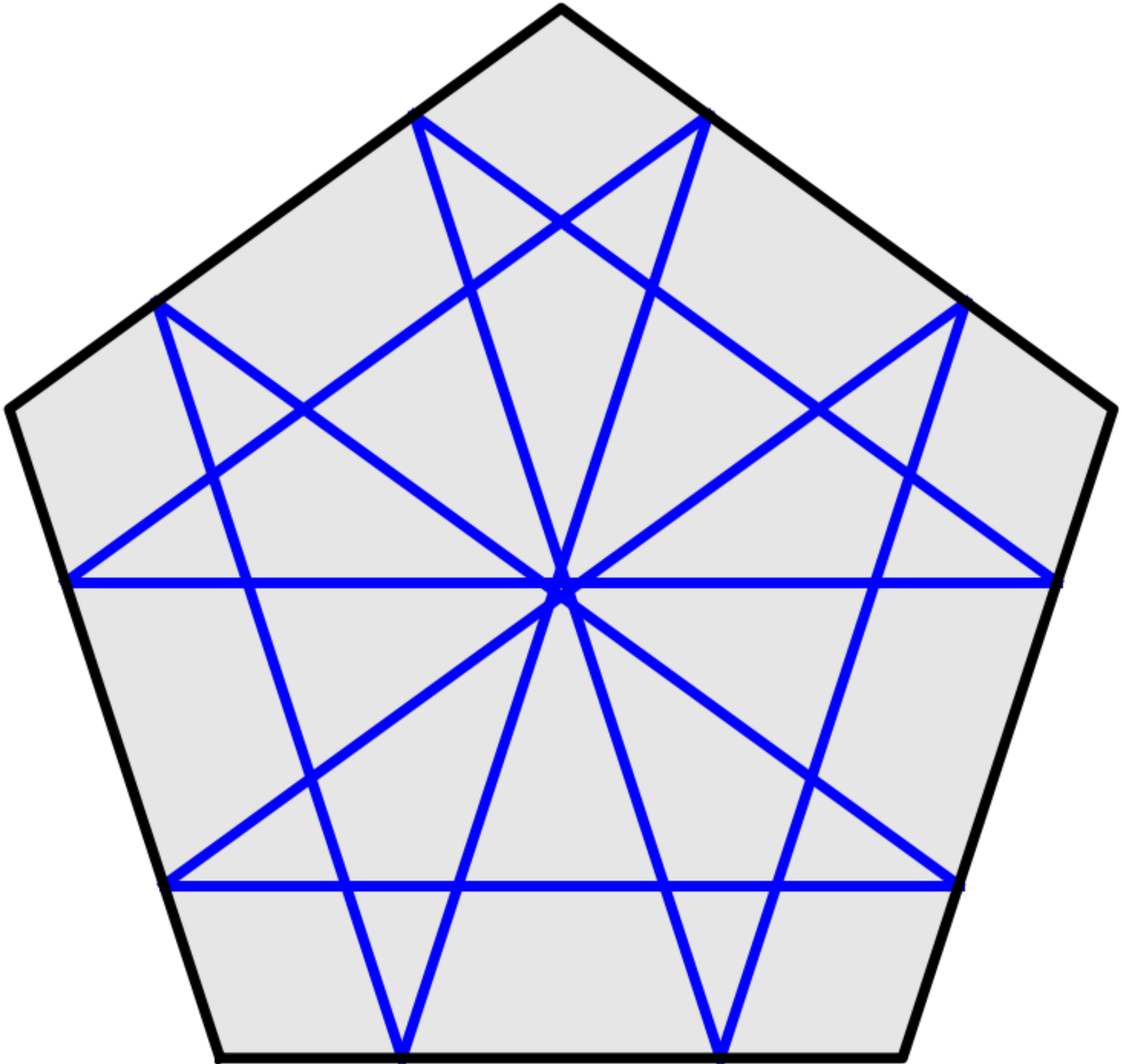


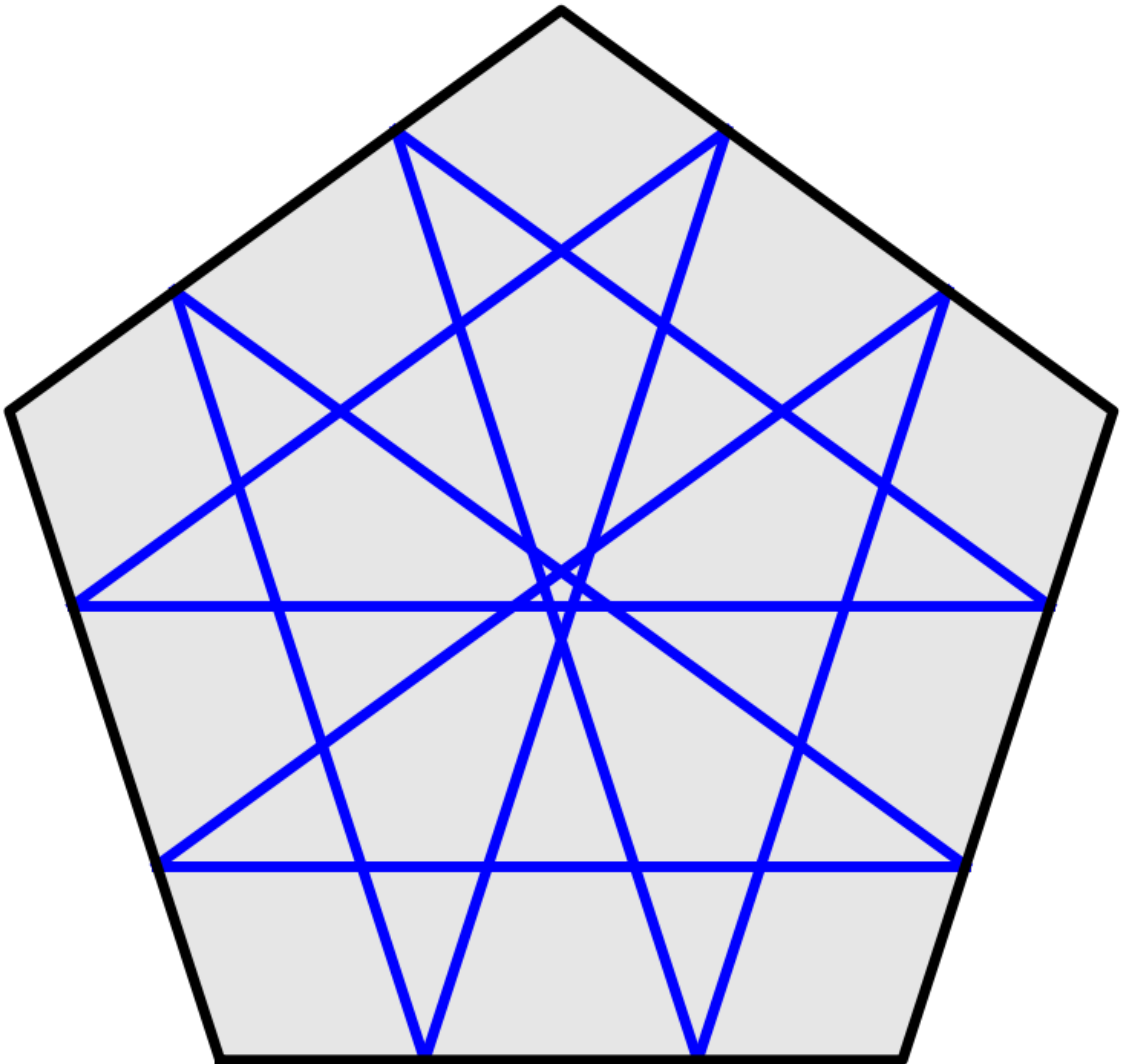


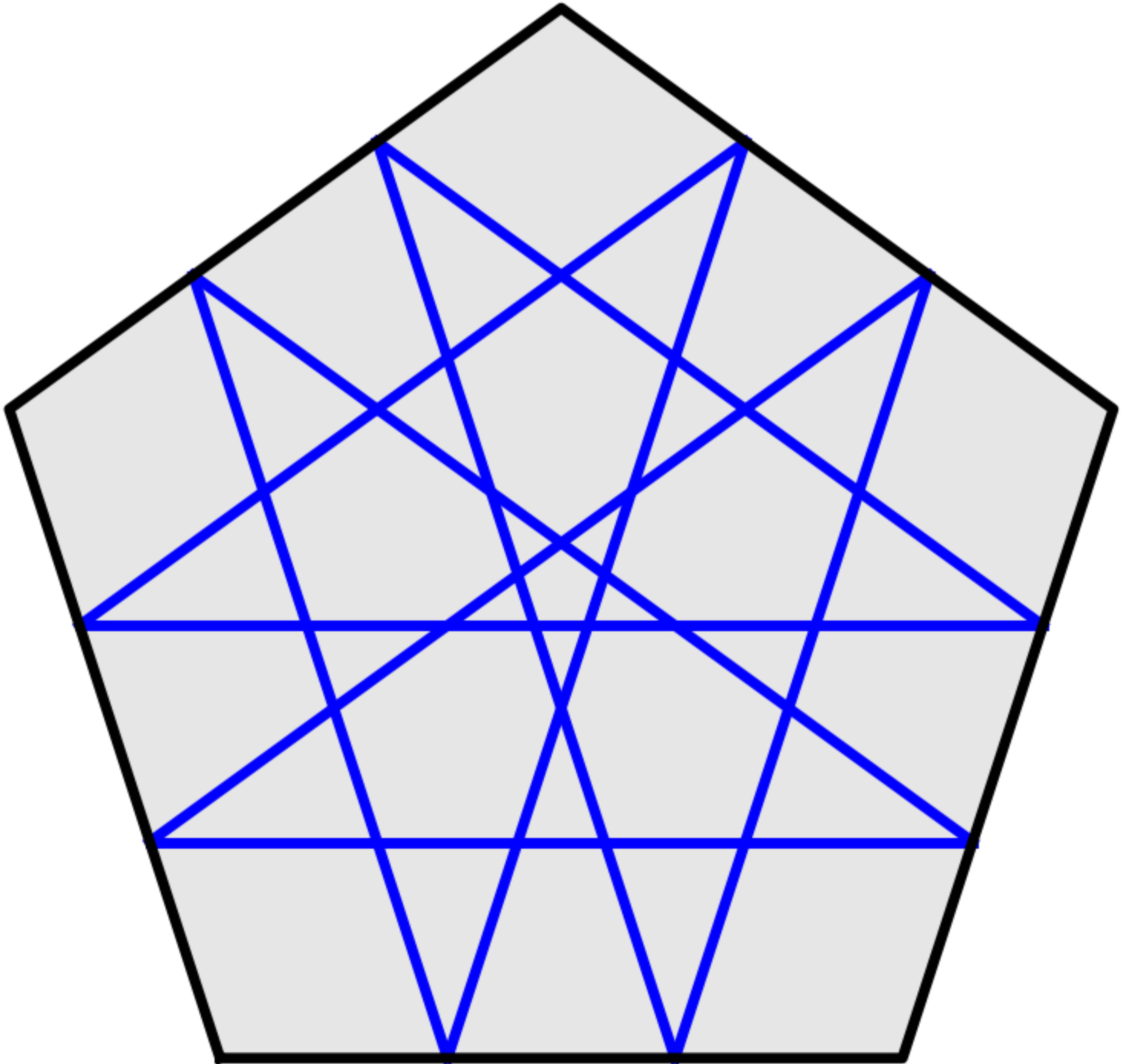


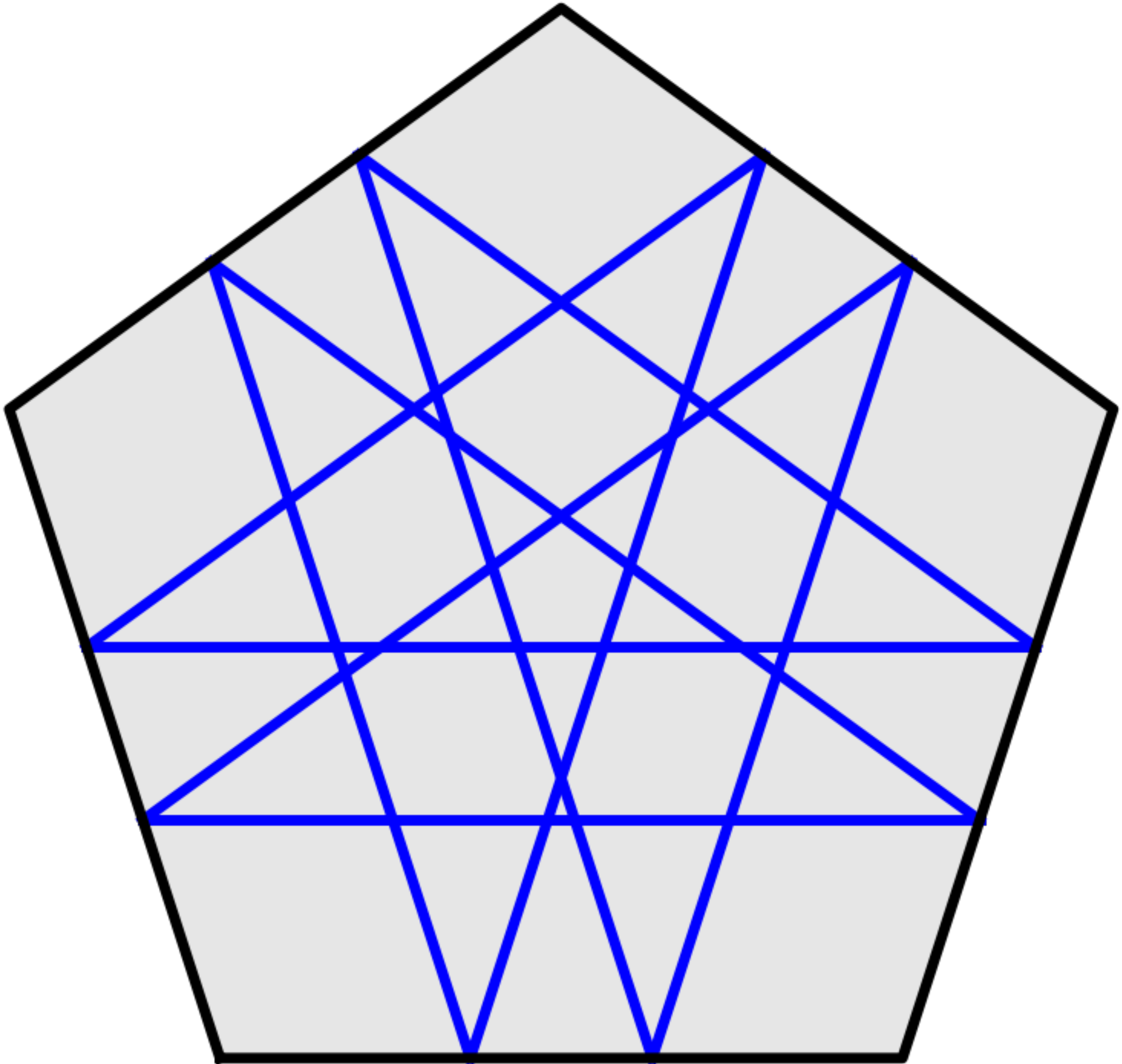


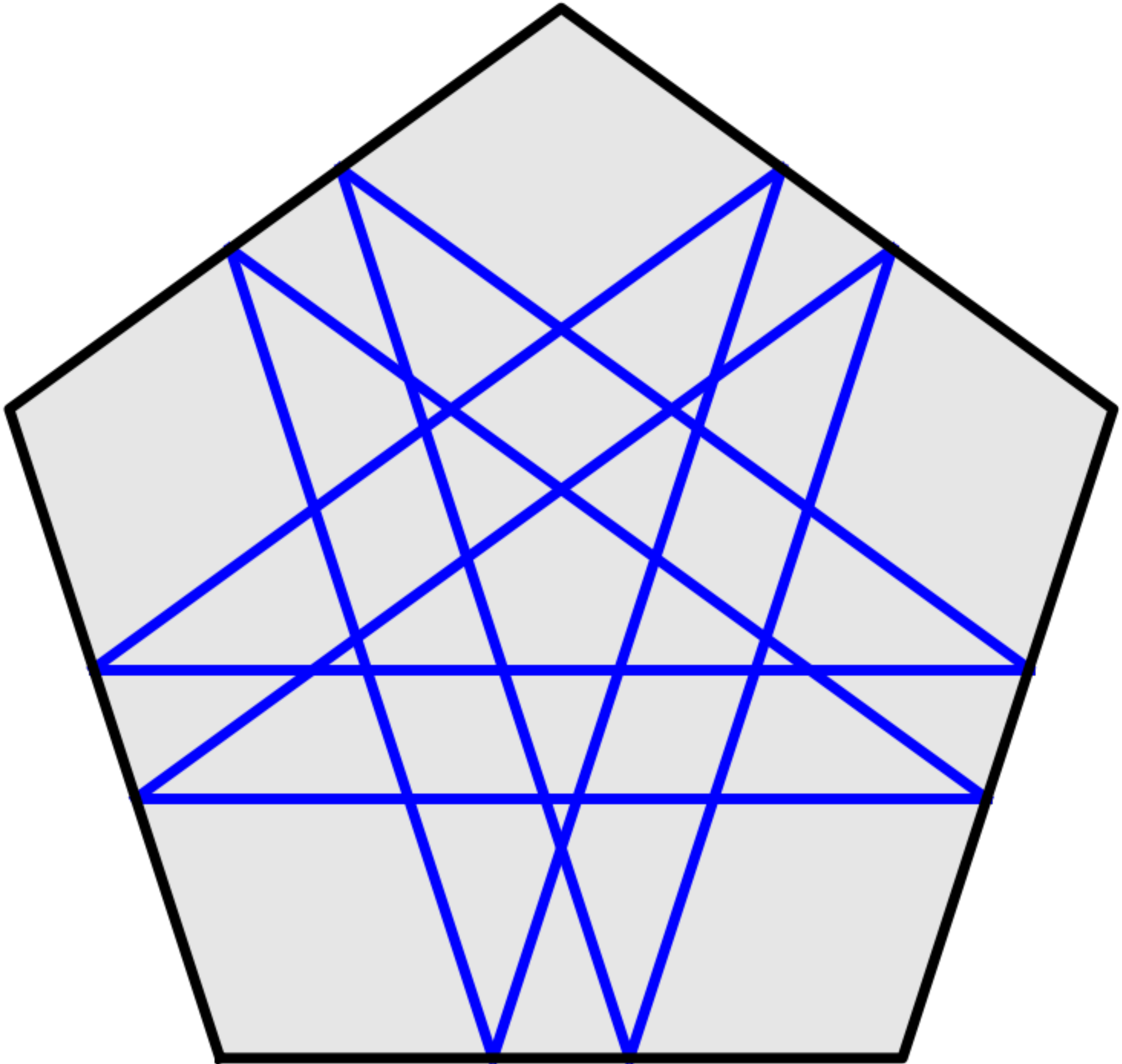


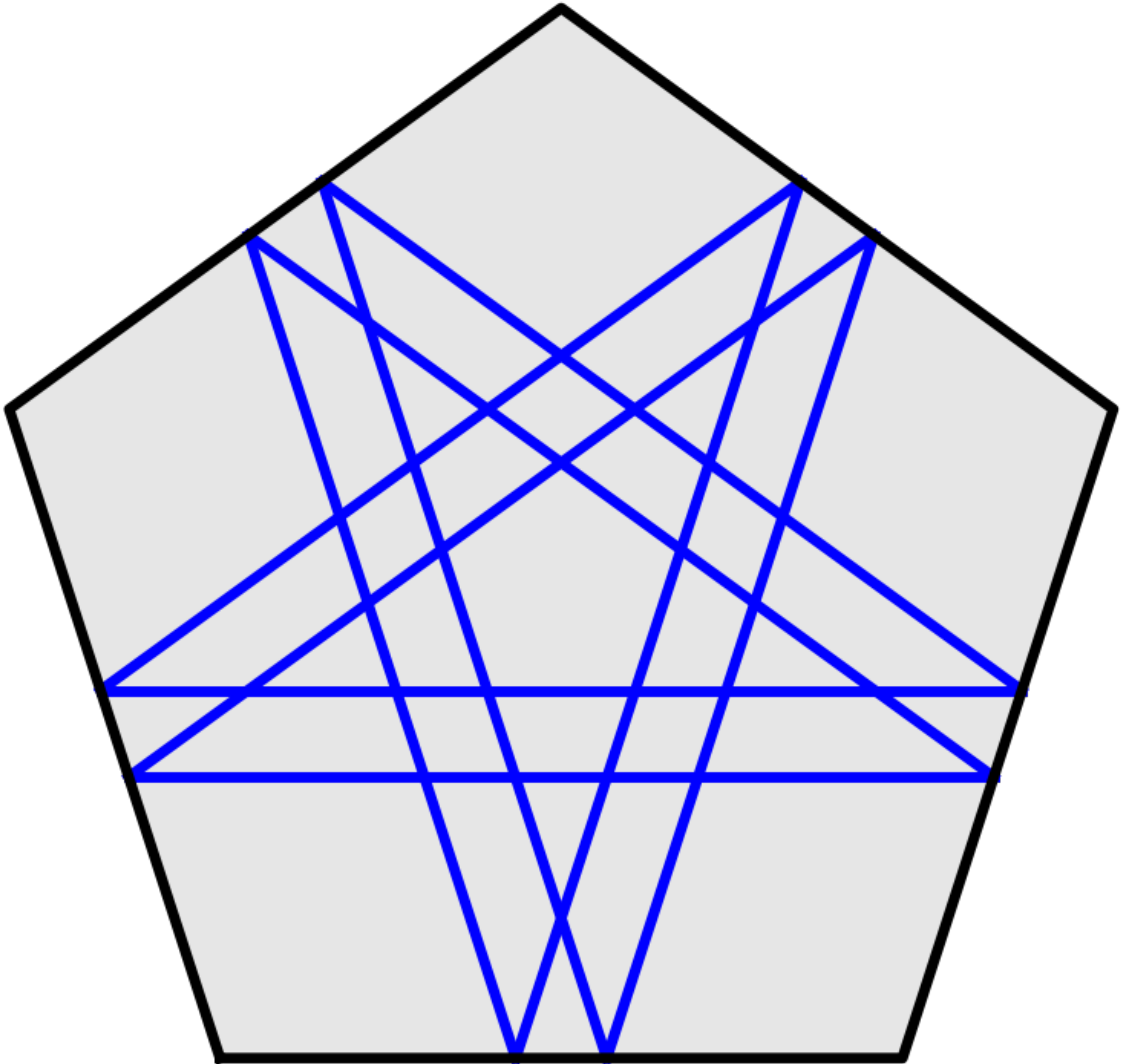


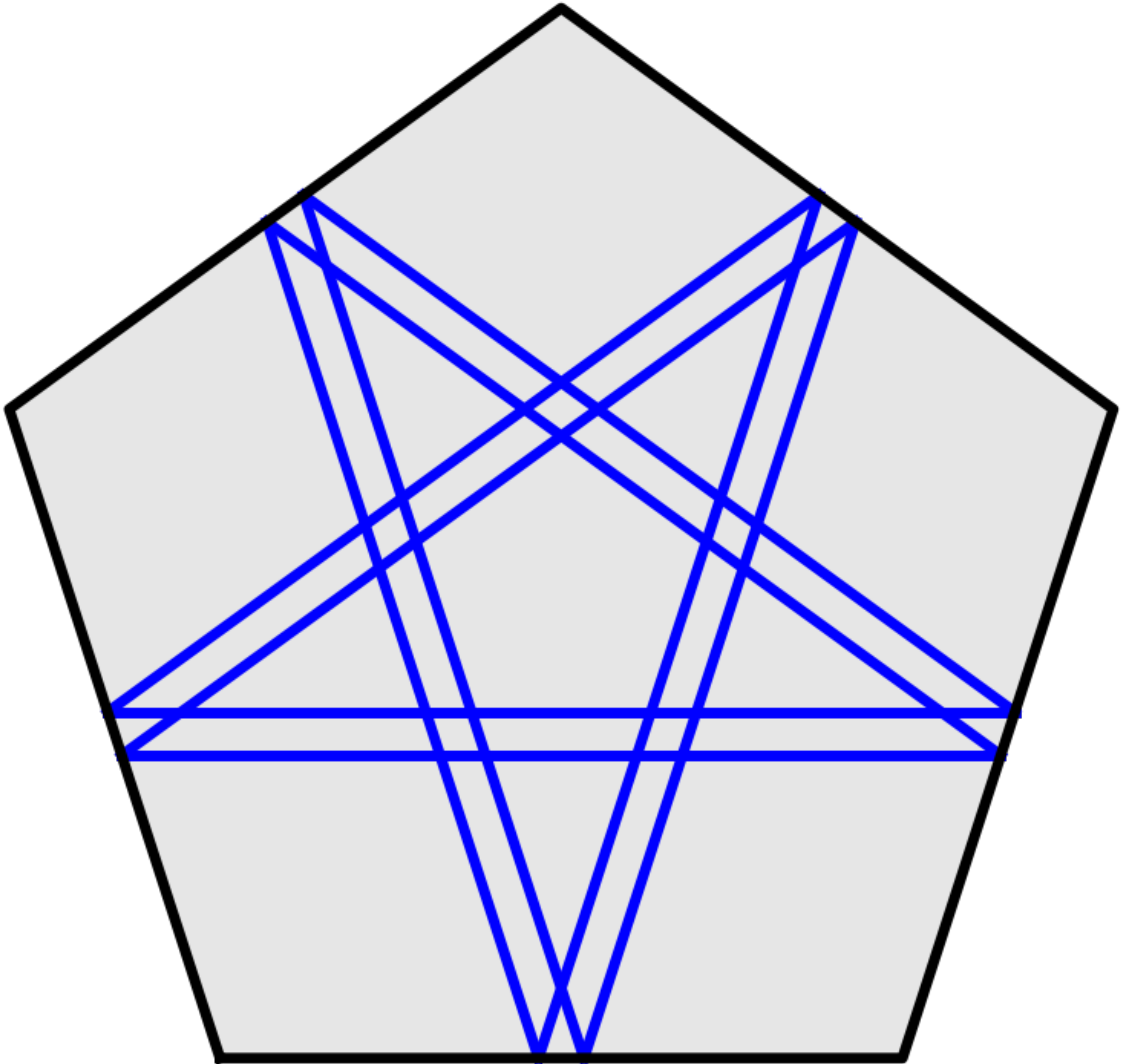


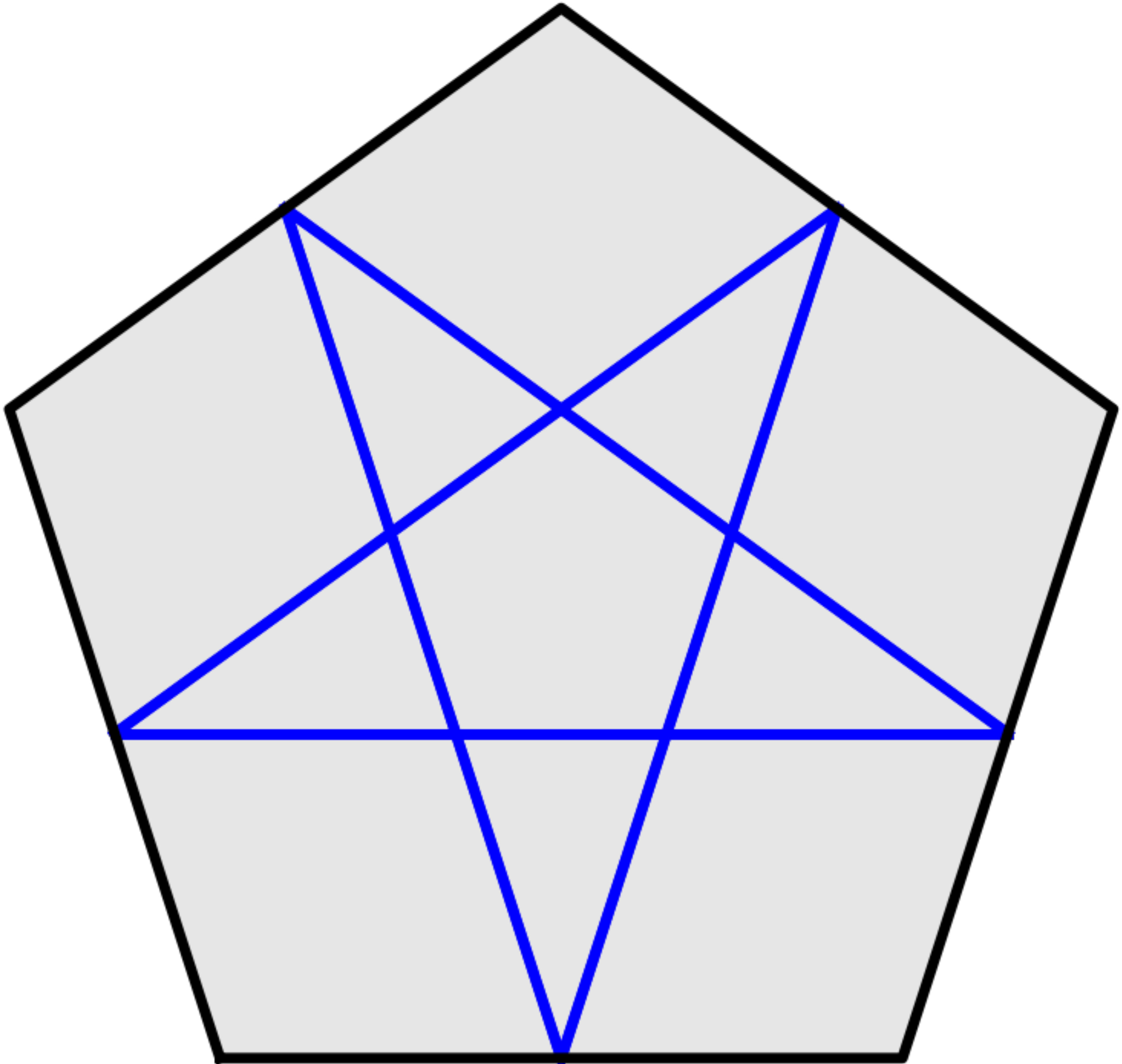




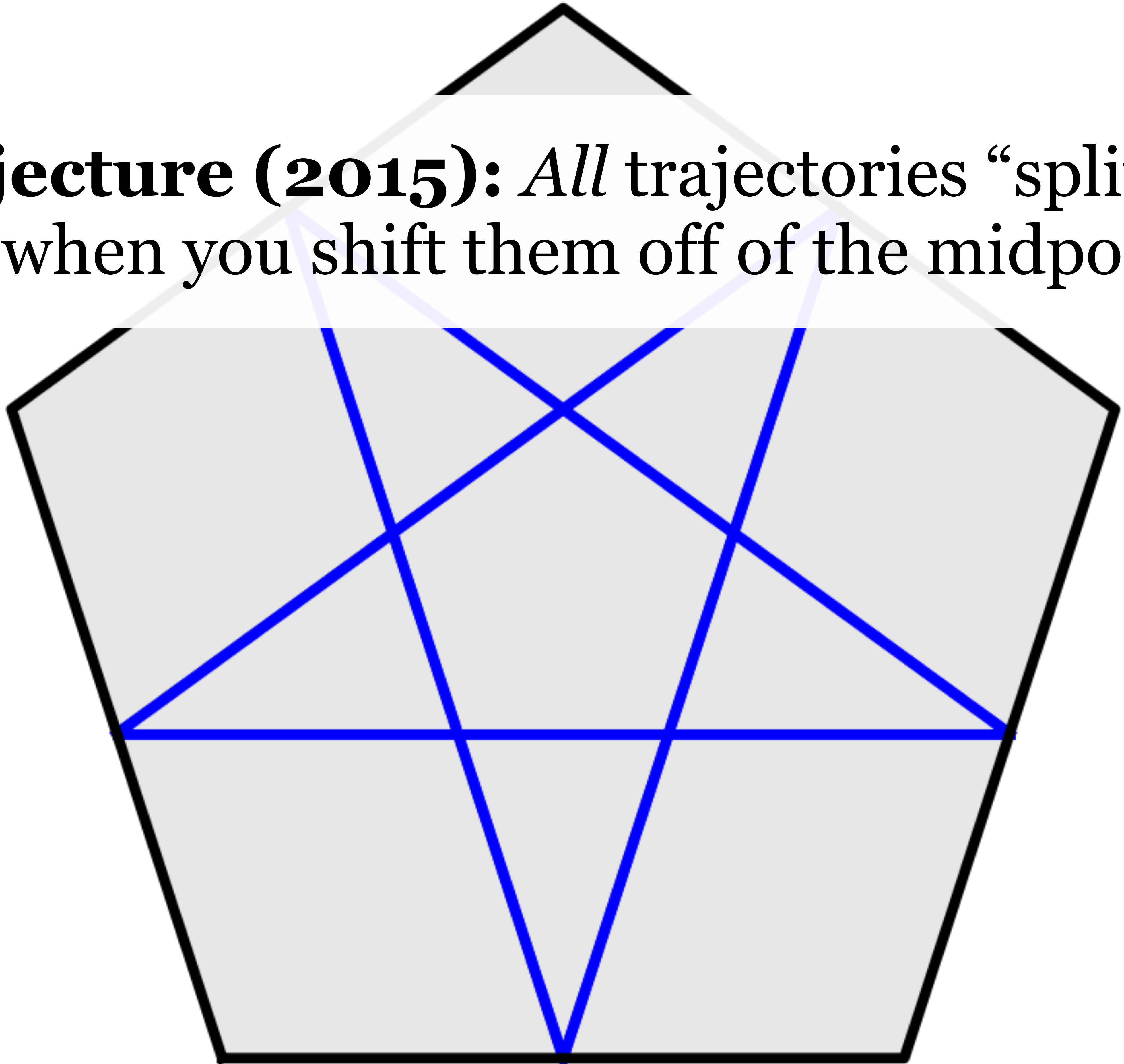


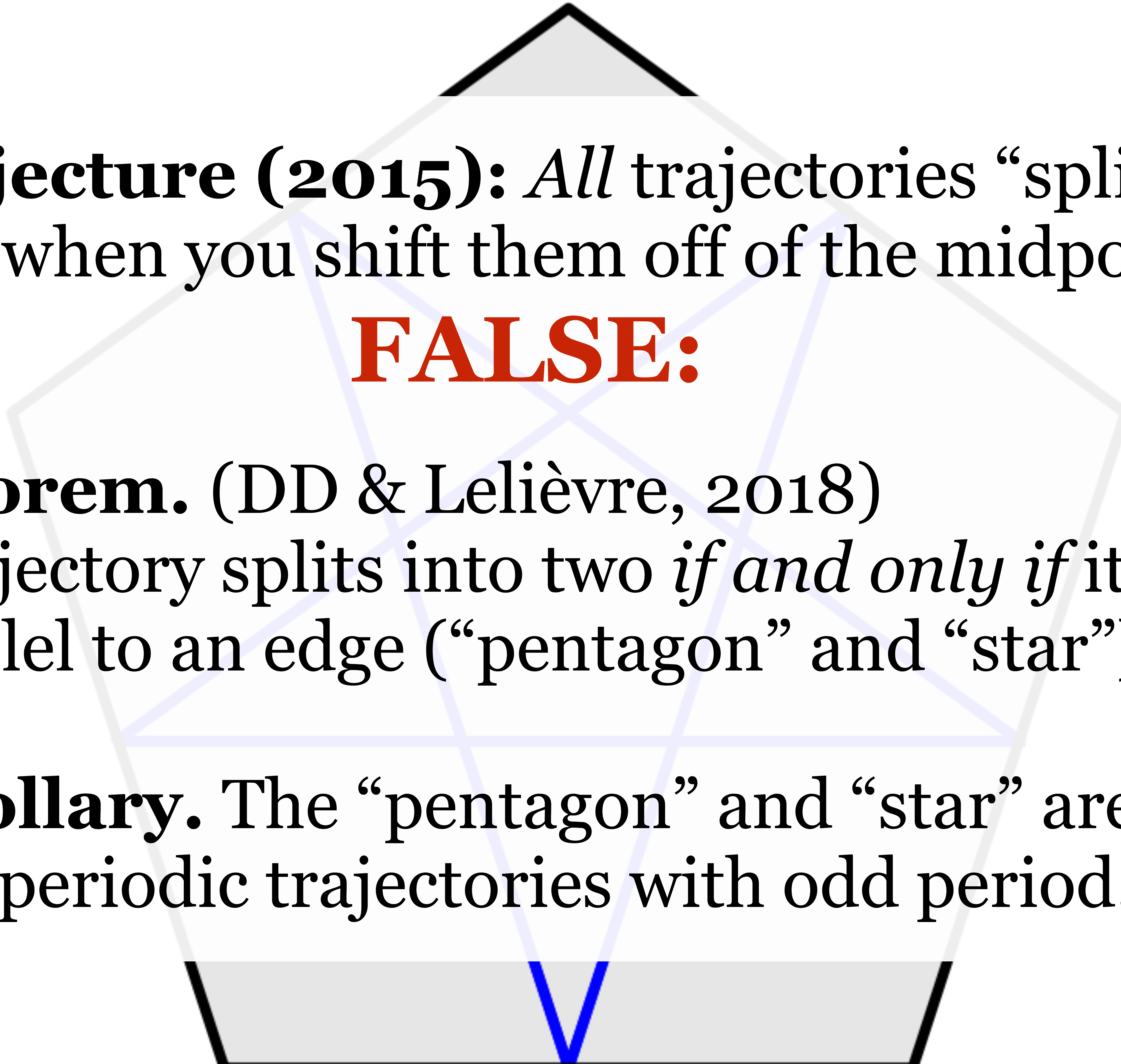






Conjecture (2015): *All* trajectories "split into two" when you shift them off of the midpoint.





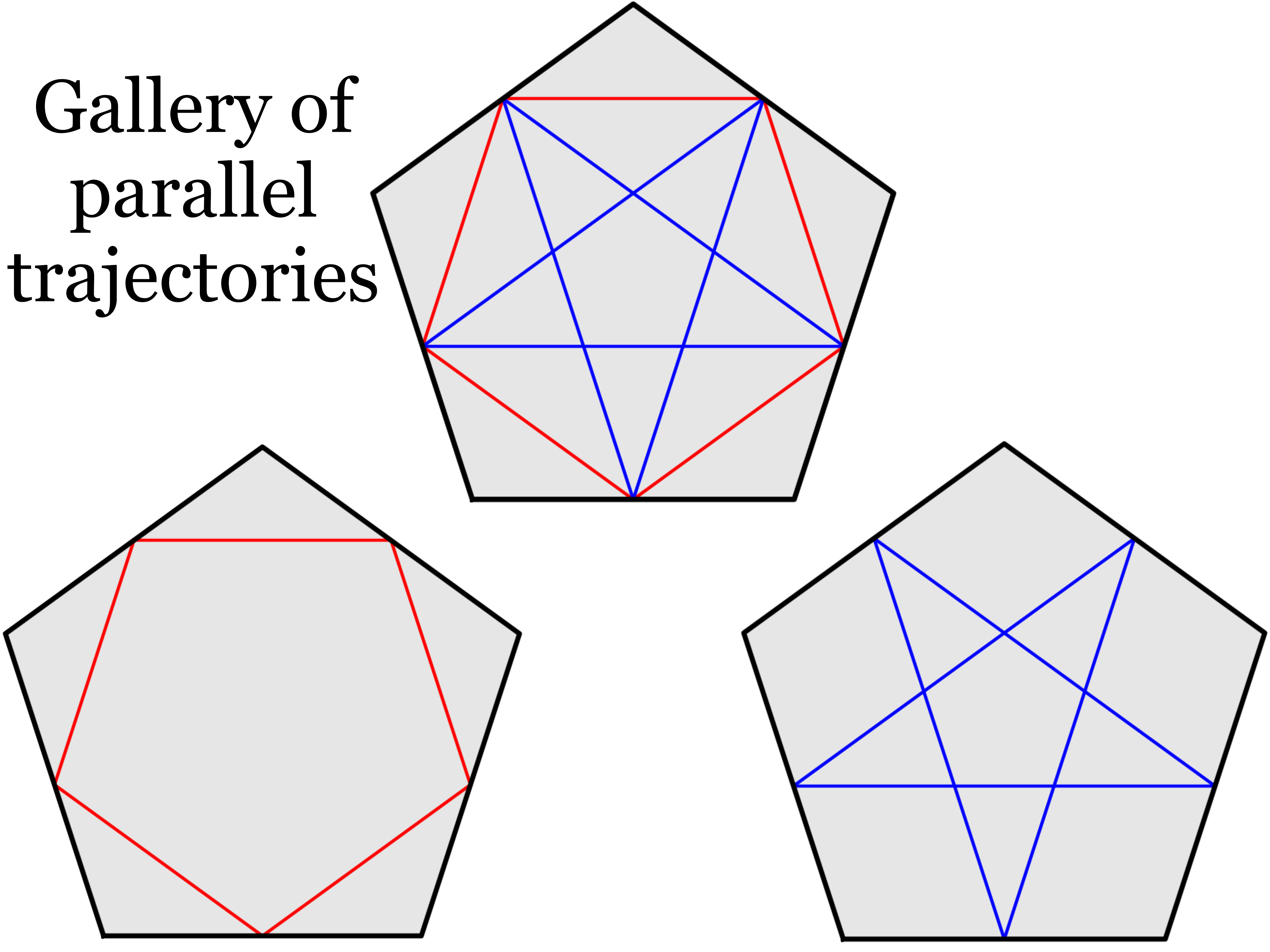
Conjecture (2015): *All* trajectories "split into two" when you shift them off of the midpoint.

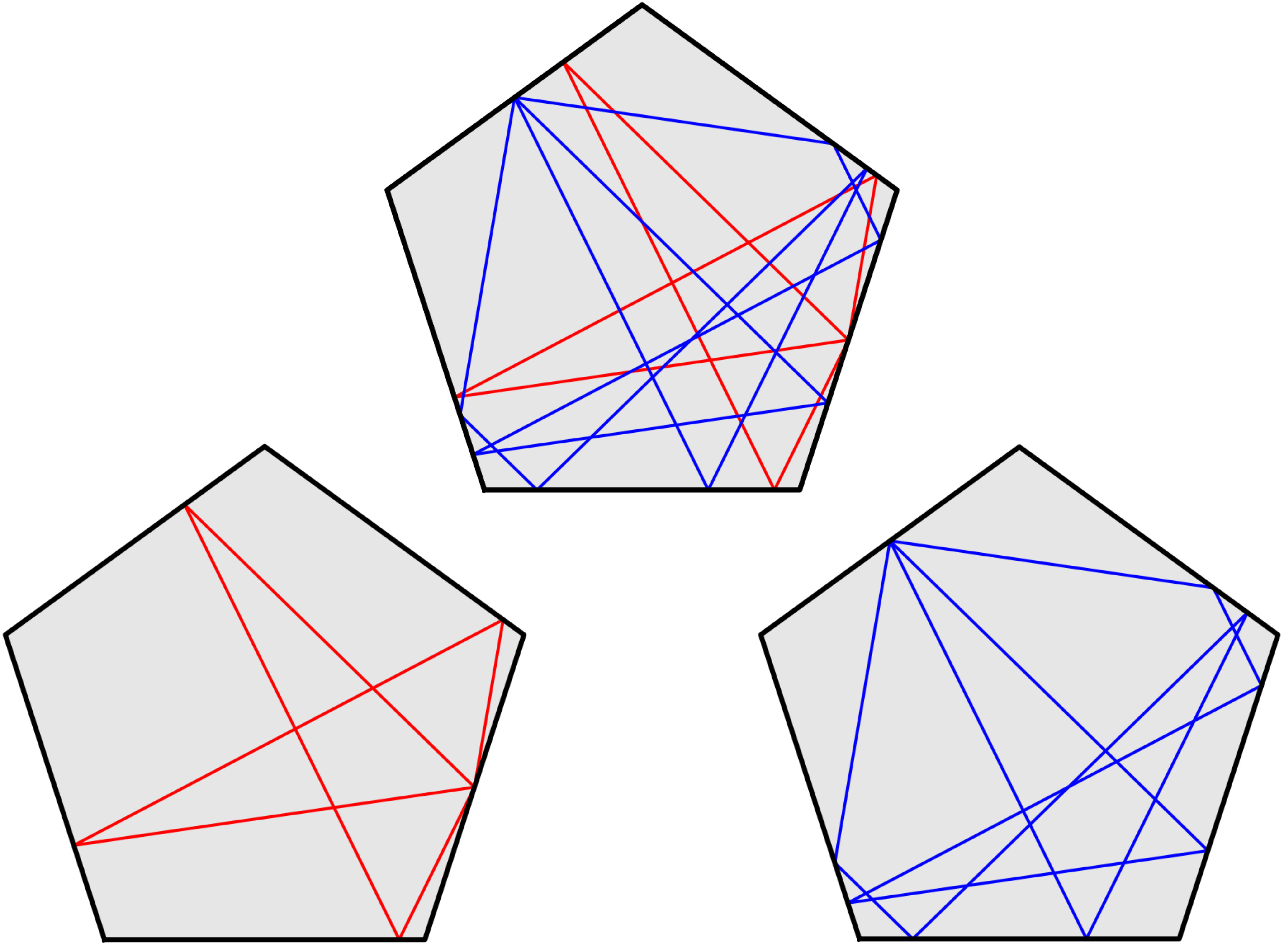
FALSE:

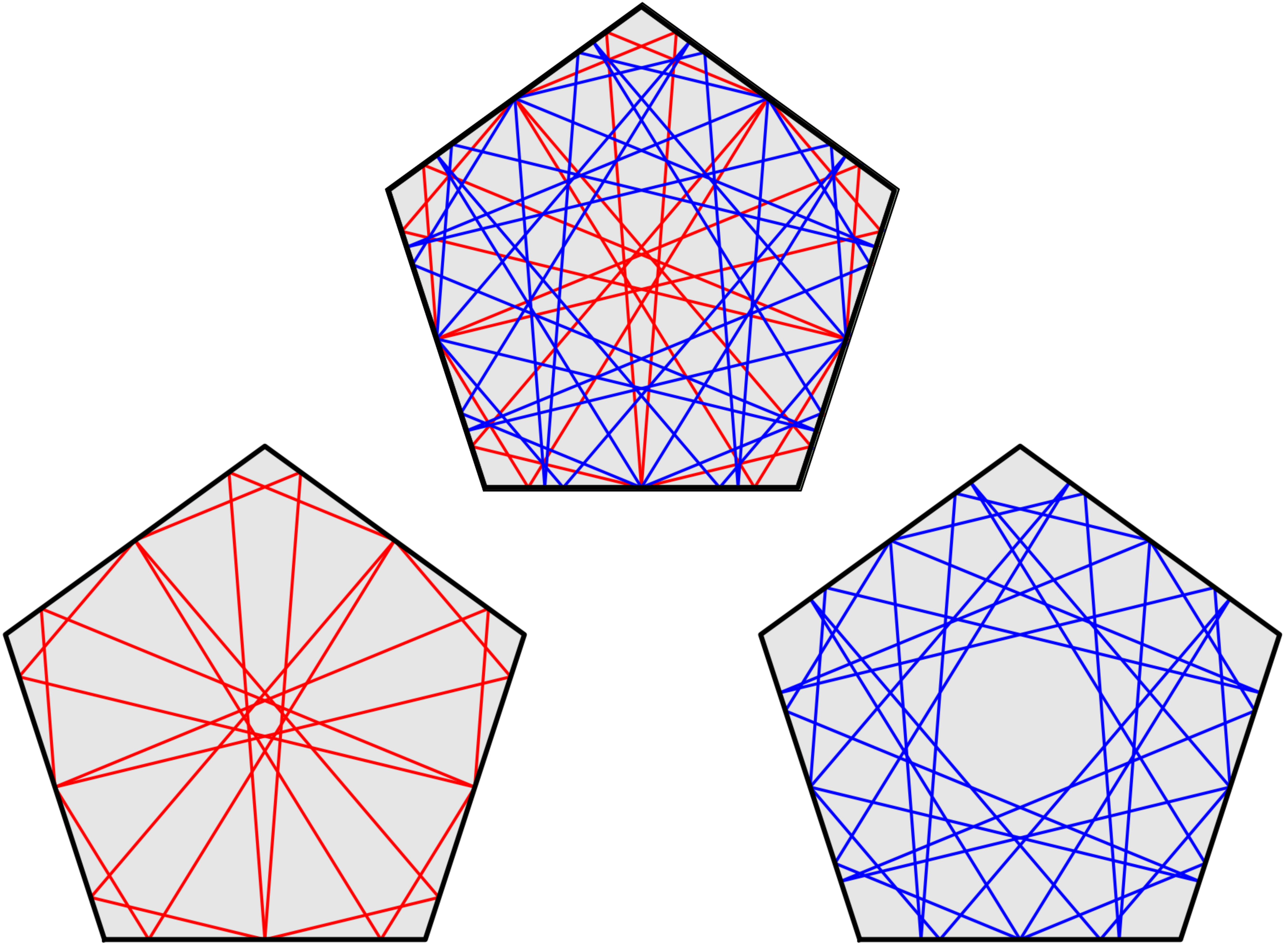
Theorem. (DD & Lelièvre, 2018)
A trajectory splits into two *if and only if* it is parallel to an edge ("pentagon" and "star").

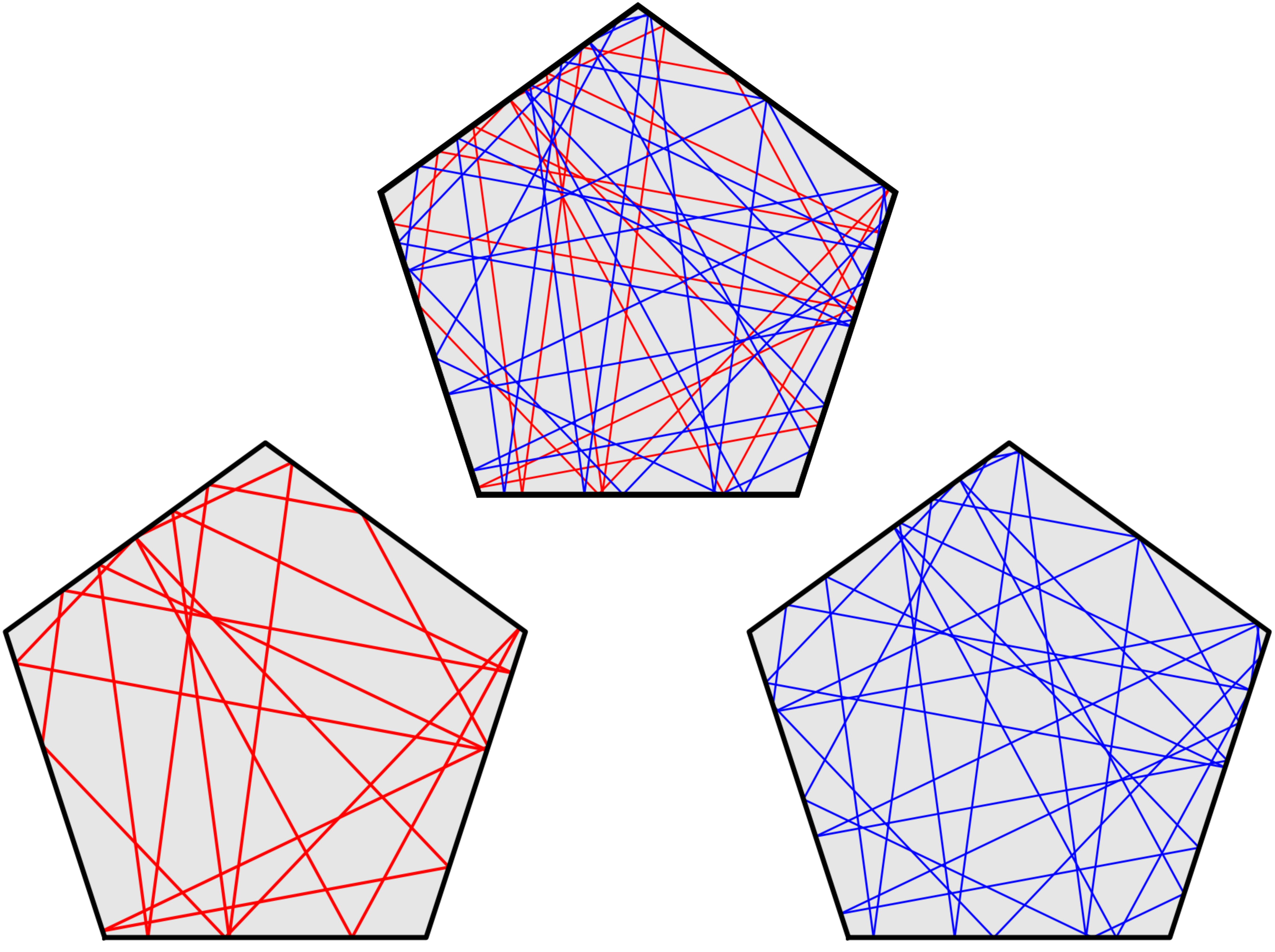
Corollary. The "pentagon" and "star" are the *only* periodic trajectories with odd period.

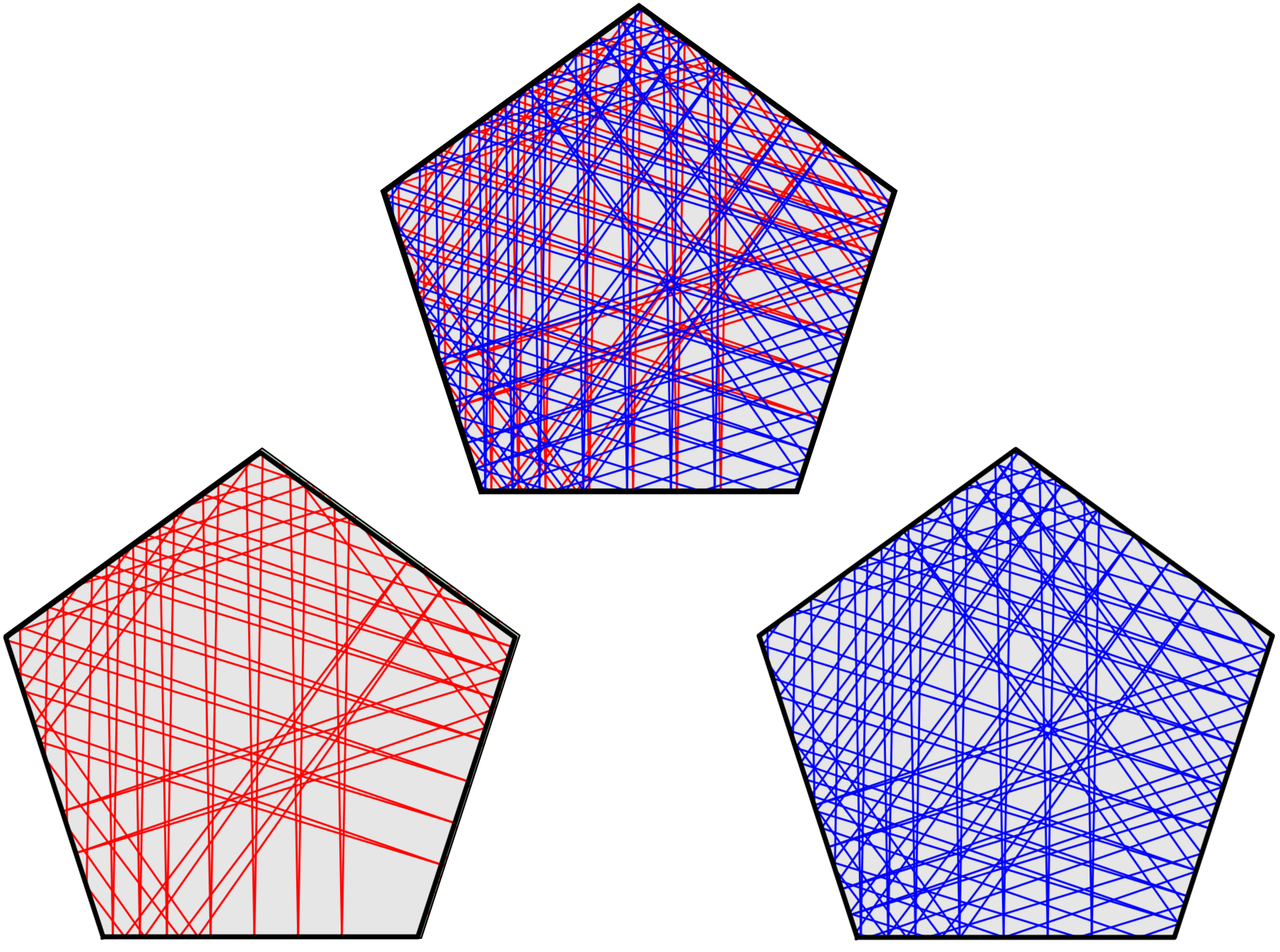
Gallery of parallel trajectories

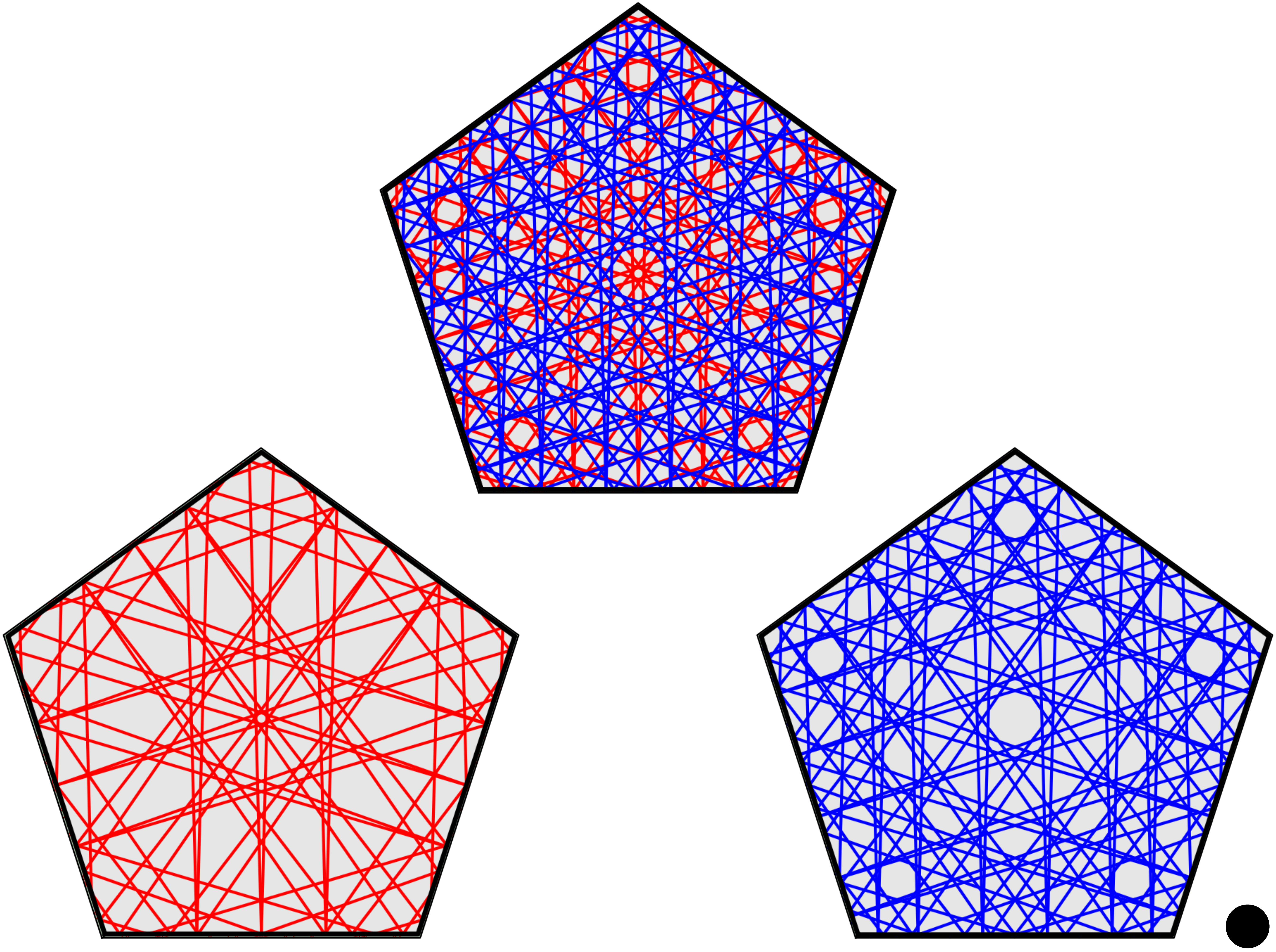








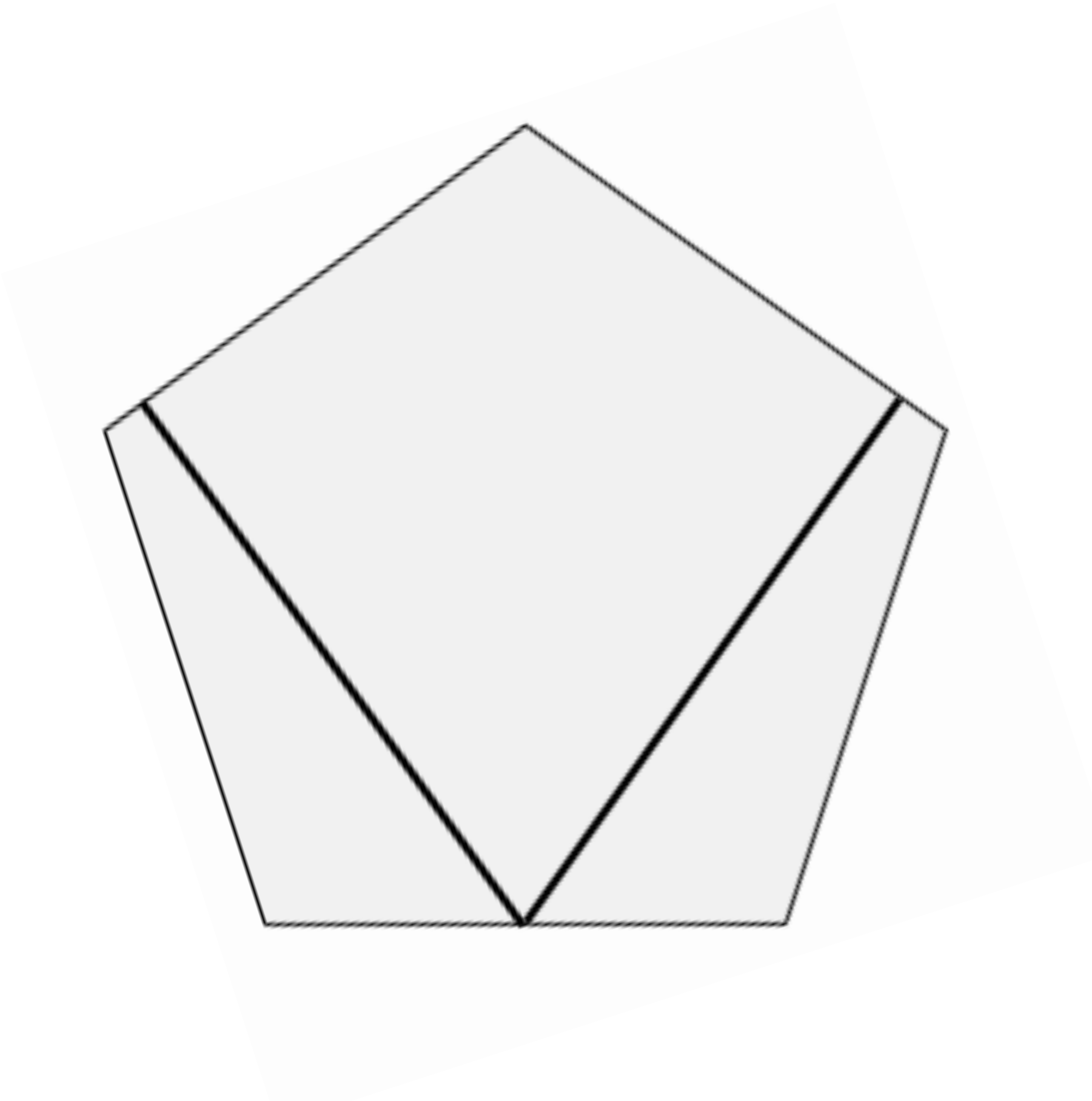


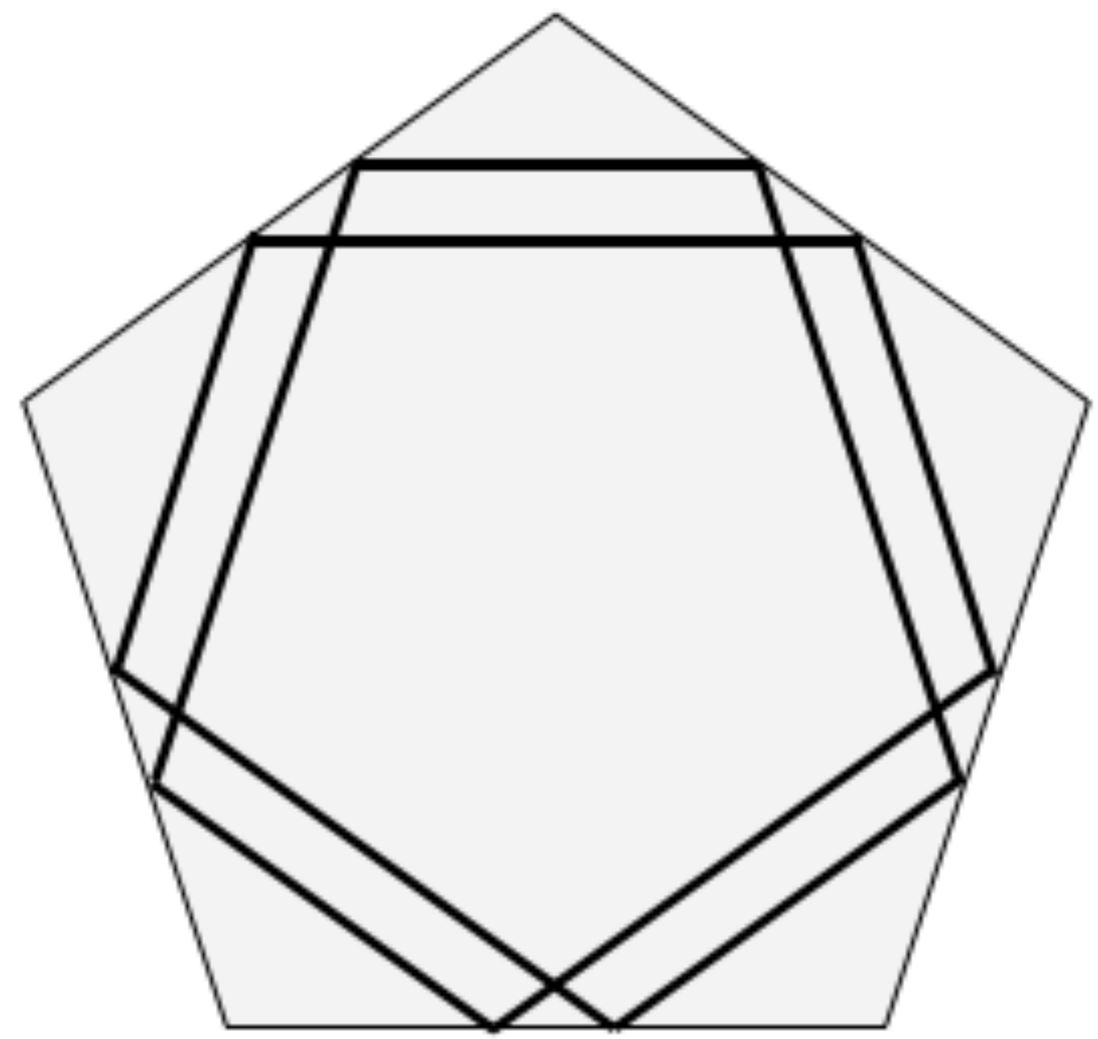


SYMMETRY

Two types of trajectories:

- Only reflection symmetry
- Rotation & reflection symmetry





Two types of trajectories:

- Only reflection symmetry
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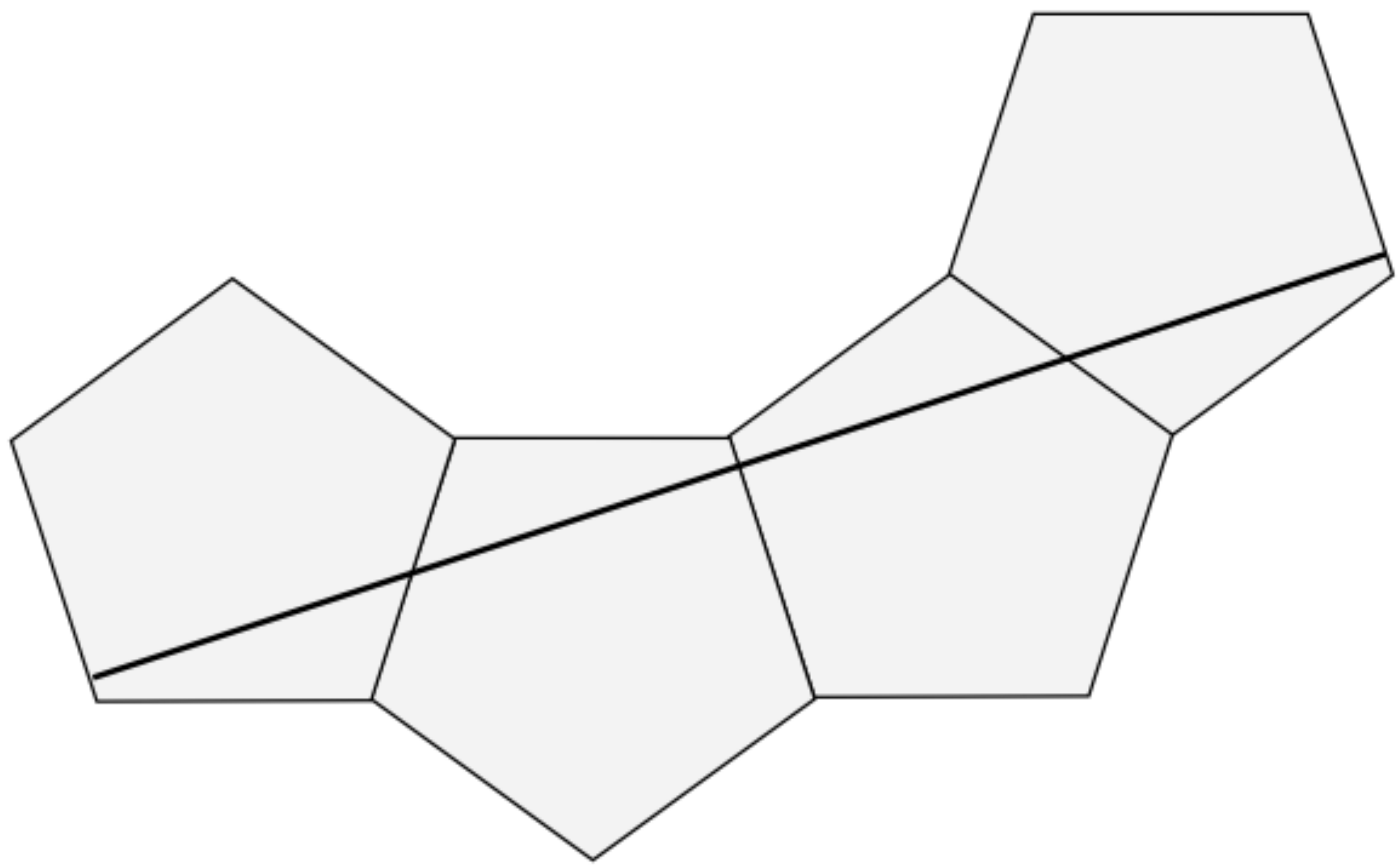
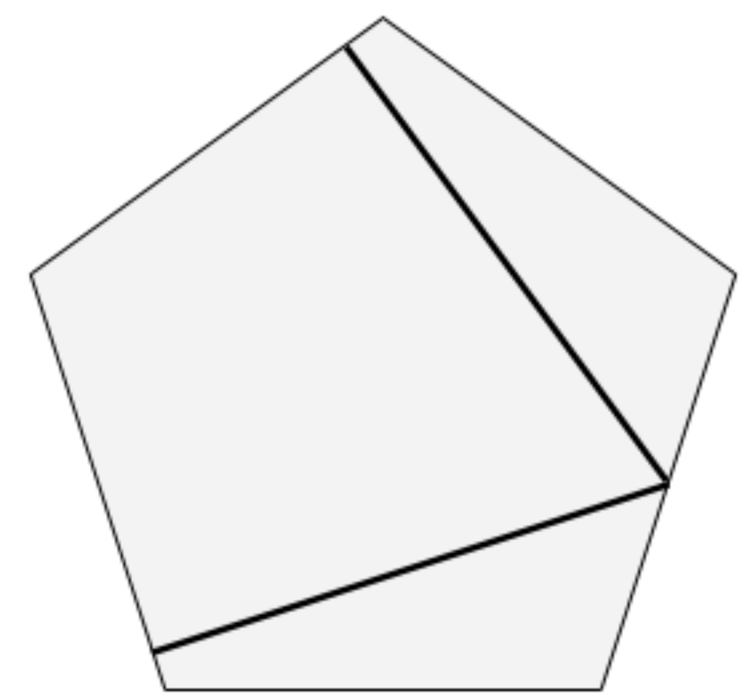


period 4 on
double pentagon



Two types of trajectories:

- Only reflection symmetry
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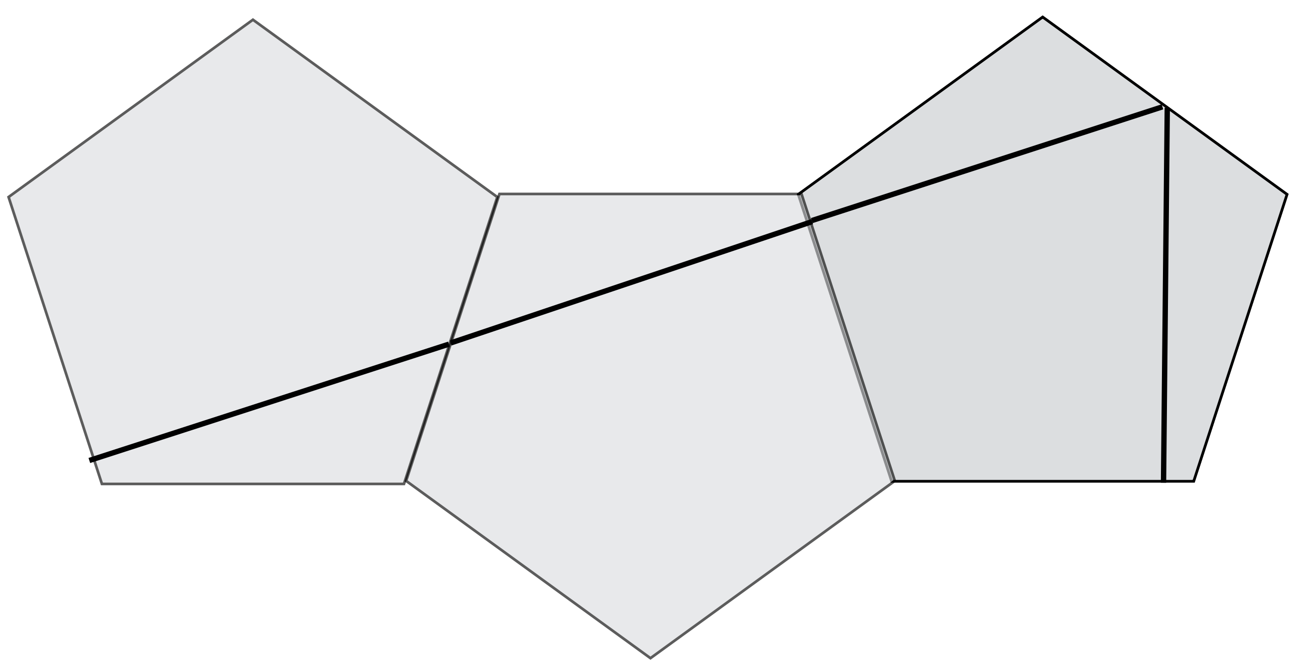
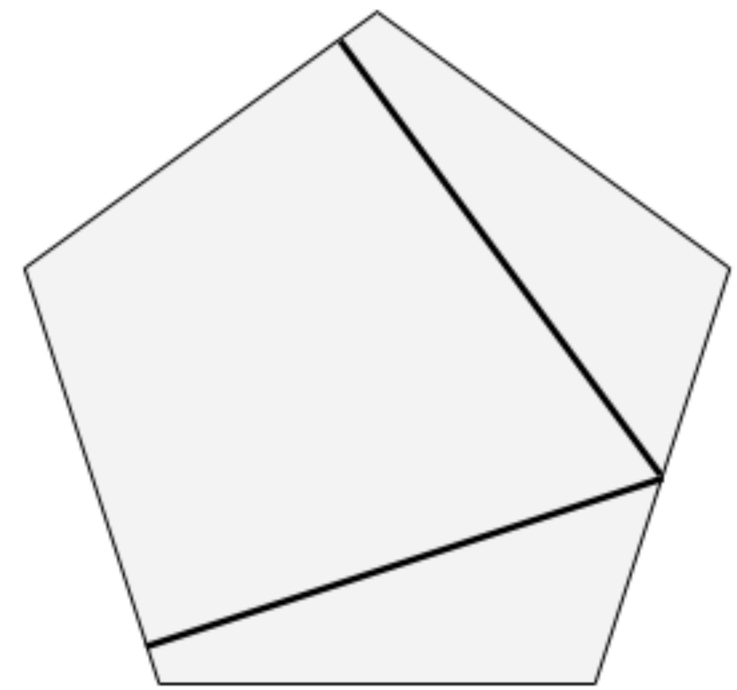


Fold each pentagon onto the previous one



Two types of trajectories:

- Only reflection symmetry
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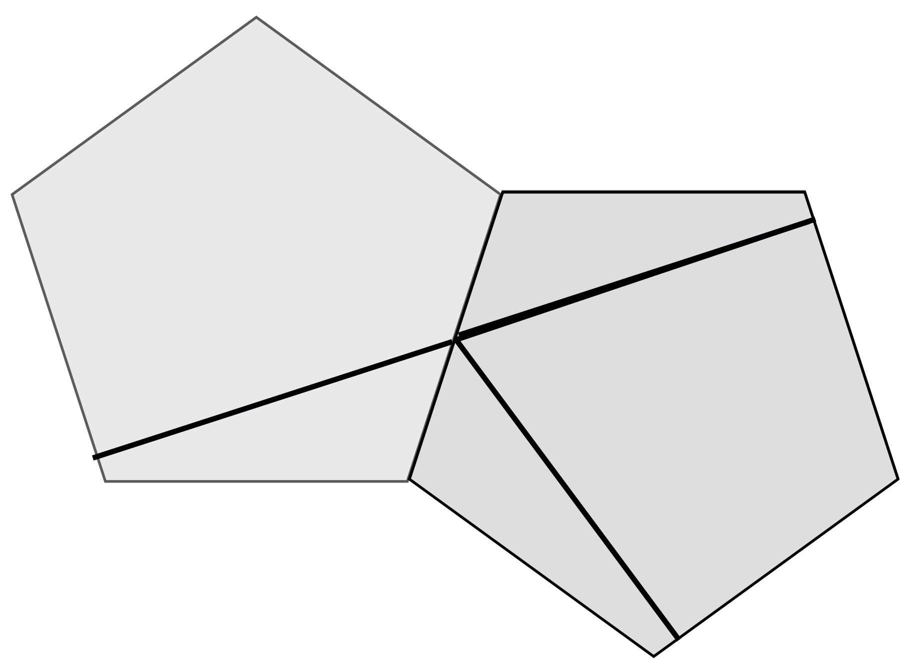
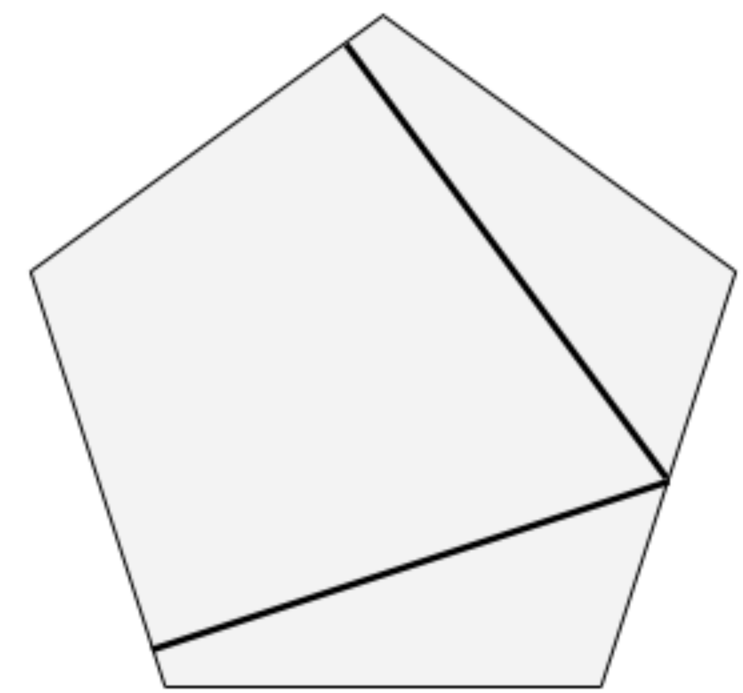


Fold each pentagon onto the previous one



Two types of trajectories:

- Only reflection symmetry
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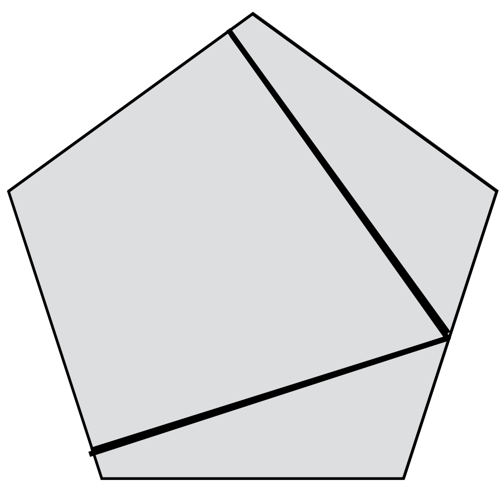
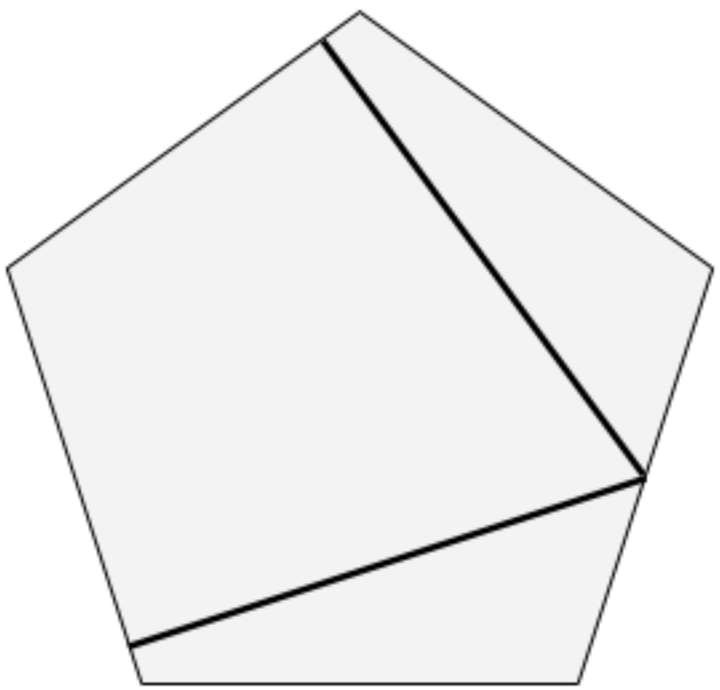




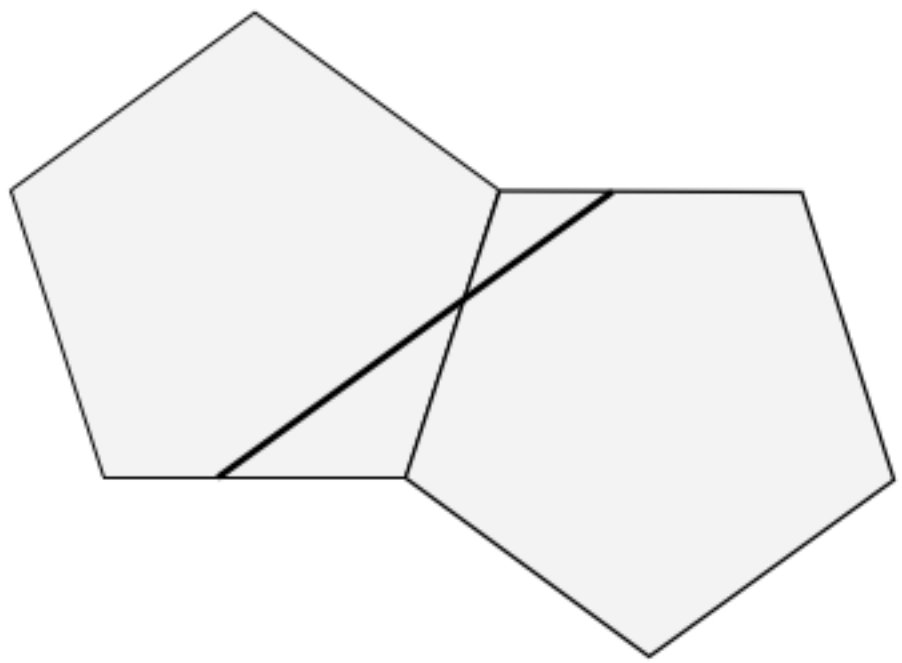
Fold each pentagon onto the previous one

Returns to where it started on the pentagon after folding

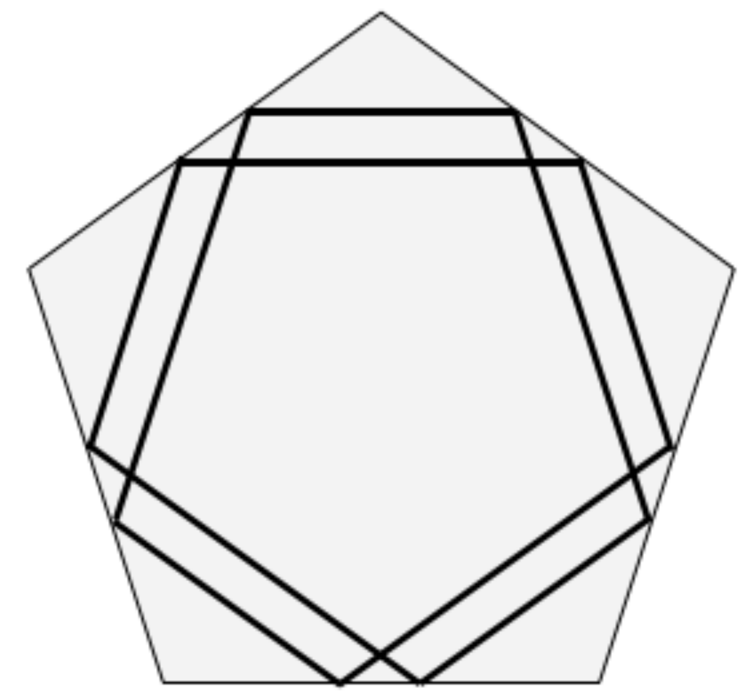
double pentagon period = billiard period

Two types of trajectories:

- Only reflection symmetry
- Rotation & reflection symmetry



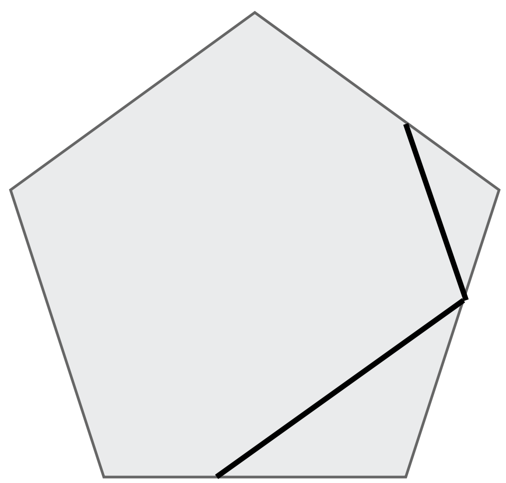
period 2 on
double pentagon



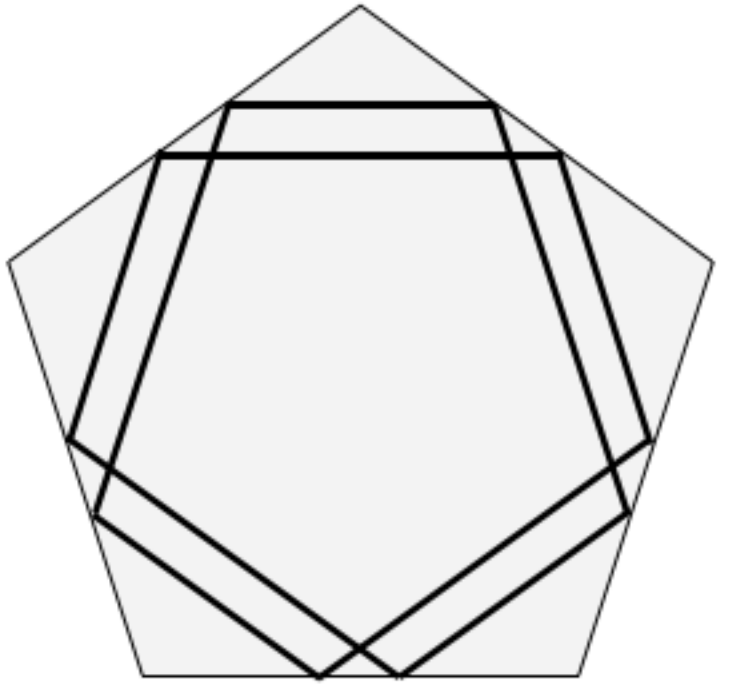
Repeat 5 times
(rotate) to get
billiard path.

Two types of trajectories:

- Only reflection symmetry
- Rotation & reflection symmetry



Isn't back to
where it started
on the pentagon
after folding



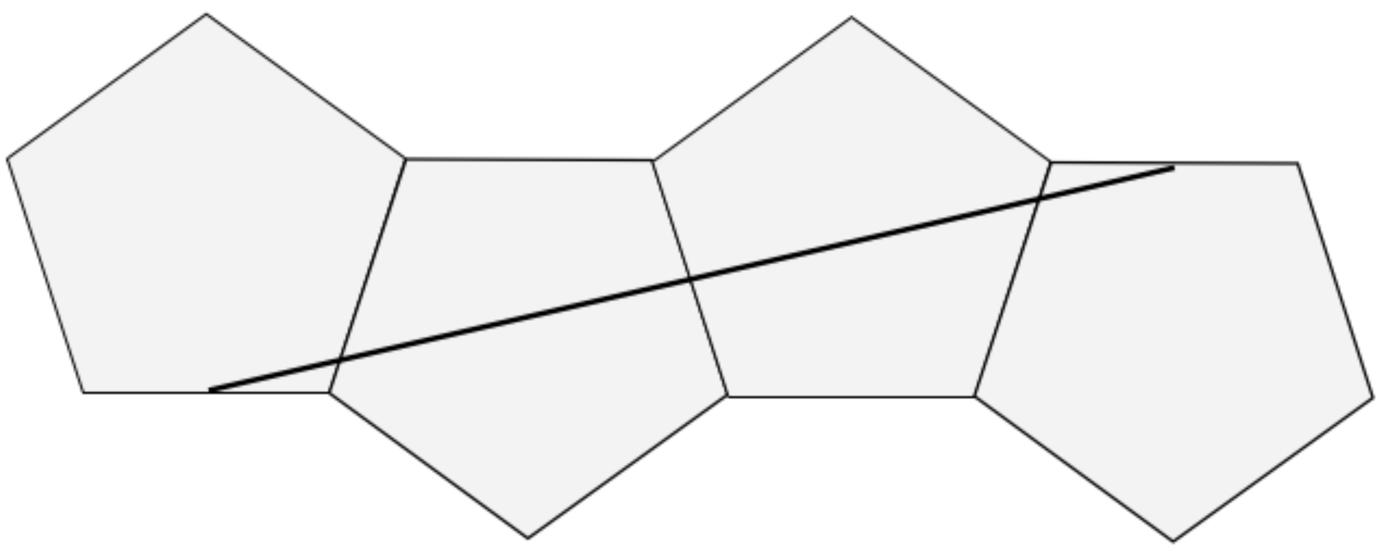
period 2 on
double pentagon

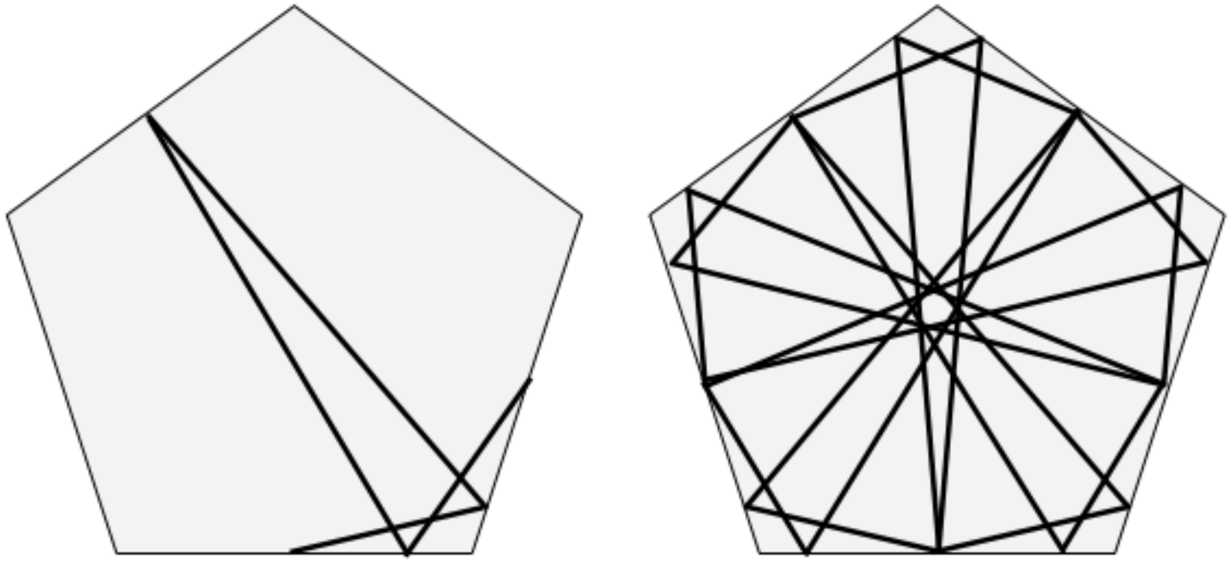
Repeat 5 times
(rotate) to get
billiard path.

5 * double pentagon period = billiard period

Two types of trajectories:

- Only reflection symmetry
- Rotation & reflection symmetry





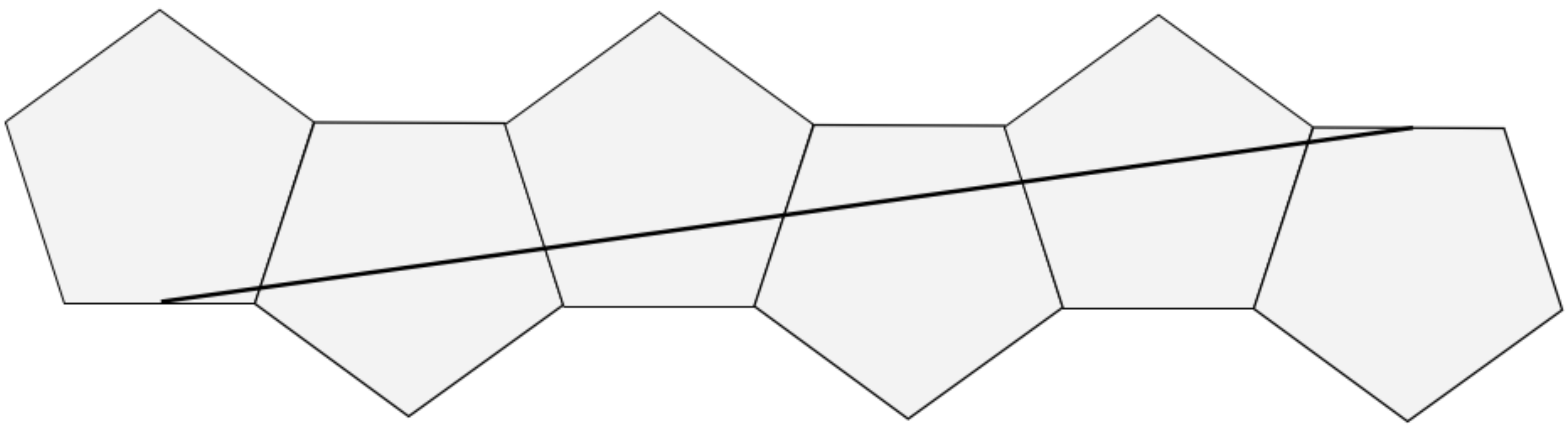
period 4 on double pentagon

Repeat 5 times (rotate) to get billiard path.

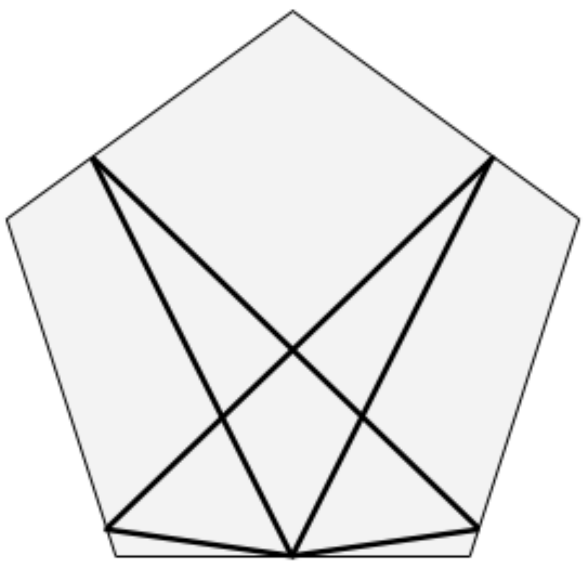
5 * double pentagon period = billiard period

Two types of trajectories:

- Only reflection symmetry
- Rotation & reflection symmetry

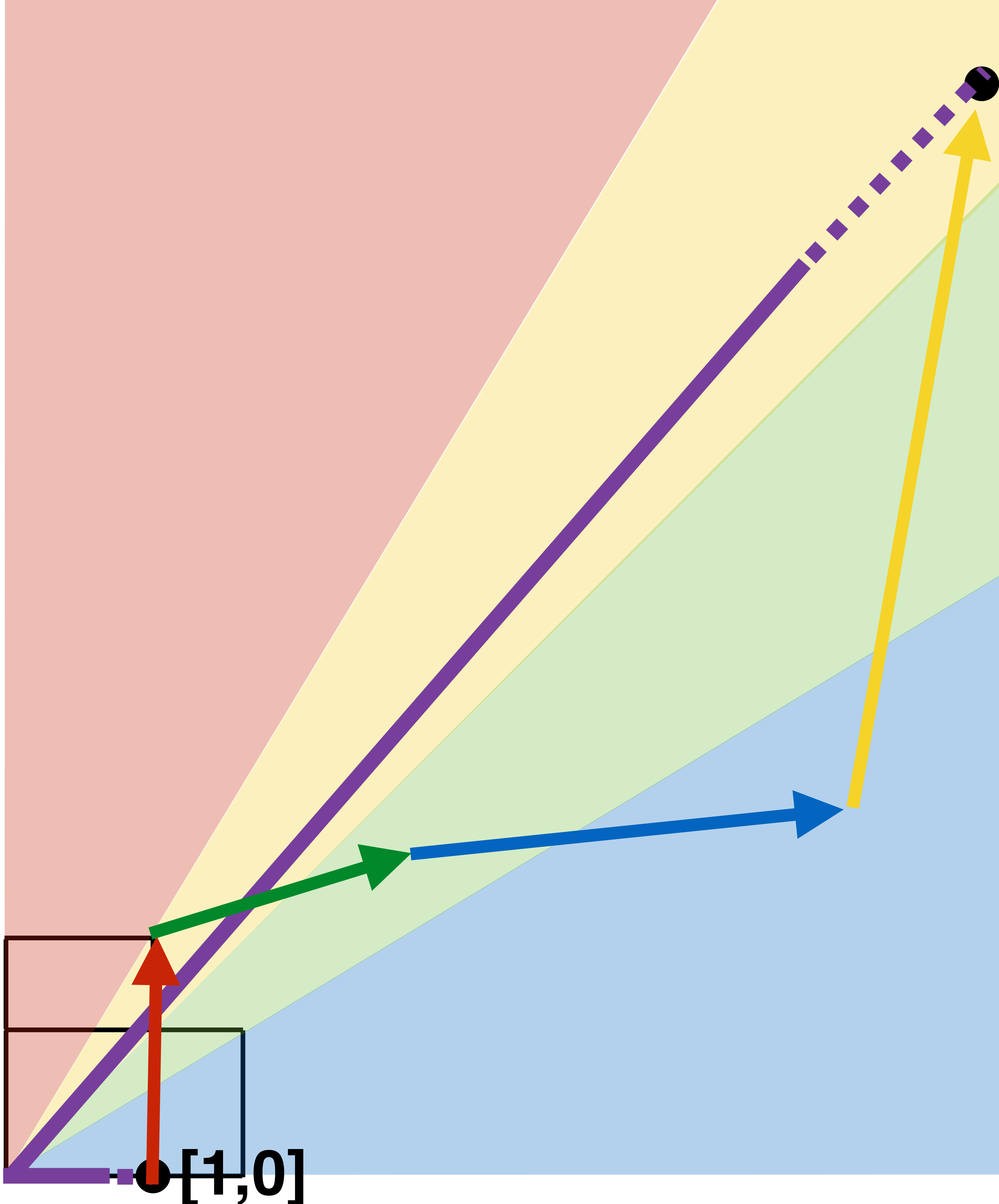


period 6 on
double pentagon



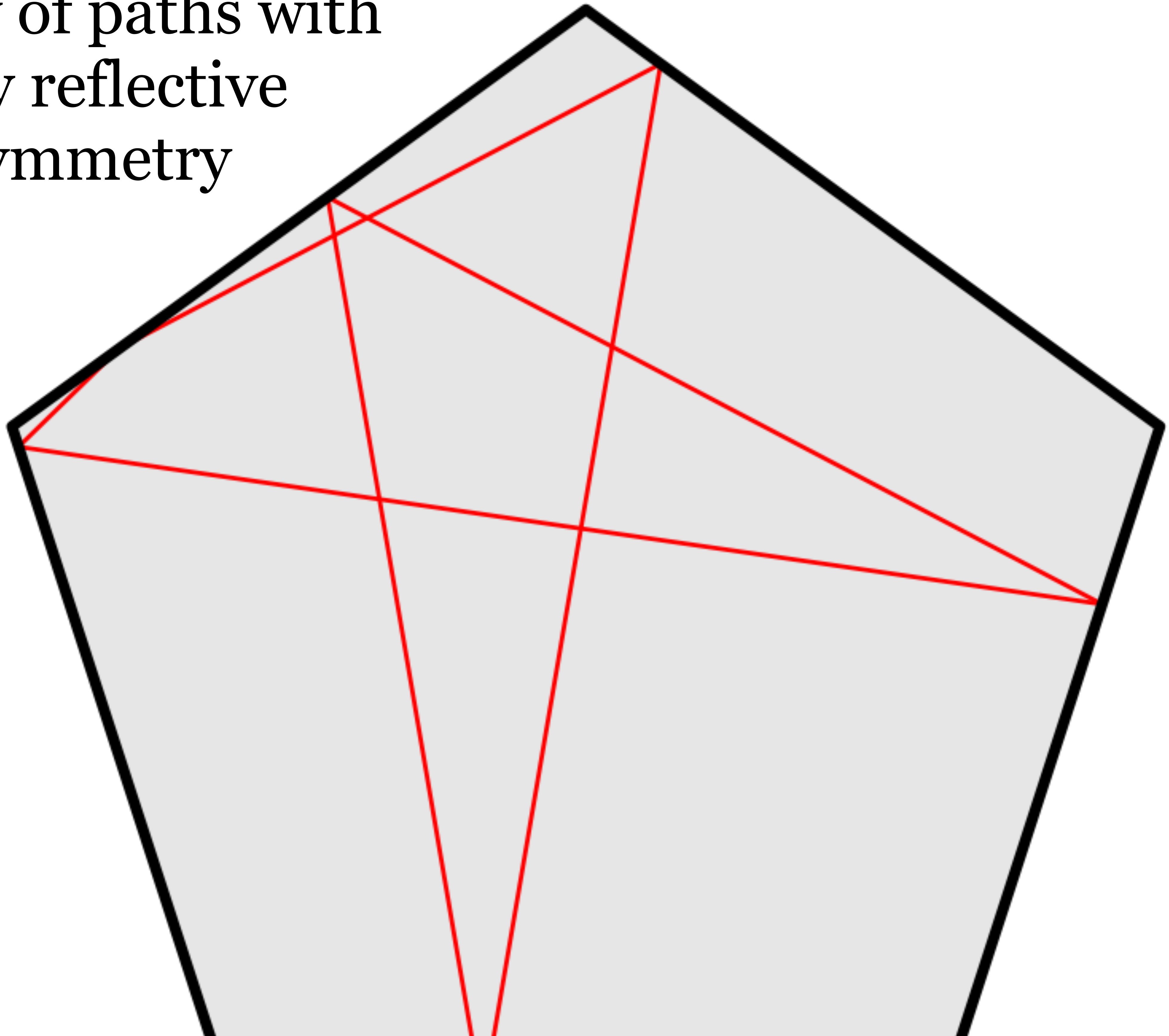
period 6 on
double pentagon

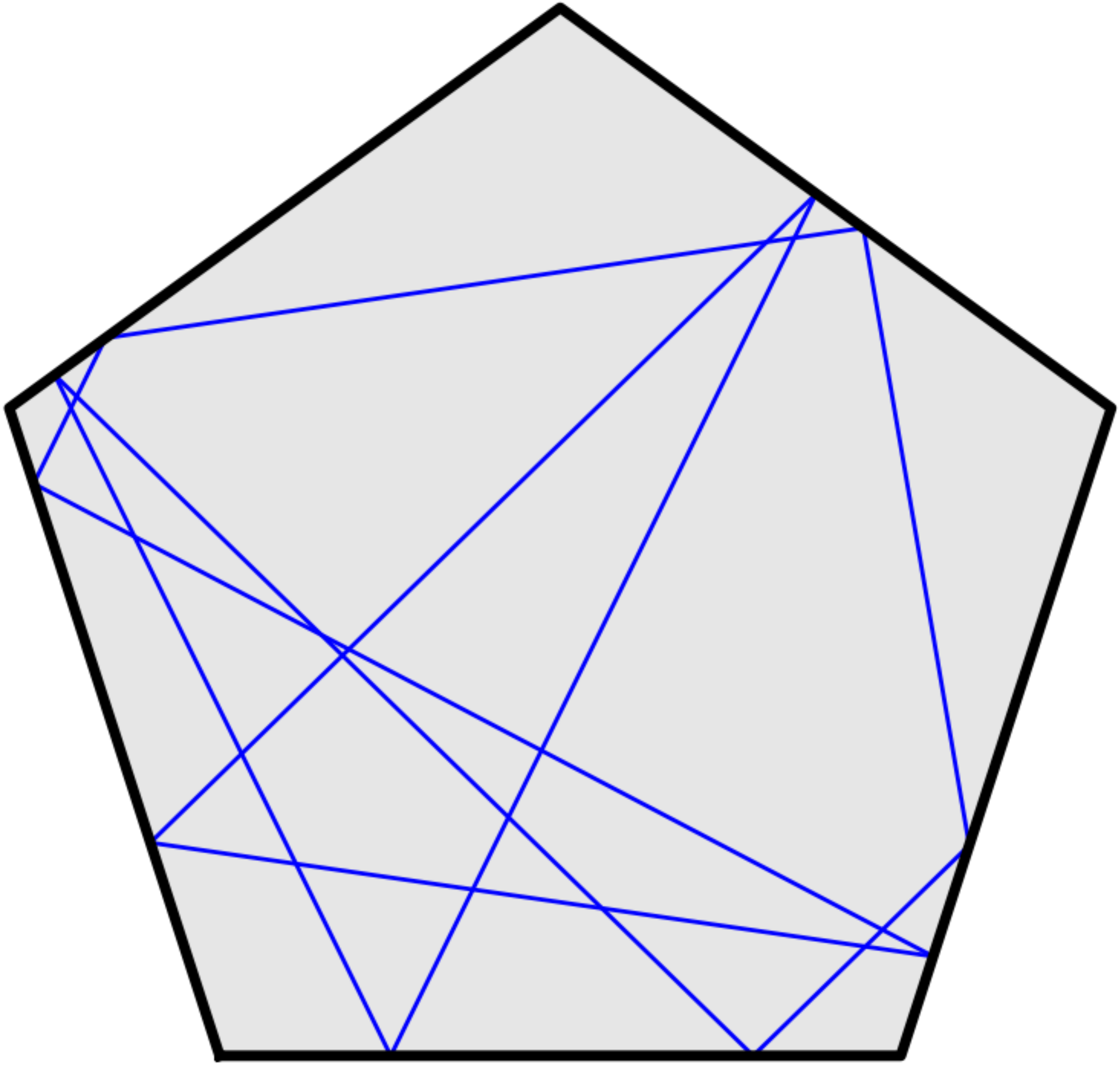
double pentagon period = billiard period

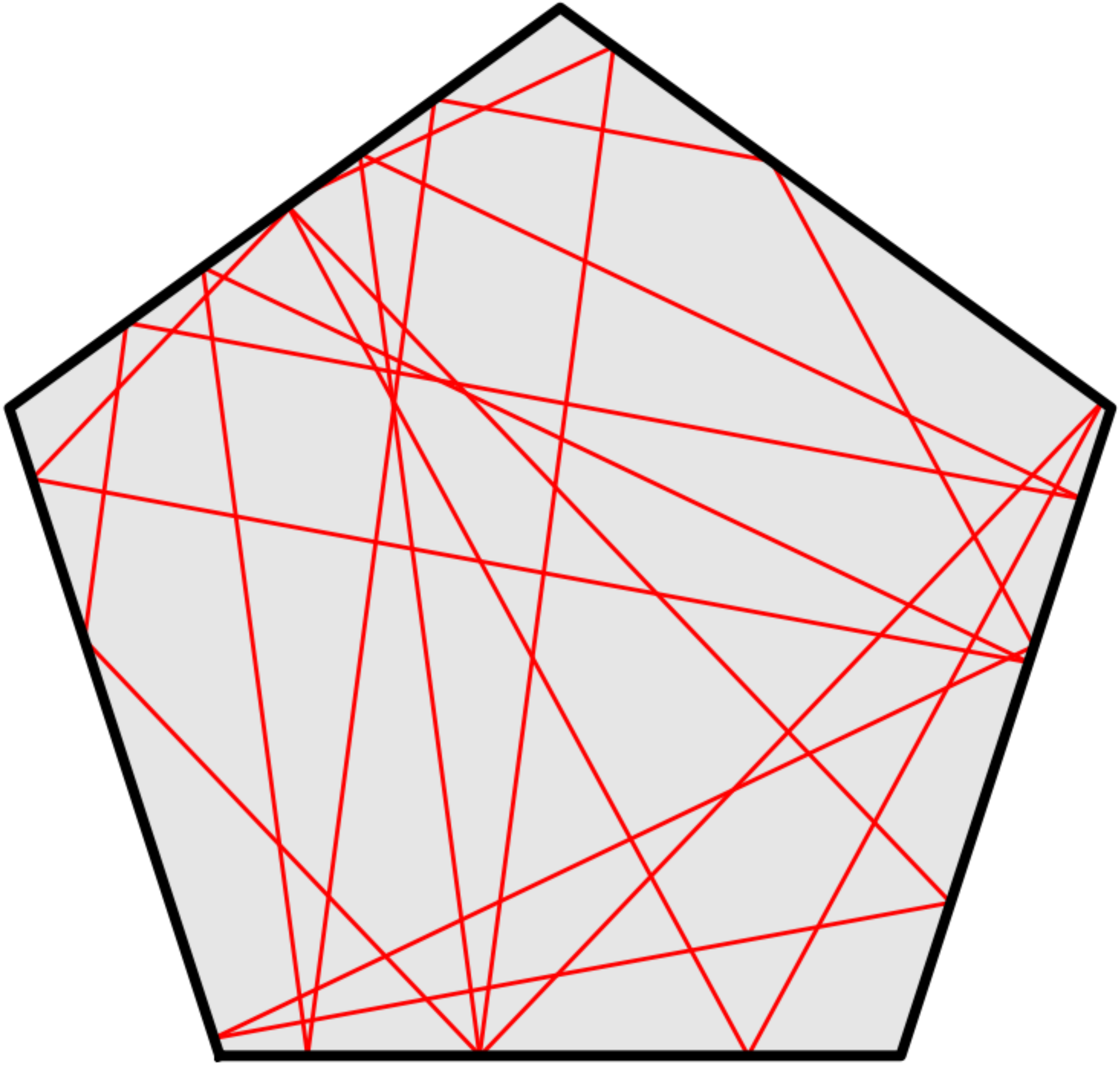


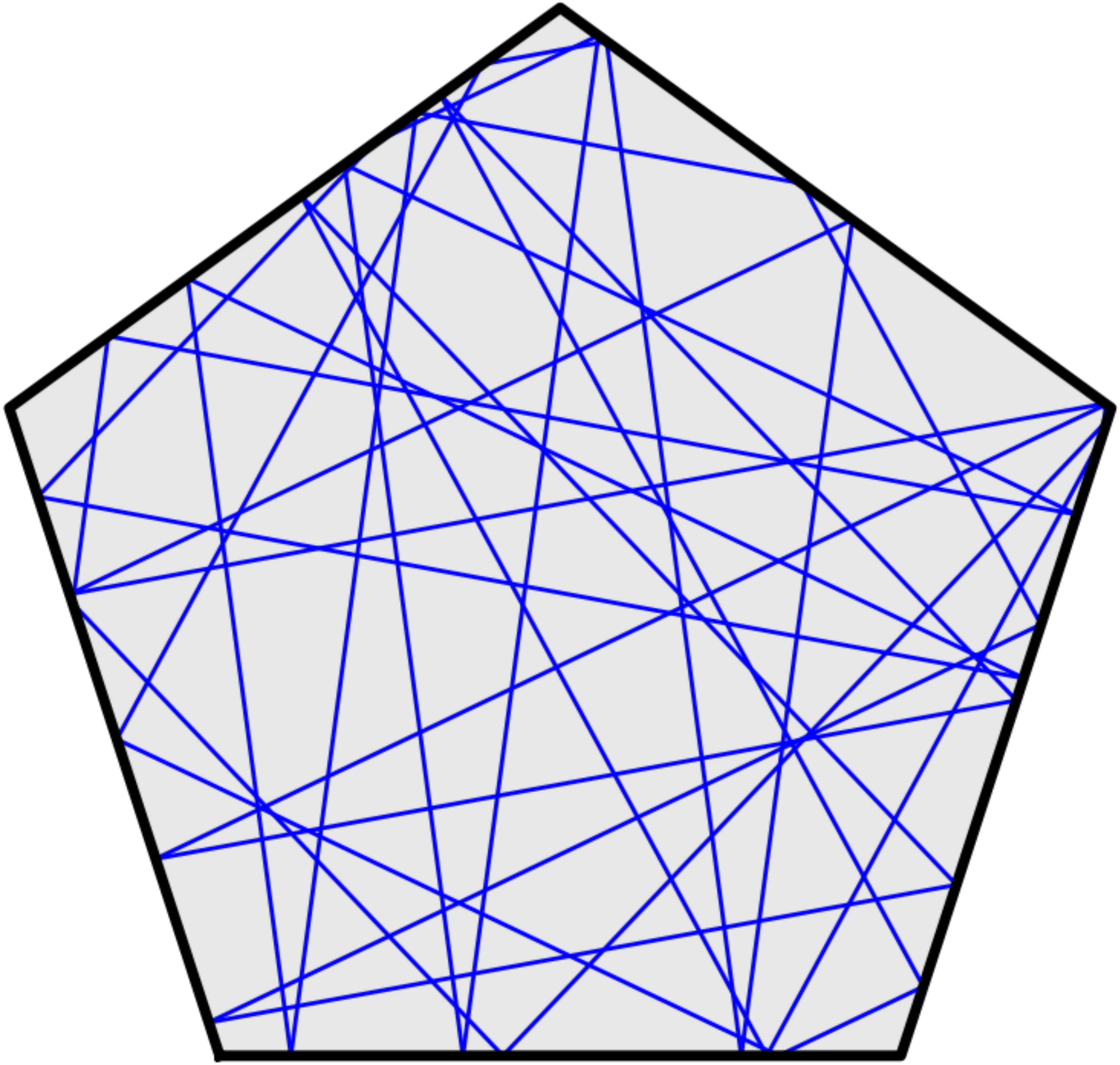
<u>Theorem</u>
<u>(DD & Lelièvre):</u>
The pentagon billiard period is 2(a+b+c+d) if the trajectory has only reflection symmetry, and 10(a+b+c+d) if it has rotation symmetry also.

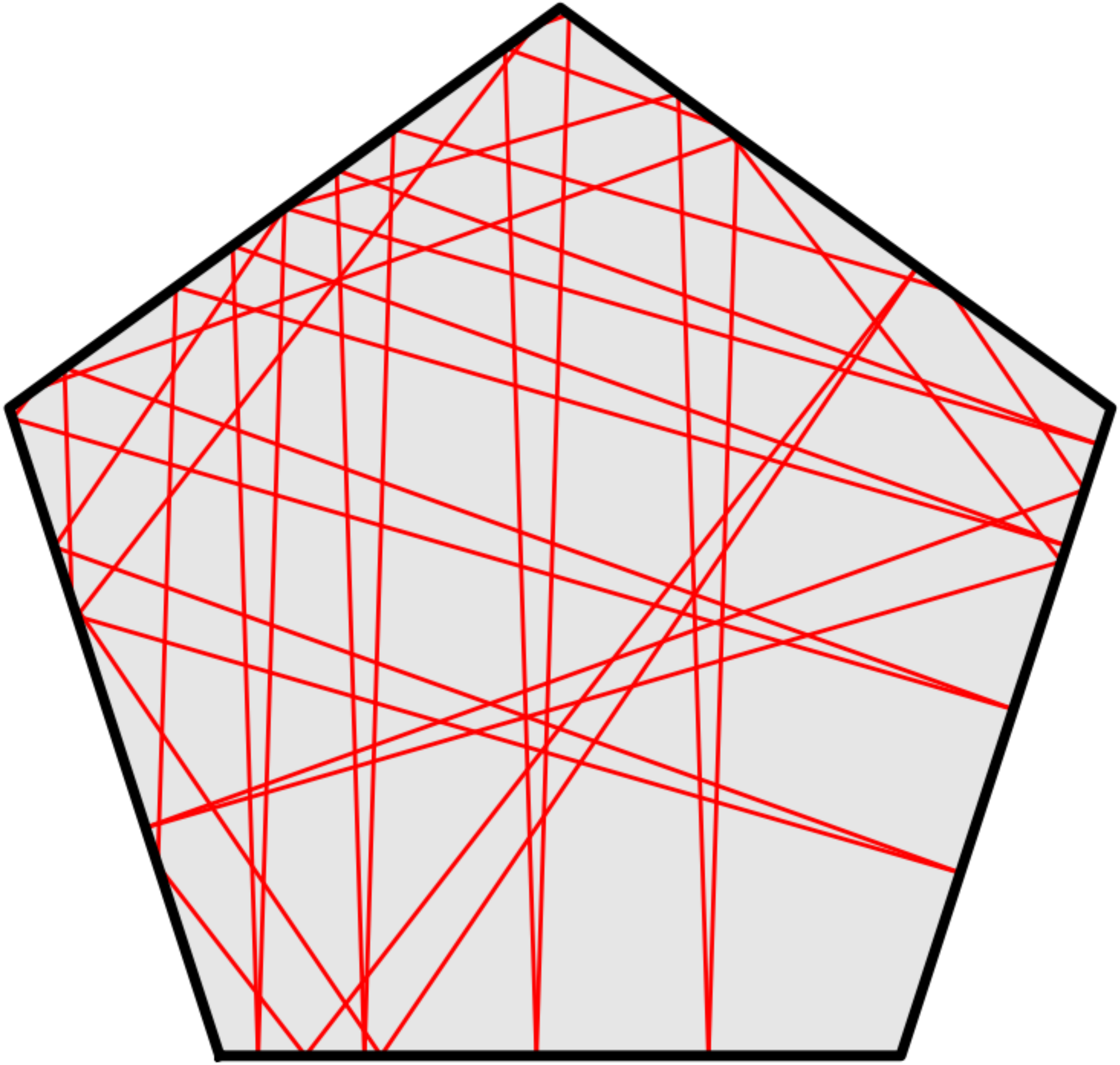
Gallery of paths with only reflective symmetry

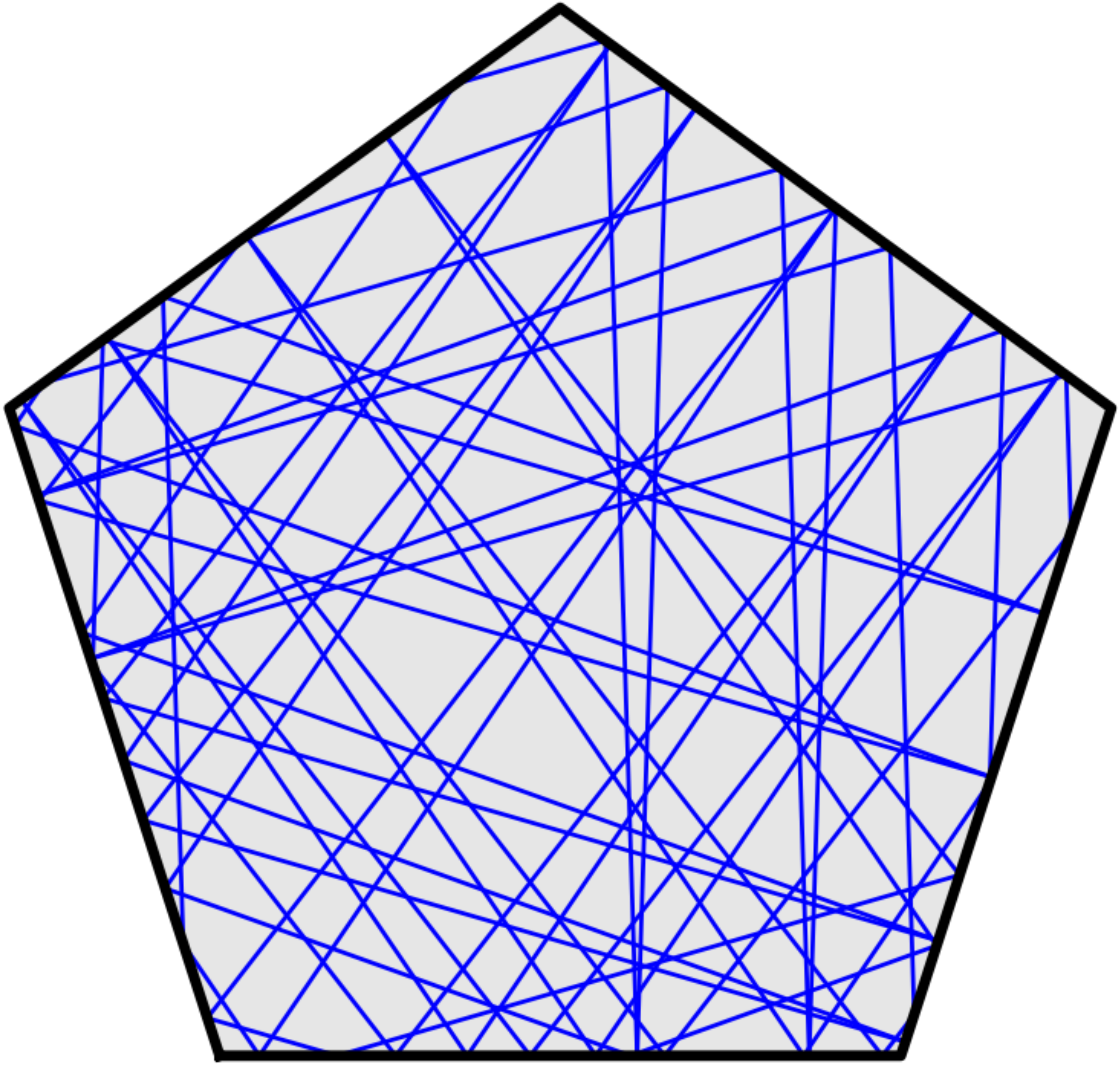


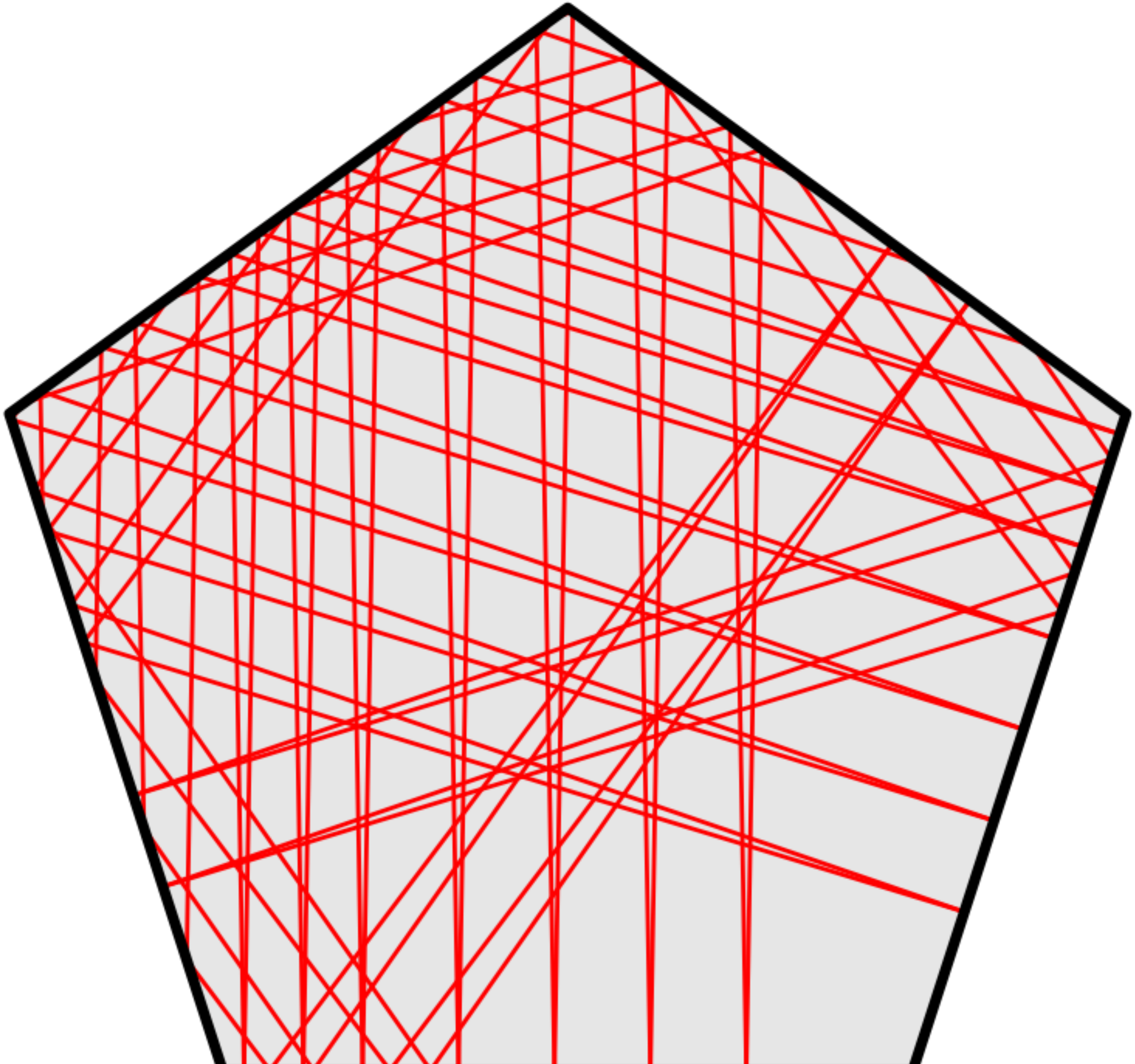


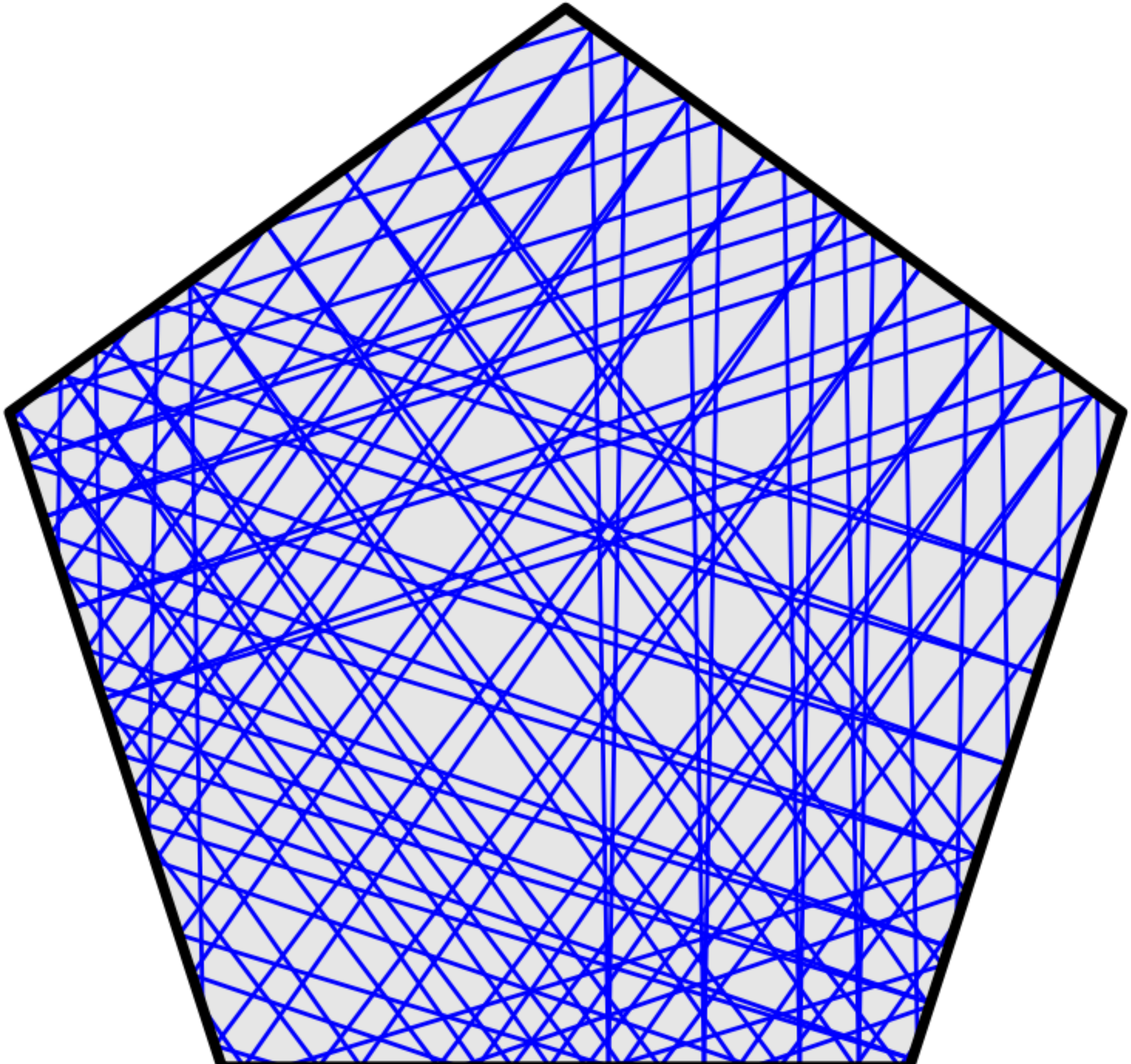


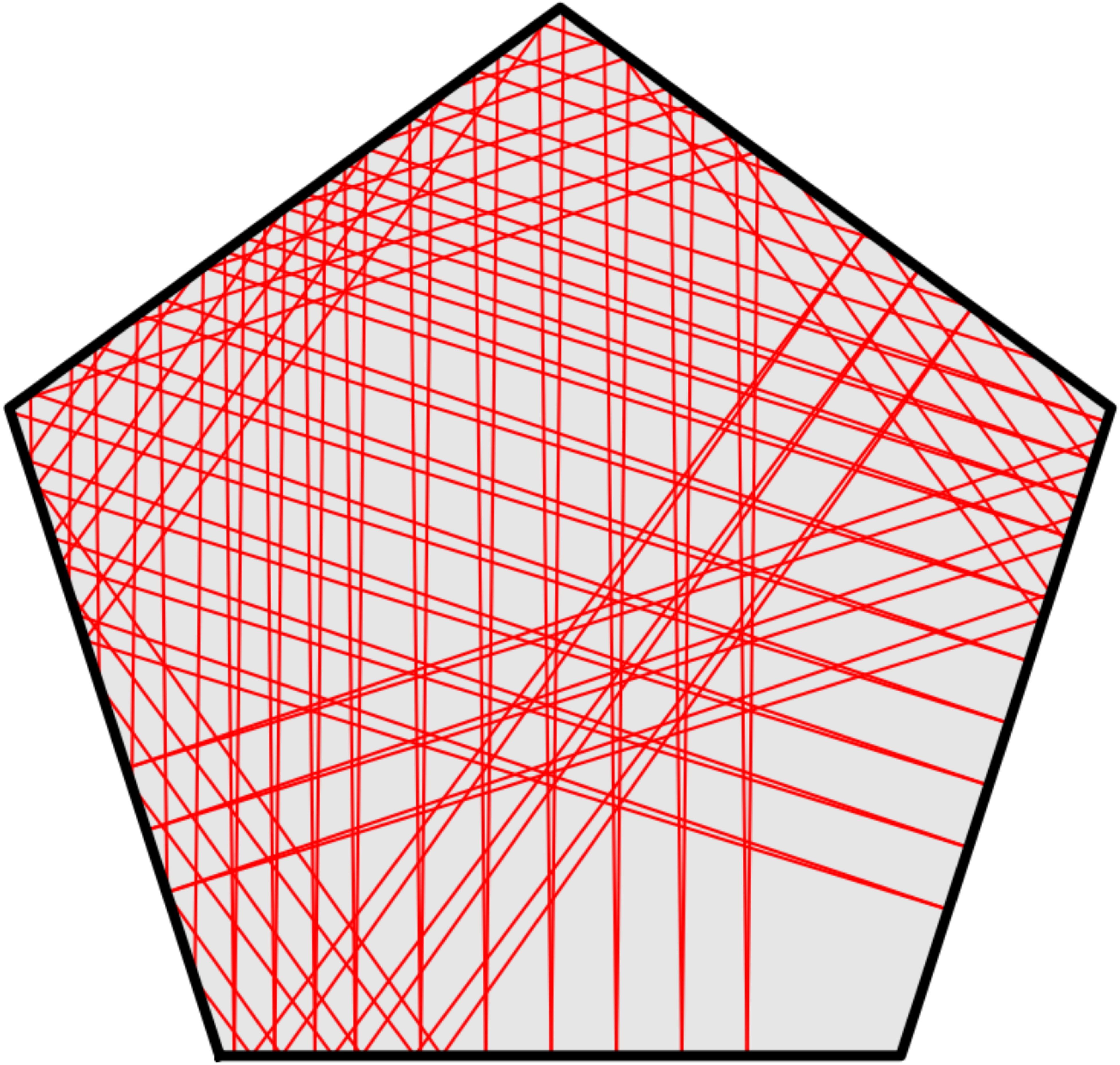


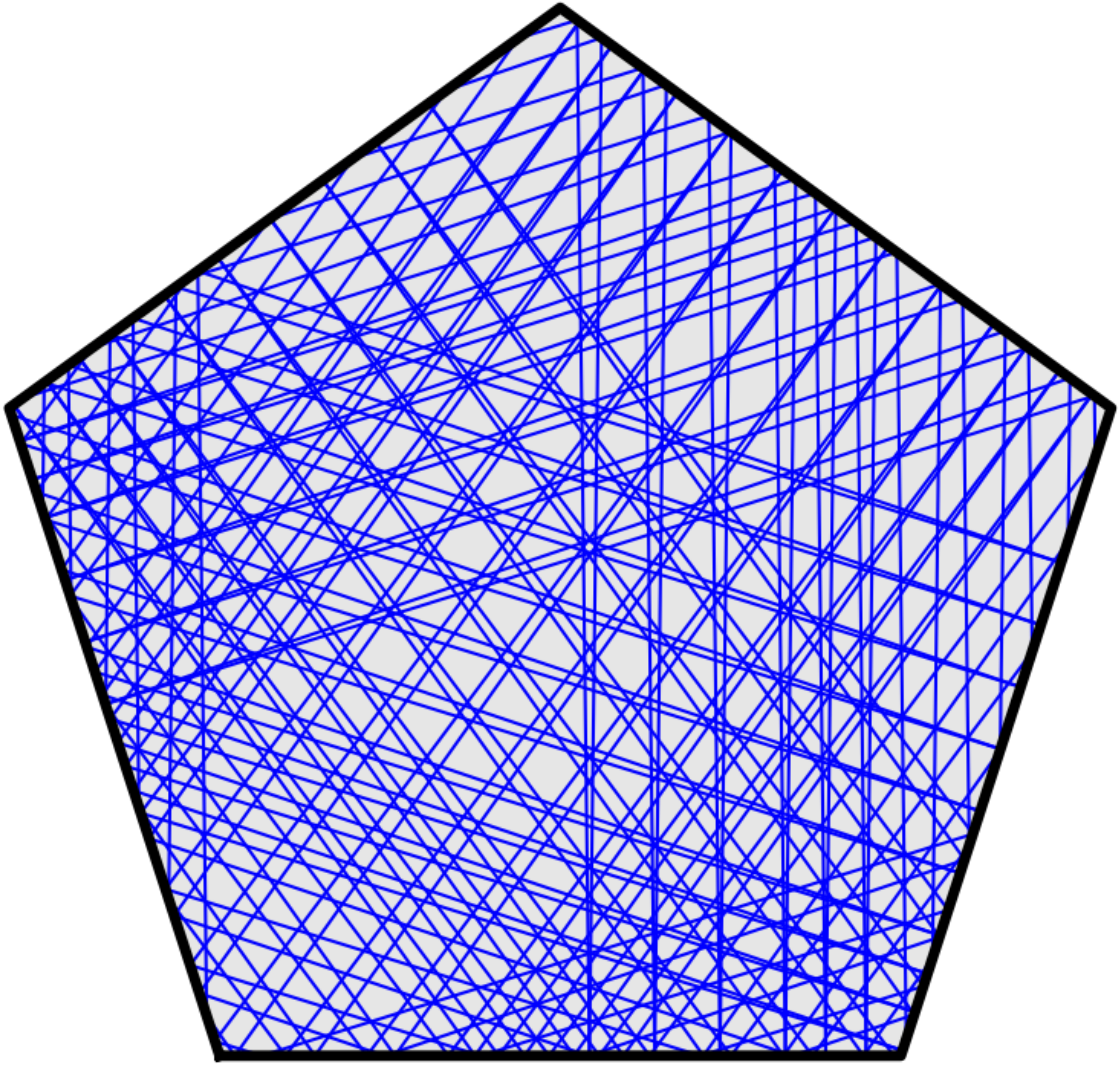


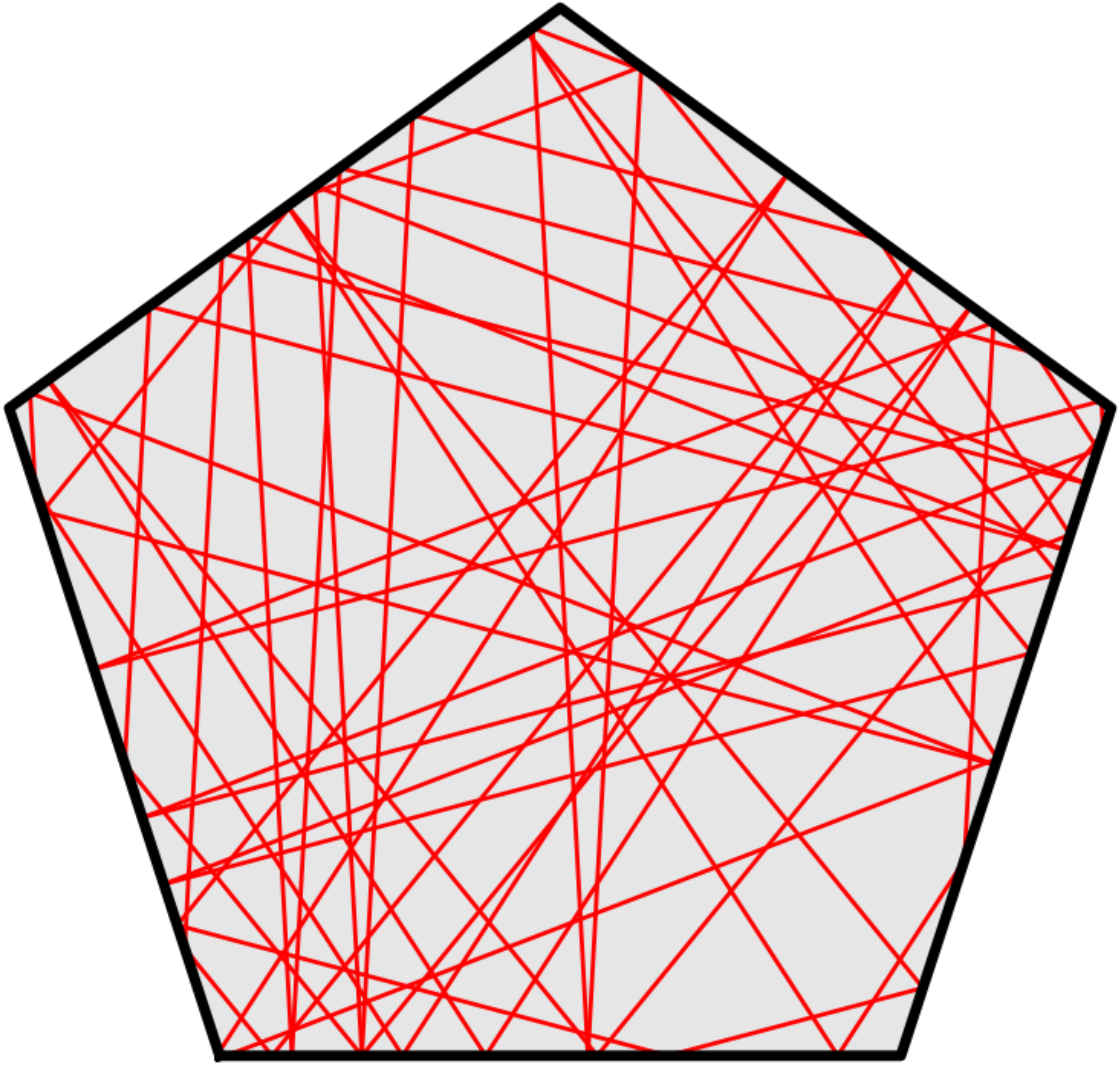


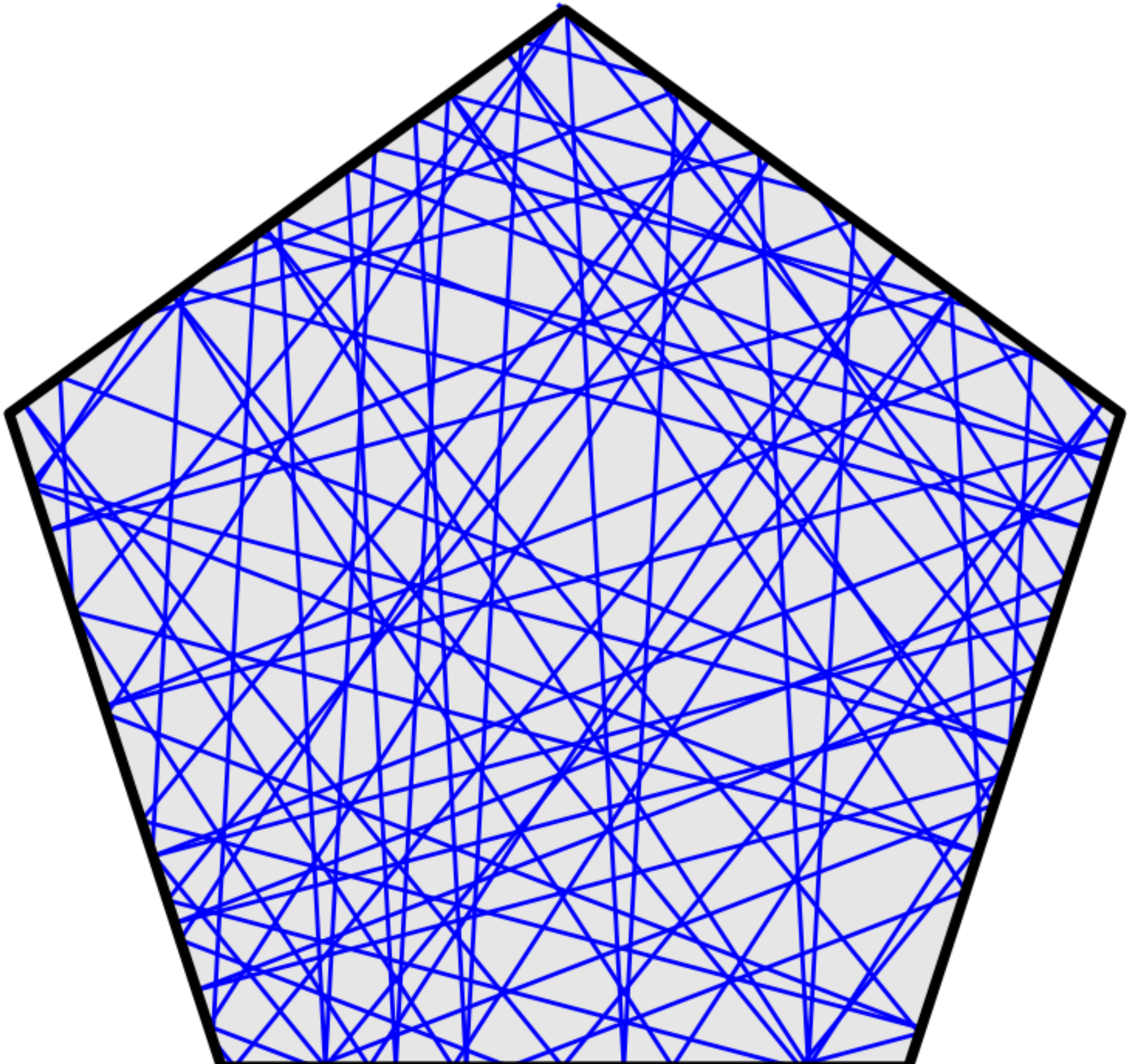


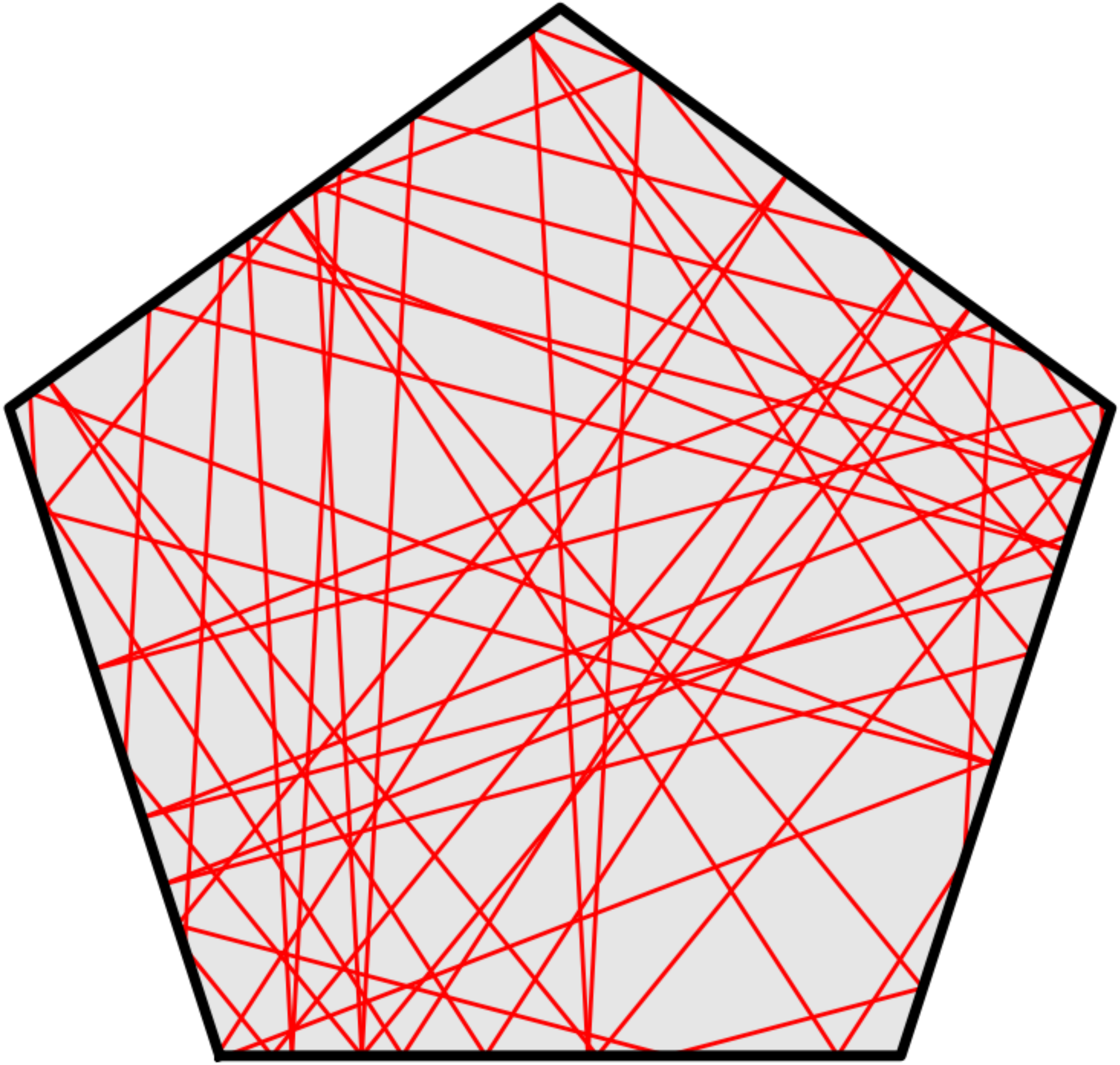


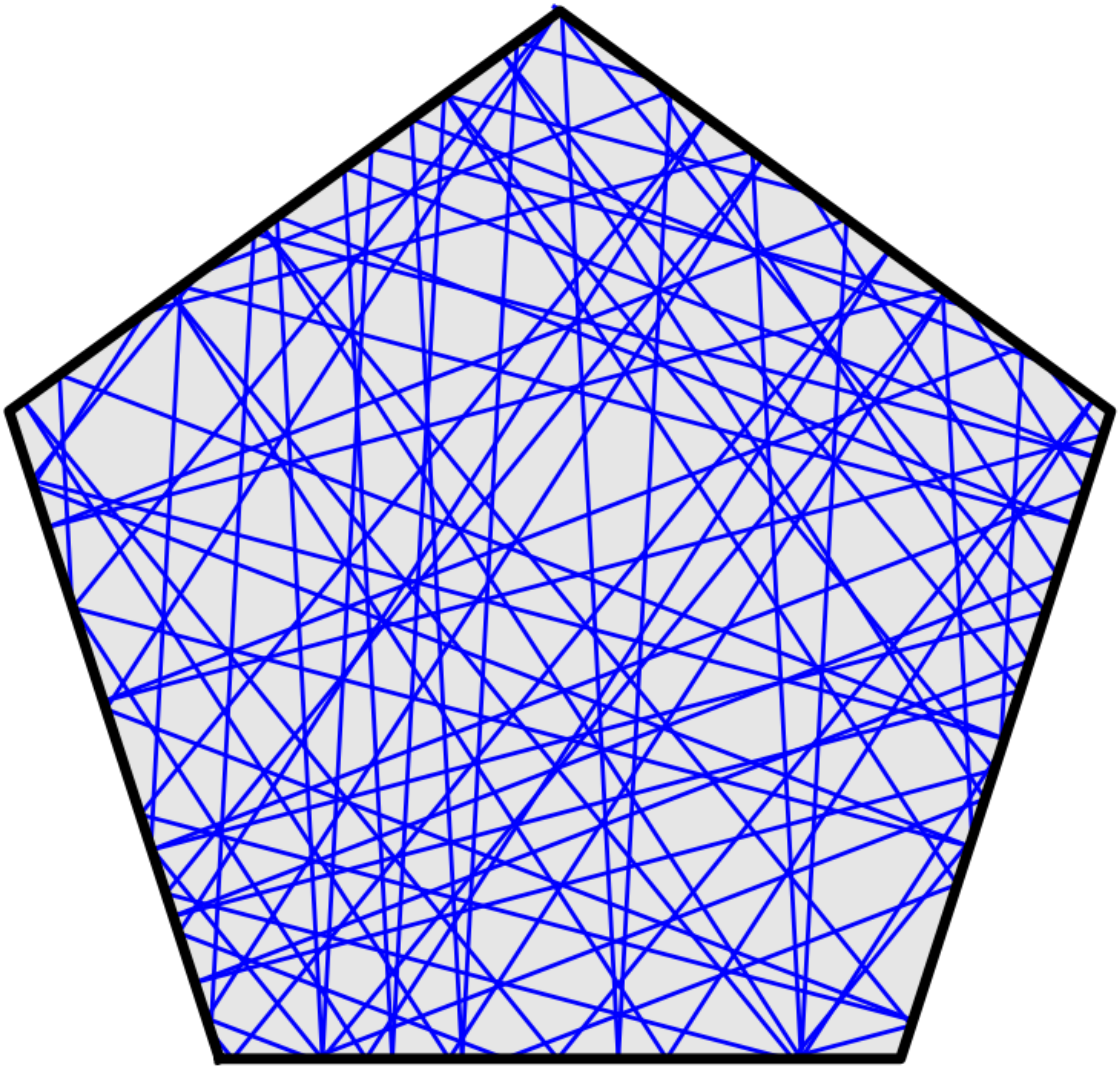


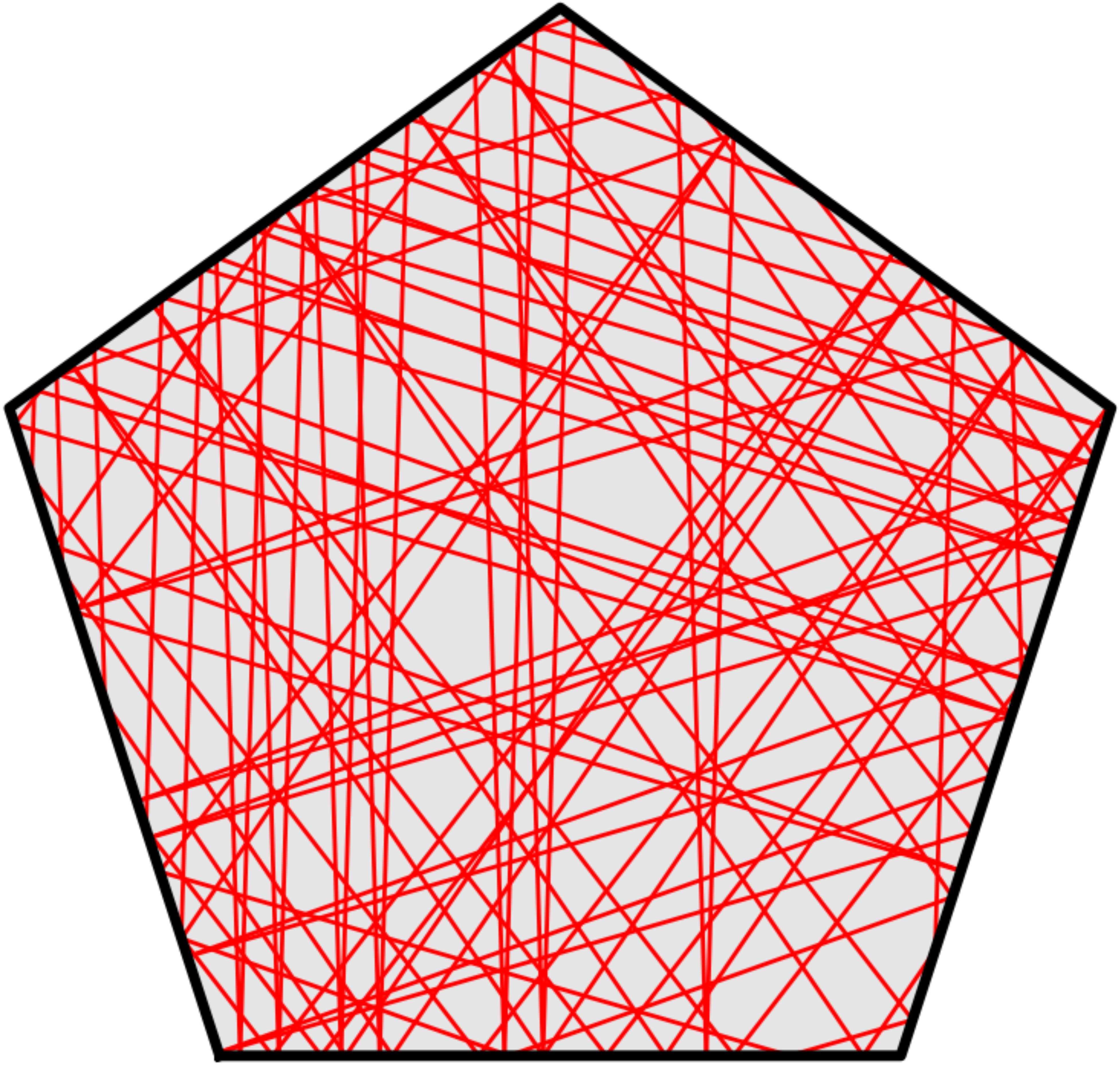


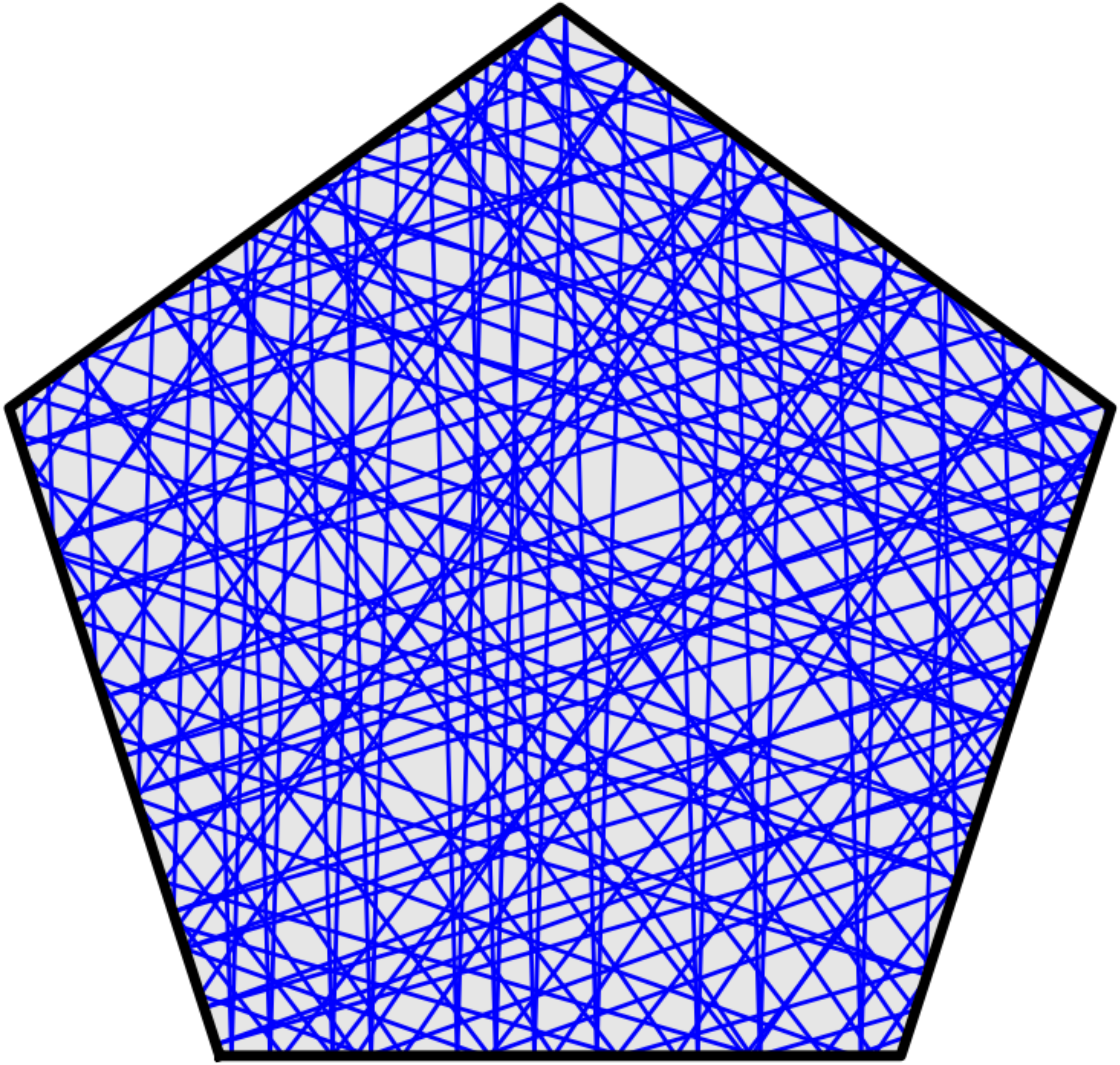


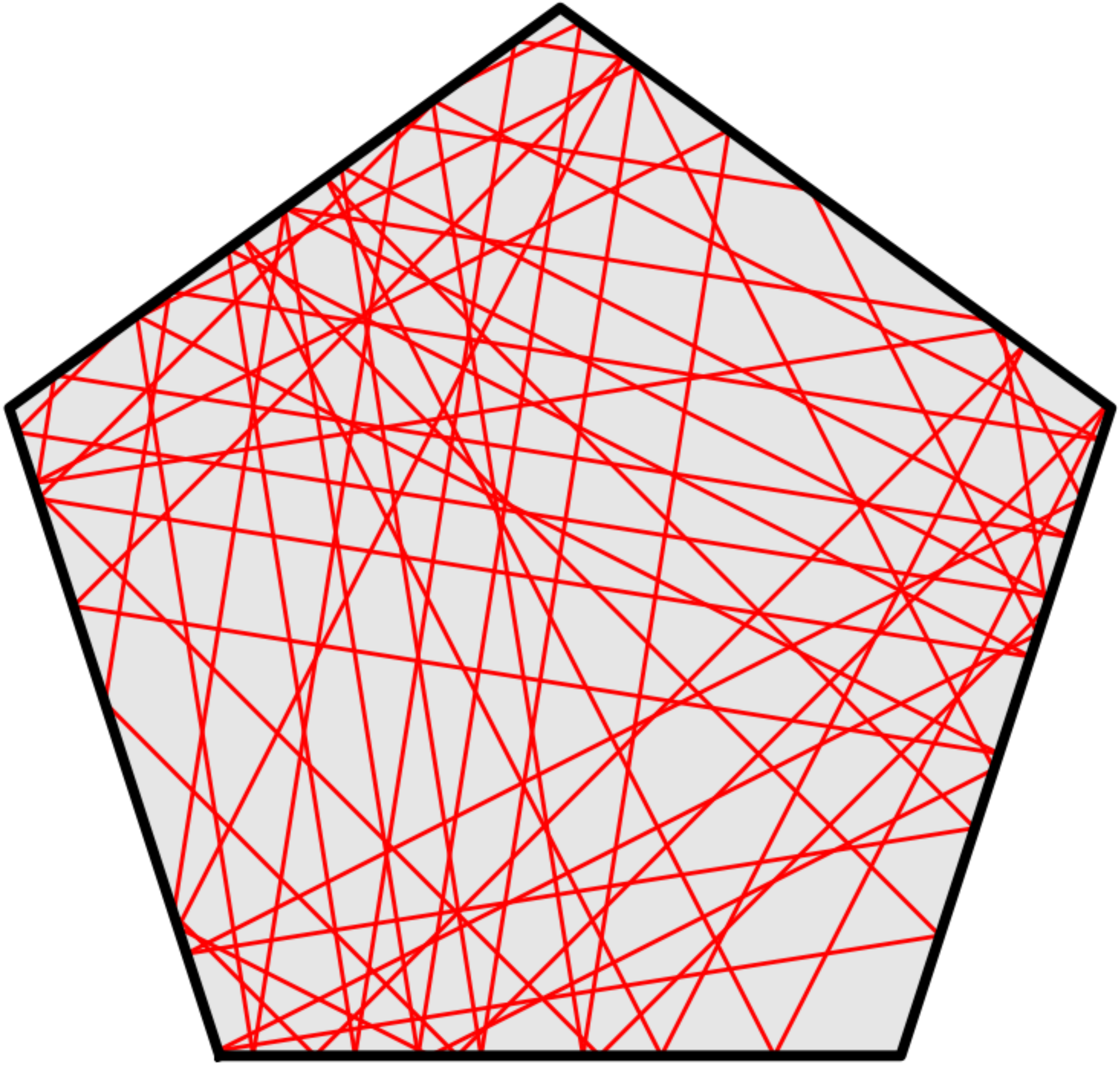


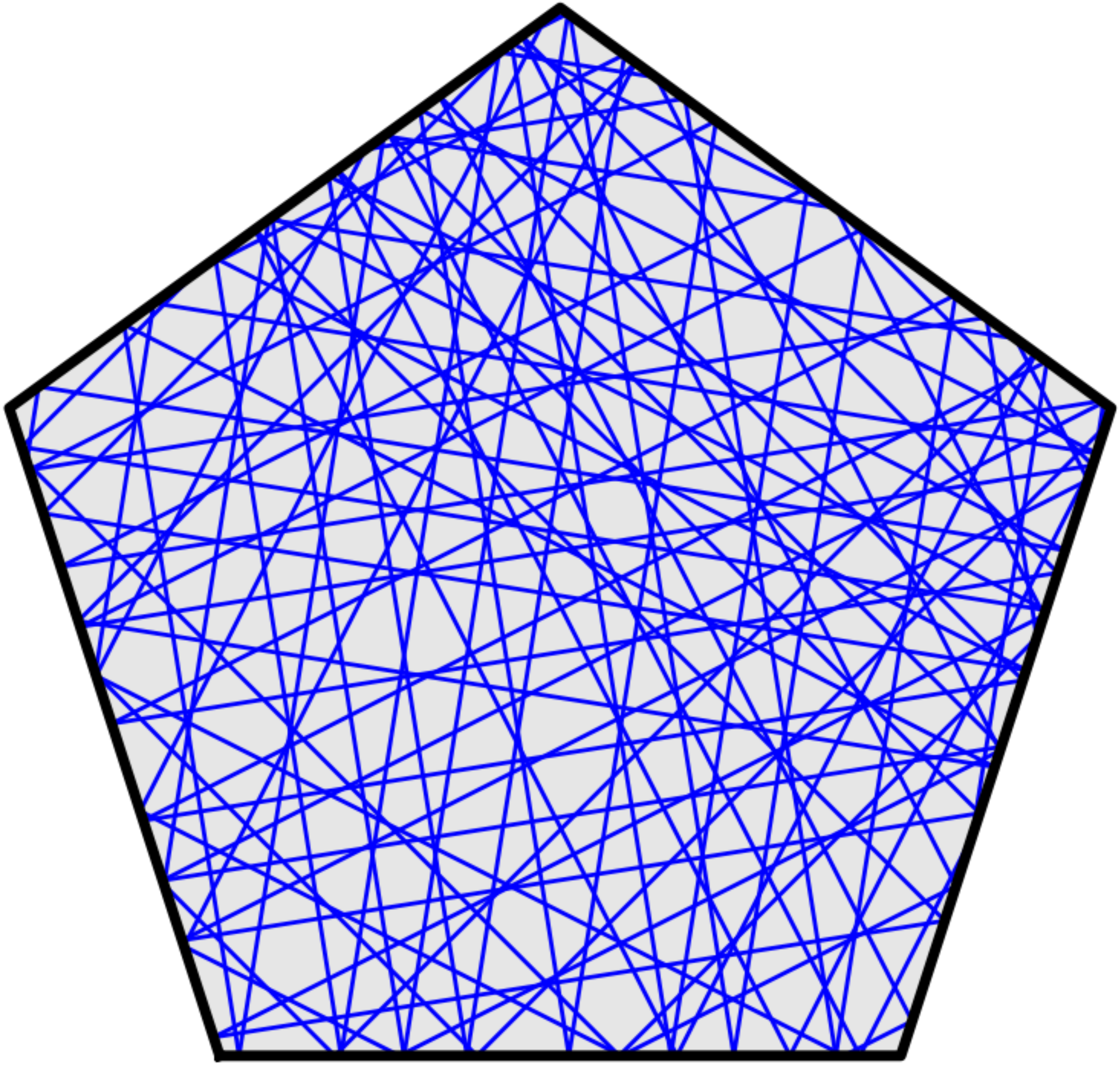


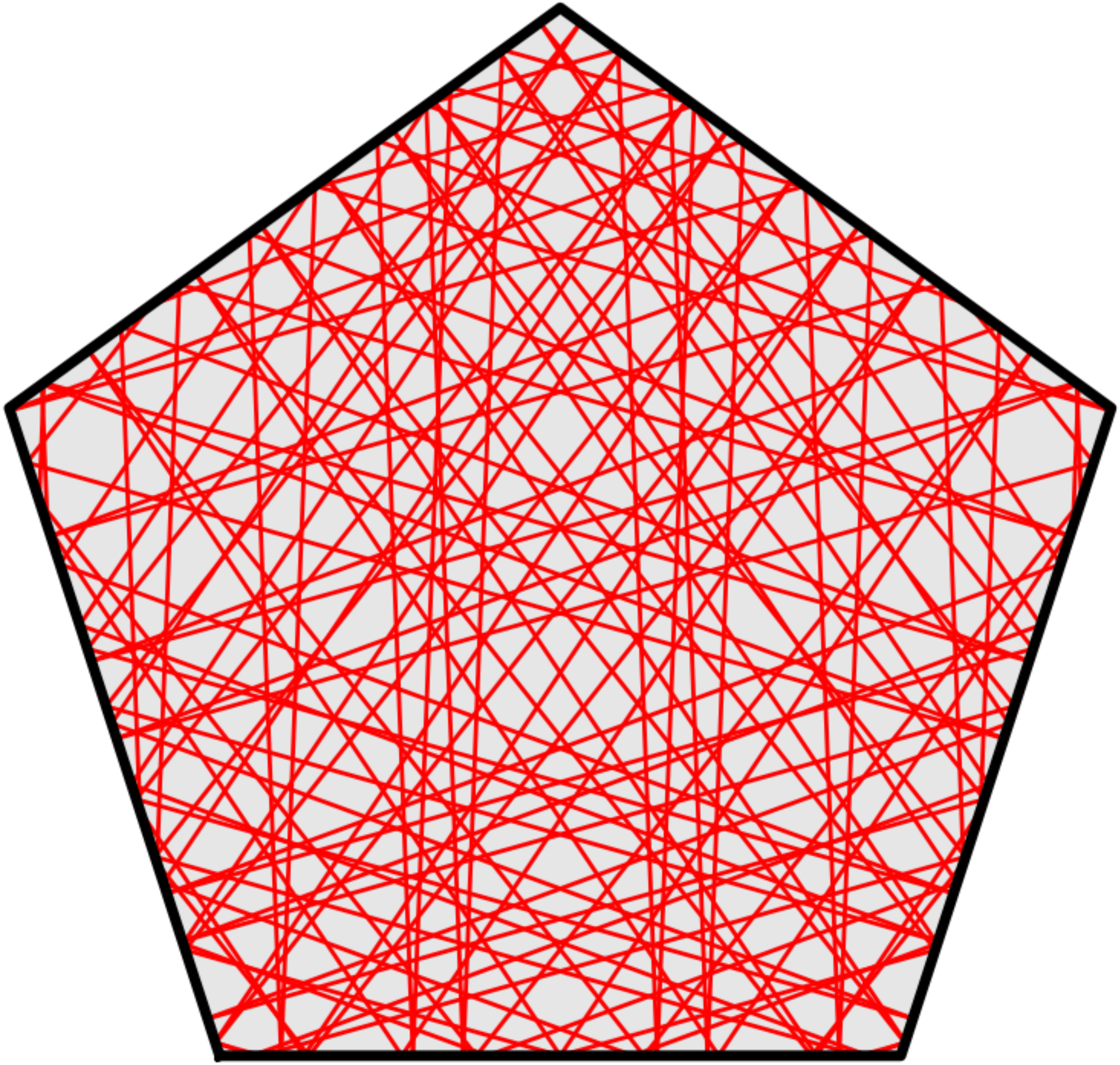


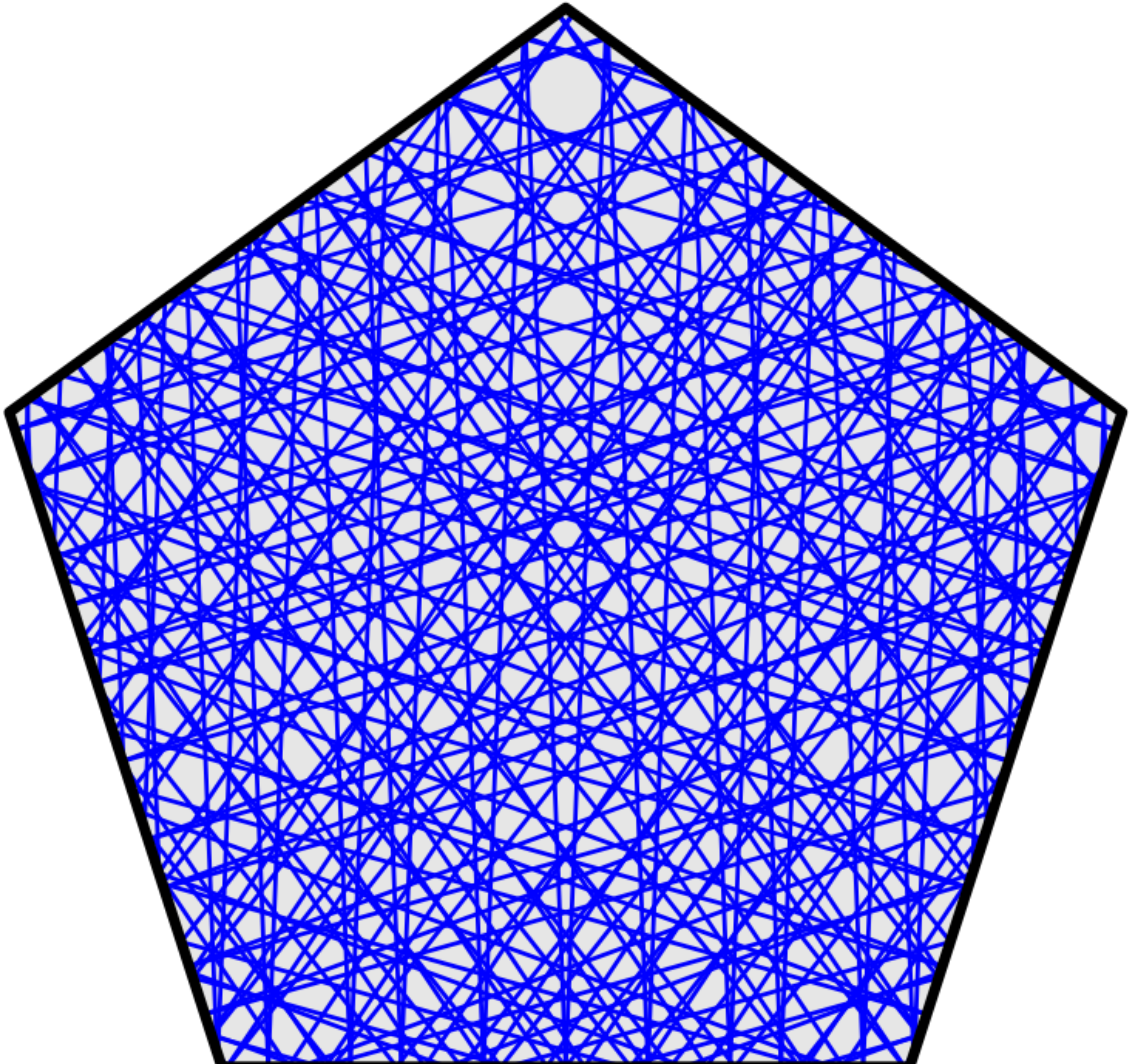


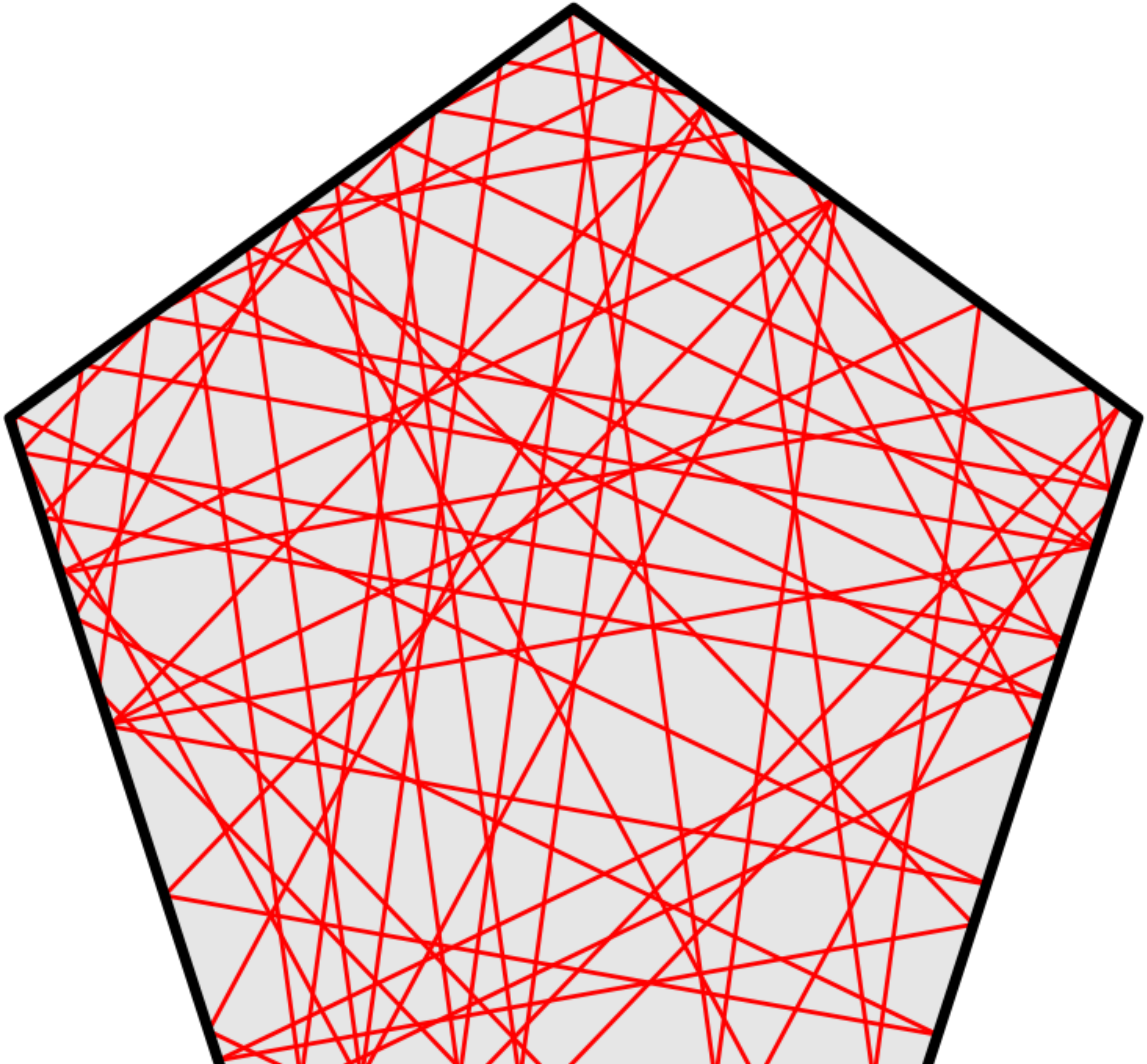


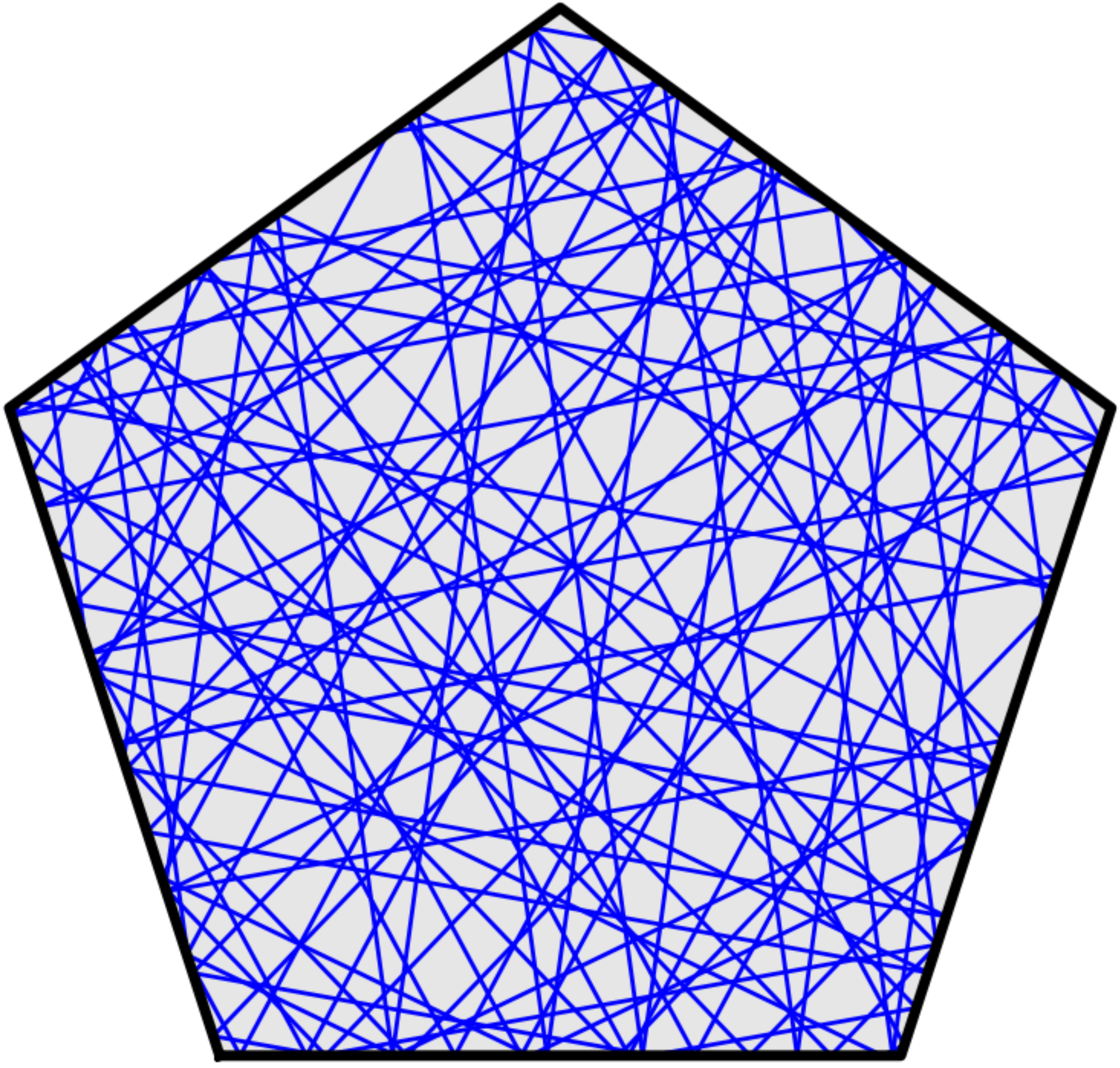






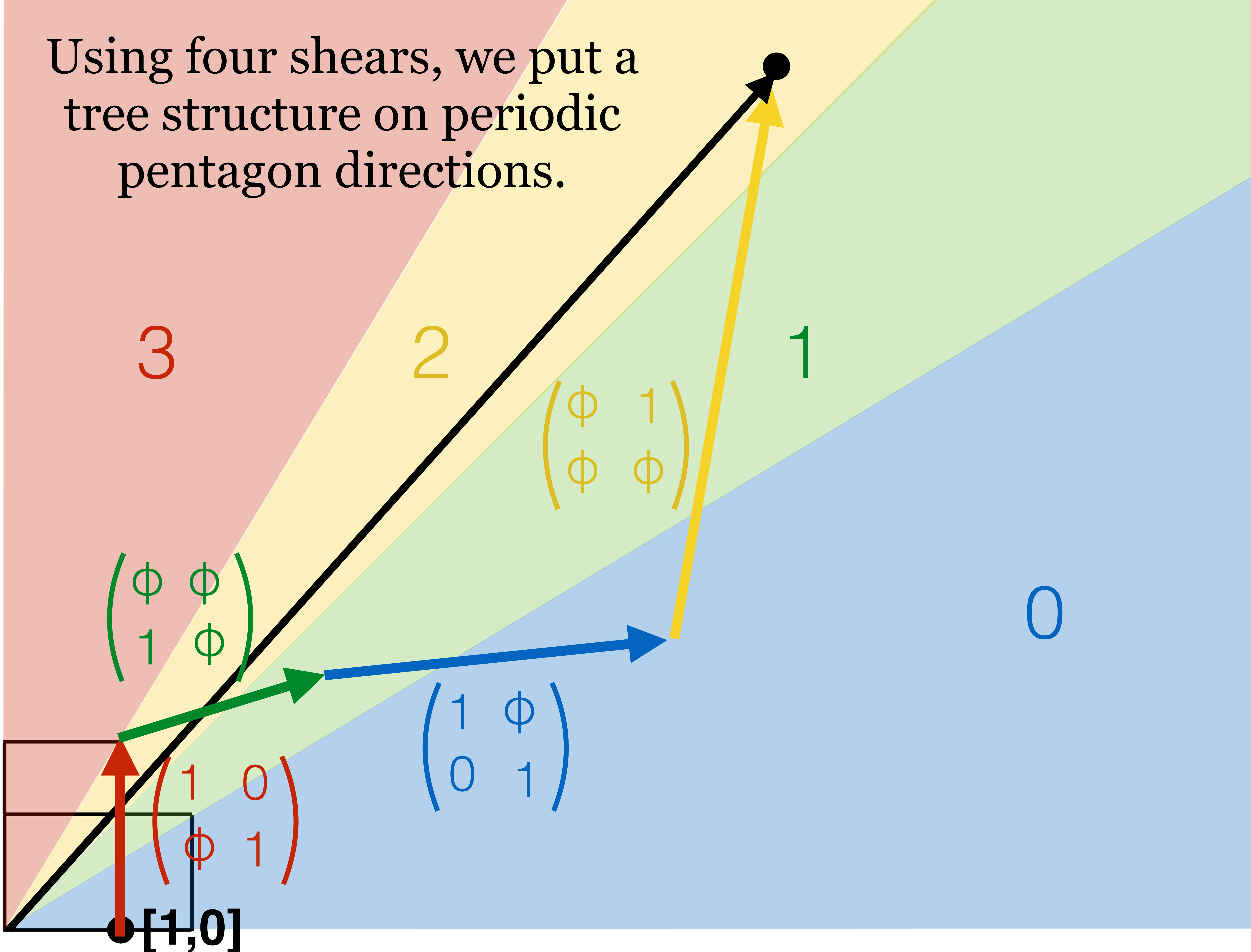






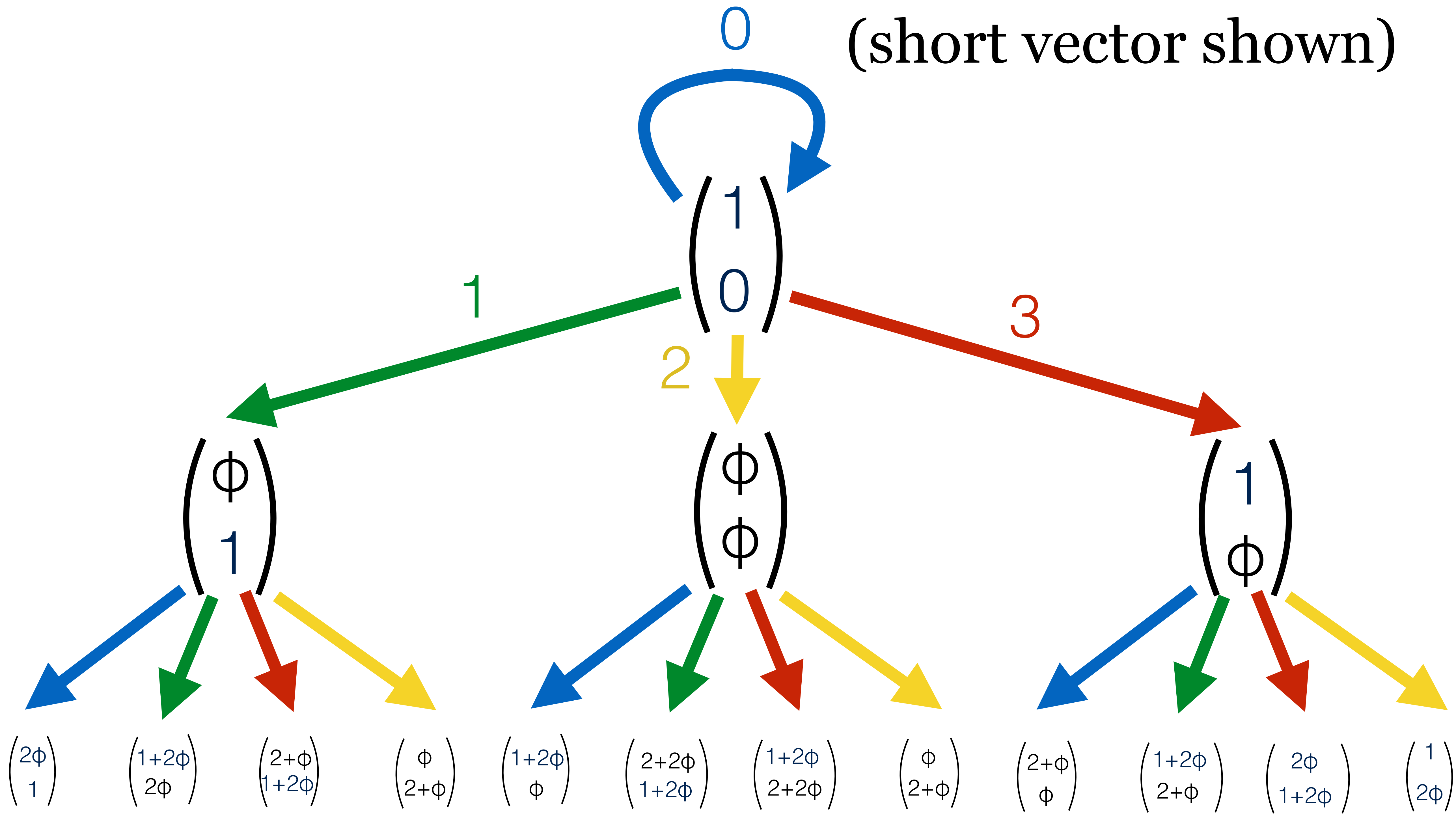
FAMILIES

Using four shears, we put a tree structure on periodic pentagon directions.



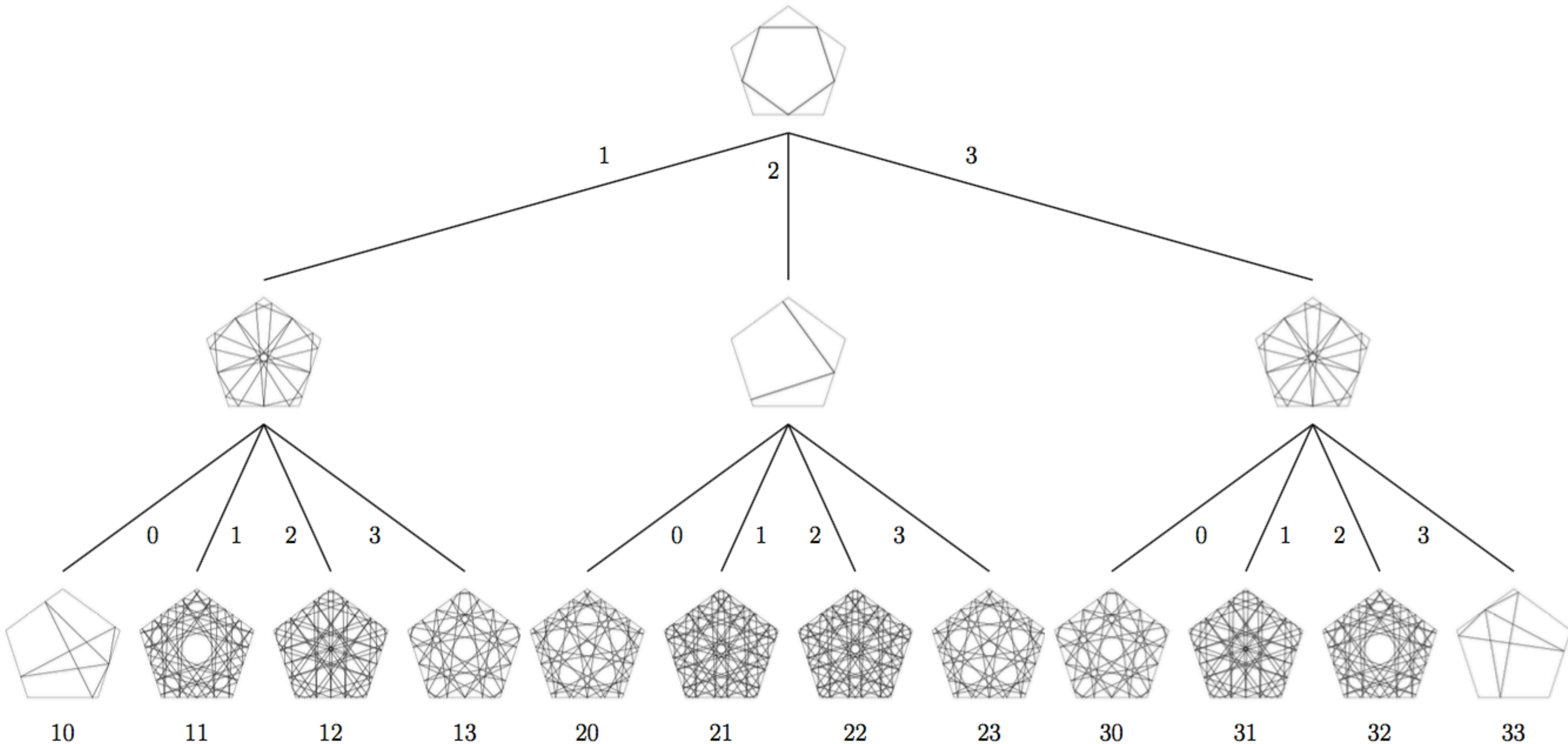
Here is the tree structure in direction vectors.

(short vector shown)



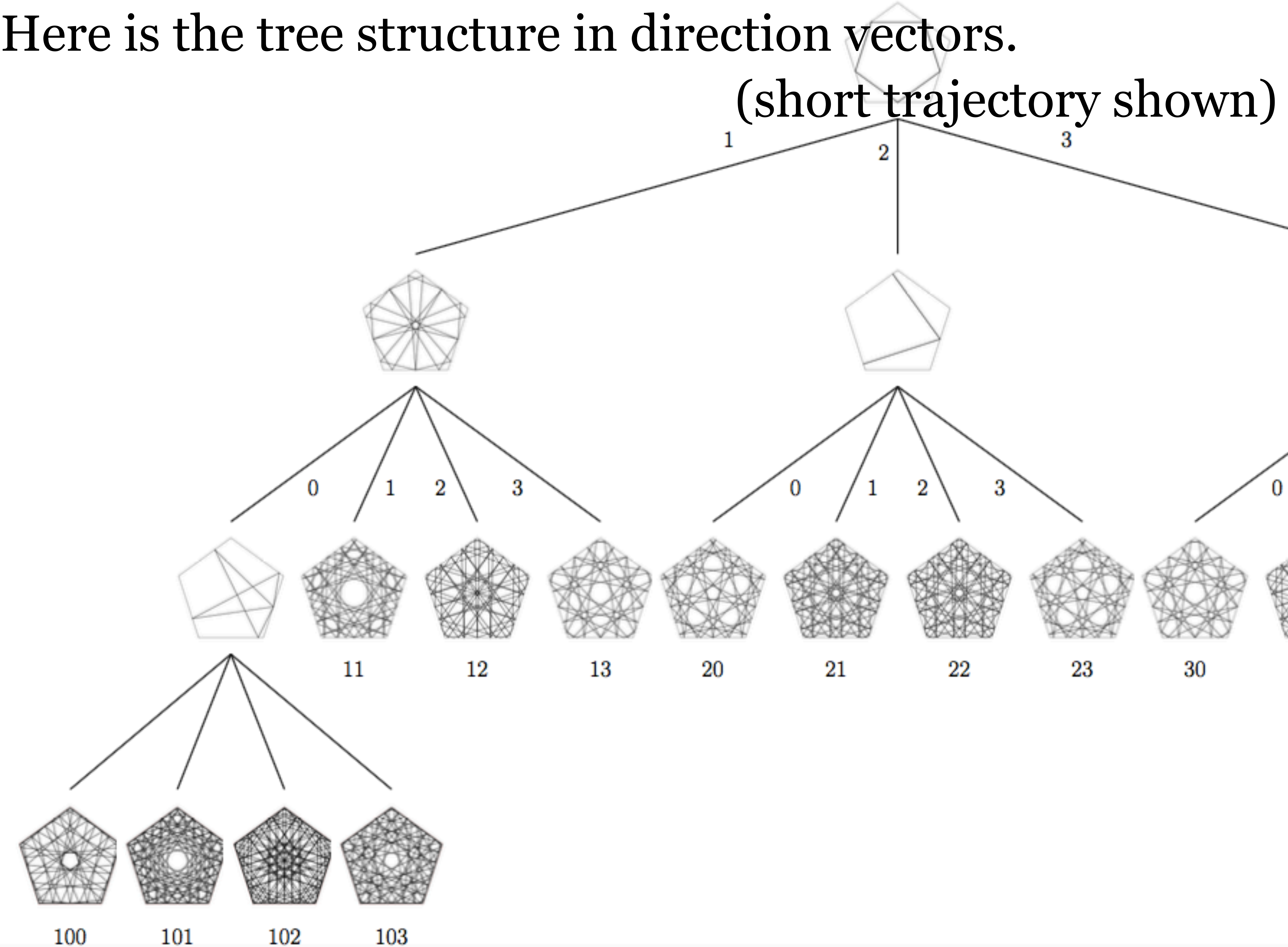
Here is the tree structure in direction vectors.

(short trajectory shown)

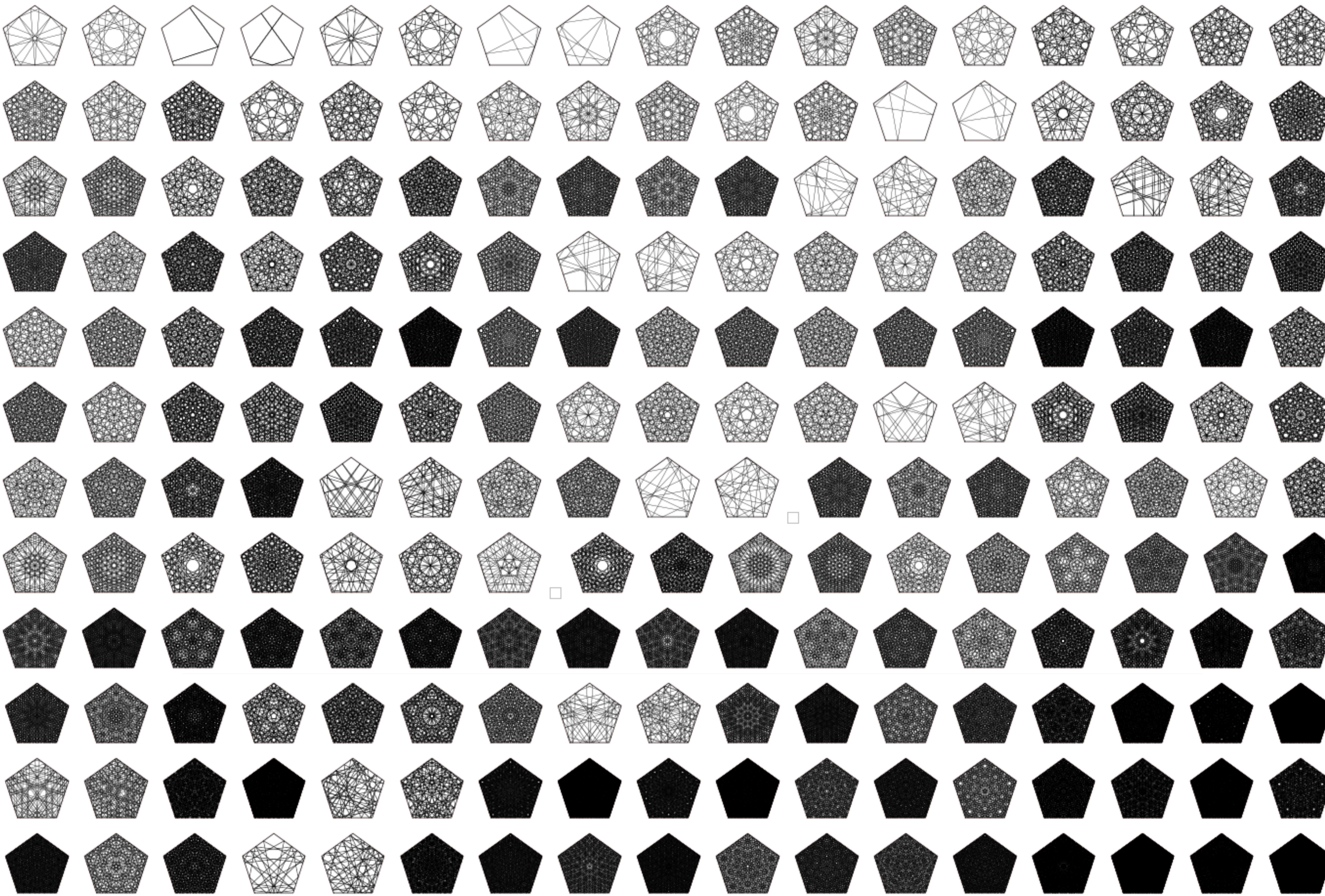


Here is the tree structure in direction vectors.

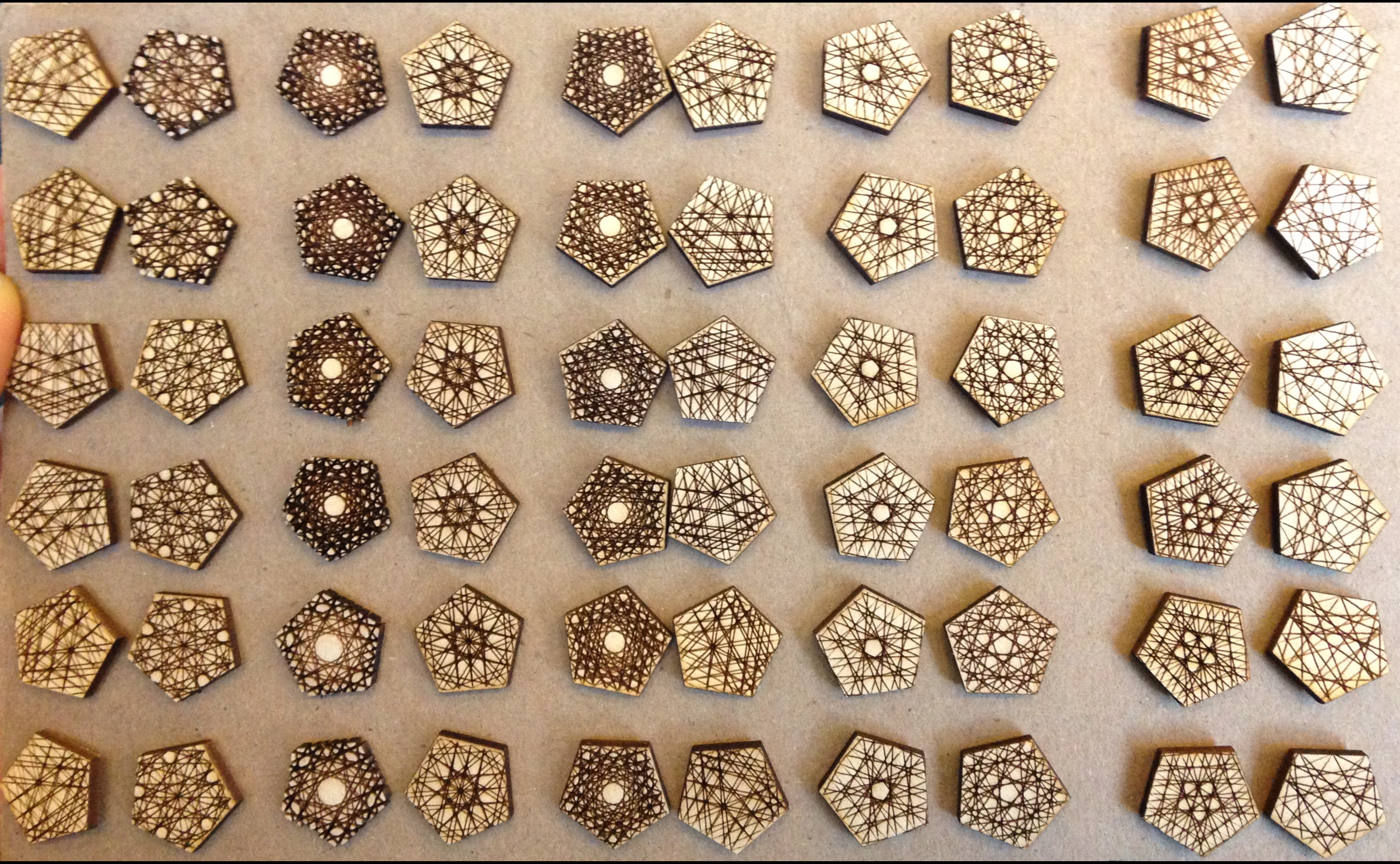
(short trajectory shown)



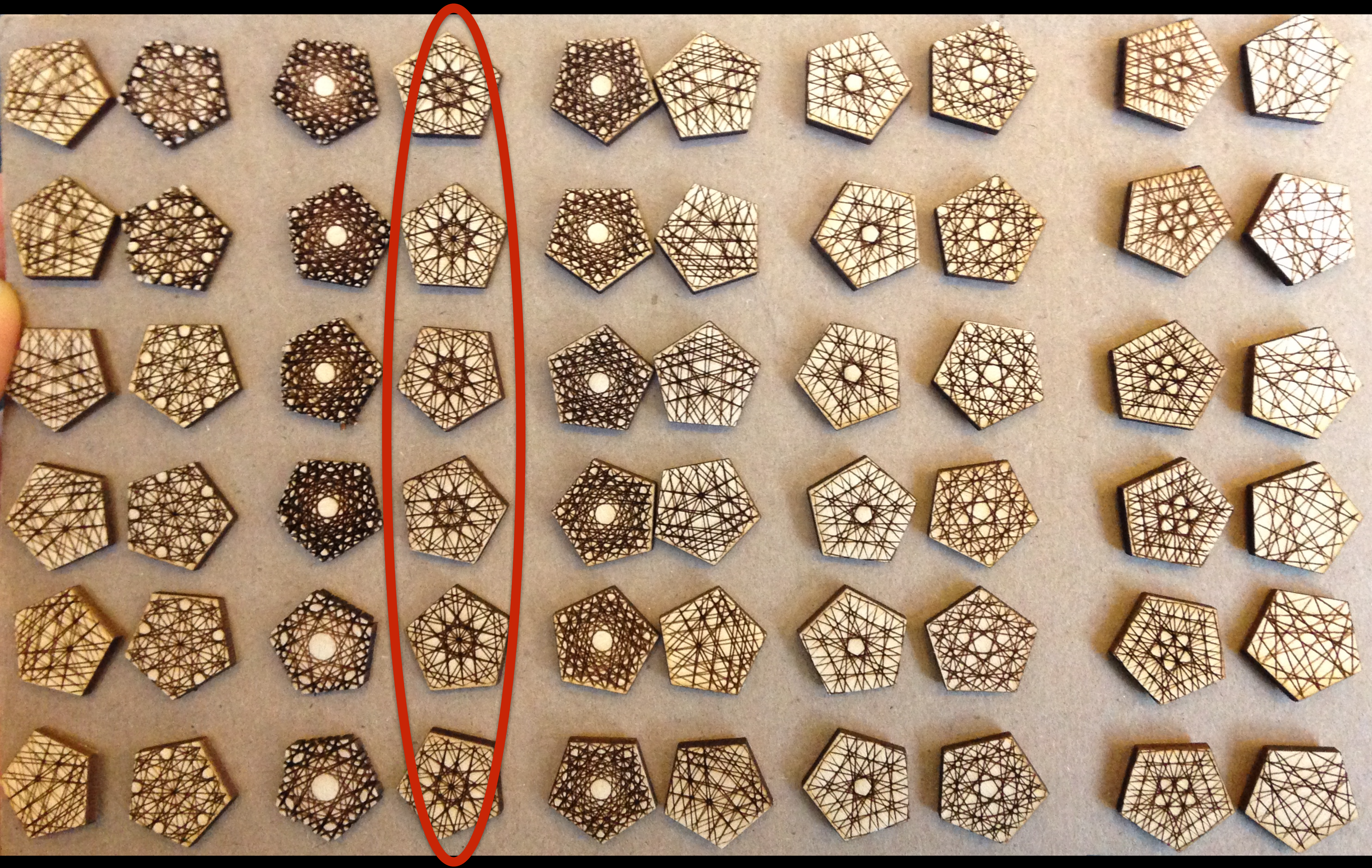
Library of pentagon billiard trajectories



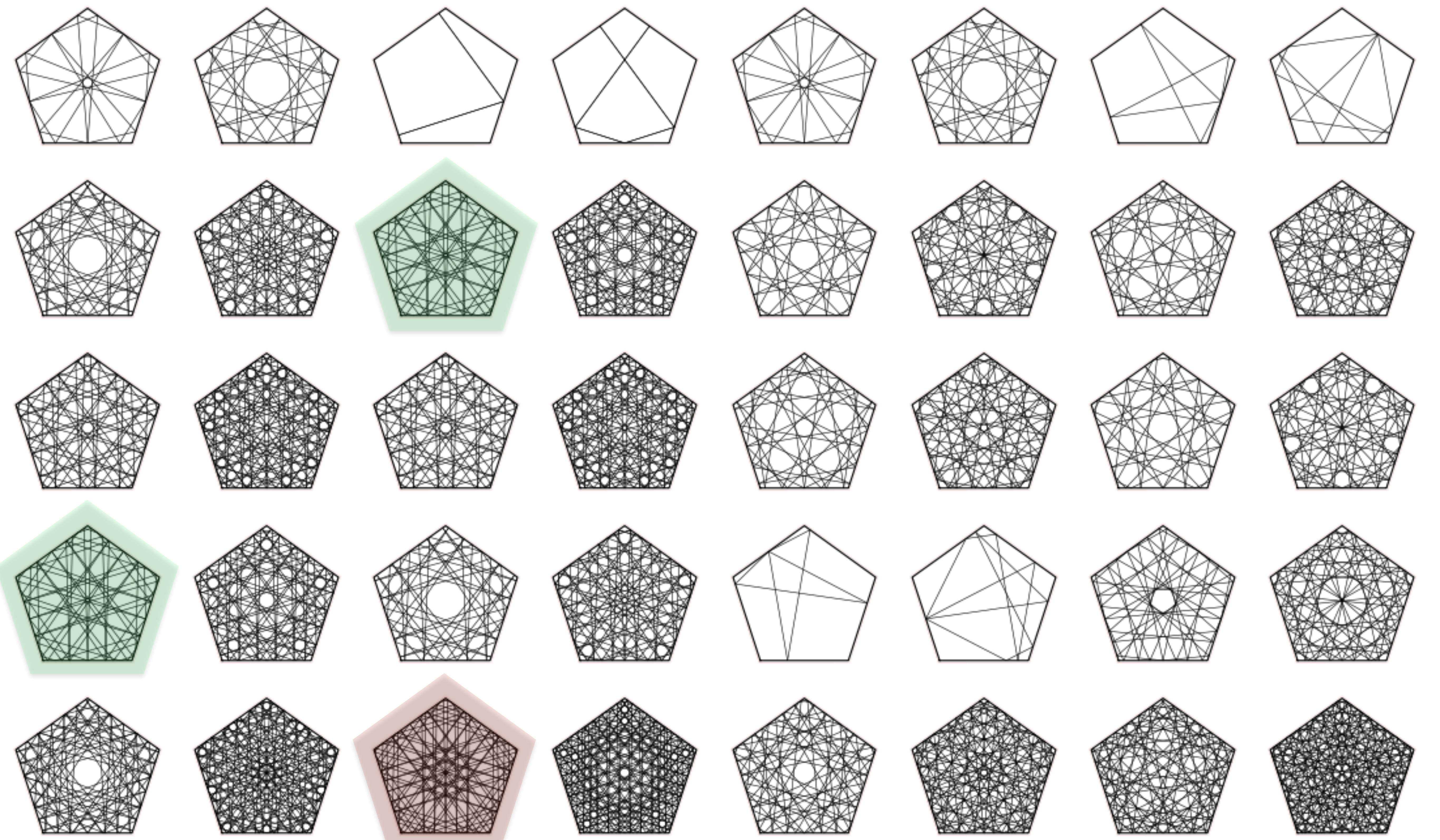
Which one is your favorite?



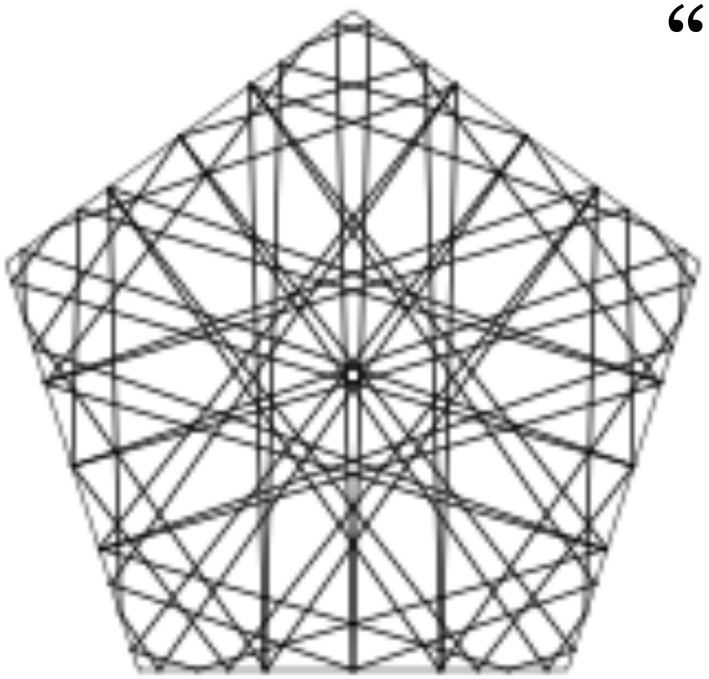
One of them is a clear favorite...



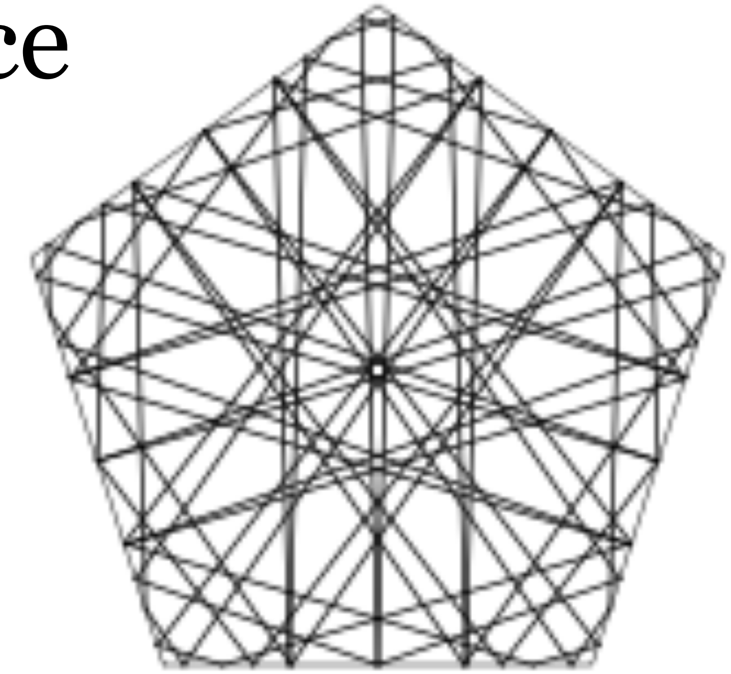
One of them is a clear favorite…
and it has a "family" of similar trajectories.



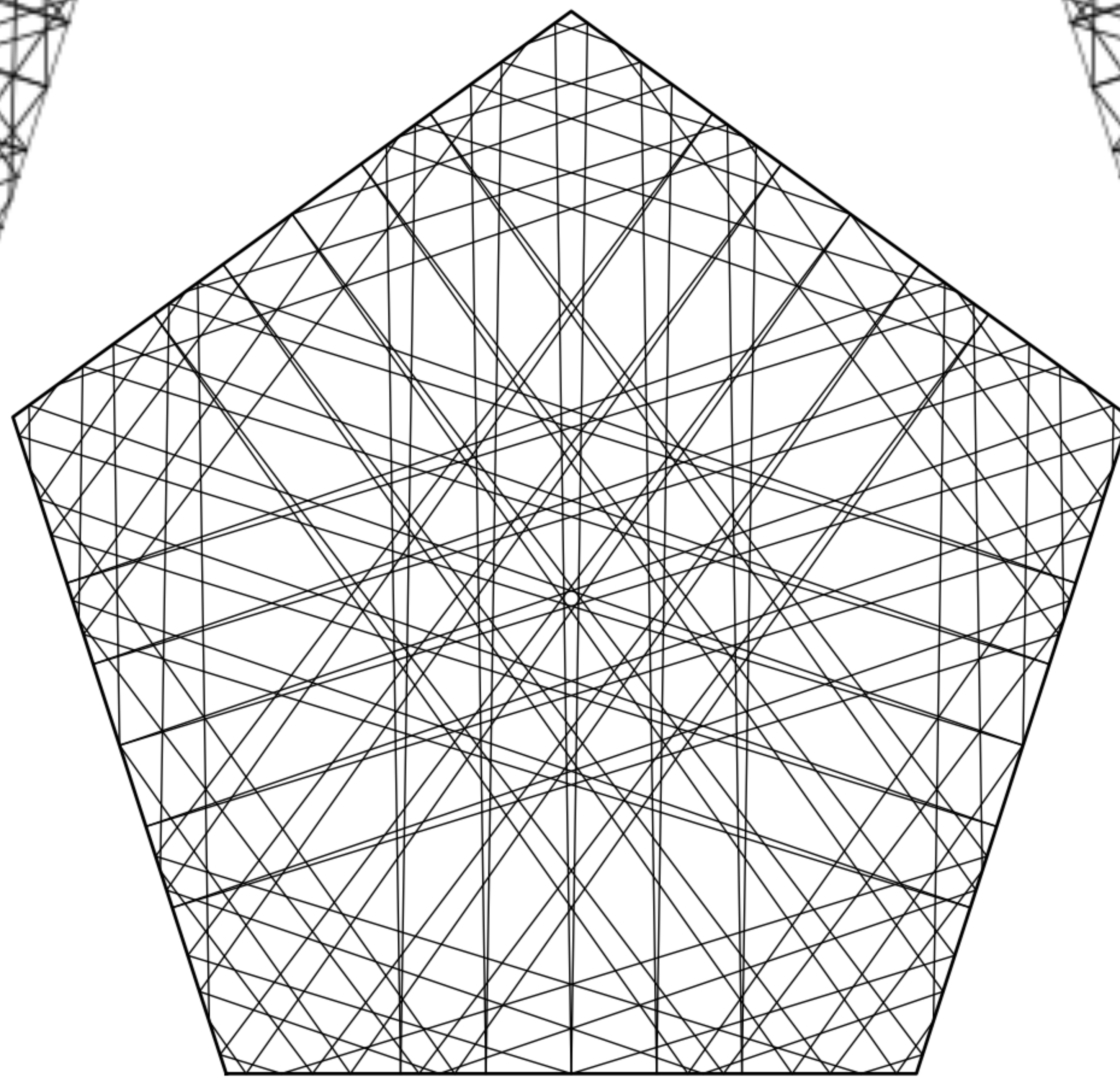
“Families” would make nice coordinating sets



12



12



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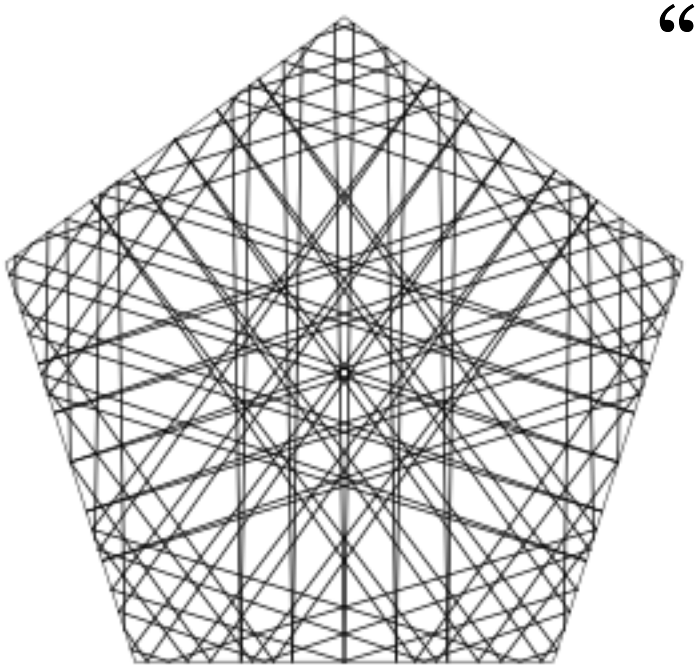
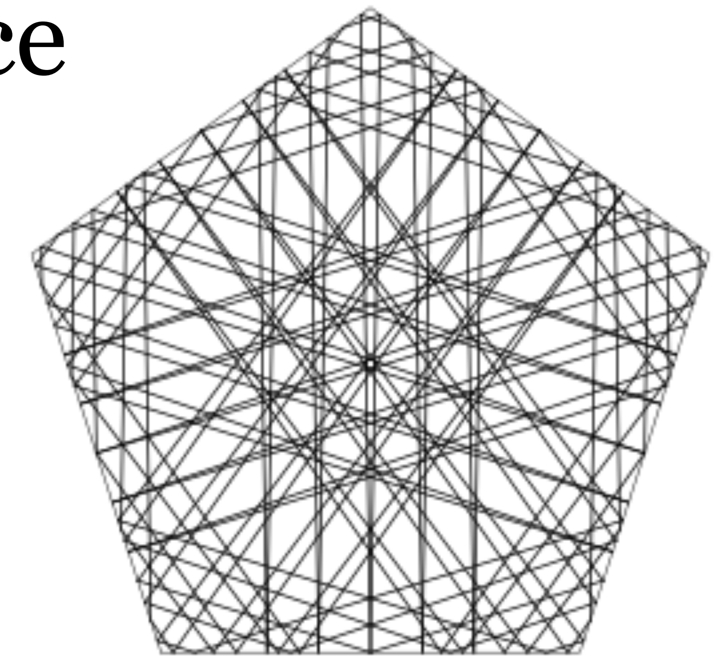
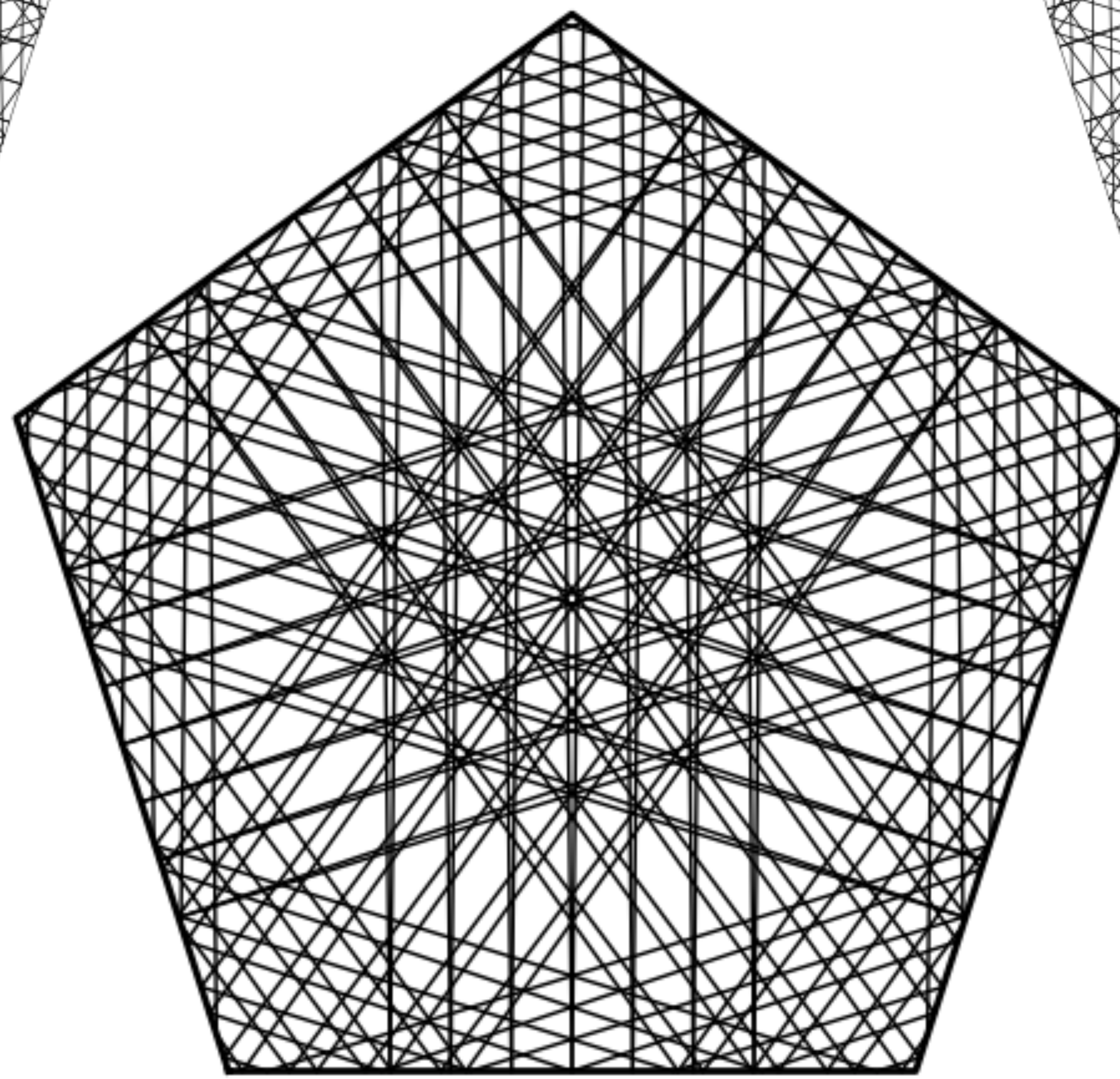
“Families” would make nice coordinating sets



102



102



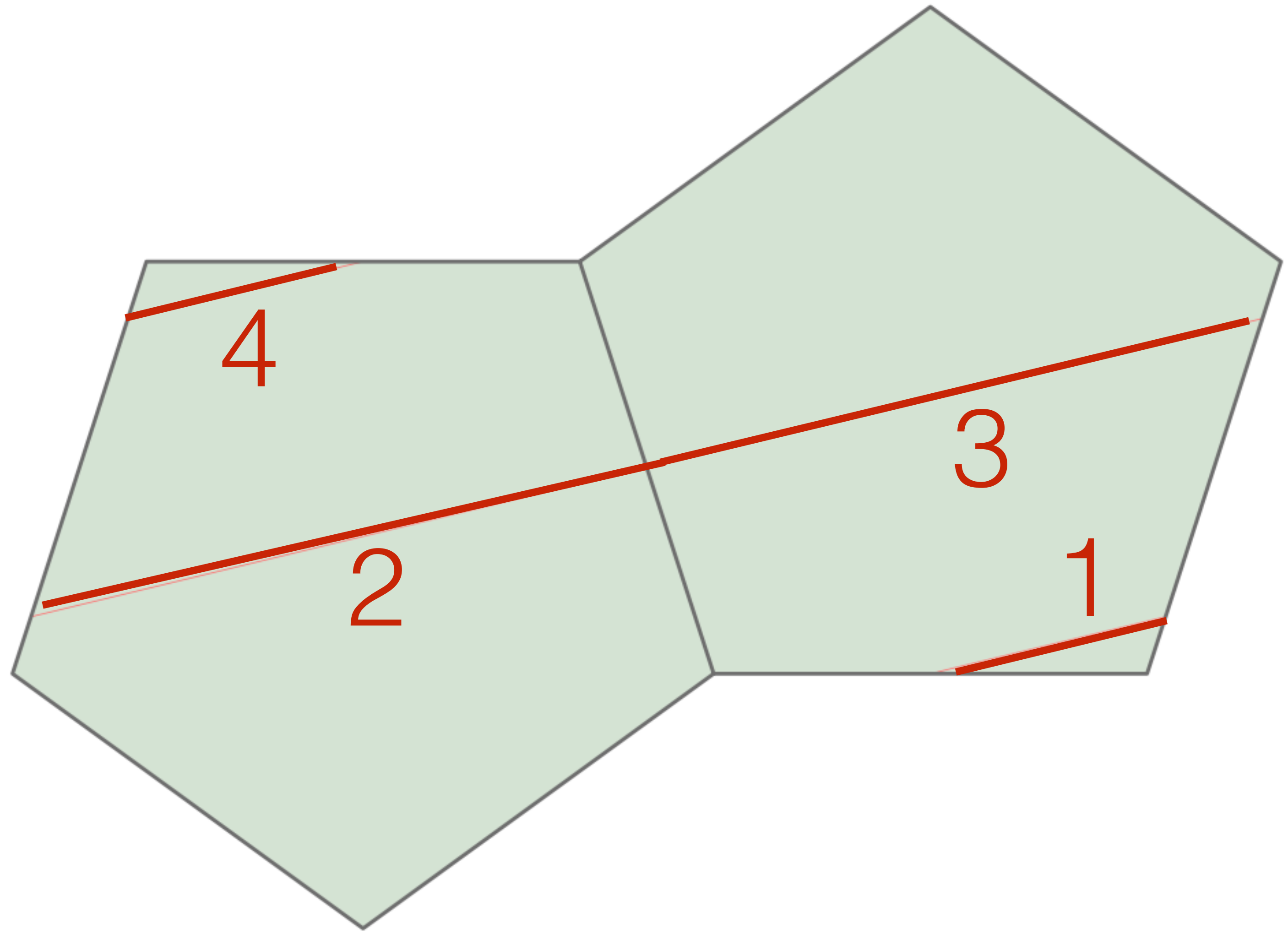
1002

Joint with

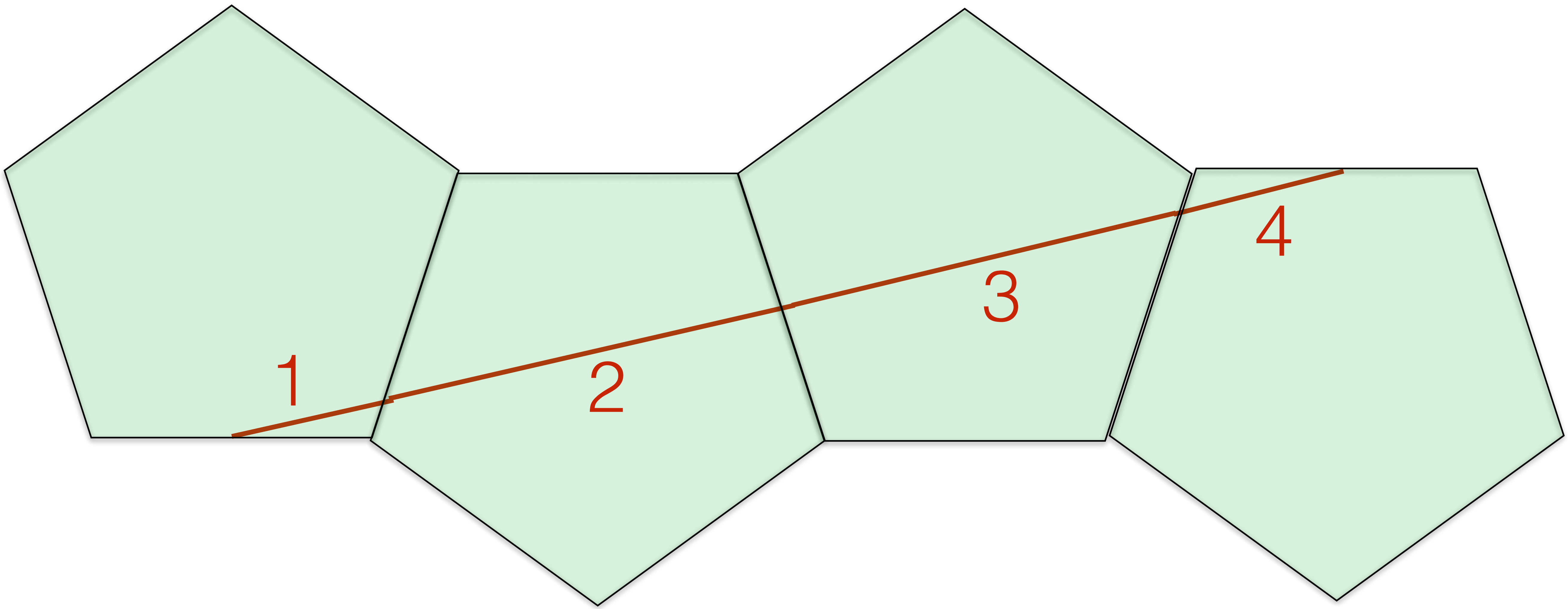
Barak Weiss
Tel Aviv University

Barak's great idea:
"twist it over and over"

Start with a trajectory on the double pentagon surface.

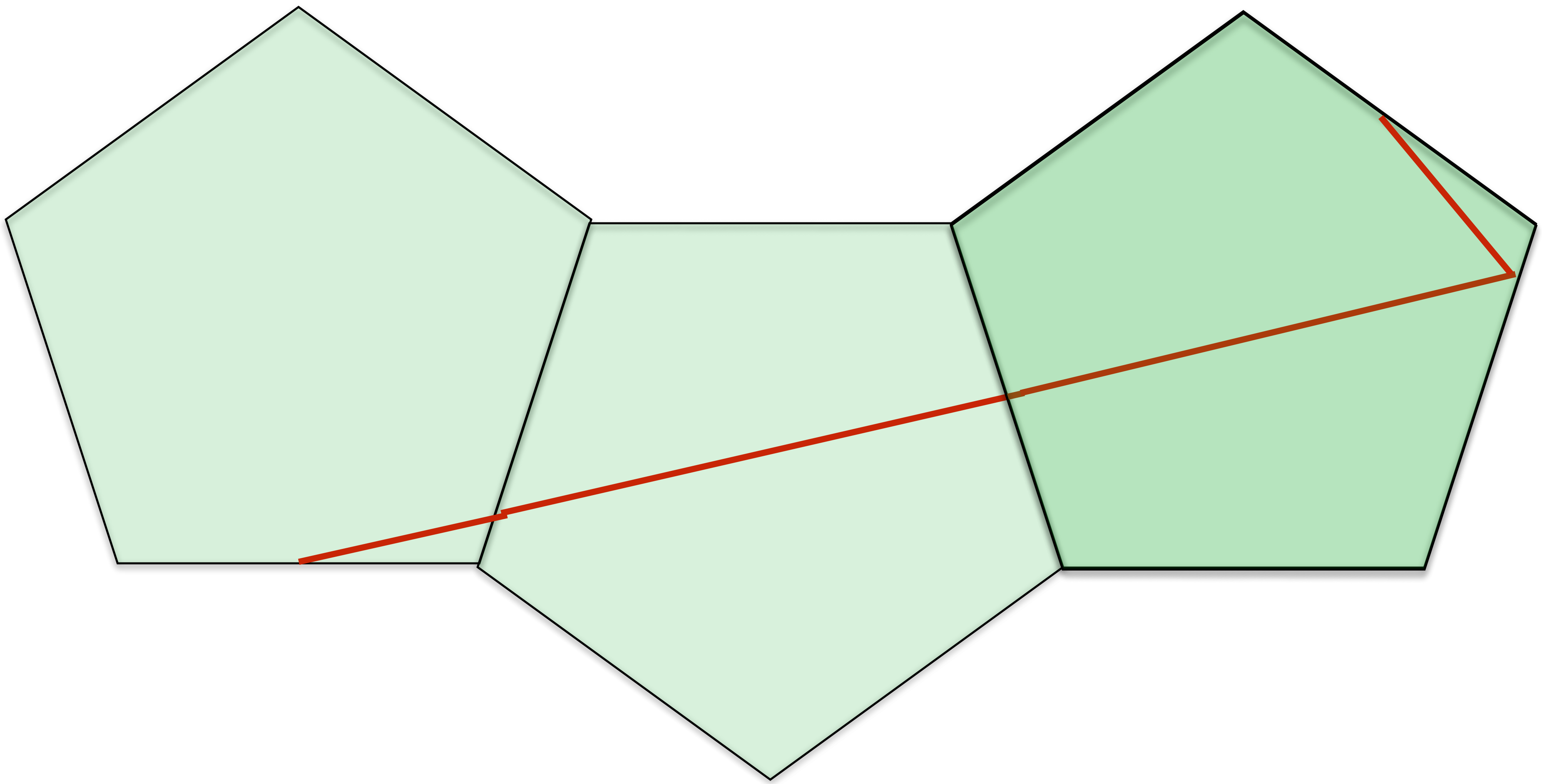


Start with a trajectory
on the double pentagon surface.



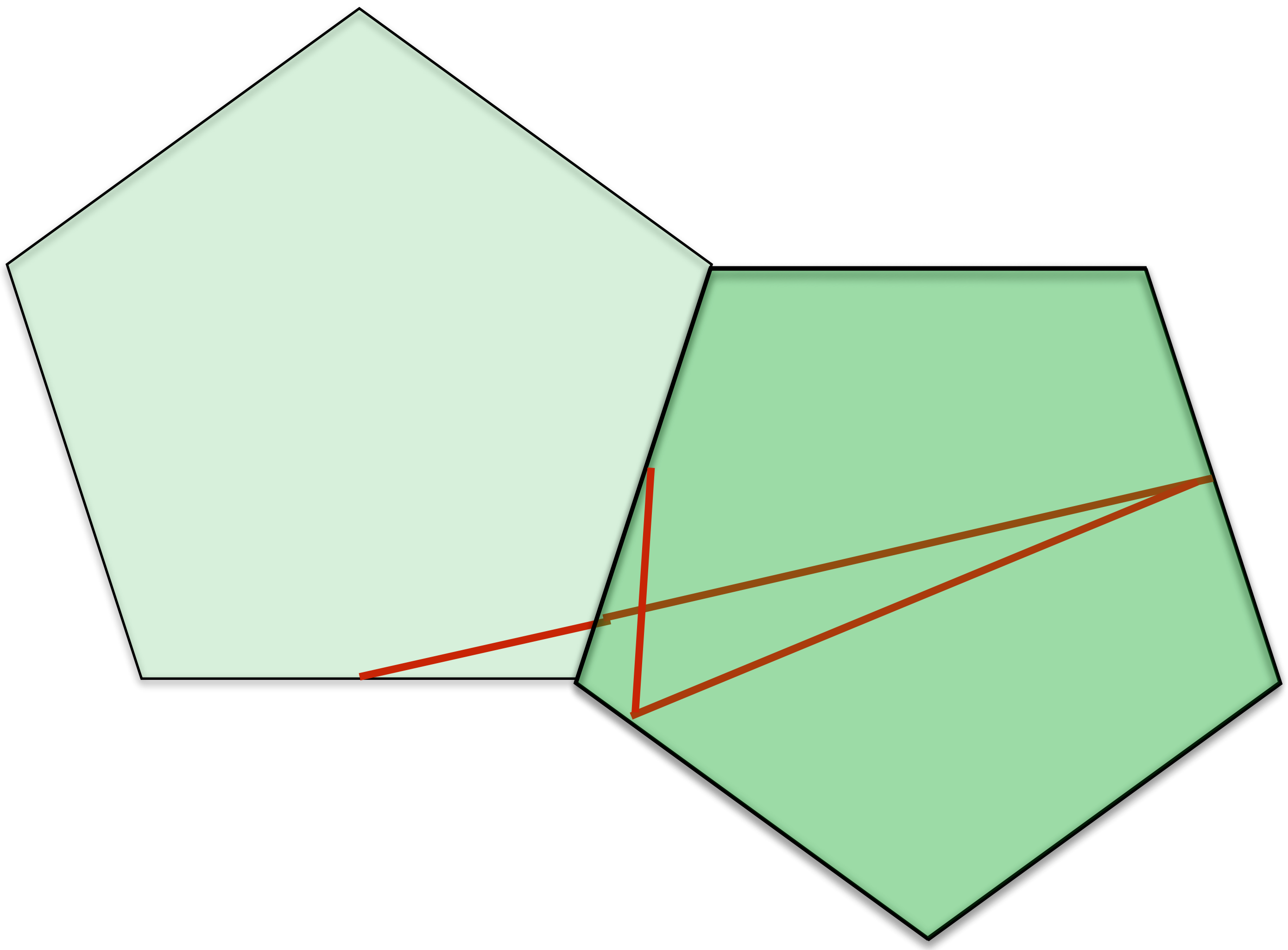
Fold each pentagon onto the previous one
to get a billiard trajectory.

Start with a trajectory
on the double pentagon surface.



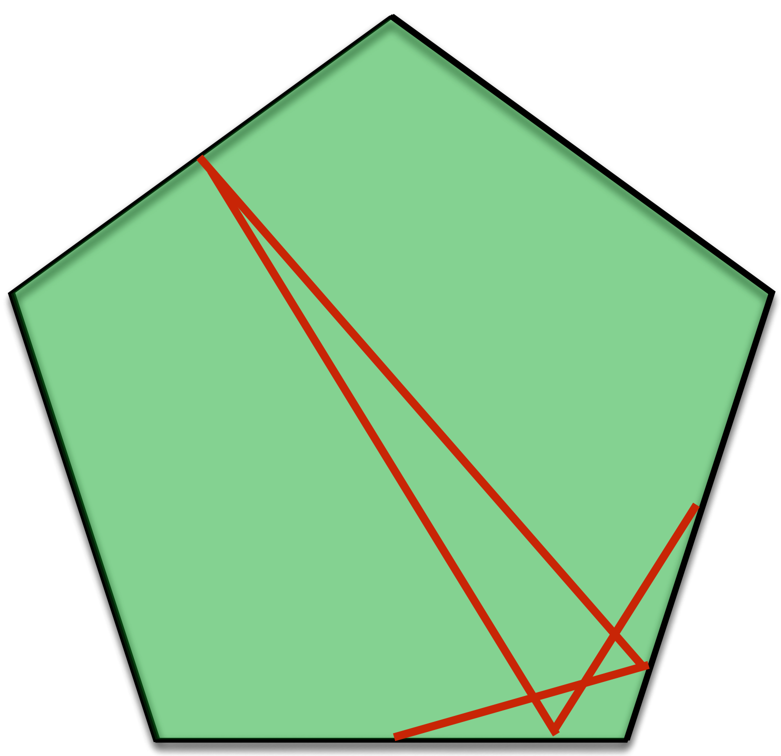
Fold each pentagon onto the previous one
to get a billiard trajectory.

Start with a trajectory
on the double pentagon surface.



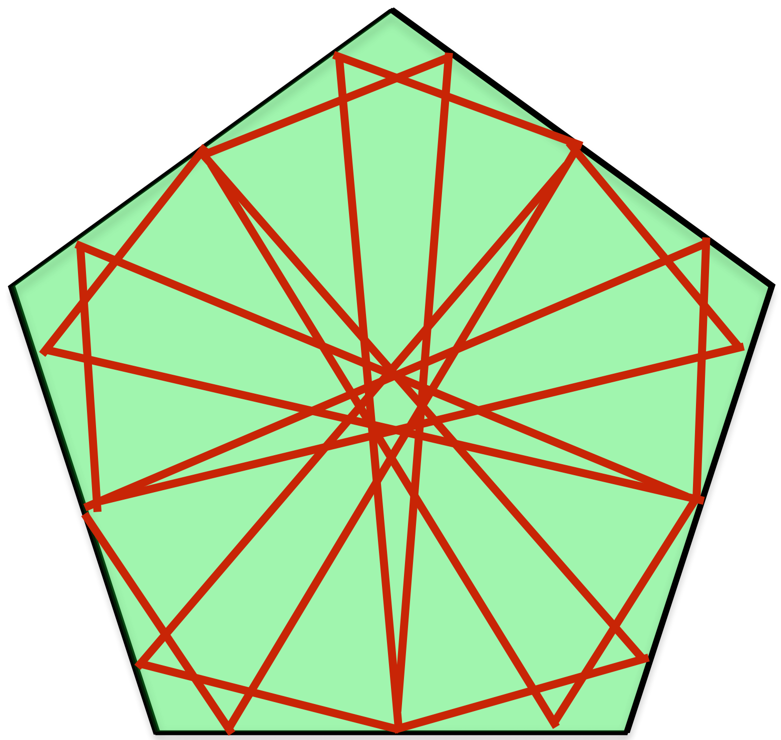
Fold each pentagon onto the previous one
to get a billiard trajectory.

Start with a trajectory
on the double pentagon surface.



Fold each pentagon onto the previous one
to get a billiard trajectory.

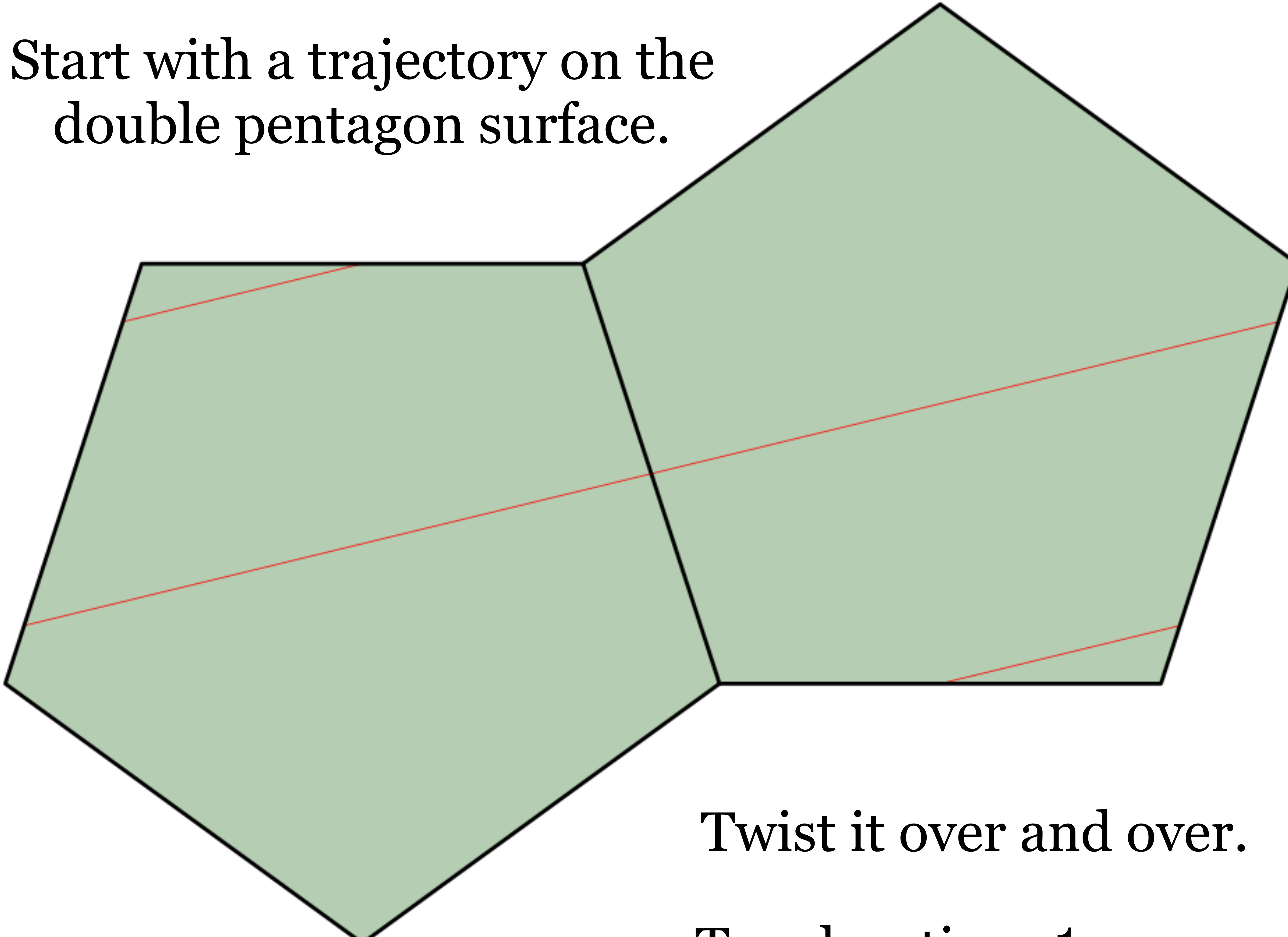
Start with a trajectory
on the double pentagon surface.



Repeat (rotate) it
5 times to get back
where we started

Fold each pentagon onto the previous one
to get a billiard trajectory.

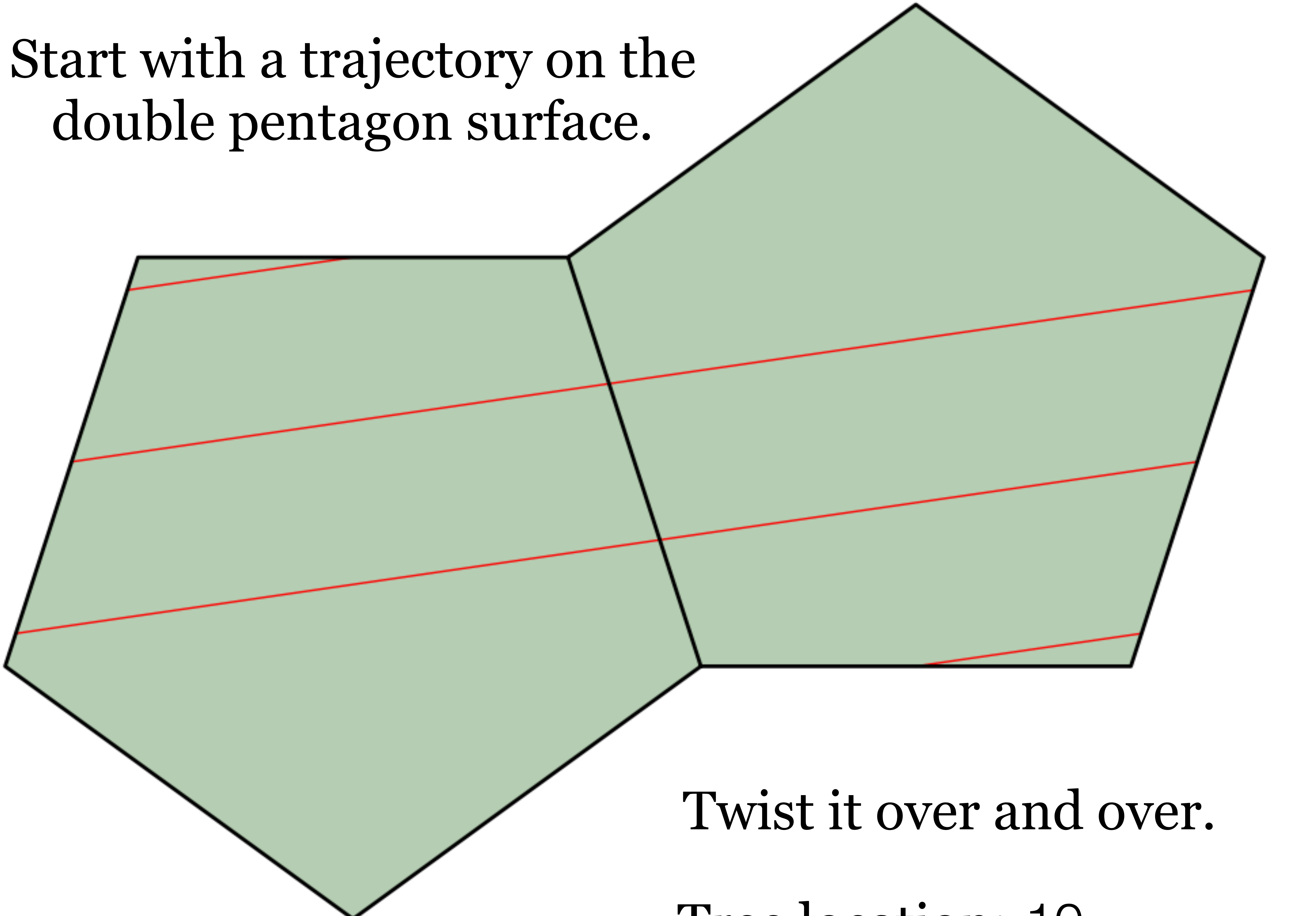
Start with a trajectory on the double pentagon surface.



Twist it over and over.

Tree location: 1

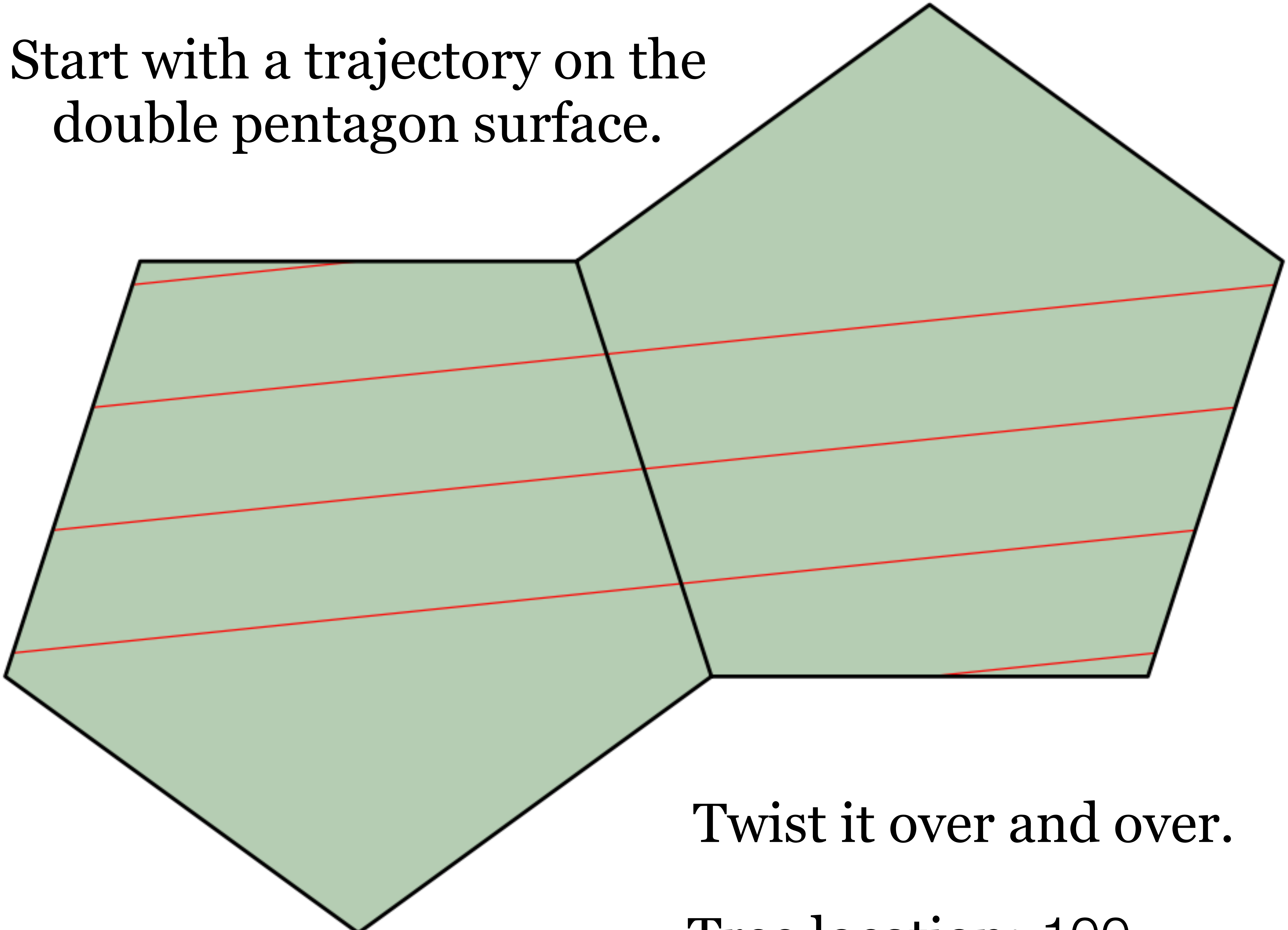
Start with a trajectory on the double pentagon surface.



Twist it over and over.

Tree location: 10

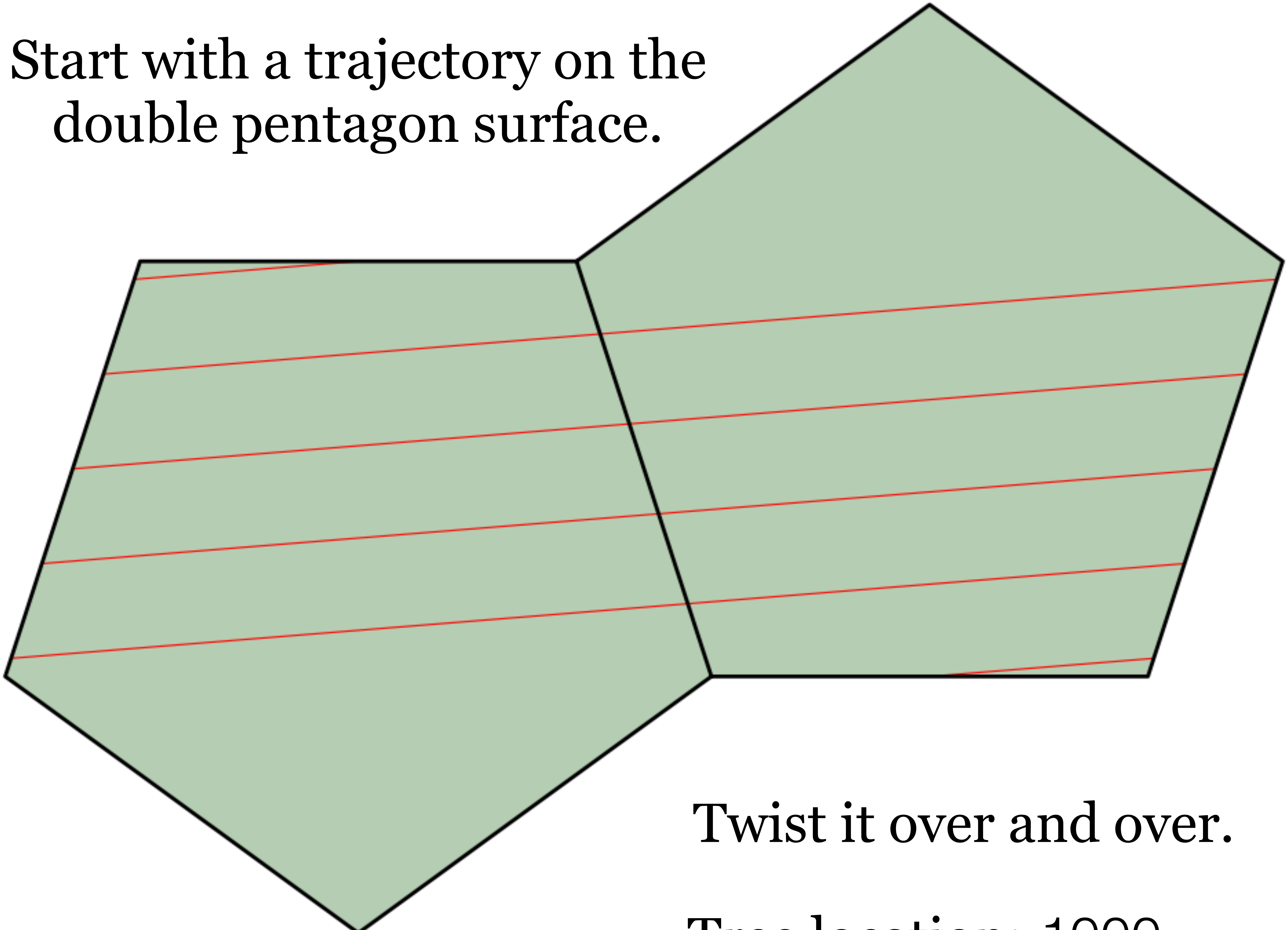
Start with a trajectory on the double pentagon surface.



Twist it over and over.

Tree location: 100

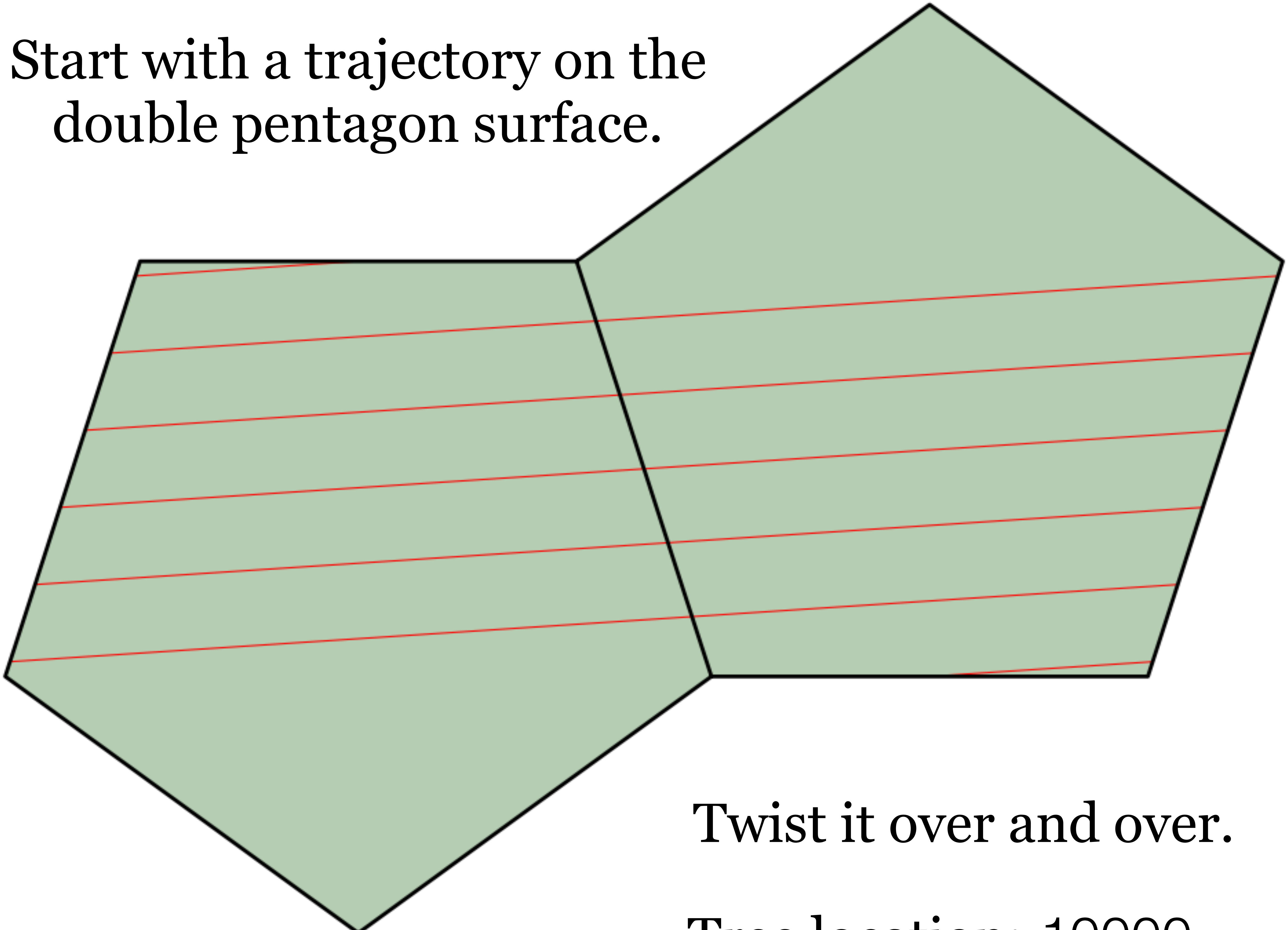
Start with a trajectory on the double pentagon surface.



Twist it over and over.

Tree location: 1000

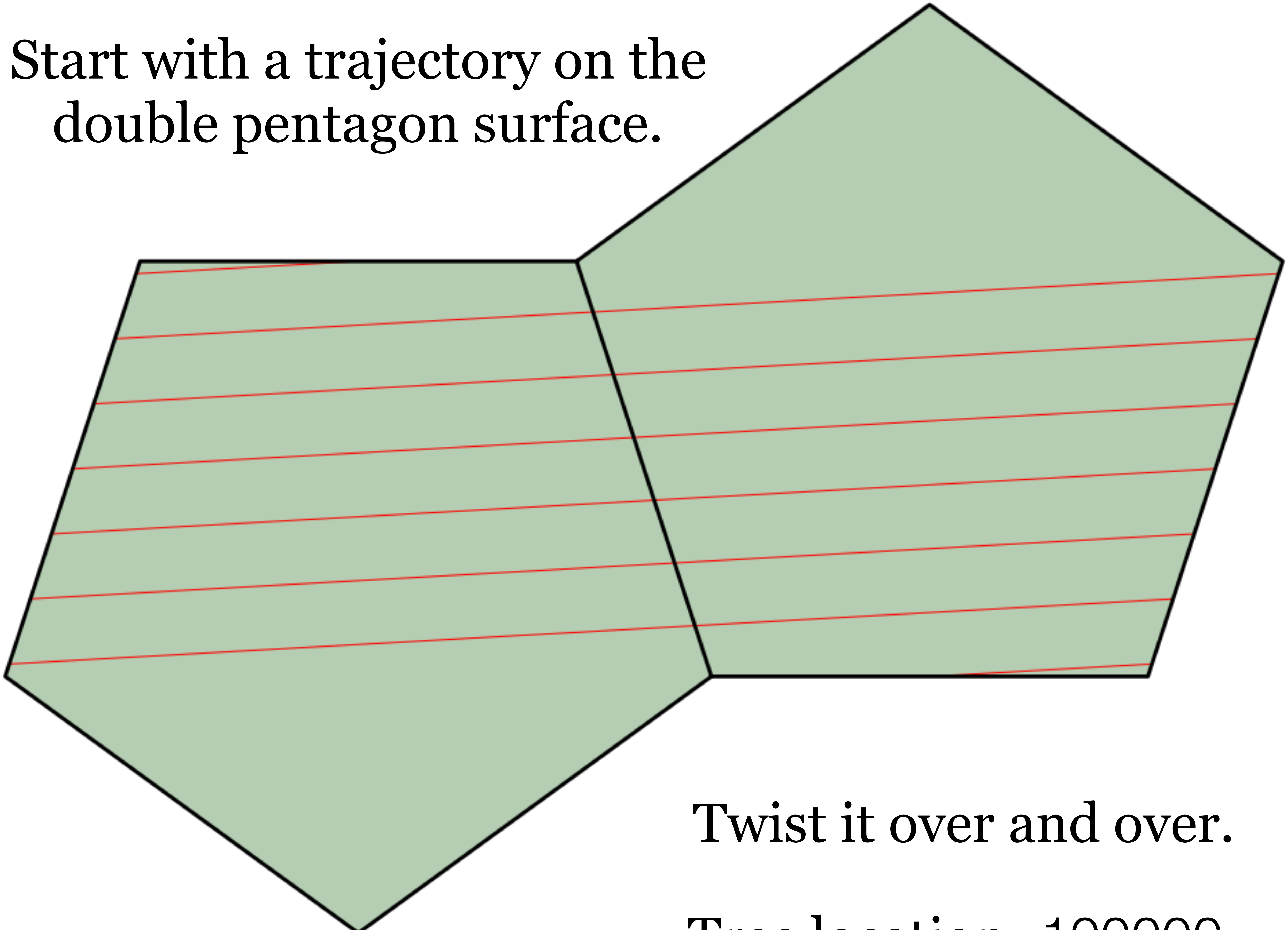
Start with a trajectory on the double pentagon surface.



Twist it over and over.

Tree location: 10000

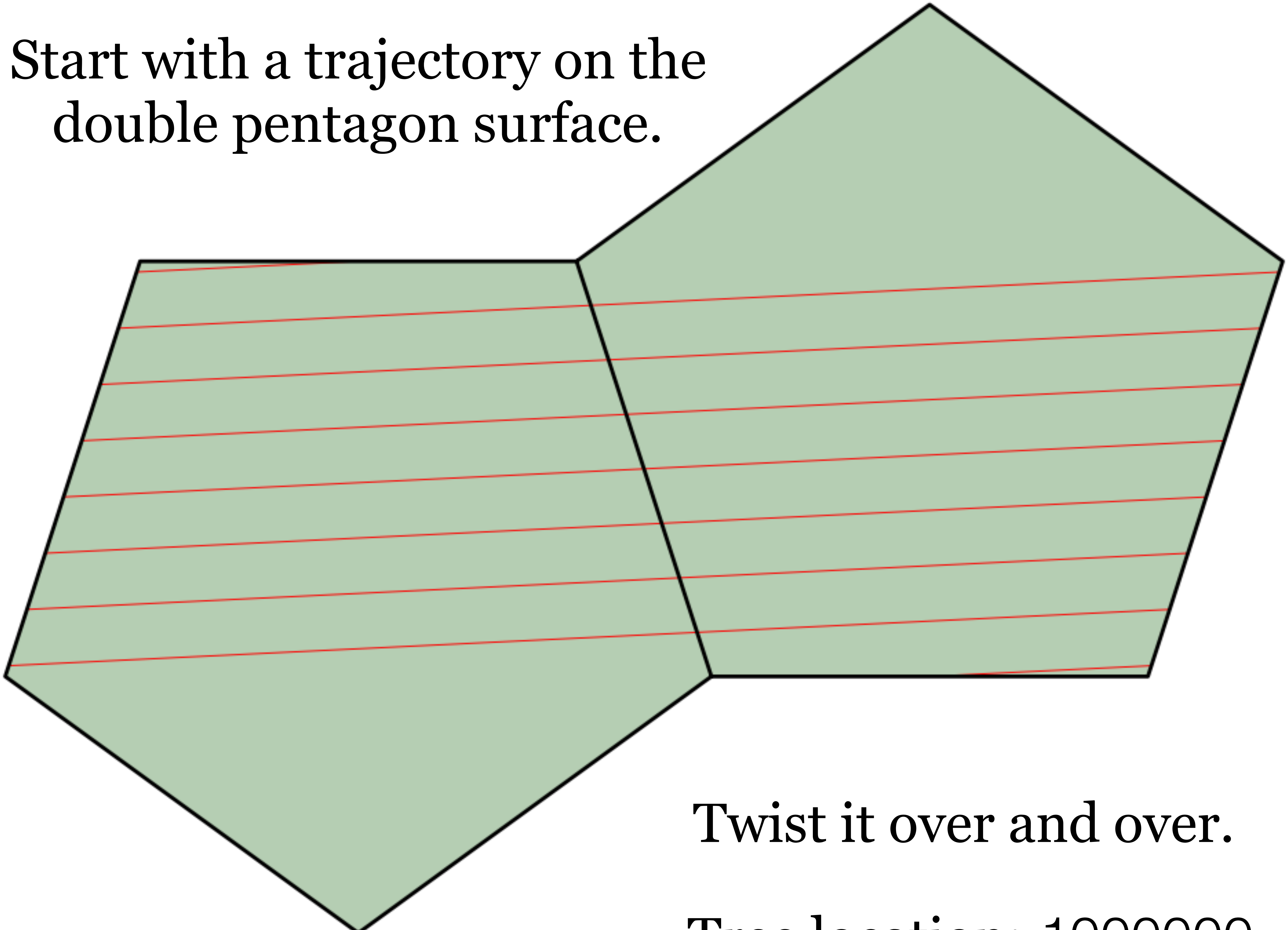
Start with a trajectory on the double pentagon surface.



Twist it over and over.

Tree location: 100000

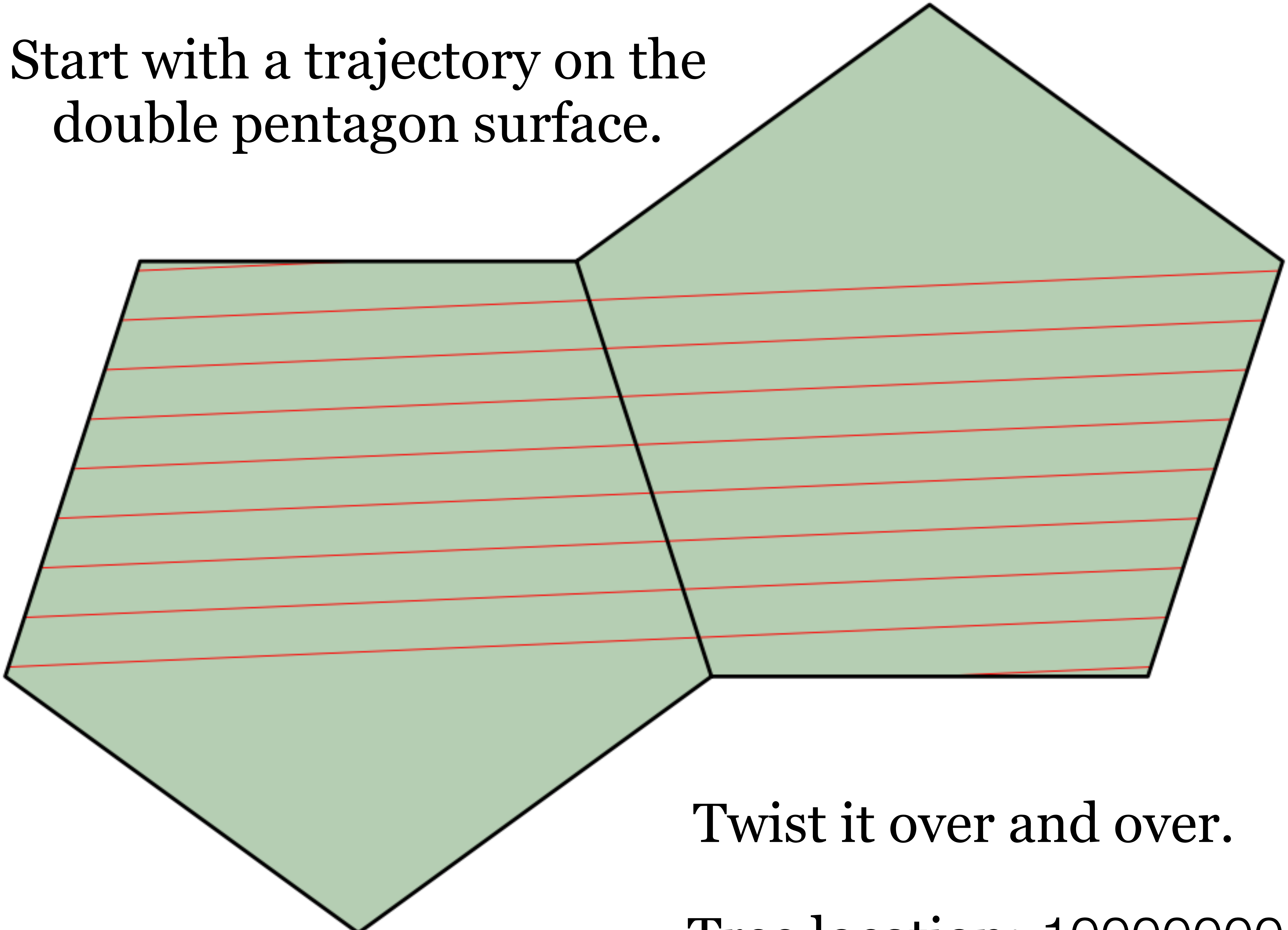
Start with a trajectory on the double pentagon surface.



Twist it over and over.

Tree location: 1000000

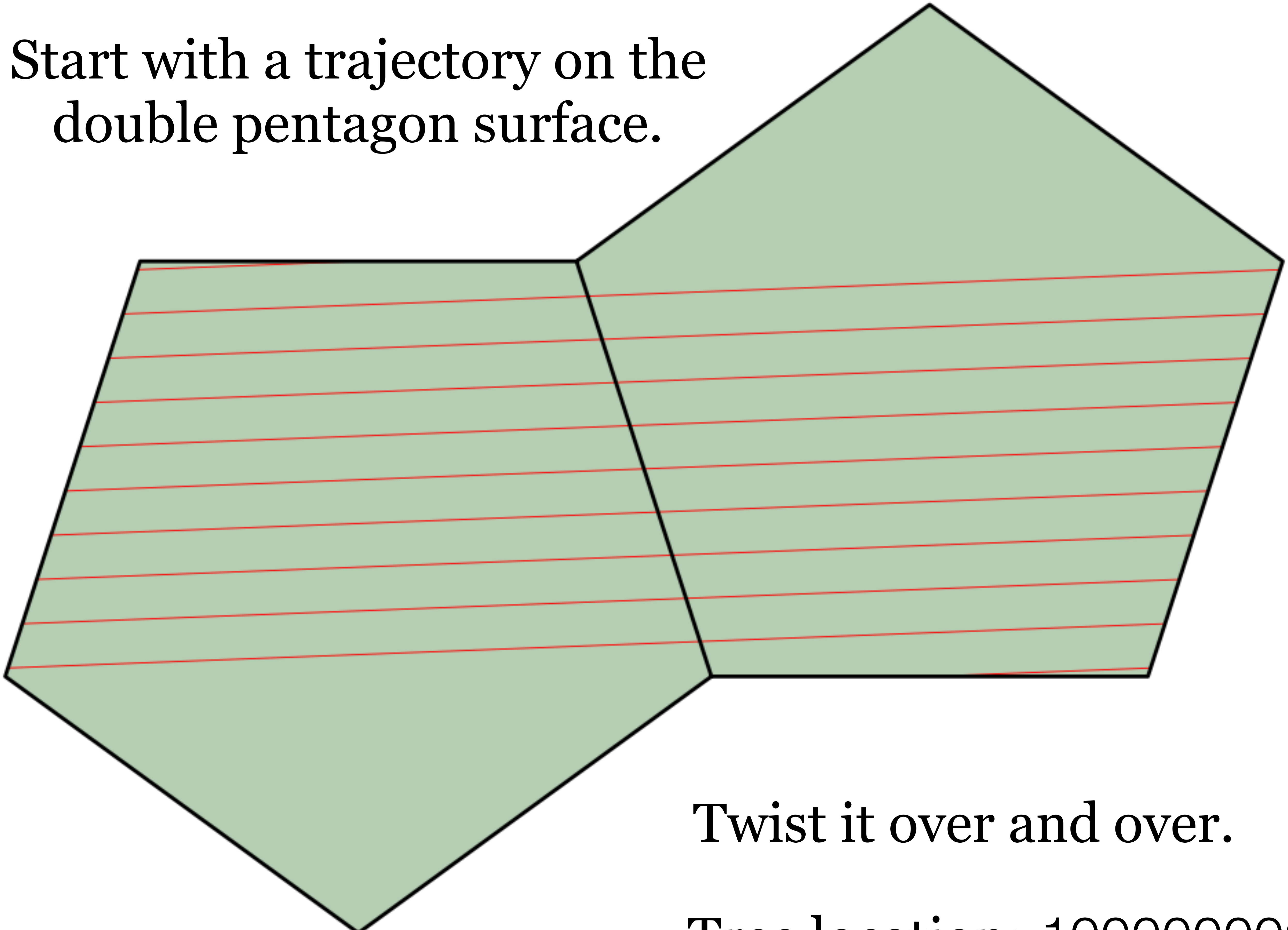
Start with a trajectory on the double pentagon surface.



Twist it over and over.

Tree location: 10000000

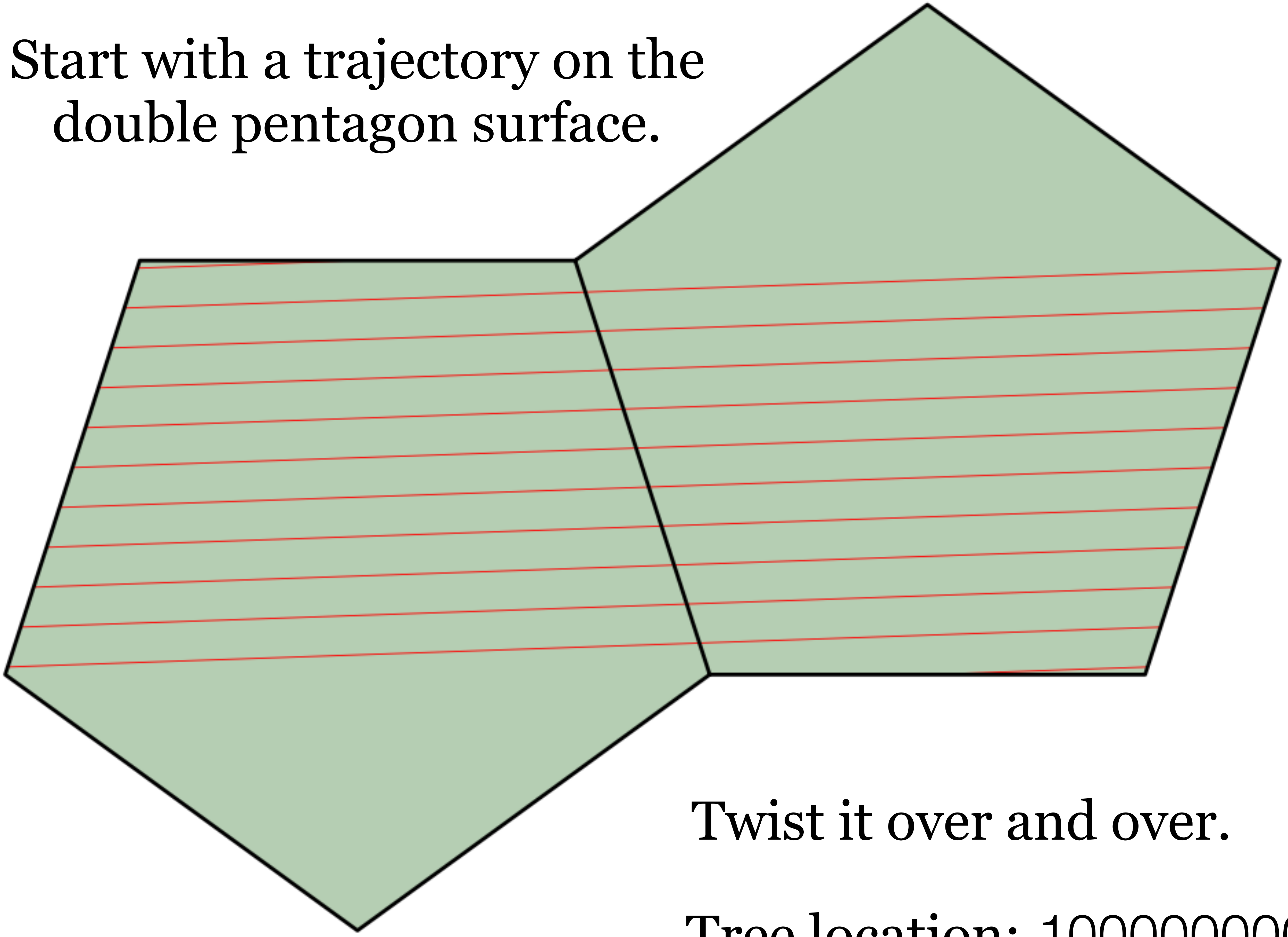
Start with a trajectory on the double pentagon surface.



Twist it over and over.

Tree location: 100000000

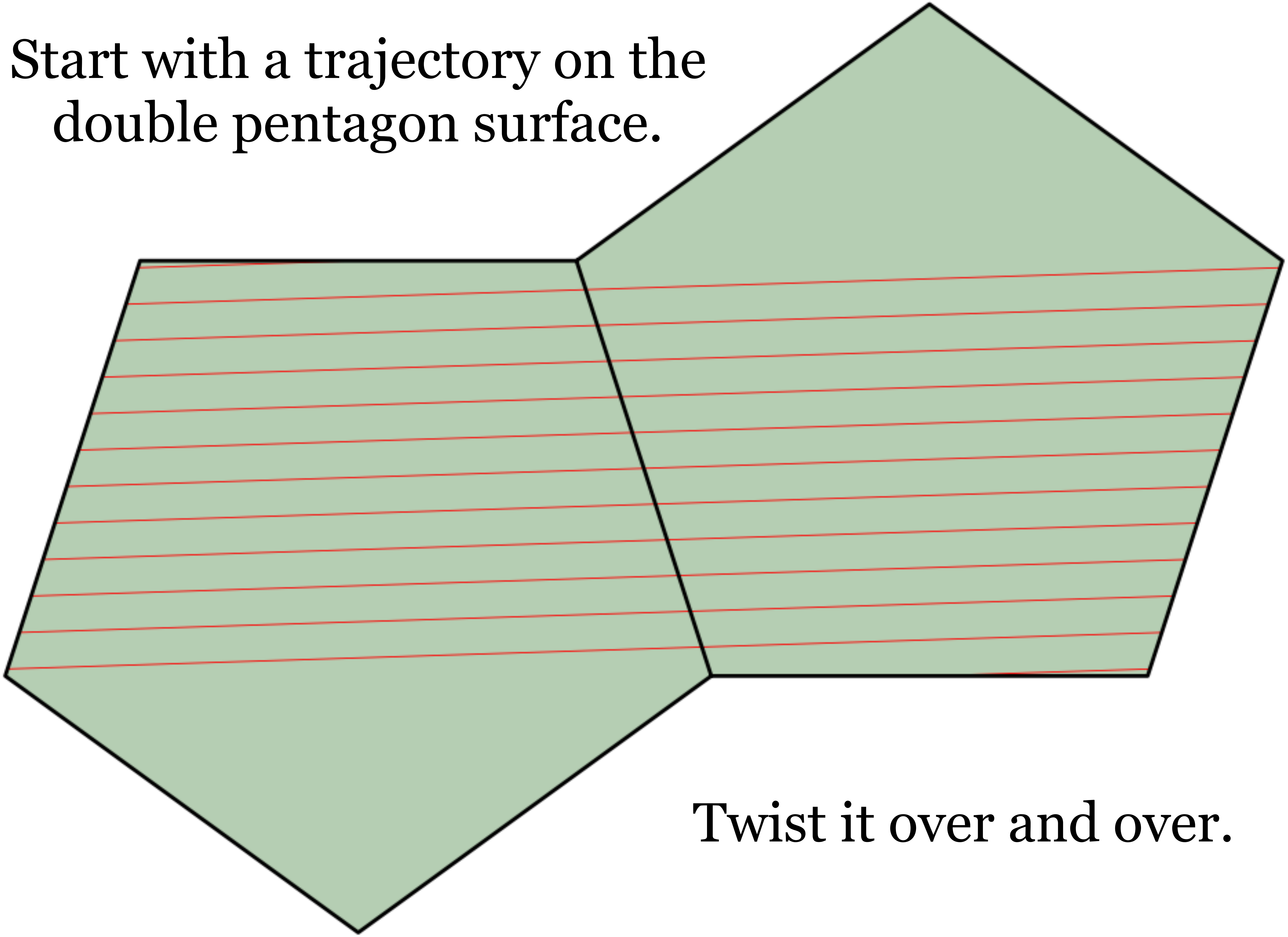
Start with a trajectory on the double pentagon surface.



Twist it over and over.

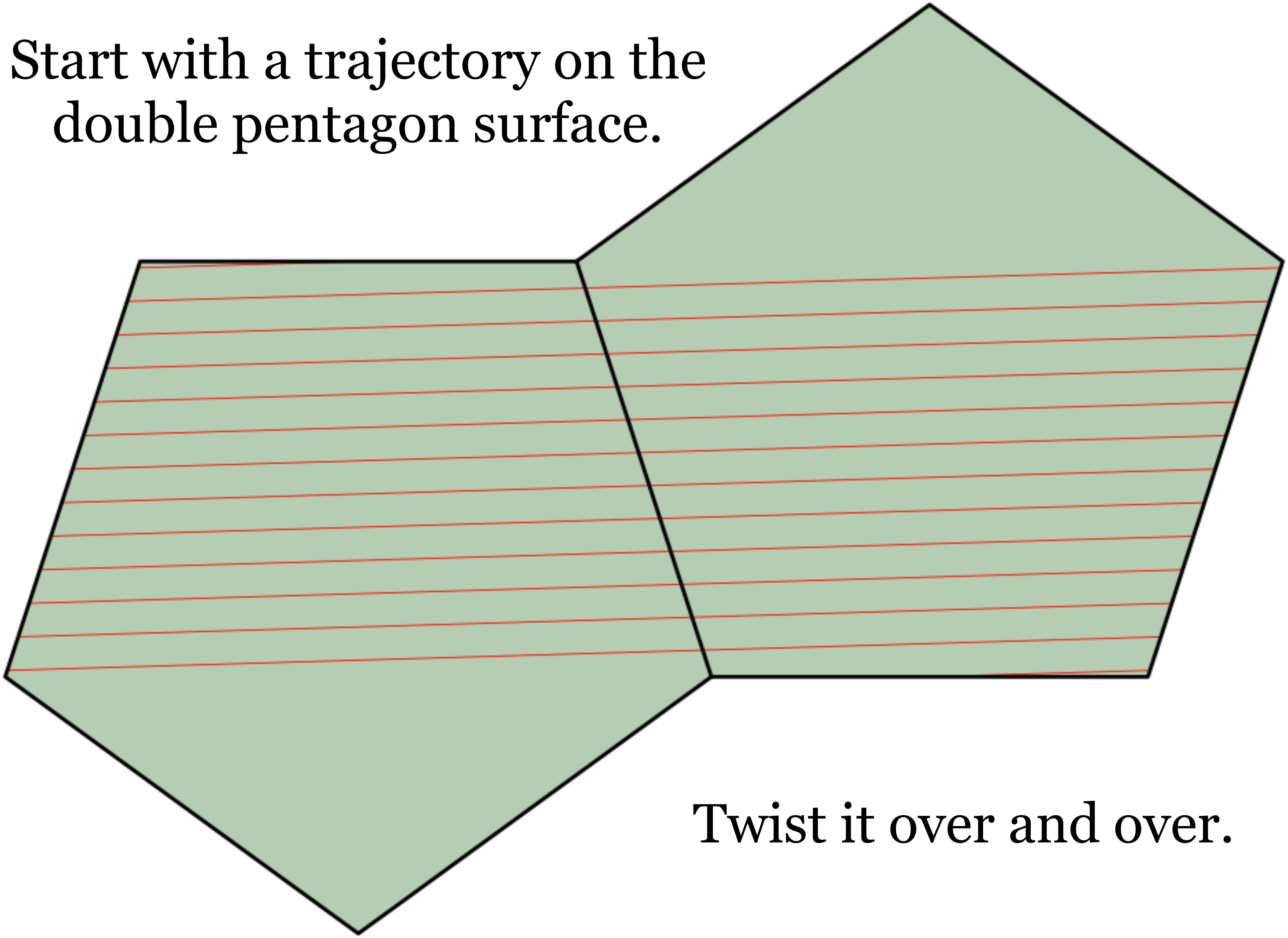
Tree location: 1000000000

Start with a trajectory on the double pentagon surface.



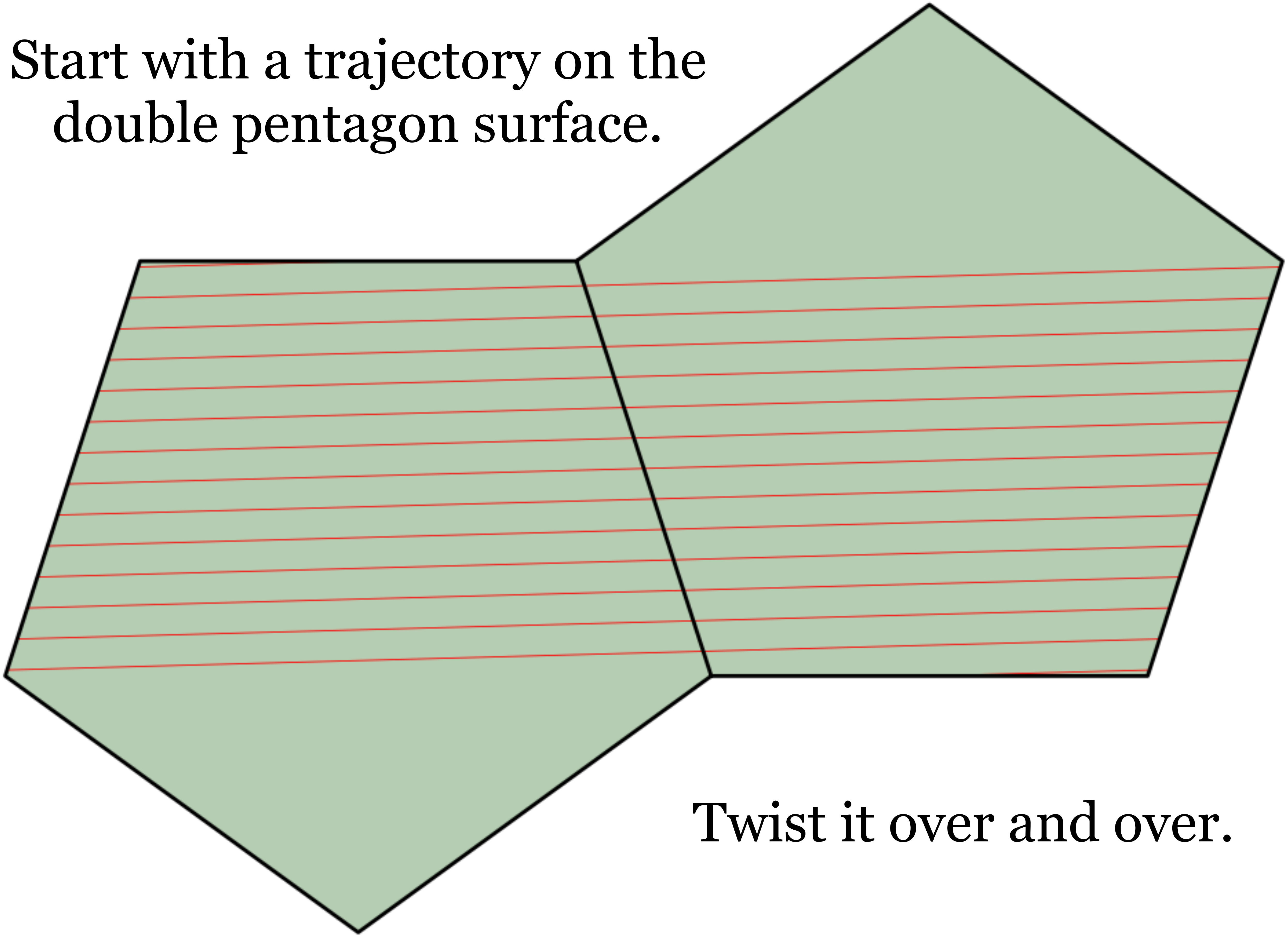
Twist it over and over.

Start with a trajectory on the double pentagon surface.



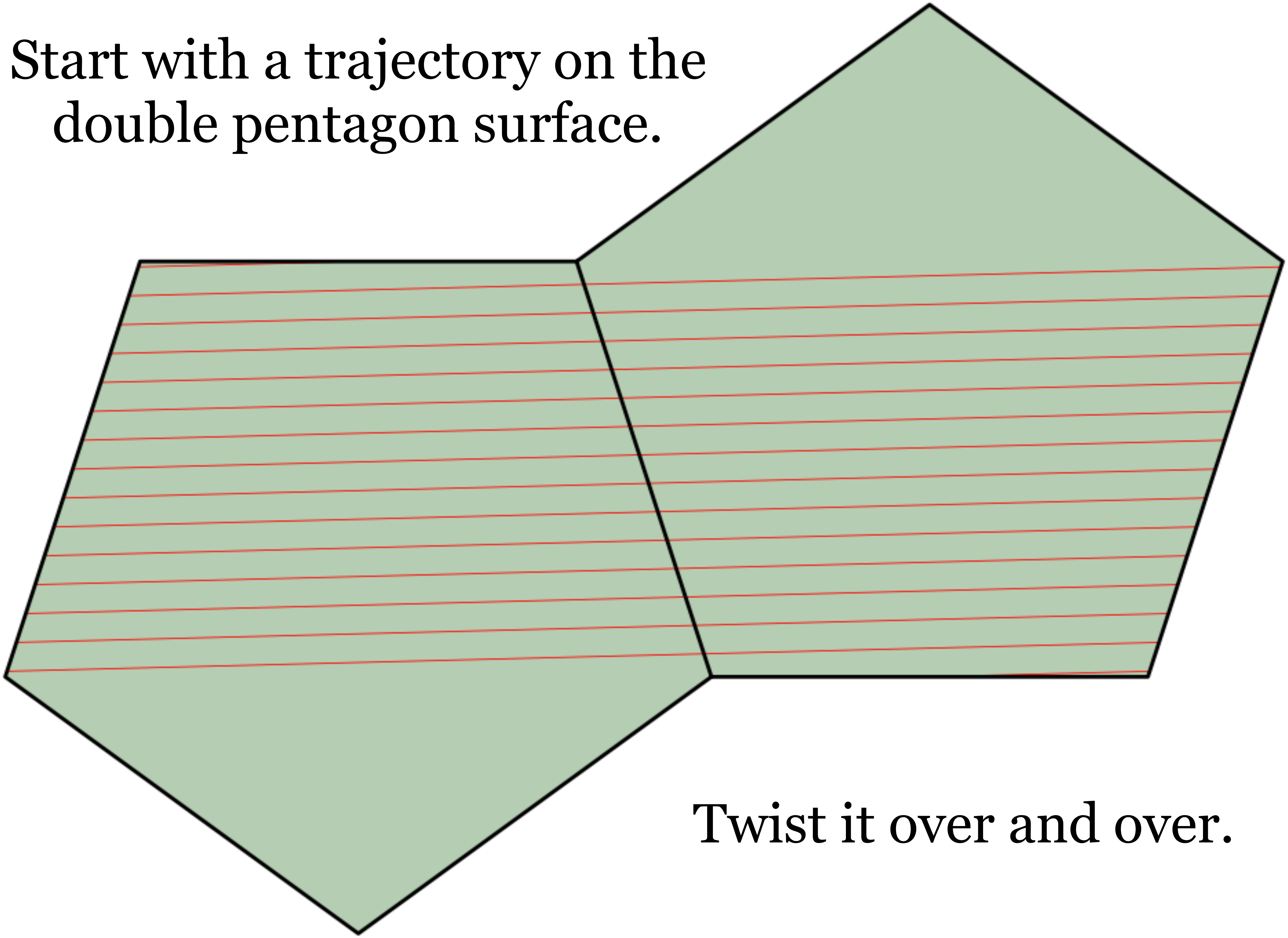
Twist it over and over.

Start with a trajectory on the double pentagon surface.



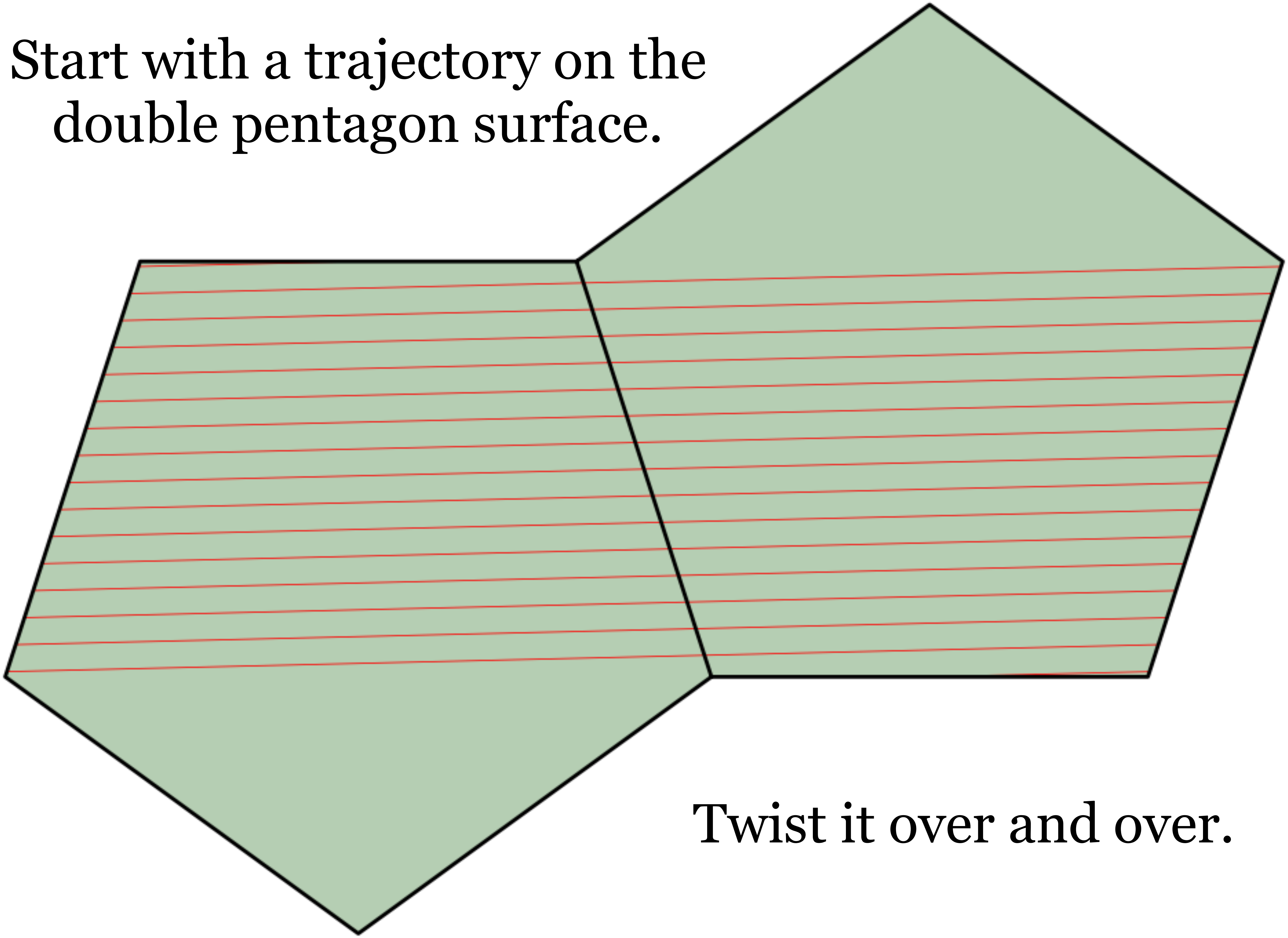
Twist it over and over.

Start with a trajectory on the double pentagon surface.



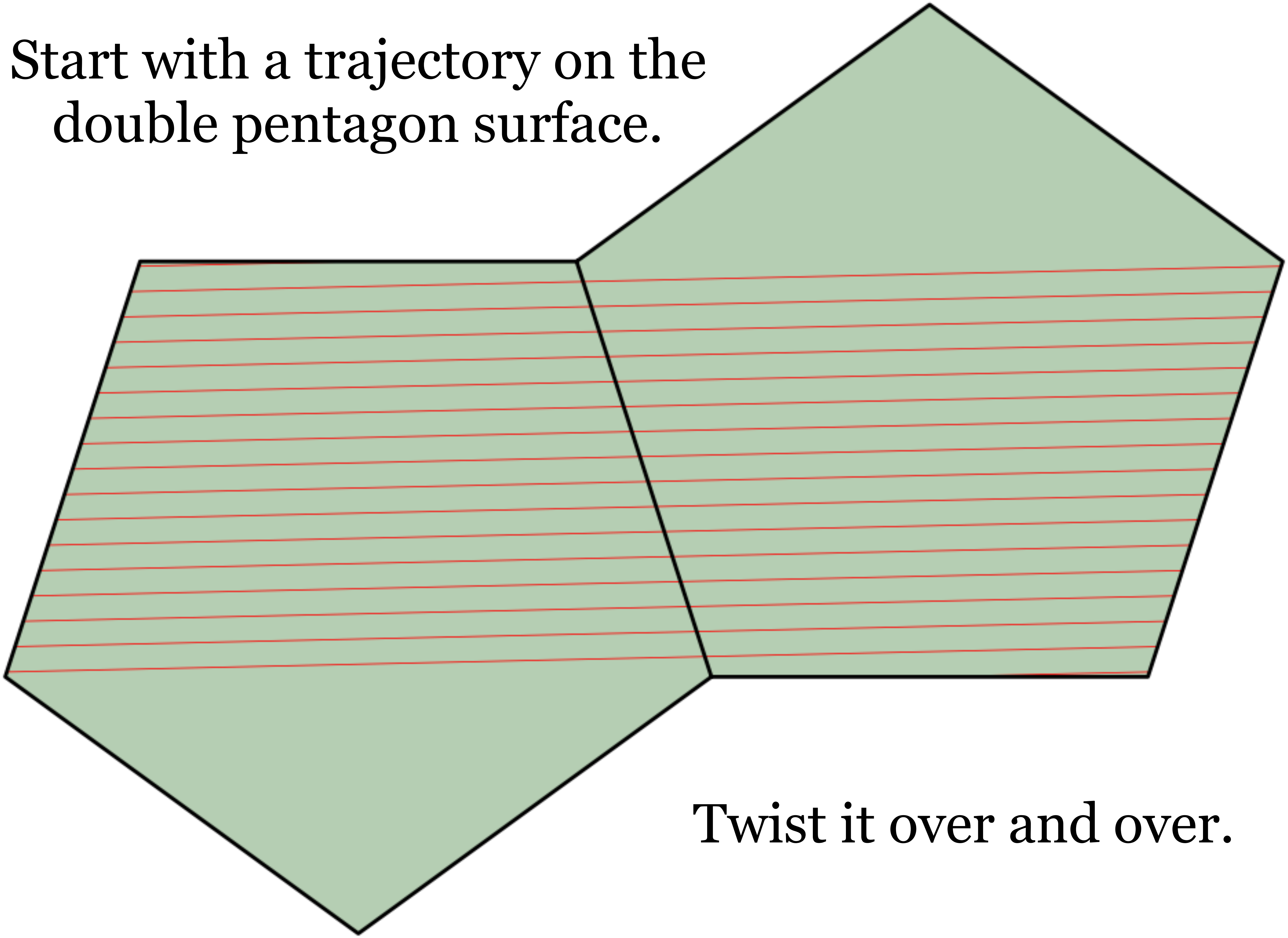
Twist it over and over.

Start with a trajectory on the double pentagon surface.



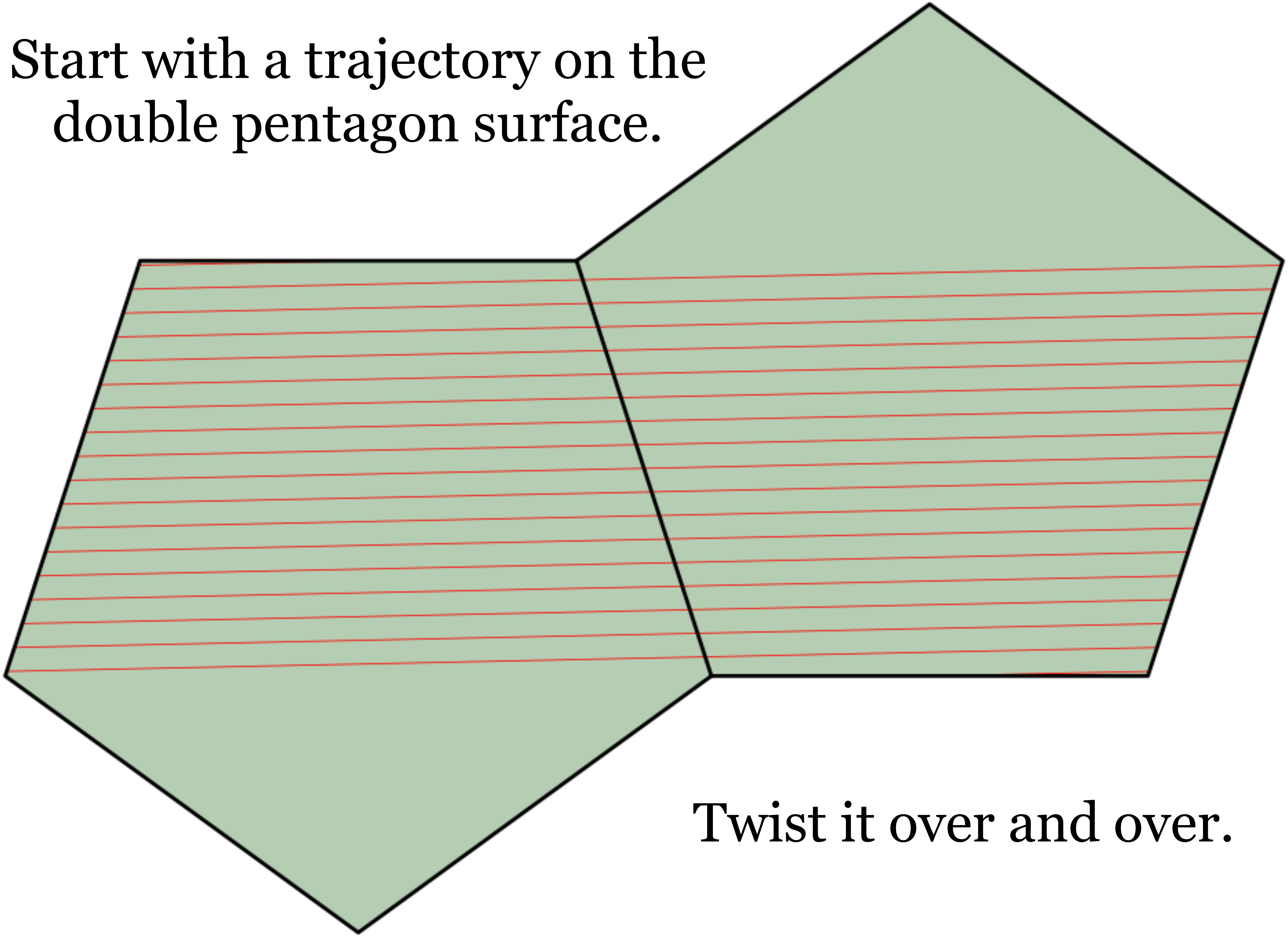
Twist it over and over.

Start with a trajectory on the double pentagon surface.



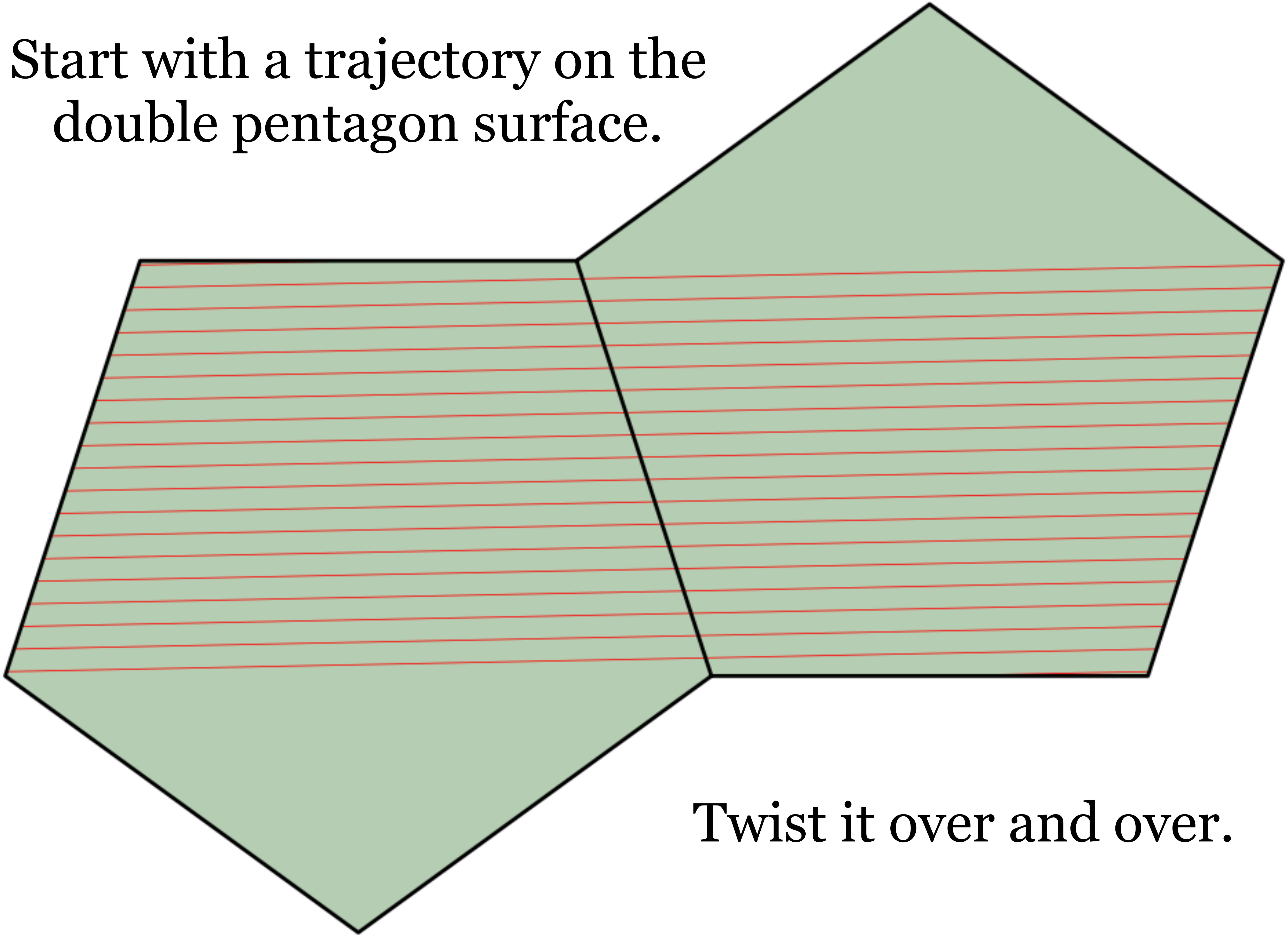
Twist it over and over.

Start with a trajectory on the double pentagon surface.



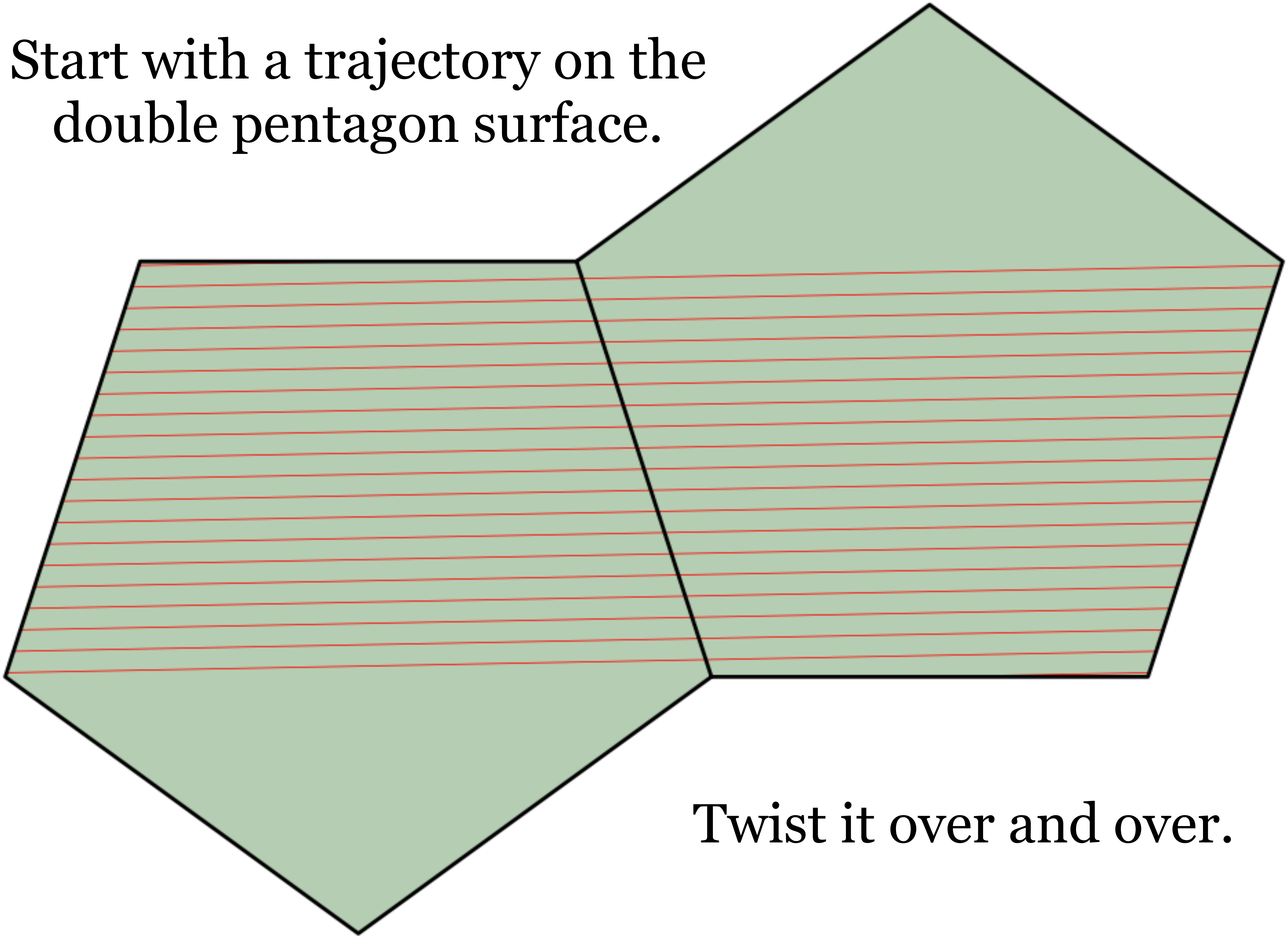
Twist it over and over.

Start with a trajectory on the double pentagon surface.



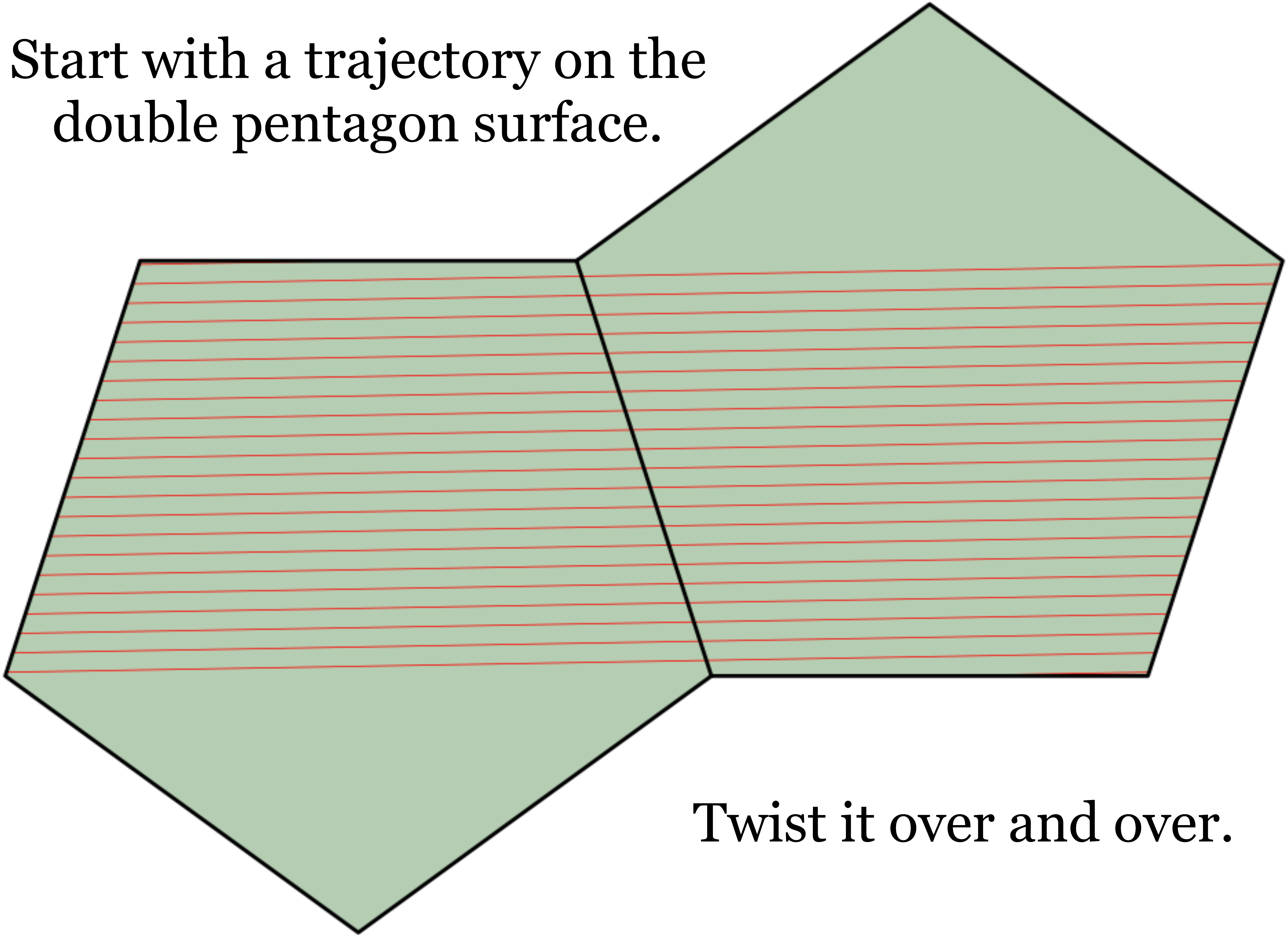
Twist it over and over.

Start with a trajectory on the double pentagon surface.



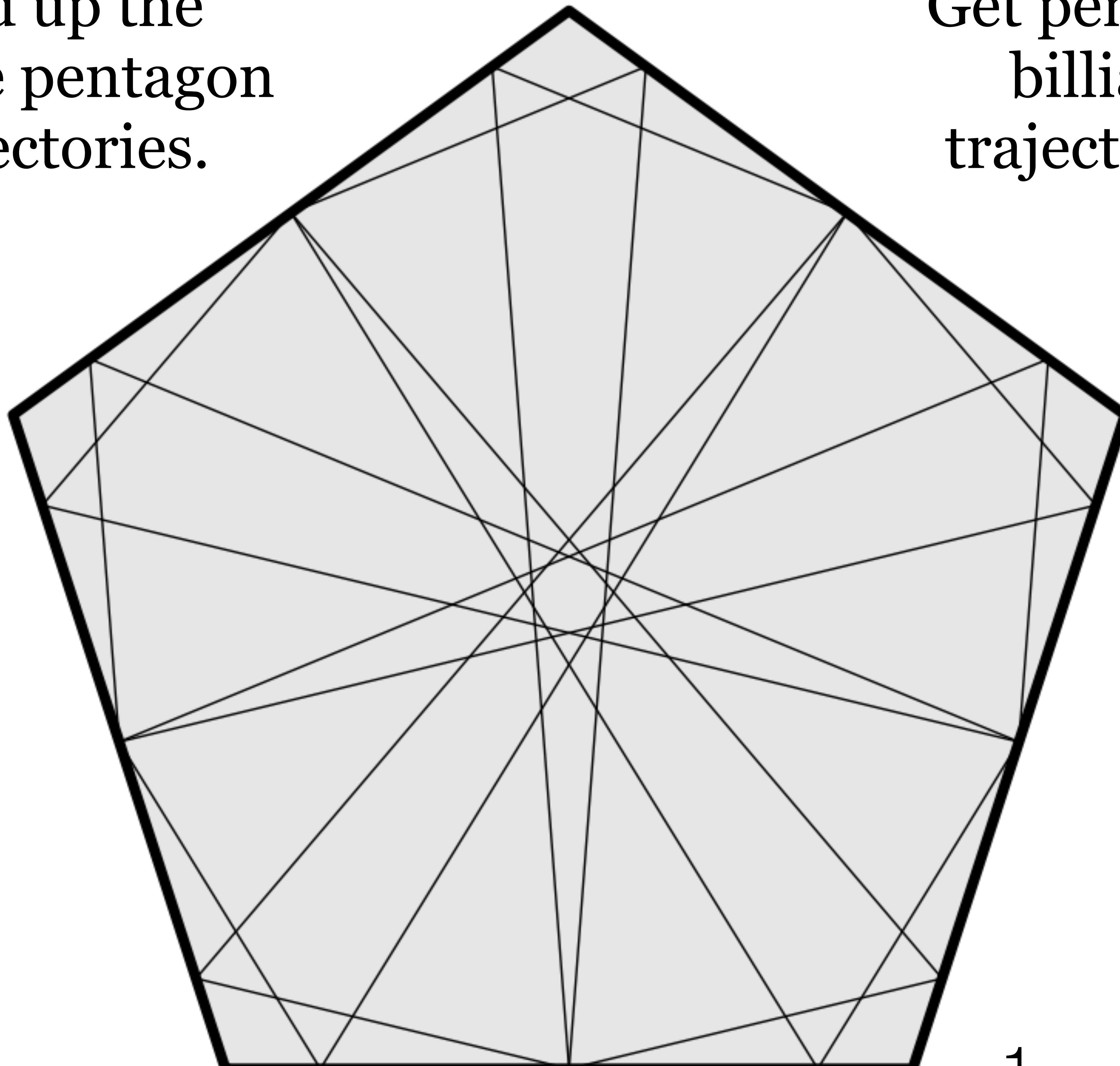
Twist it over and over.

Start with a trajectory on the double pentagon surface.



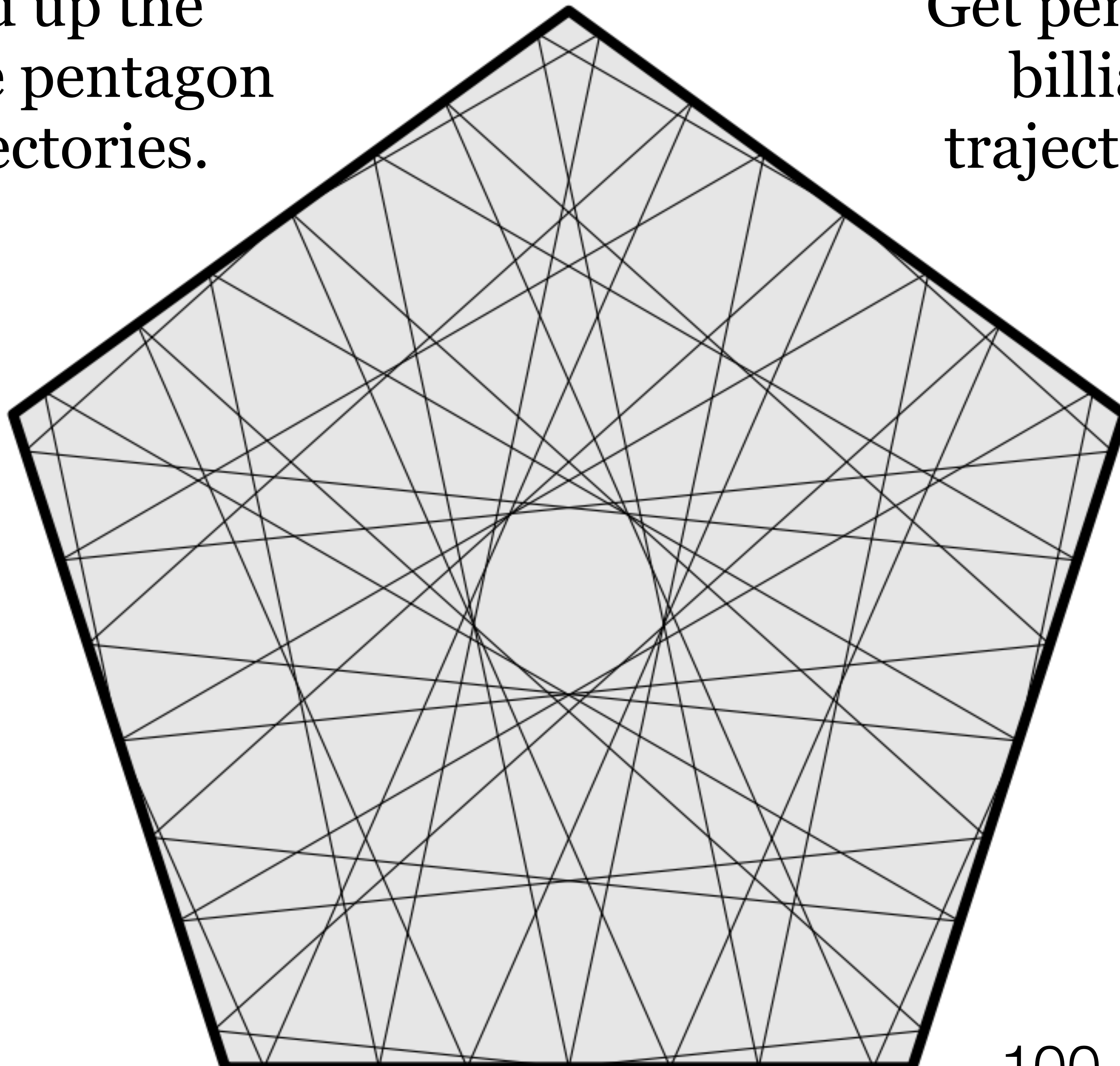
Twist it over and over.

Fold up the double pentagon trajectories.



Get pentagon billiard trajectories!

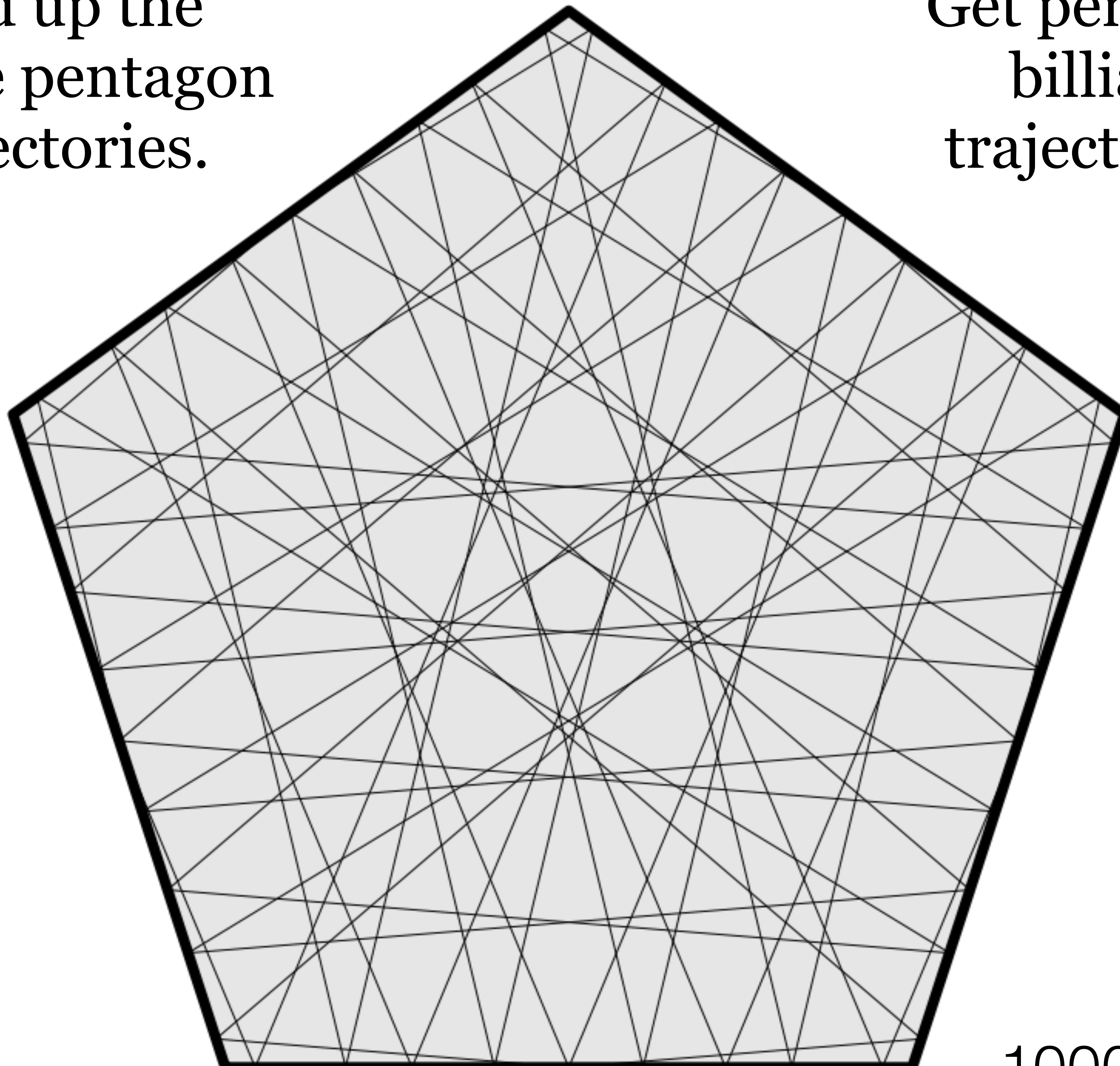
Fold up the double pentagon trajectories.



Get pentagon billiard trajectories!

Fold up the double pentagon trajectories.

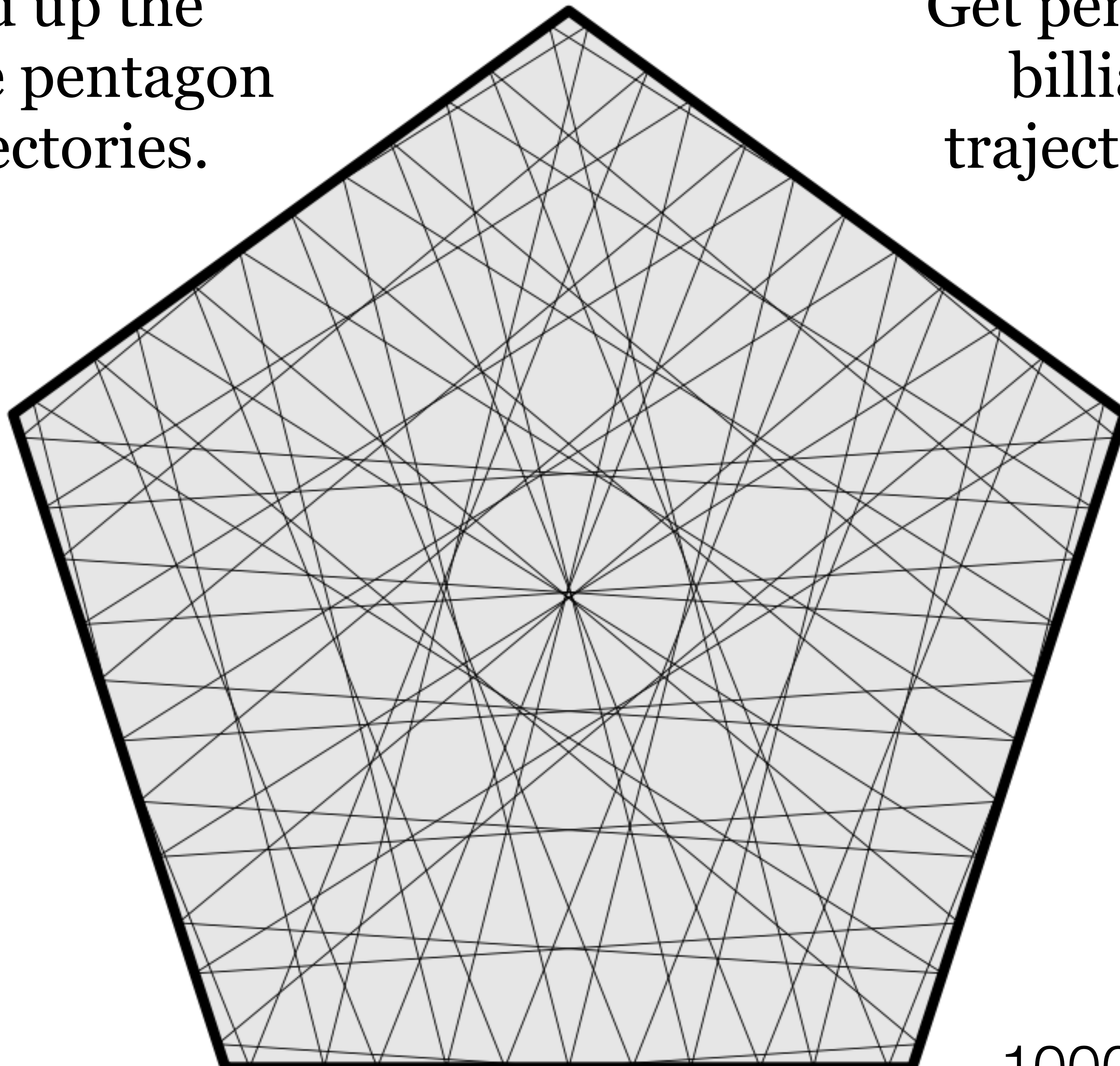
Get pentagon billiard trajectories!



1000

Fold up the double pentagon trajectories.

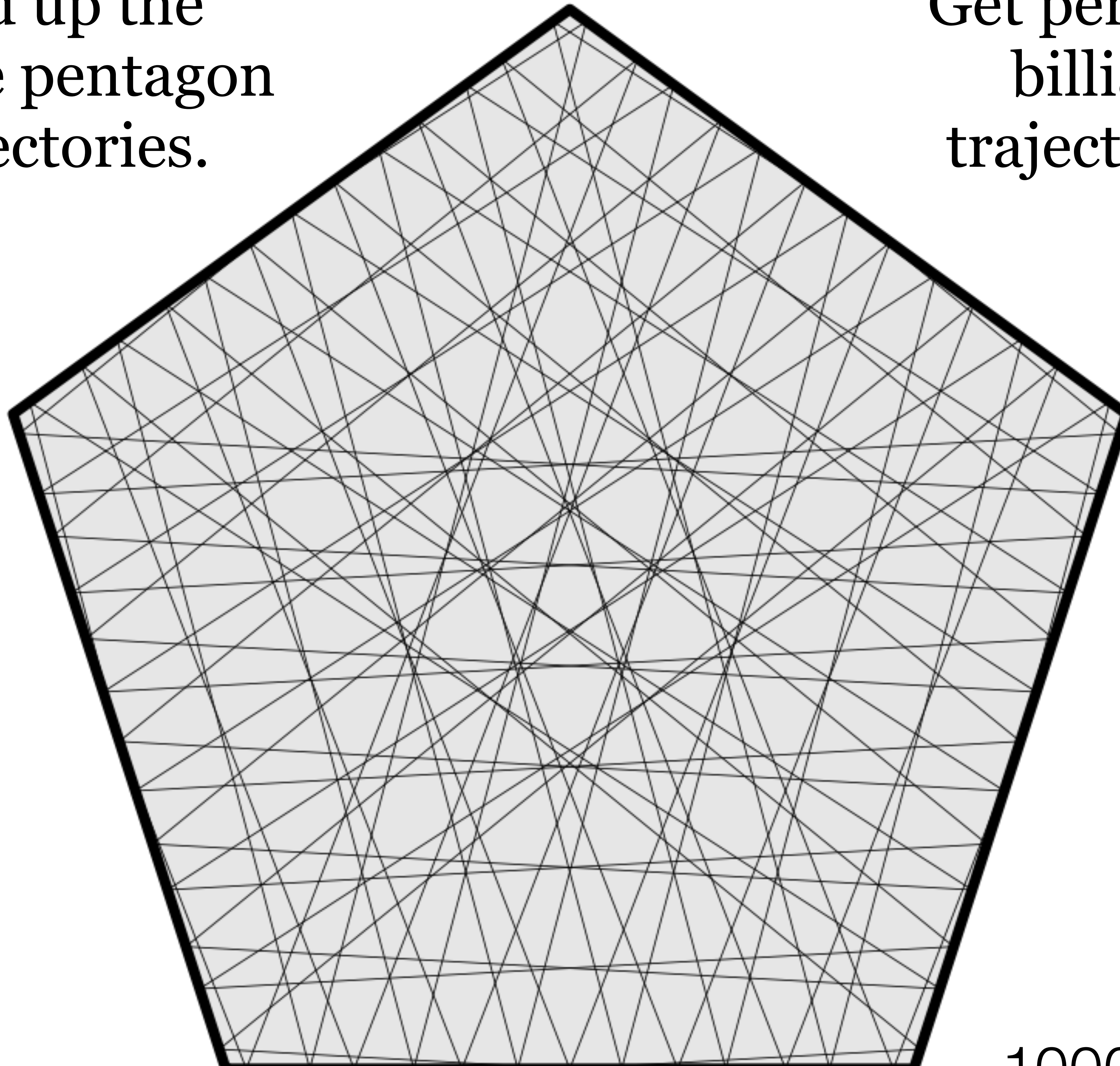
Get pentagon billiard trajectories!



10000

Fold up the double pentagon trajectories.

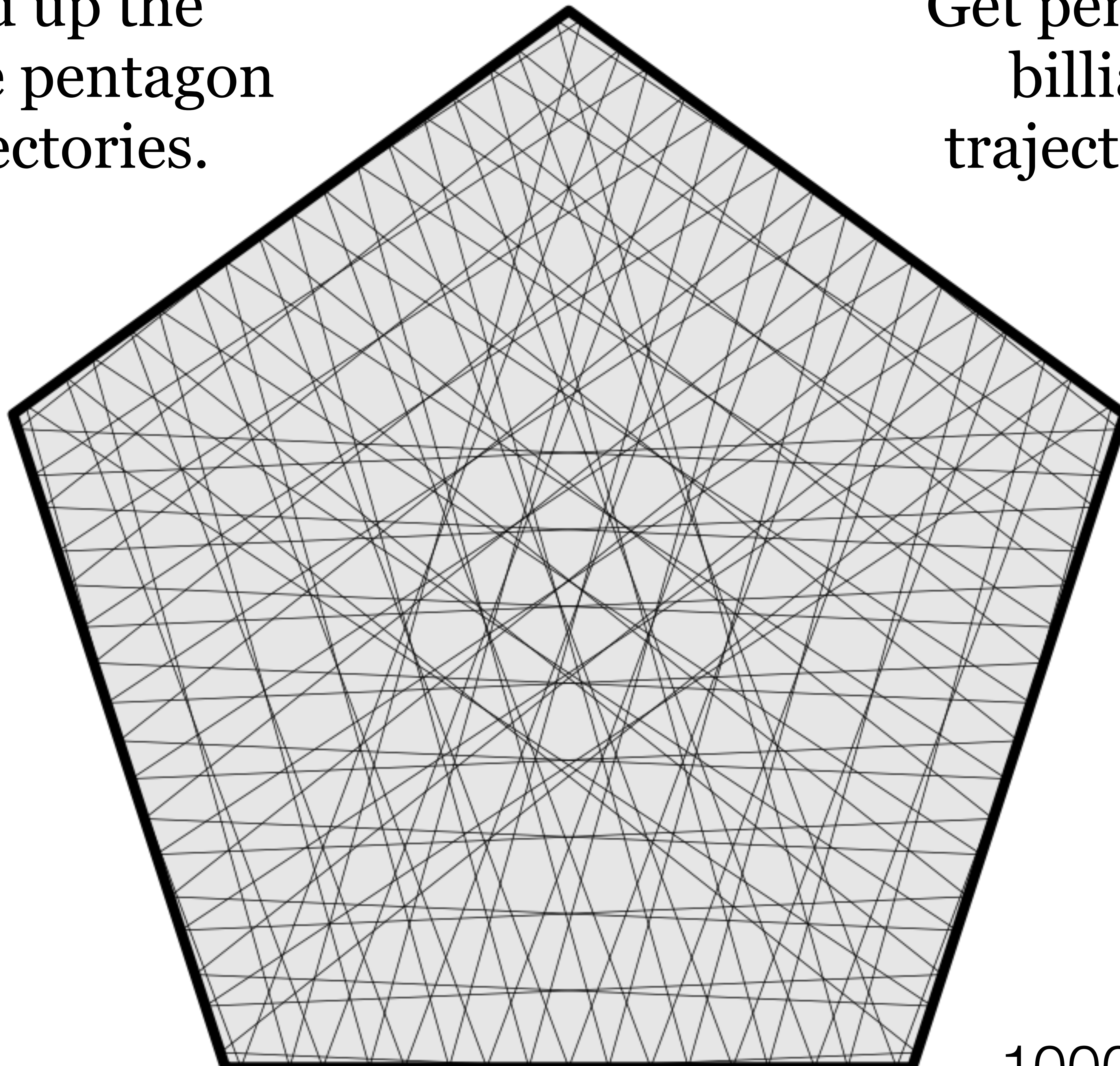
Get pentagon billiard trajectories!



100000

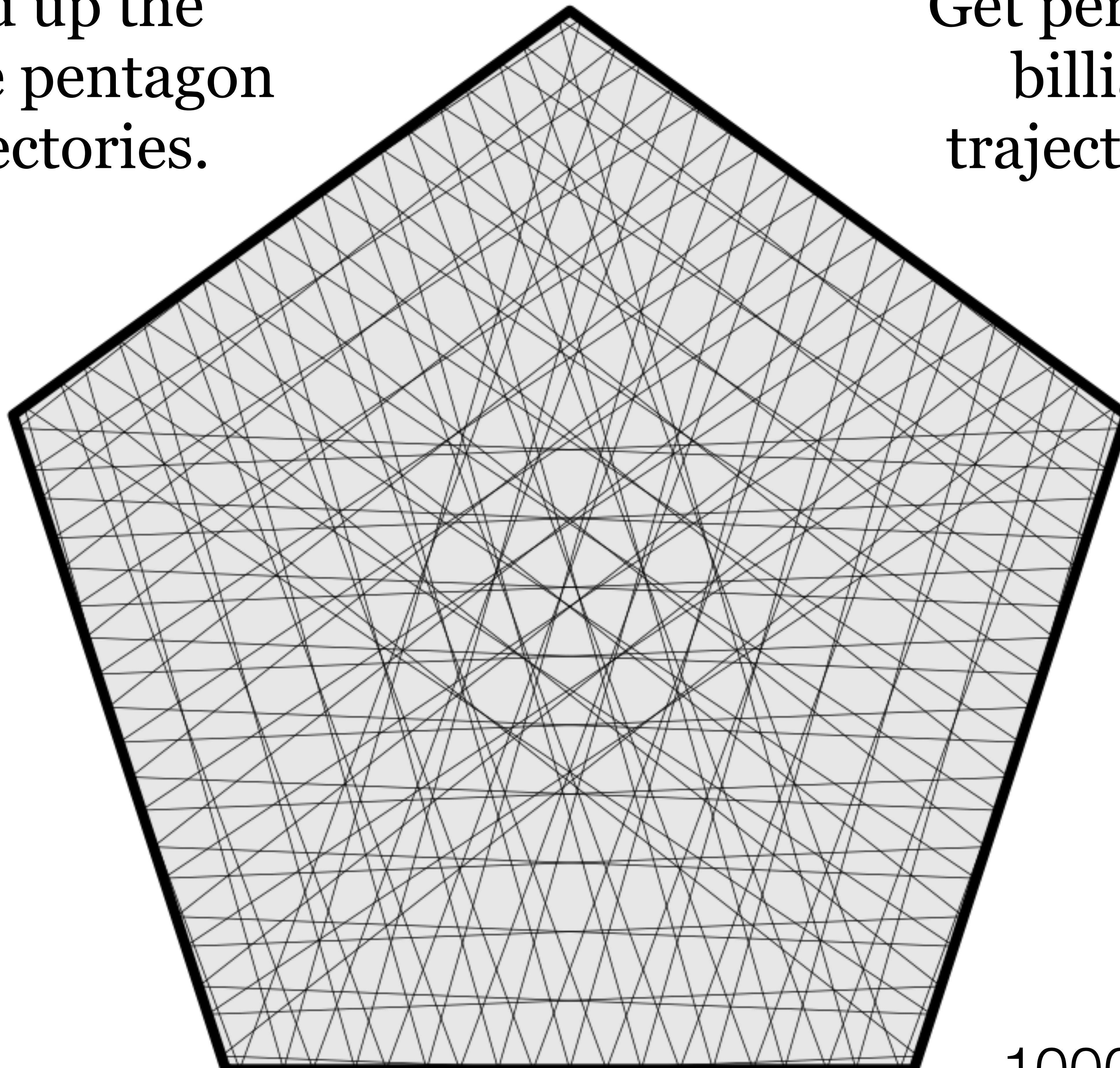
Fold up the double pentagon trajectories.

Get pentagon billiard trajectories!



10000000

Fold up the double pentagon trajectories.

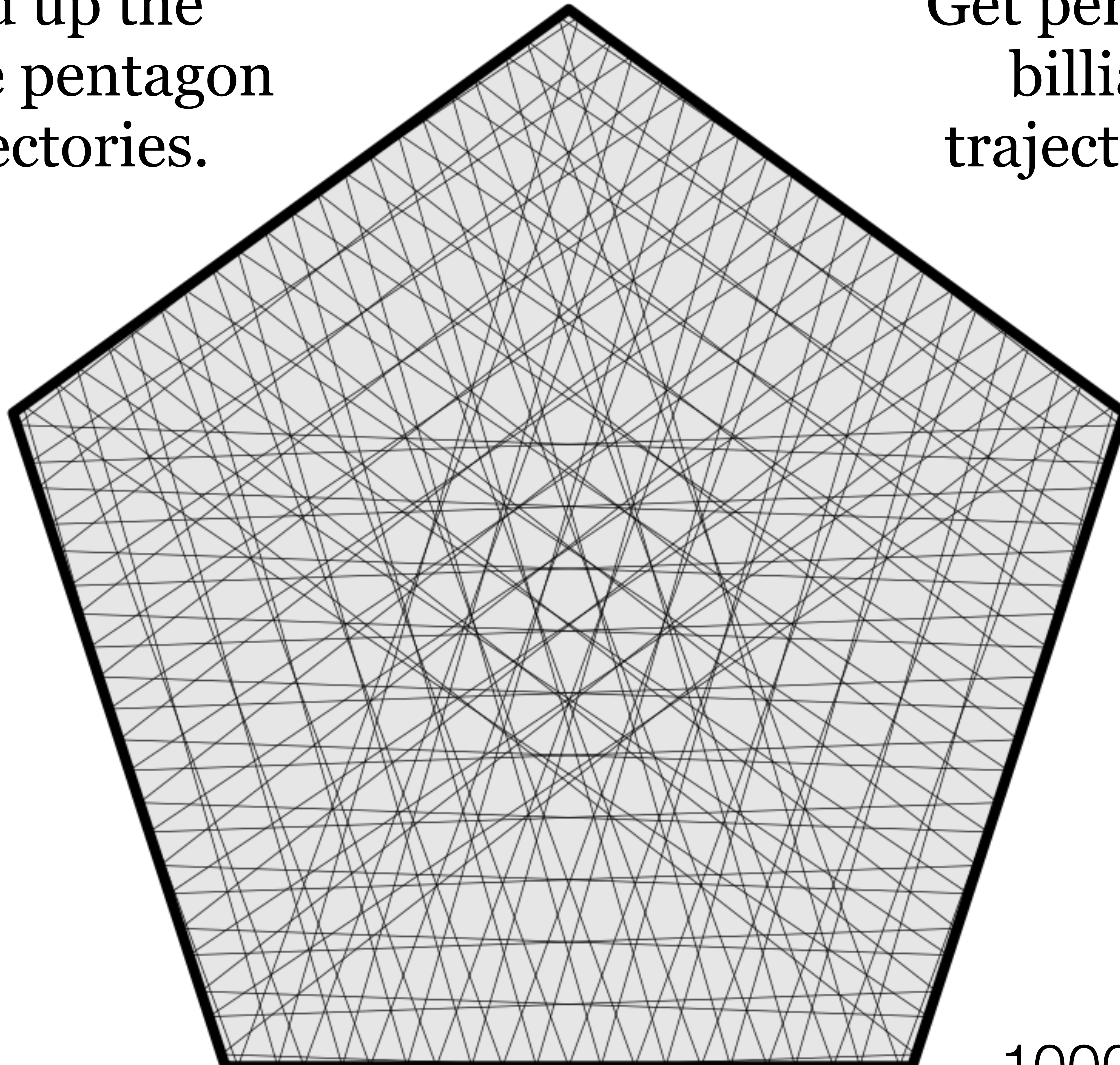


Get pentagon billiard trajectories!

100000000

Fold up the double pentagon trajectories.

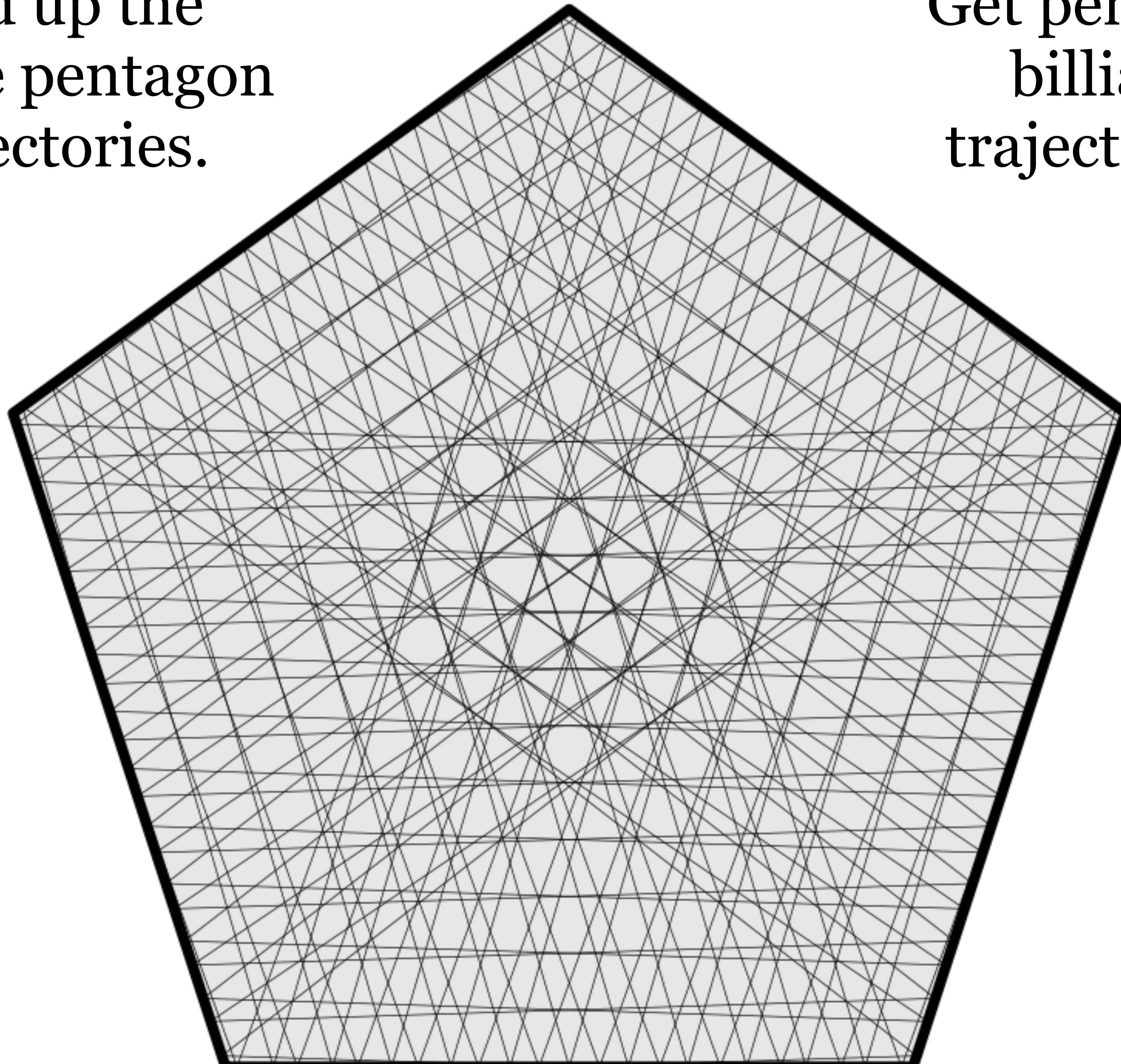
Get pentagon billiard trajectories!



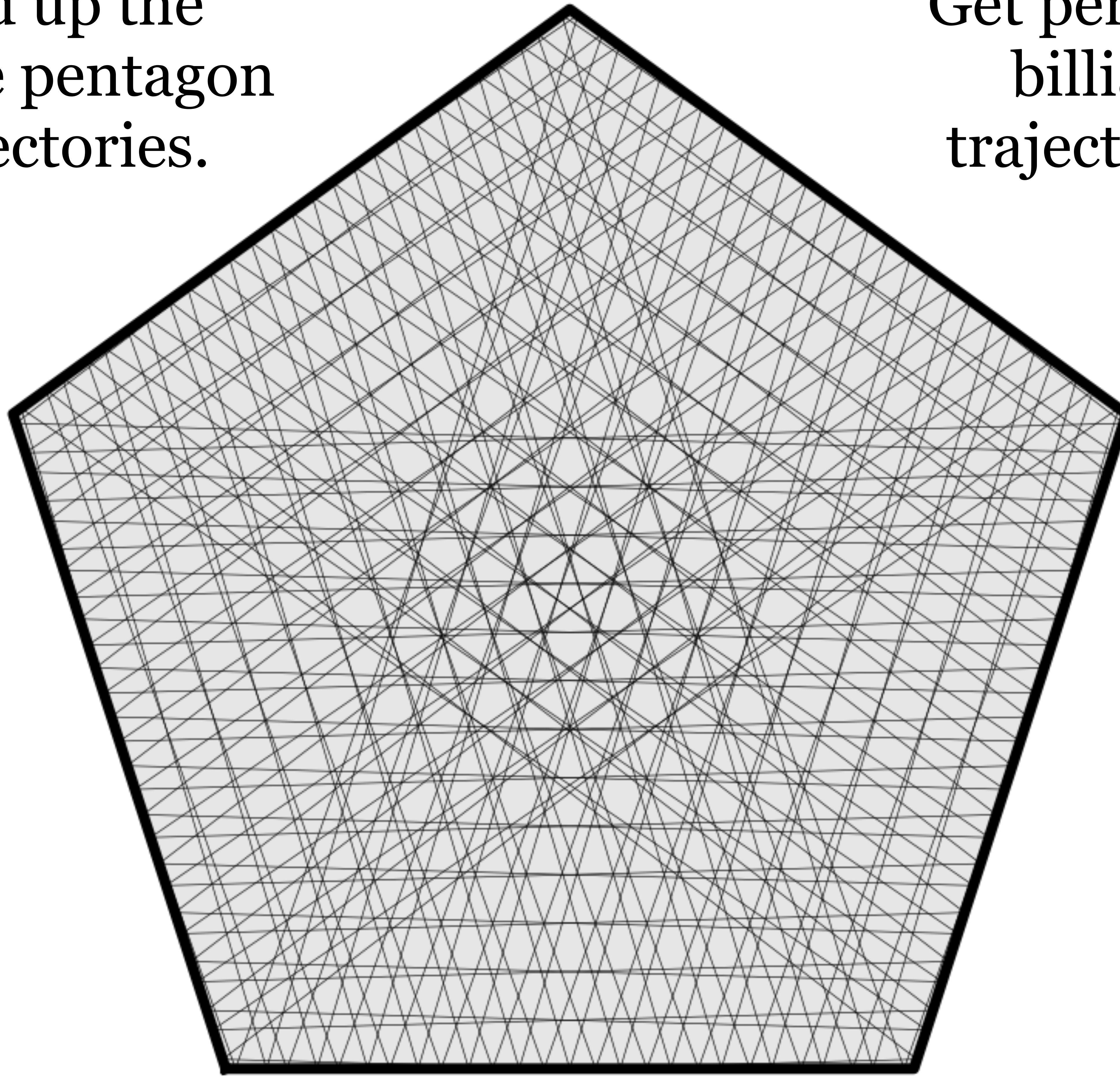
1000000000

Fold up the double pentagon trajectories.

Get pentagon billiard trajectories!

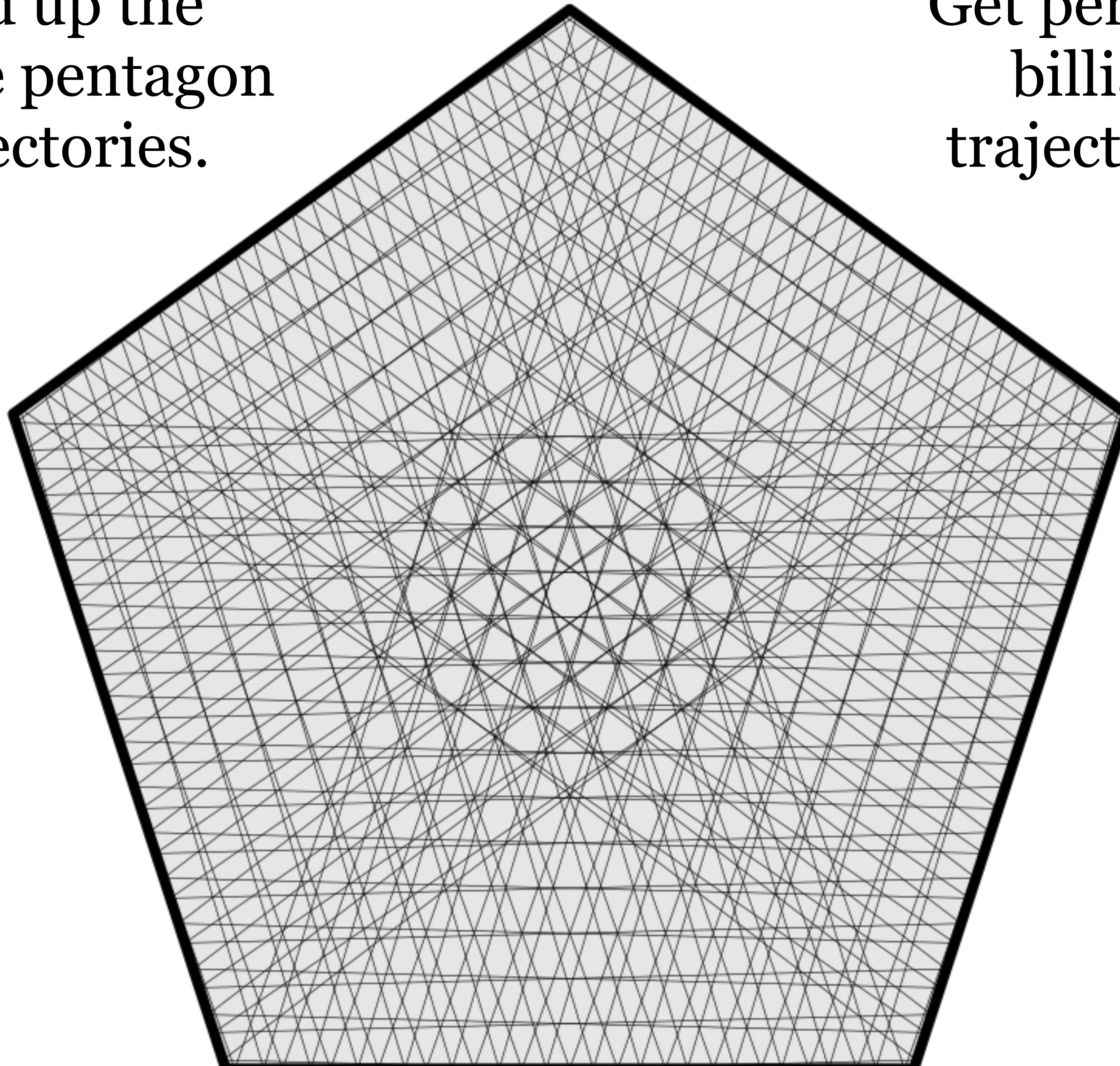


Fold up the double pentagon trajectories.



Get pentagon billiard trajectories!

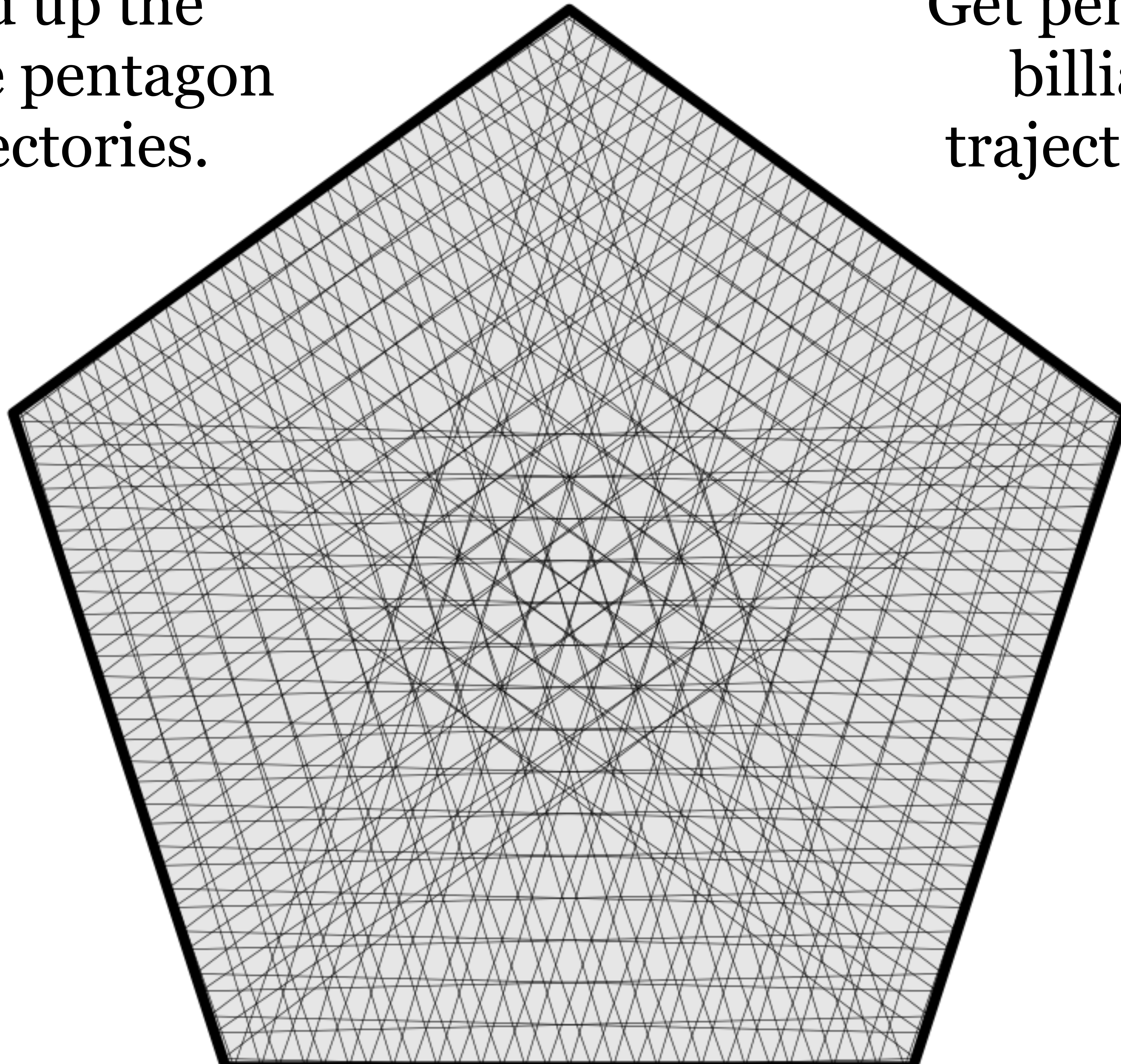
Fold up the double pentagon trajectories.



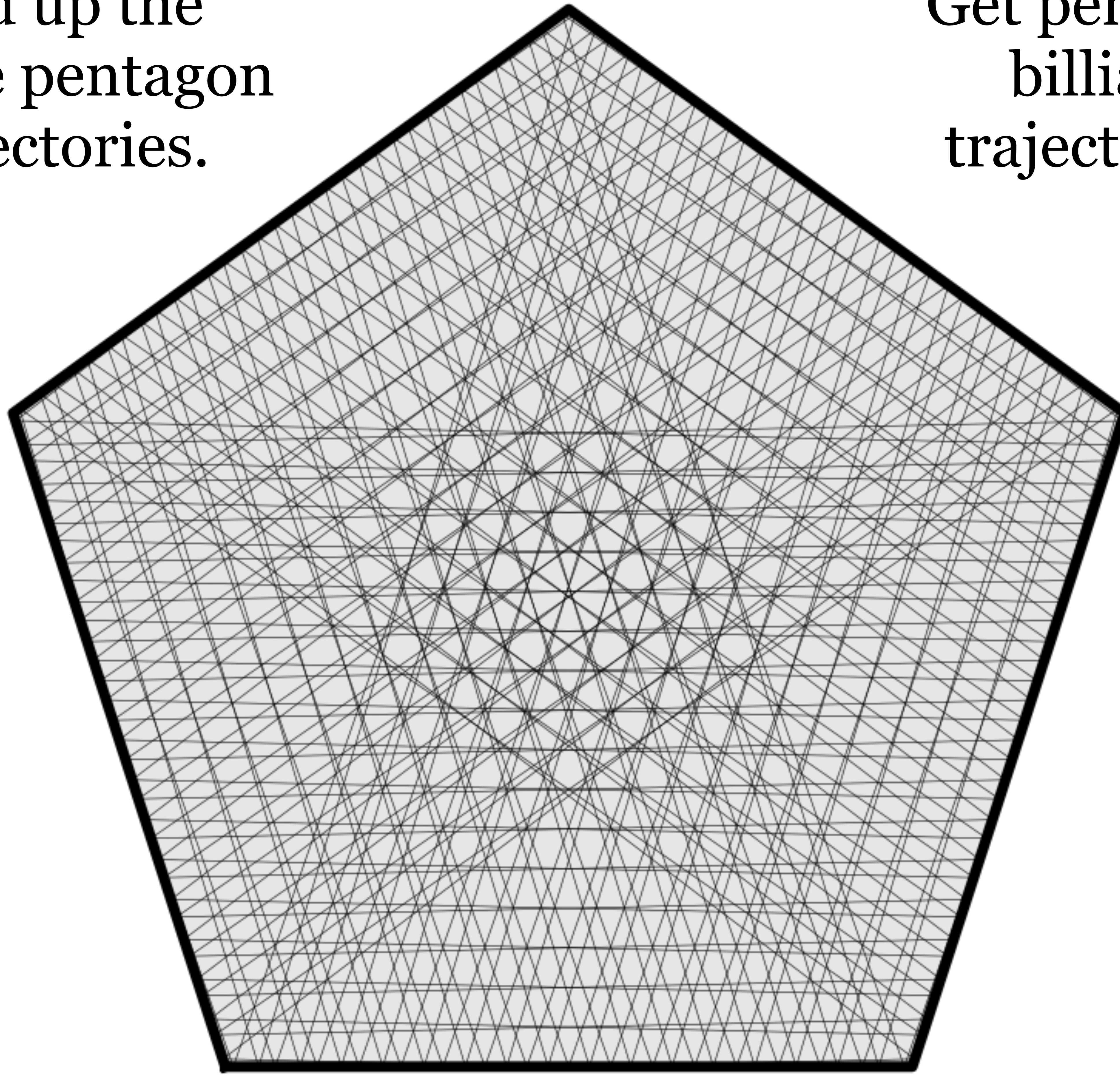
Get pentagon billiard trajectories!

Fold up the double pentagon trajectories.

Get pentagon billiard trajectories!

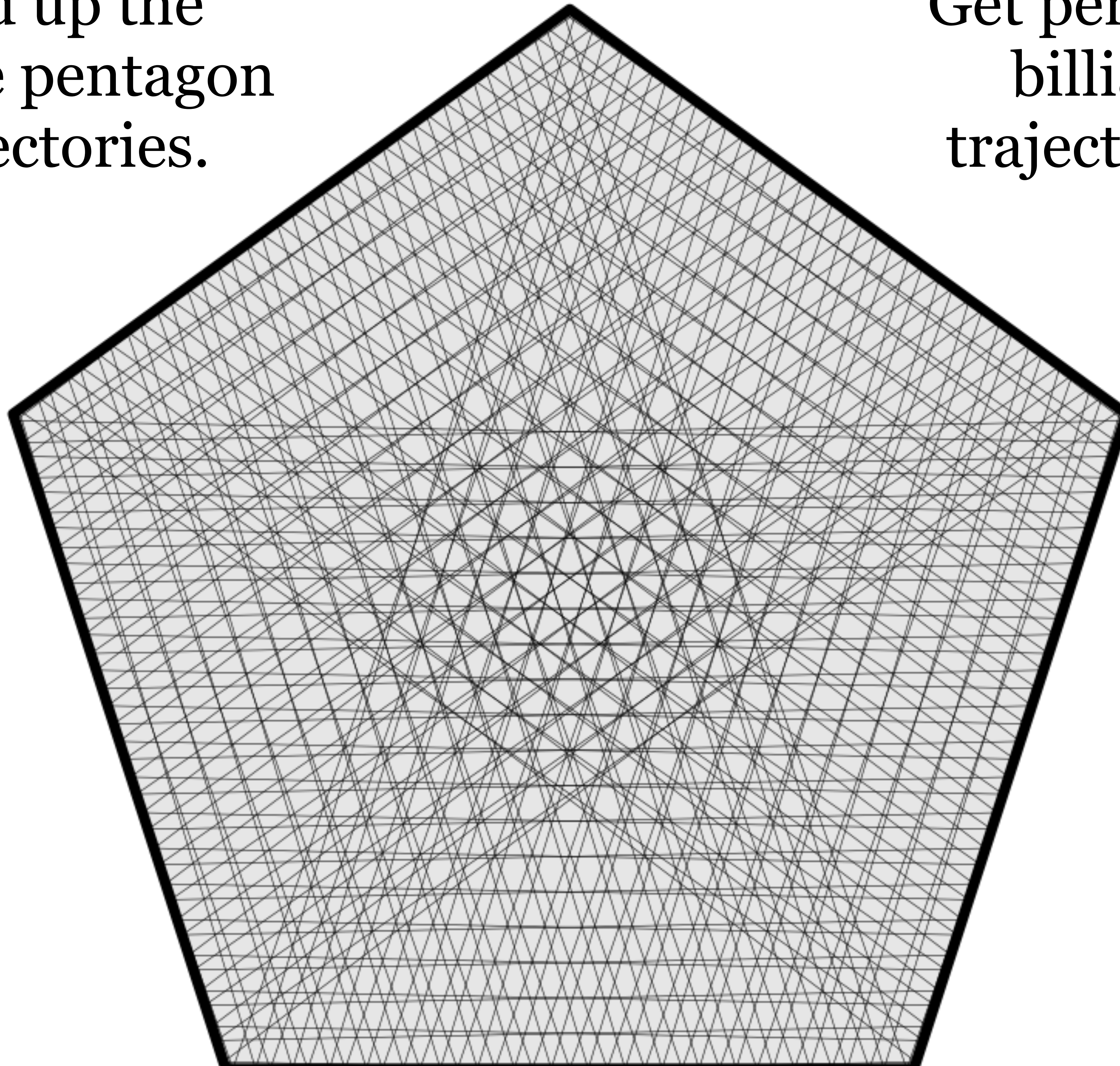


Fold up the double pentagon trajectories.



Get pentagon billiard trajectories!

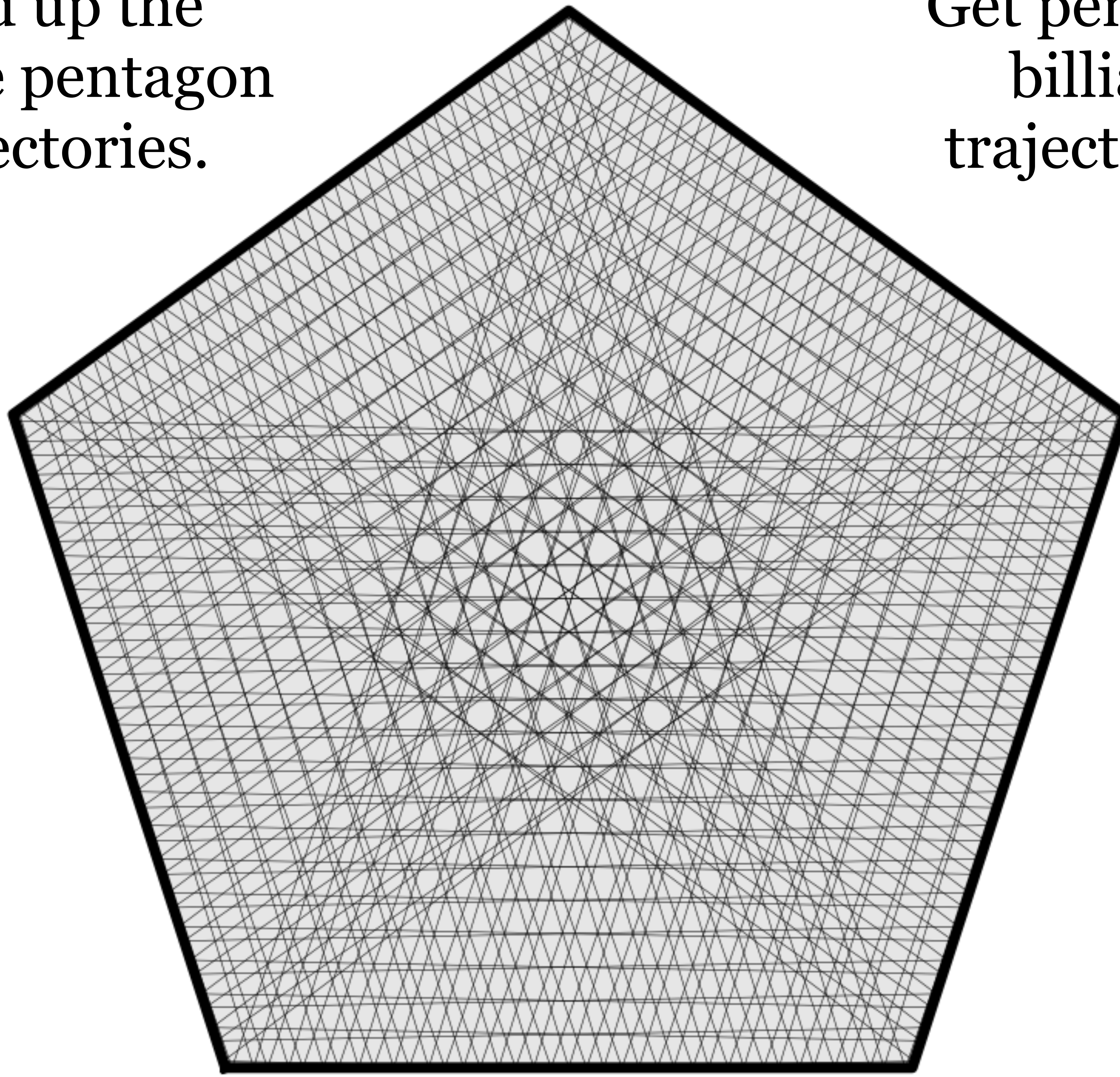
Fold up the double pentagon trajectories.



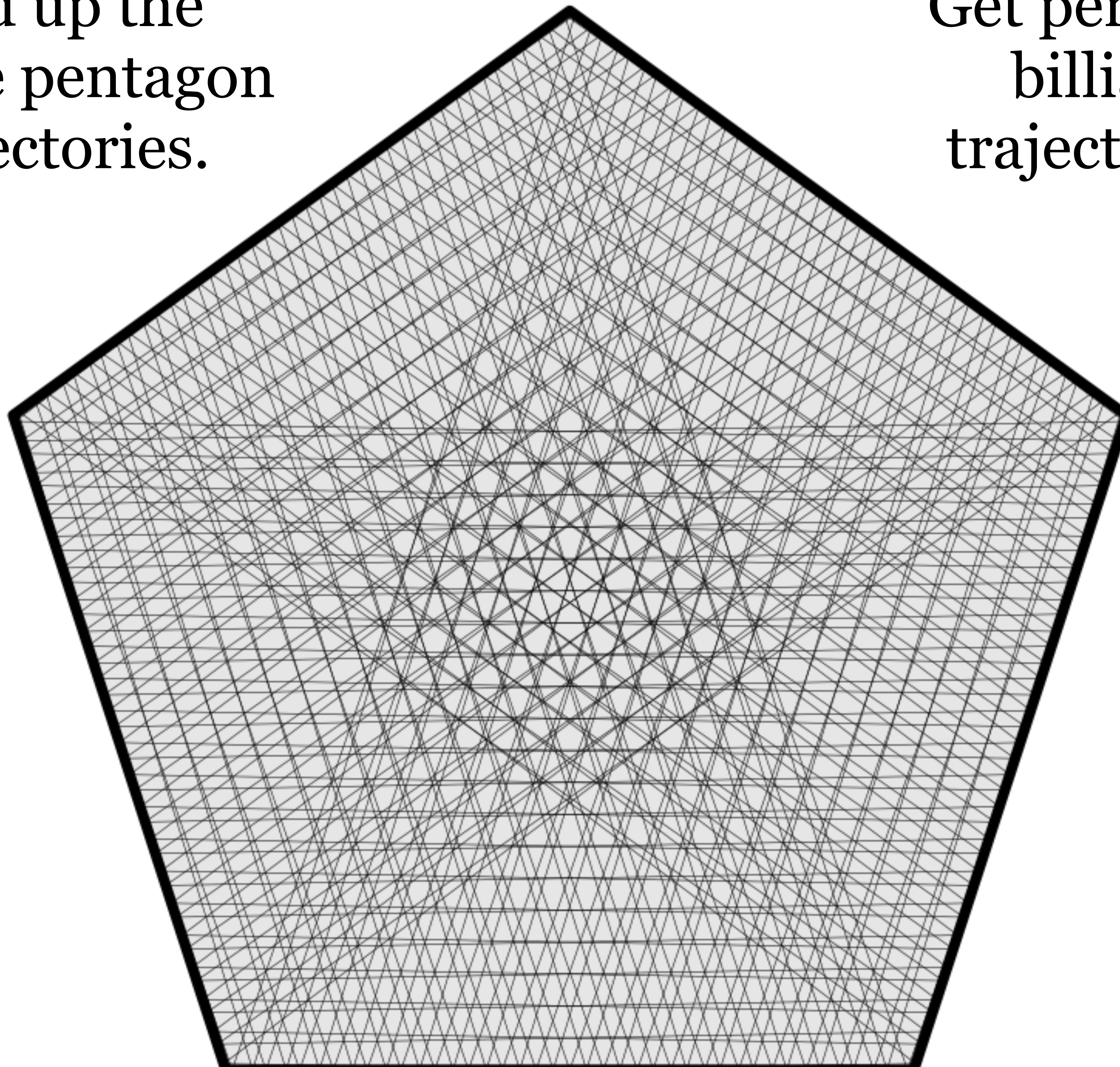
Get pentagon billiard trajectories!

Fold up the double pentagon trajectories.

Get pentagon billiard trajectories!

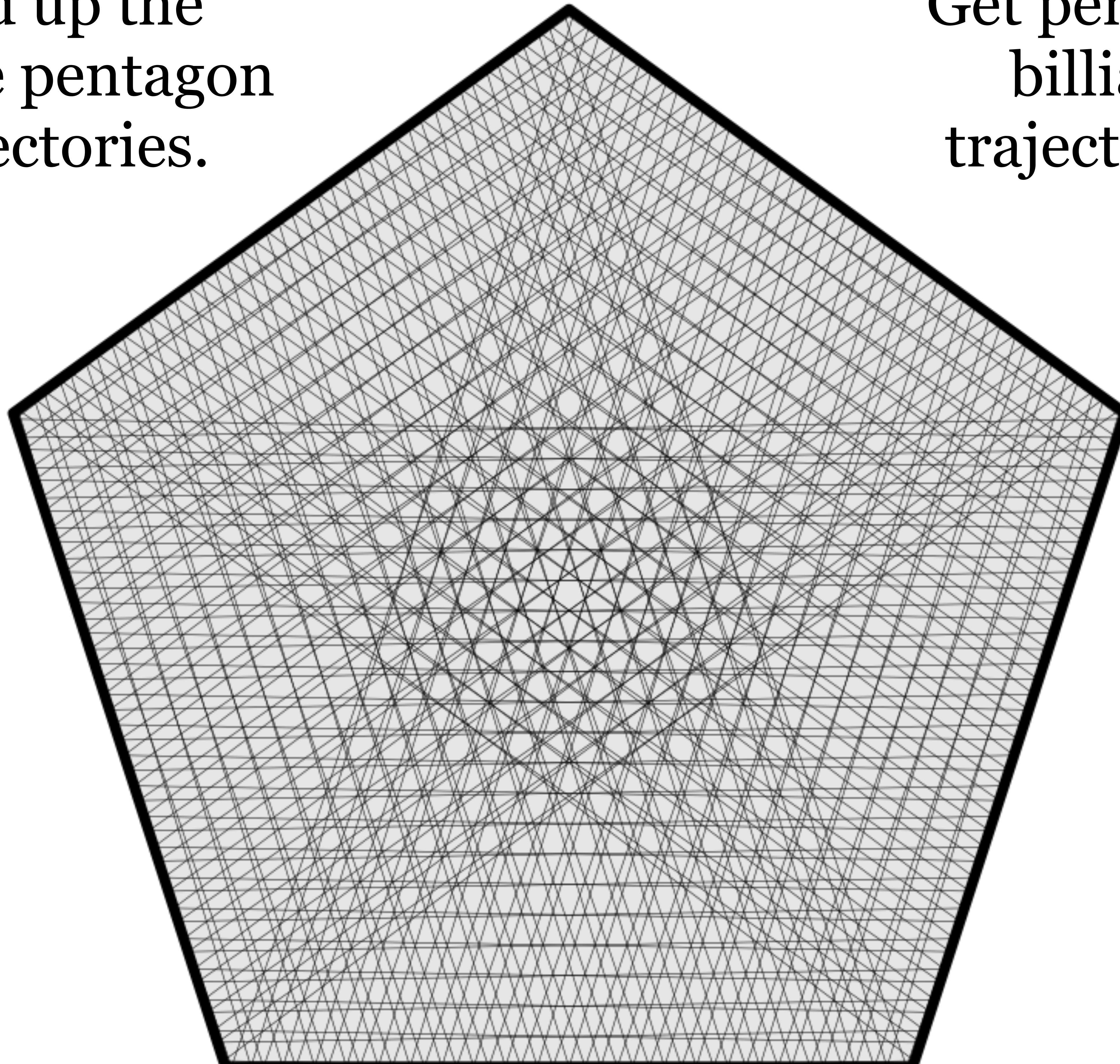


Fold up the
double pentagon
trajectories.



Get pentagon
billiard
trajectories!

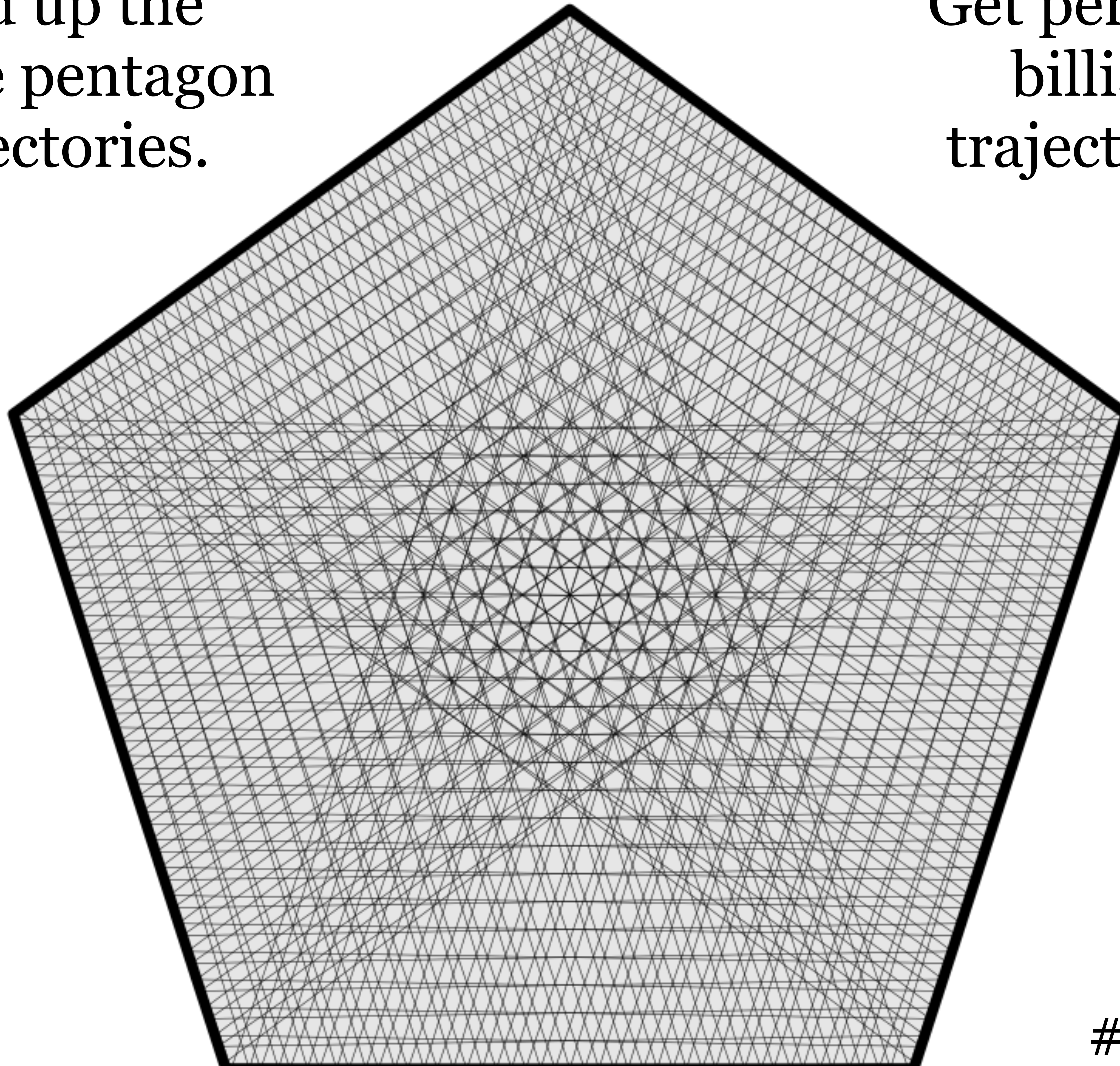
Fold up the double pentagon trajectories.



Get pentagon billiard trajectories!

Fold up the double pentagon trajectories.

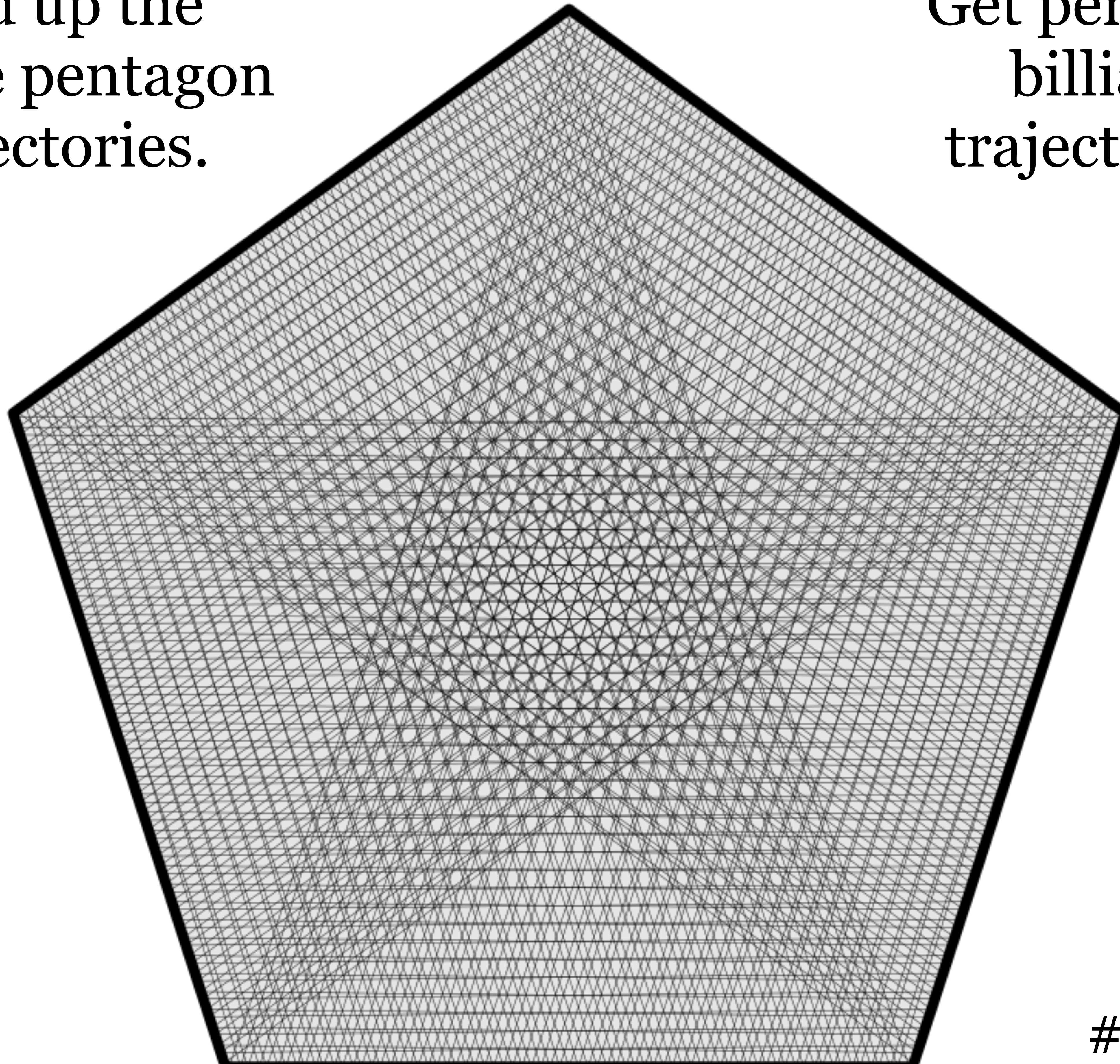
Get pentagon billiard trajectories!



#22

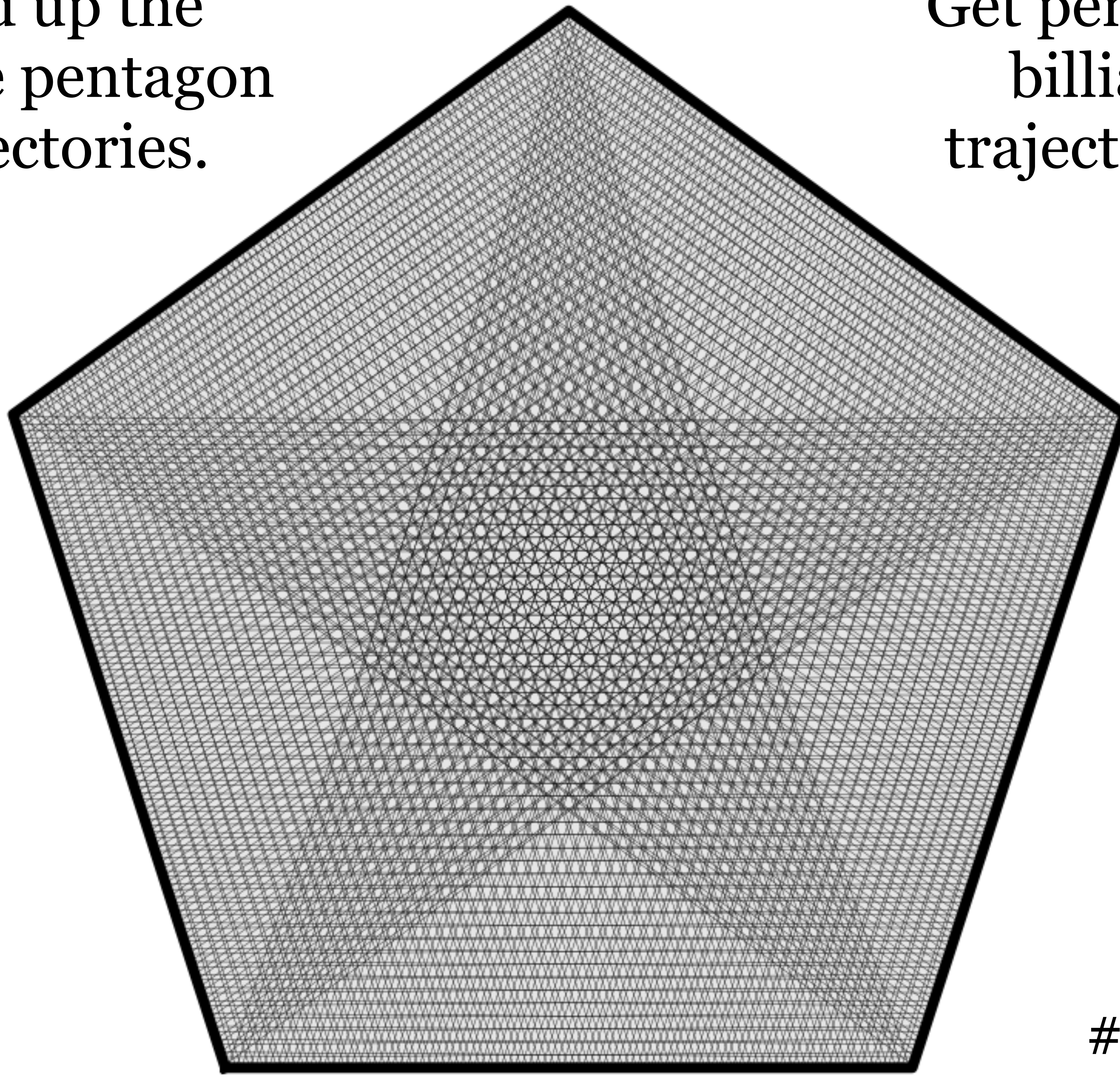
Fold up the double pentagon trajectories.

Get pentagon billiard trajectories!



#35

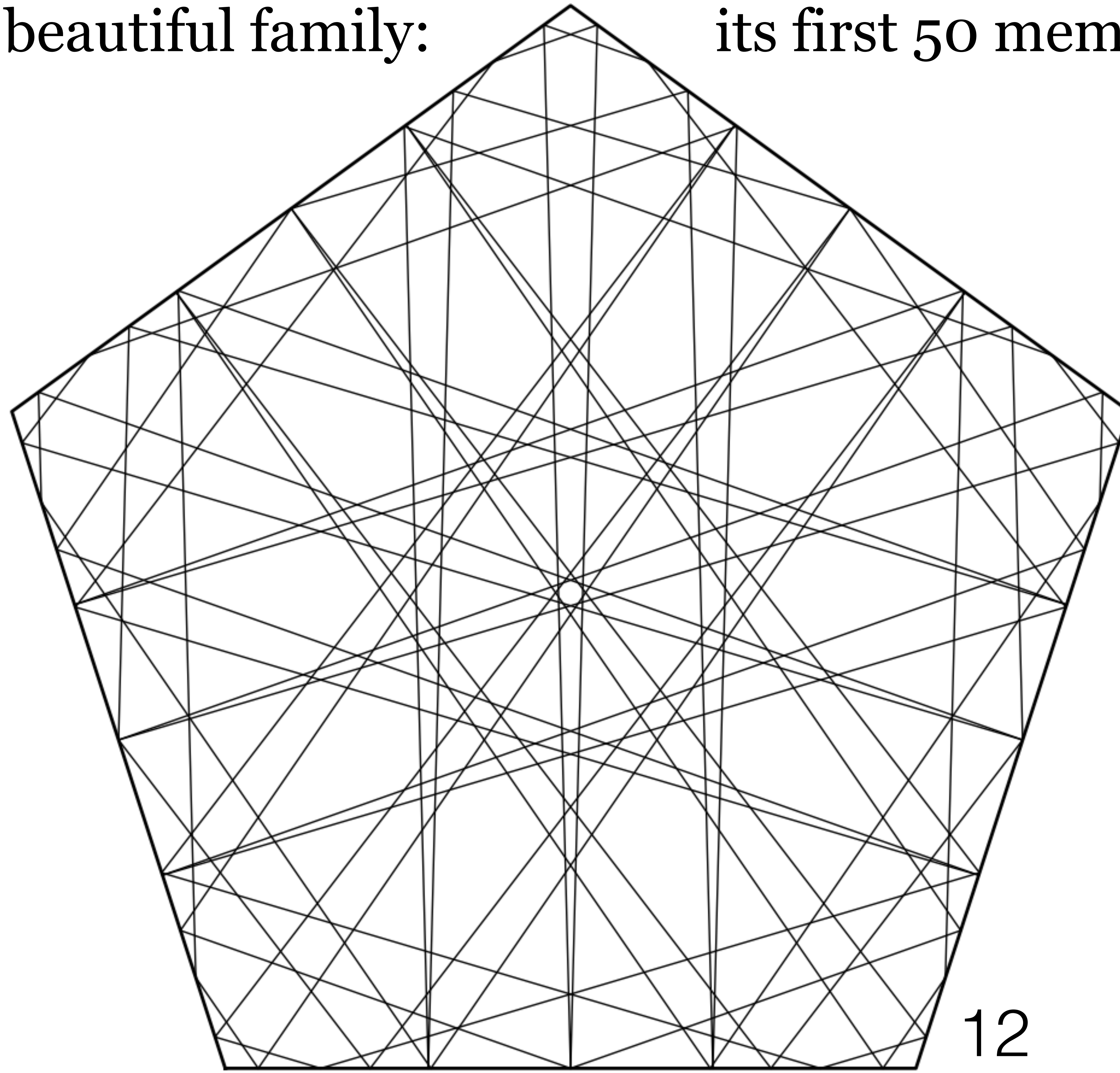
Fold up the double pentagon trajectories.

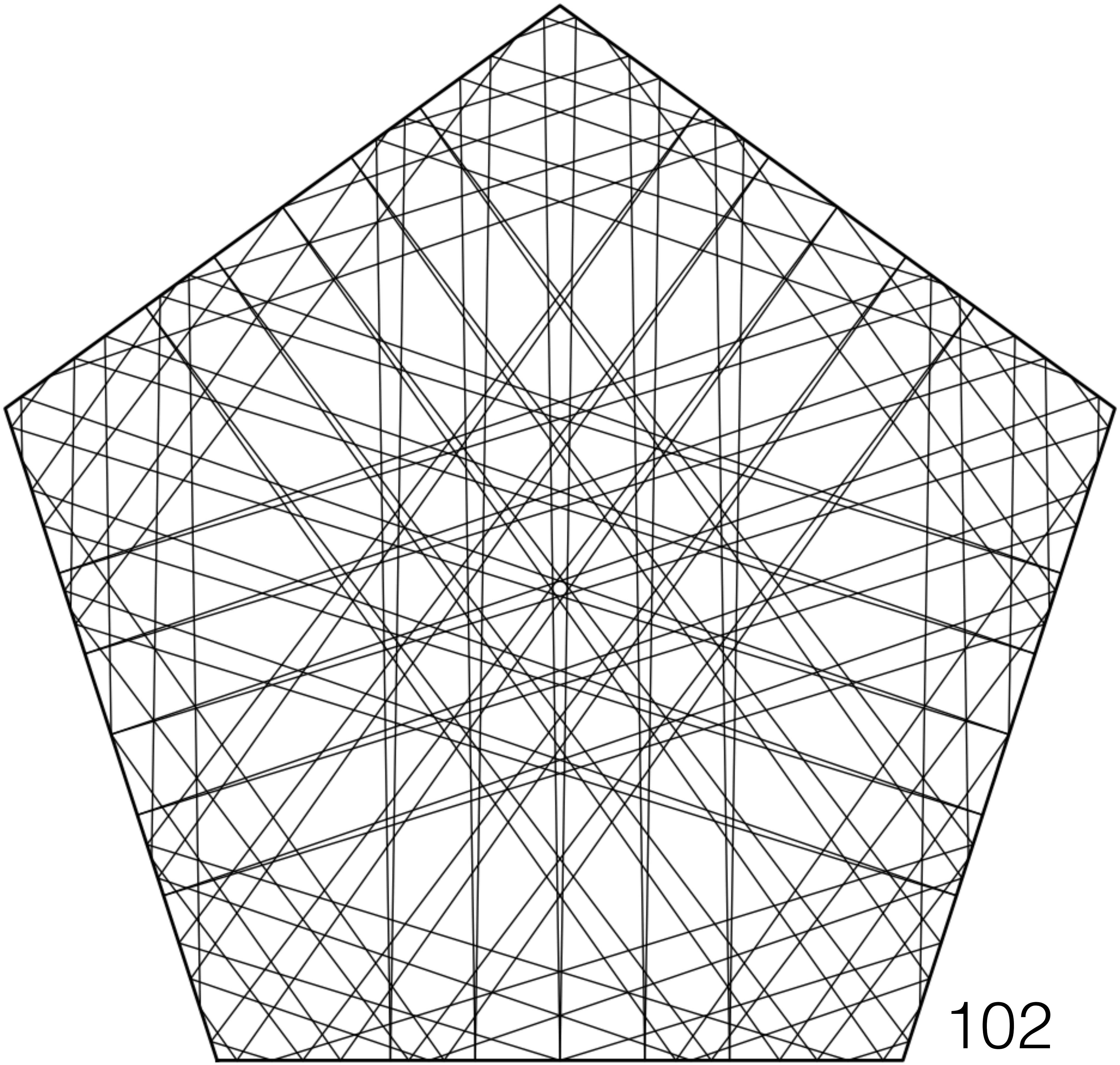


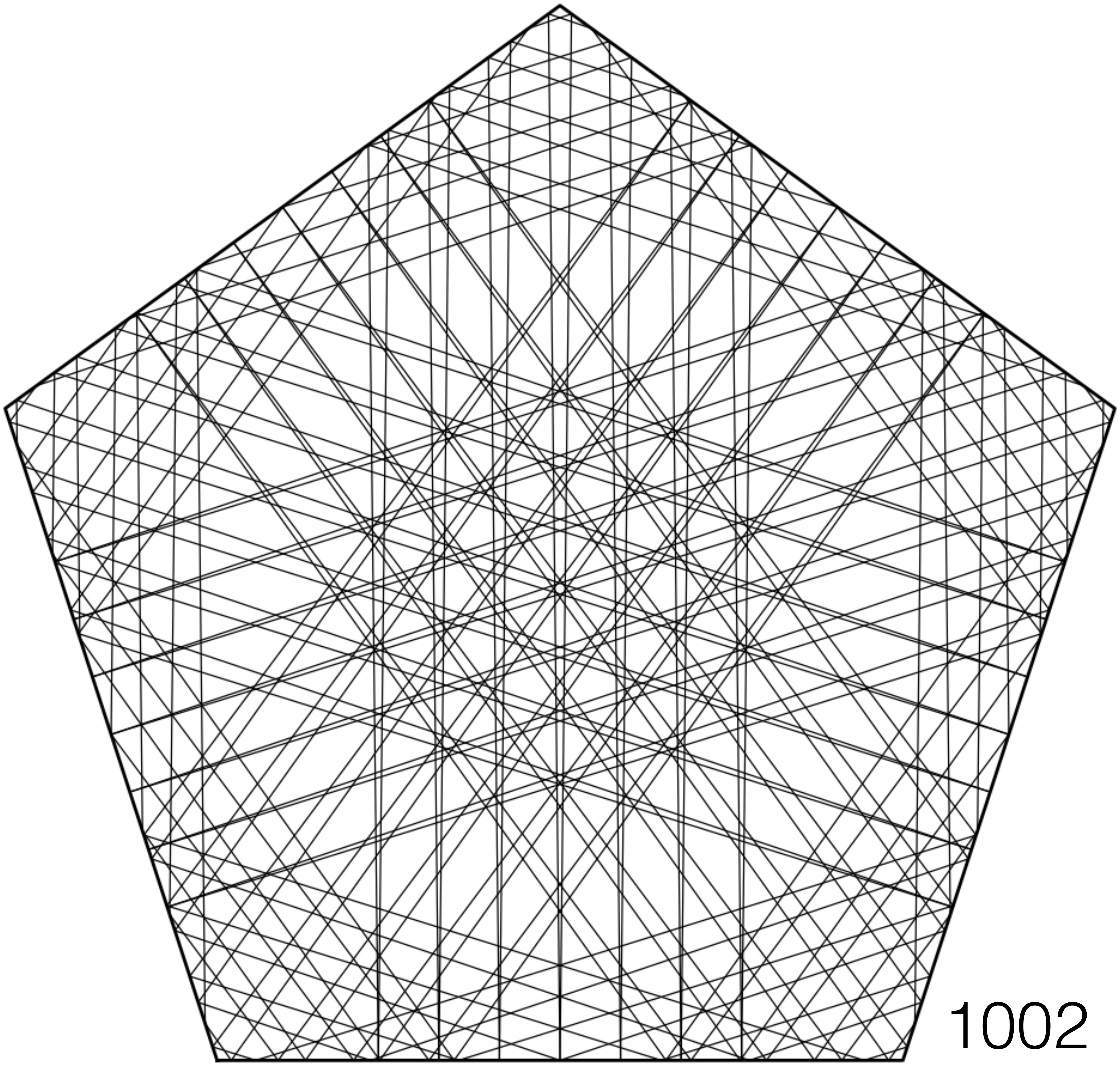
Get pentagon billiard trajectories!

#49

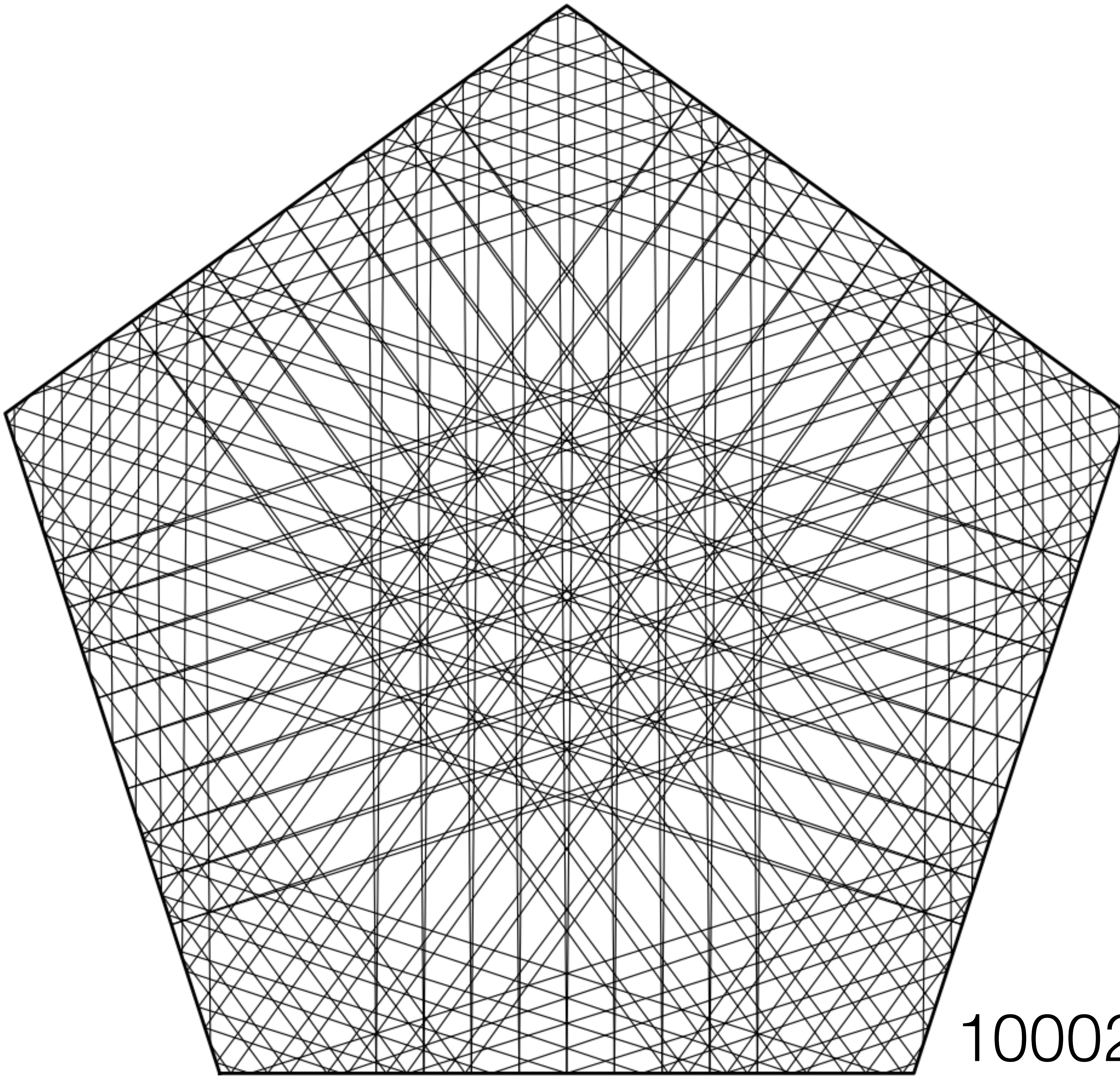
The beautiful family: its first 50 members



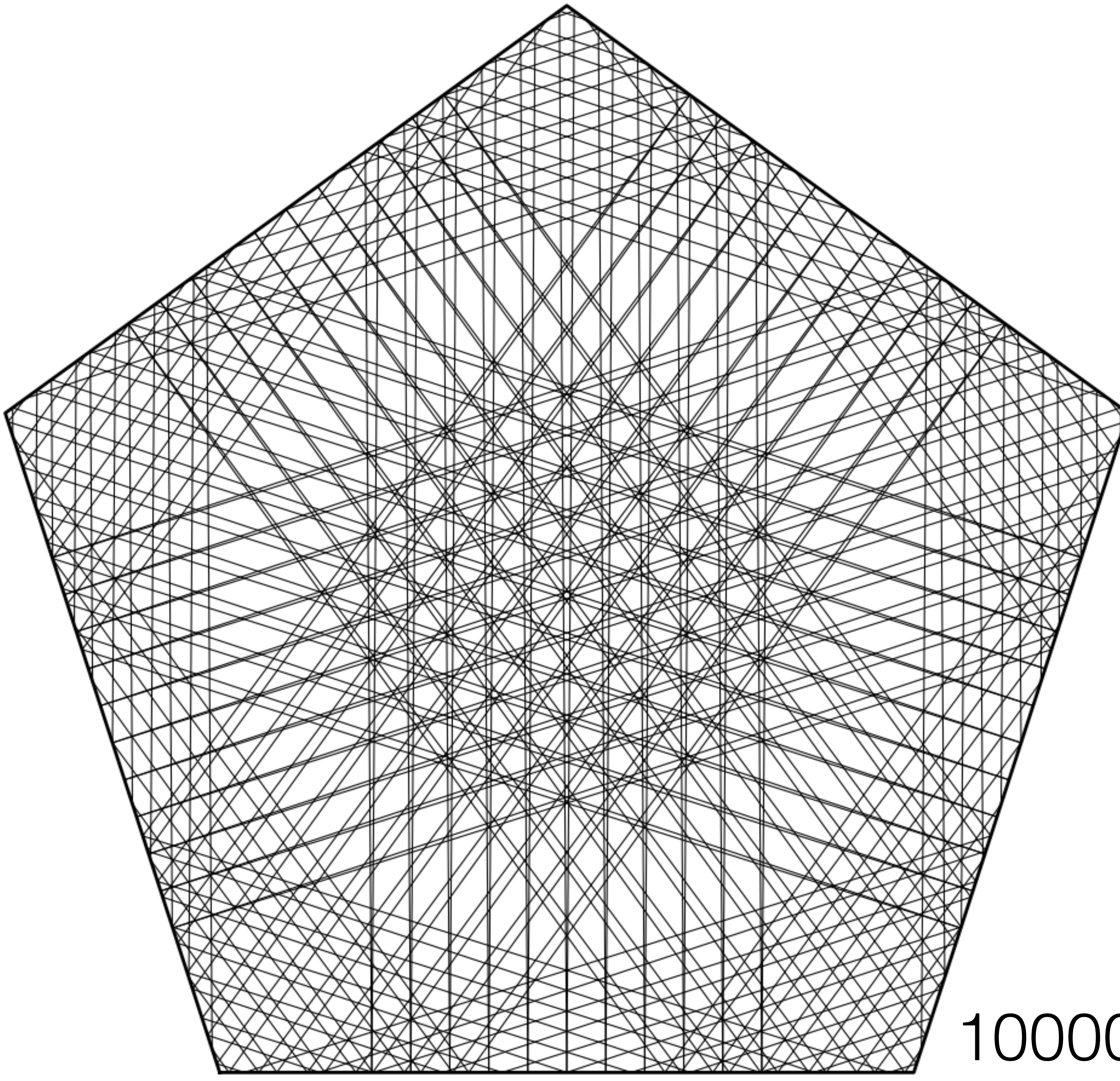




1002



10002



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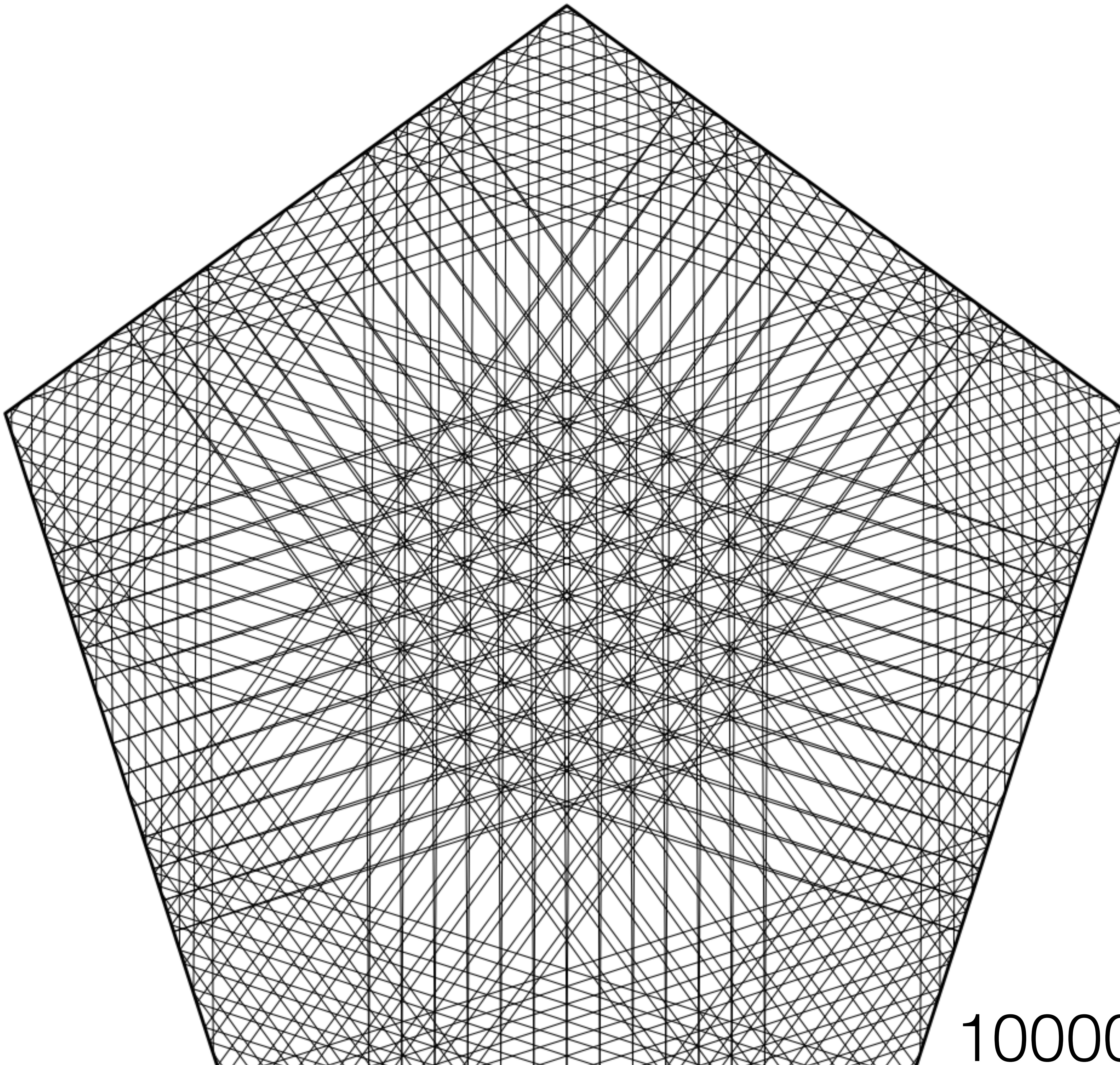

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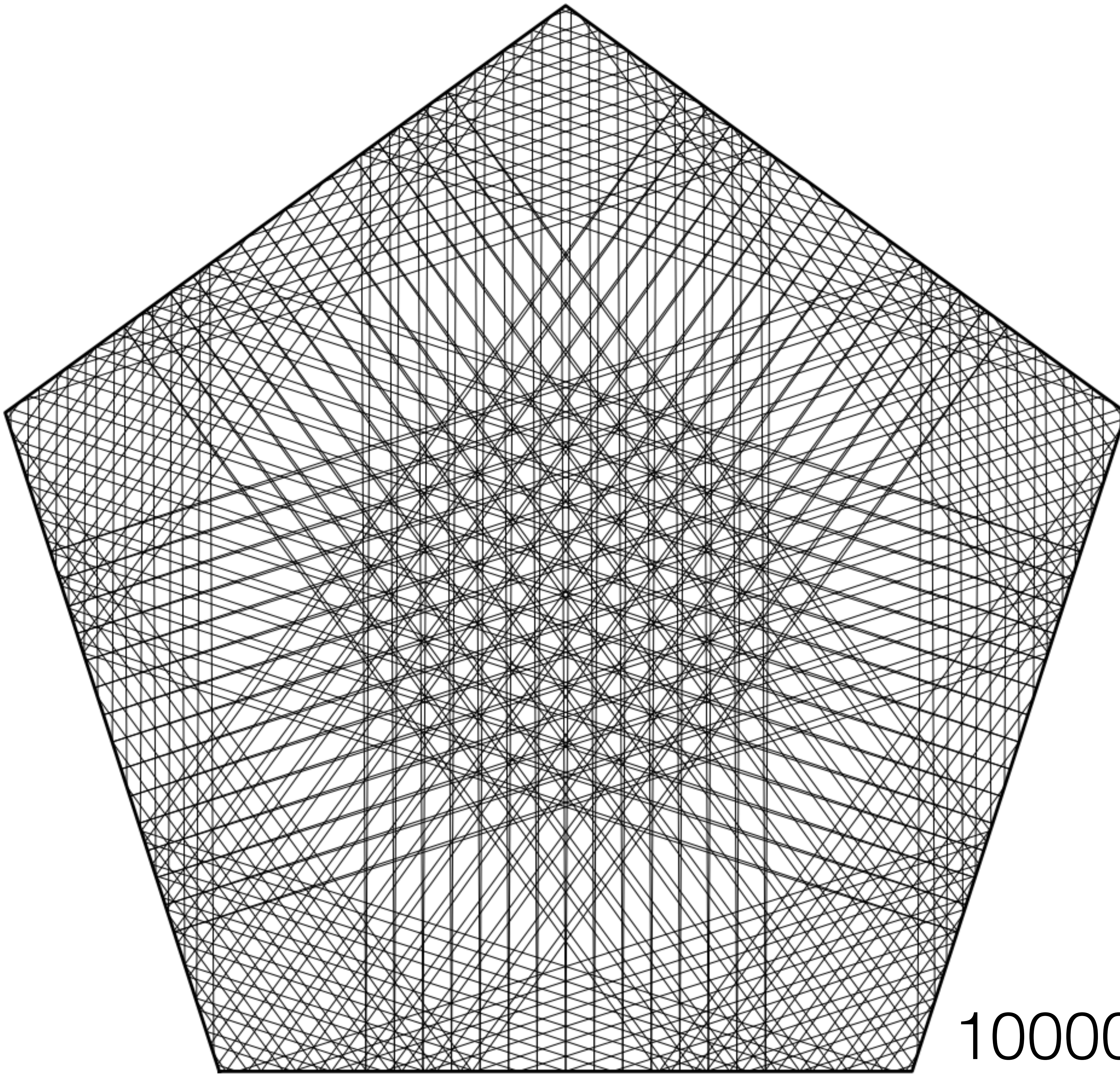

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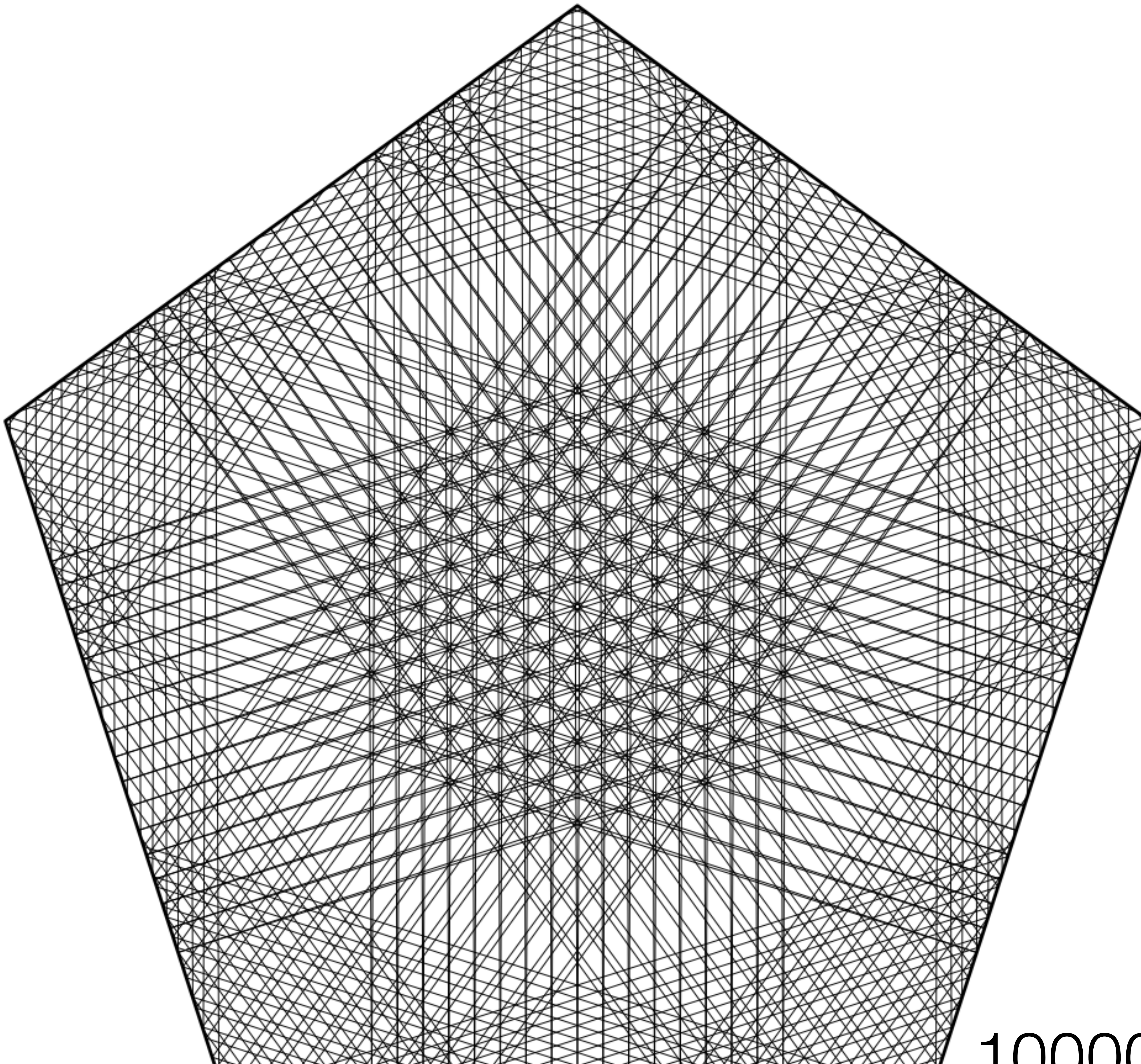

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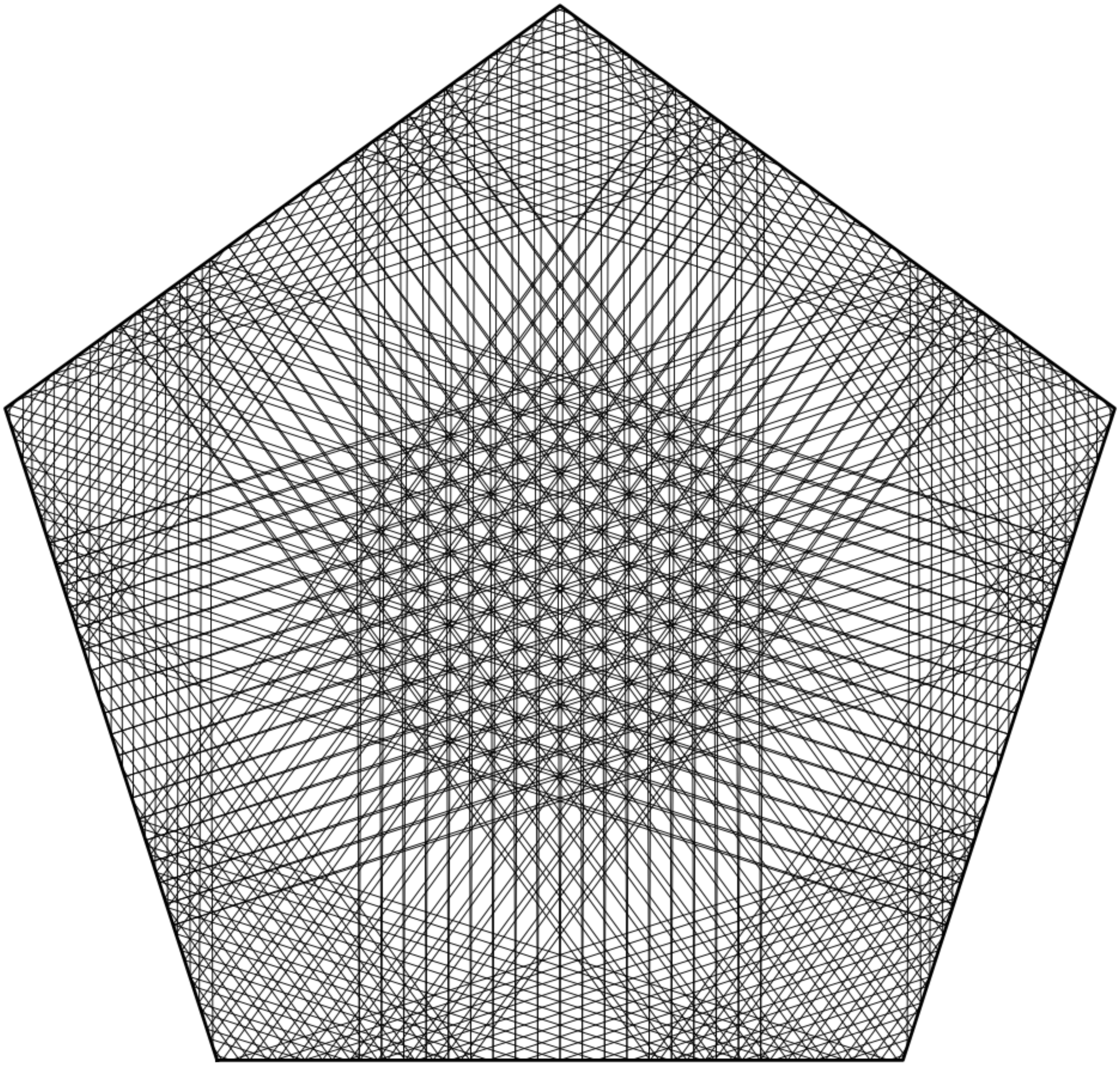
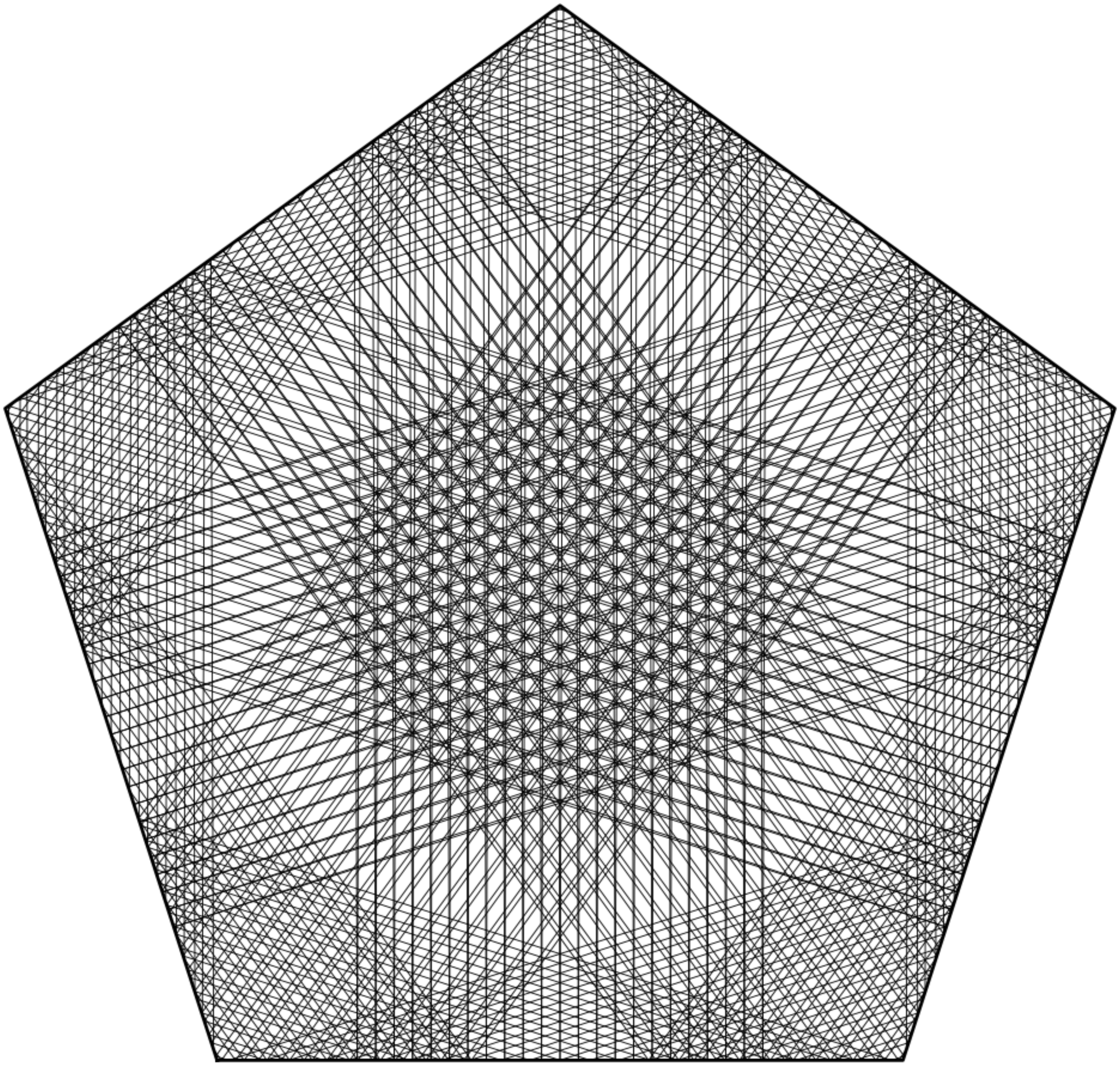
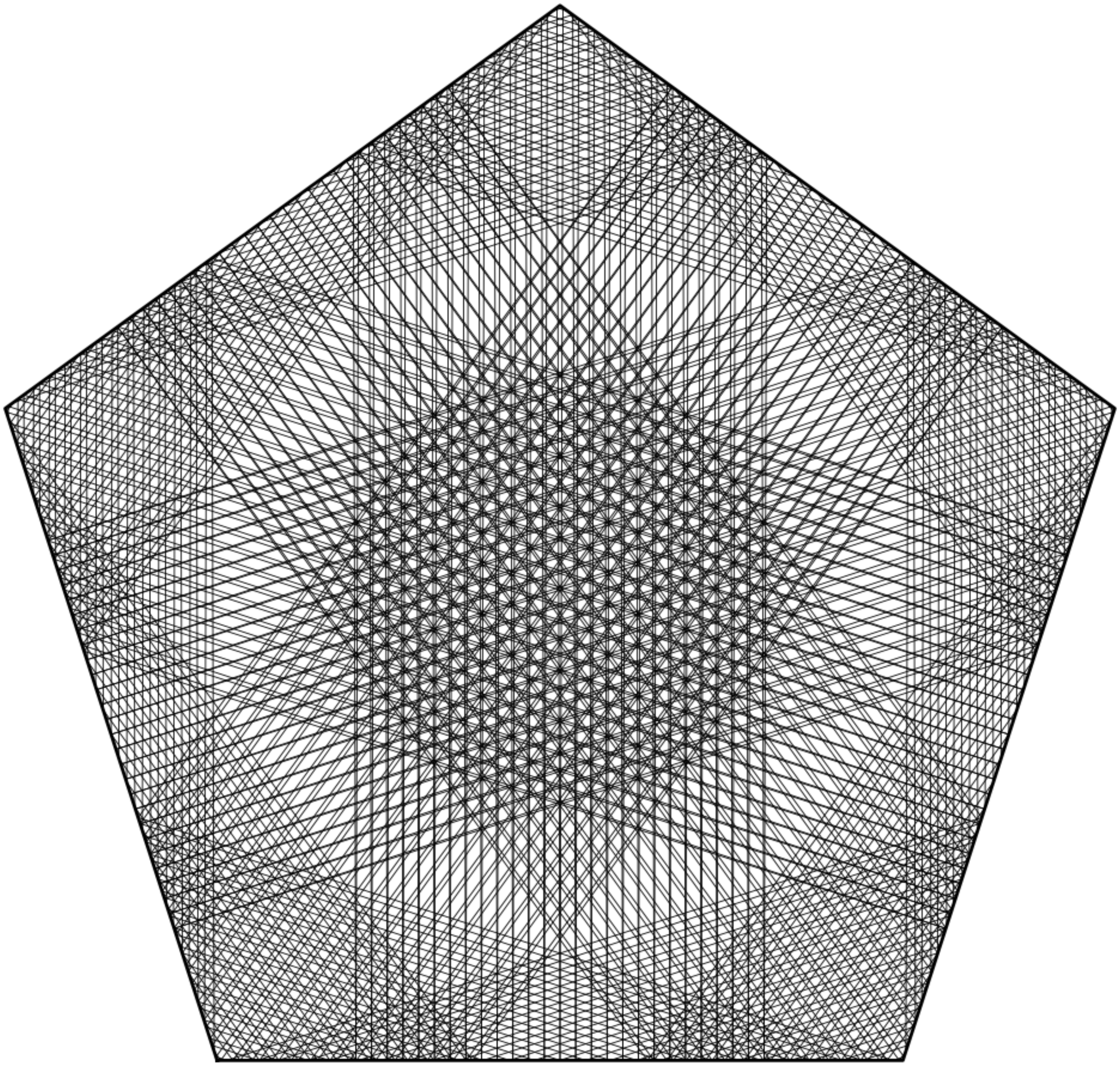
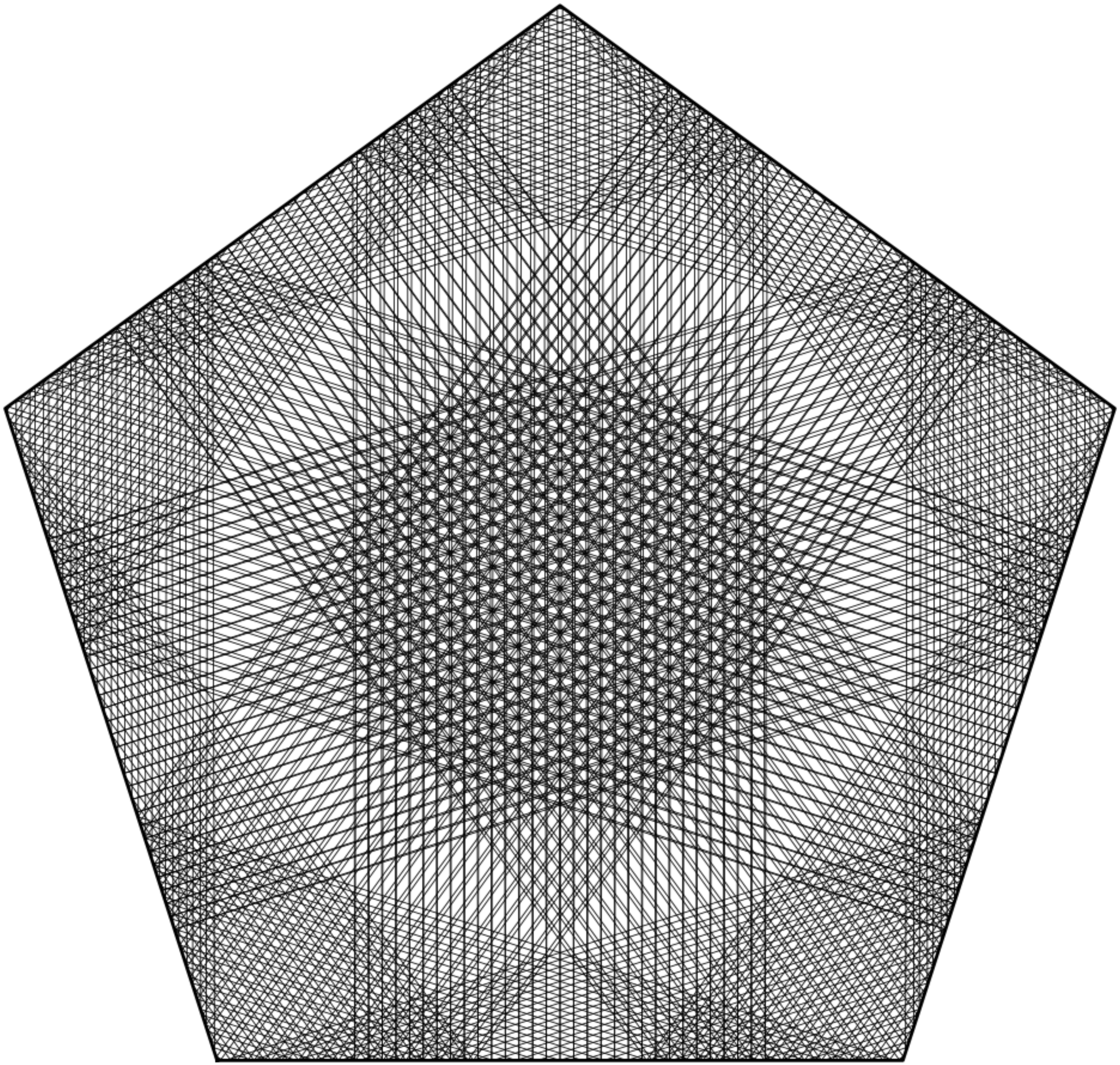
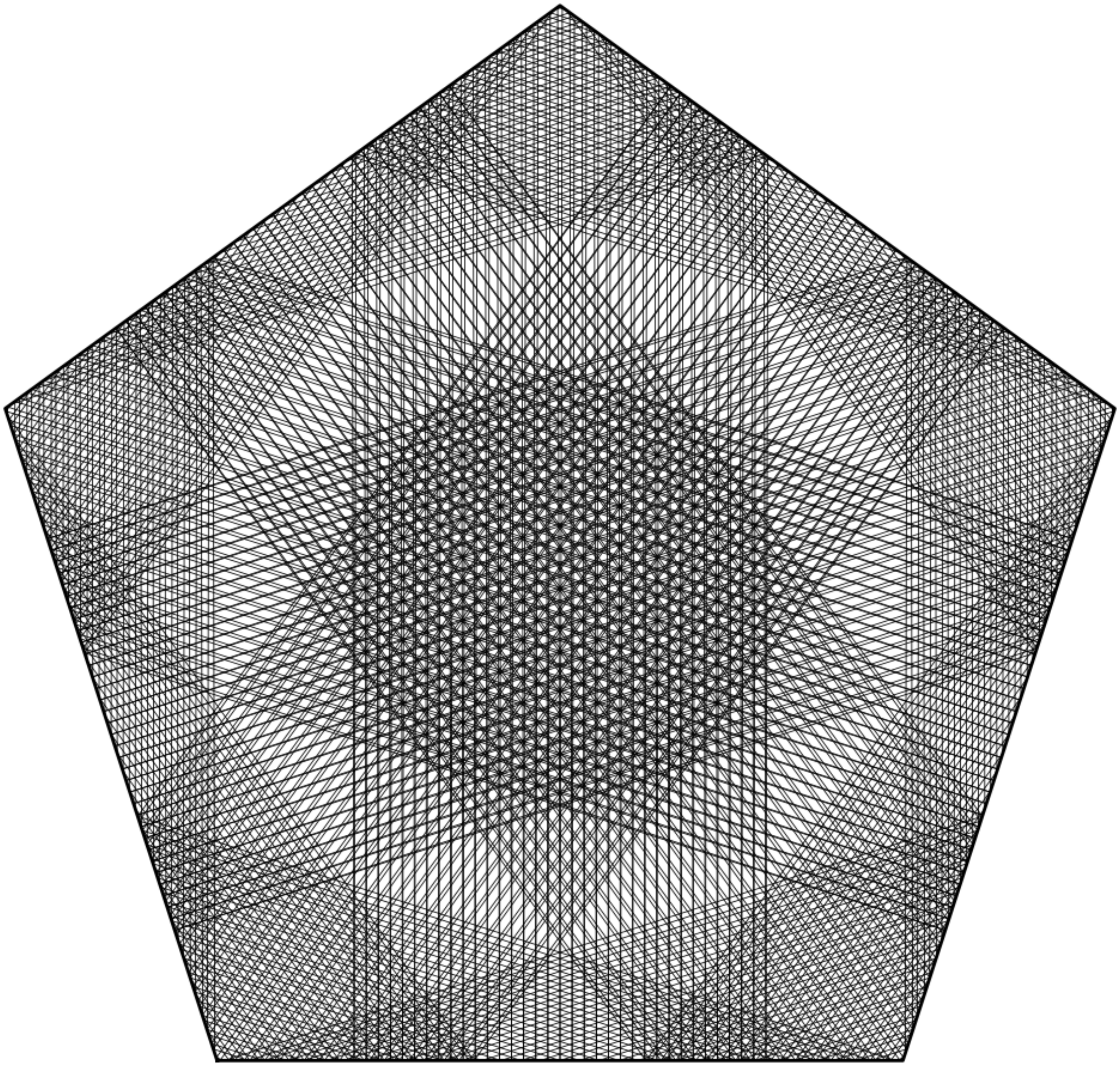
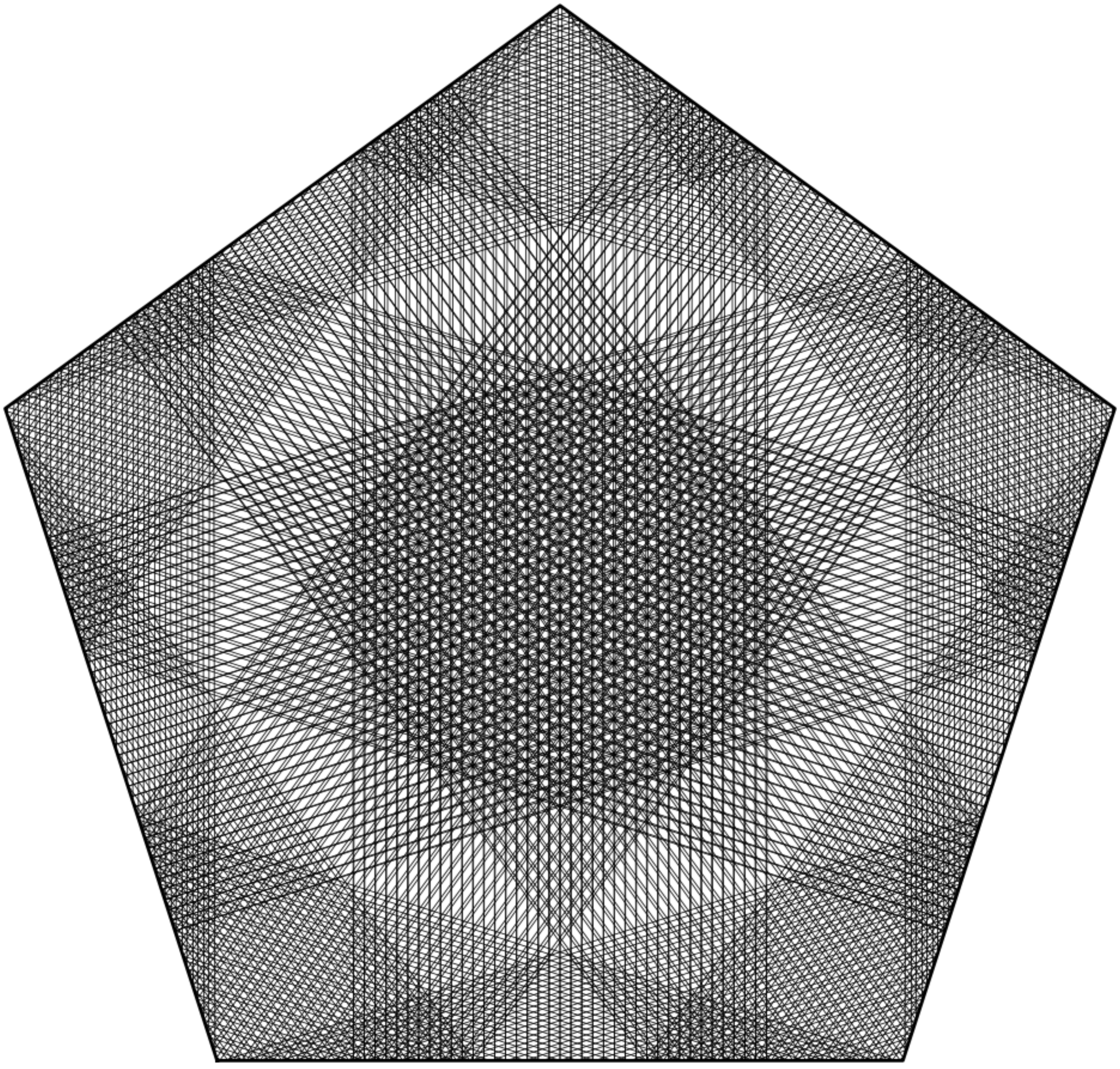


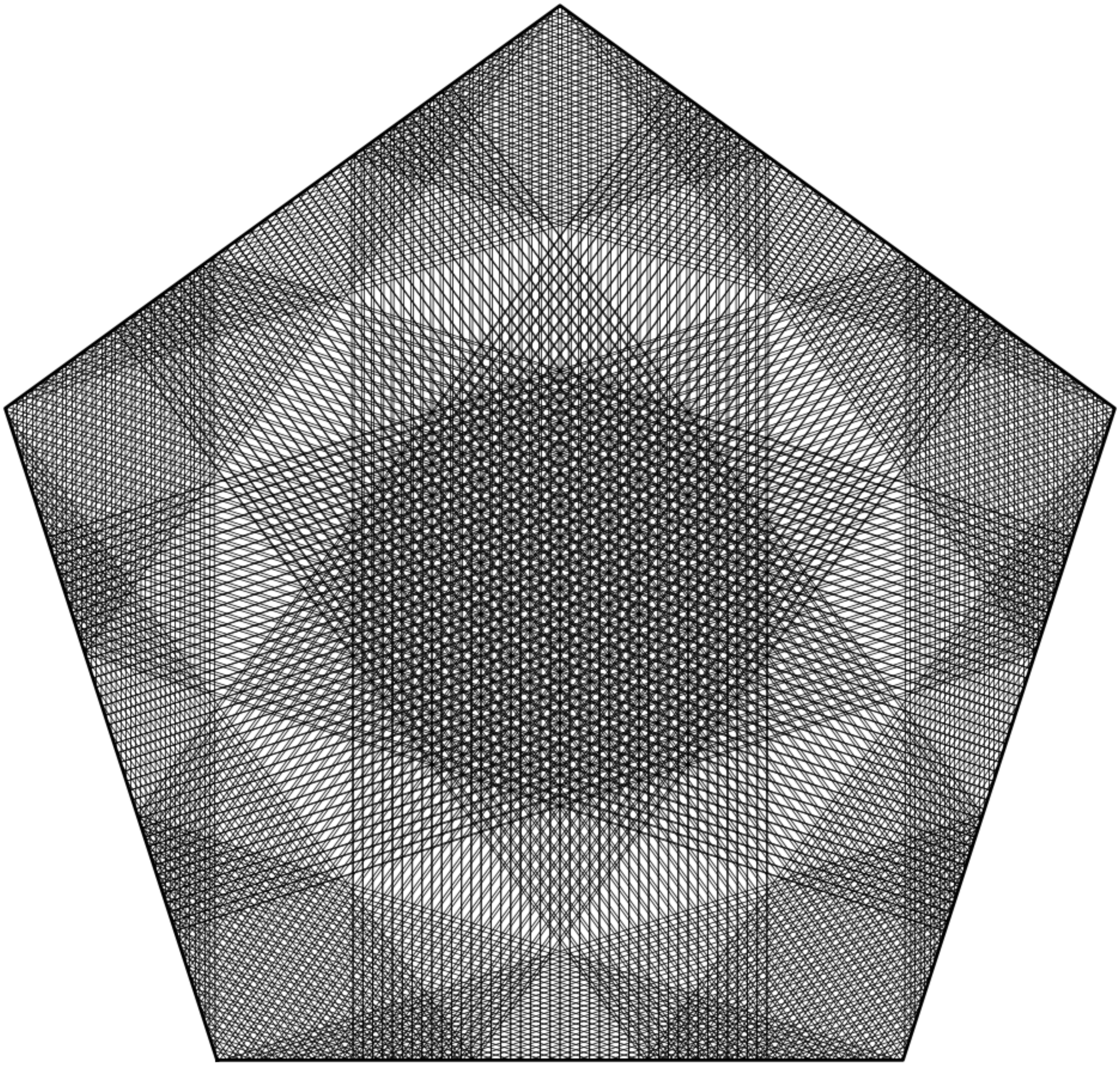


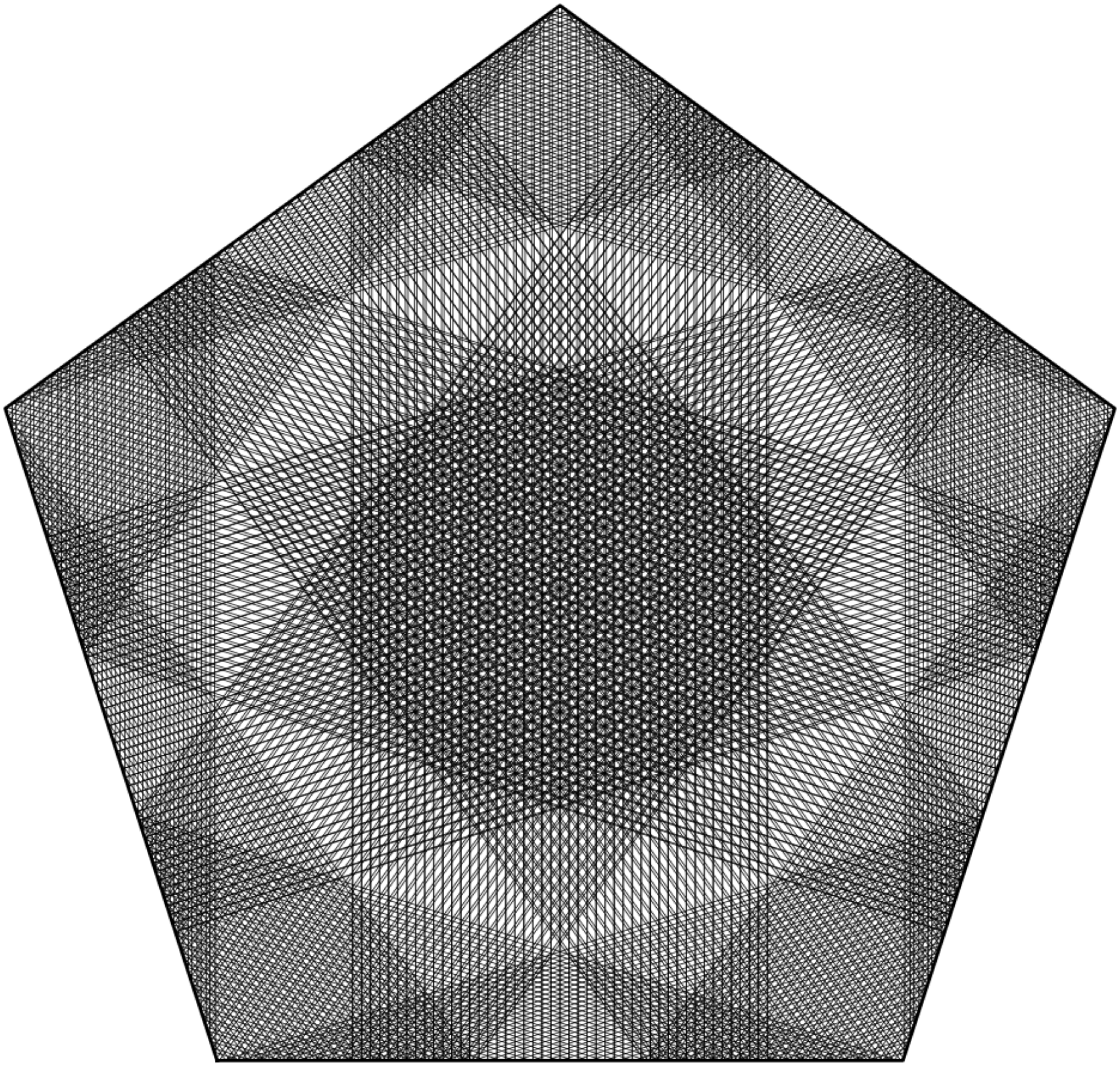


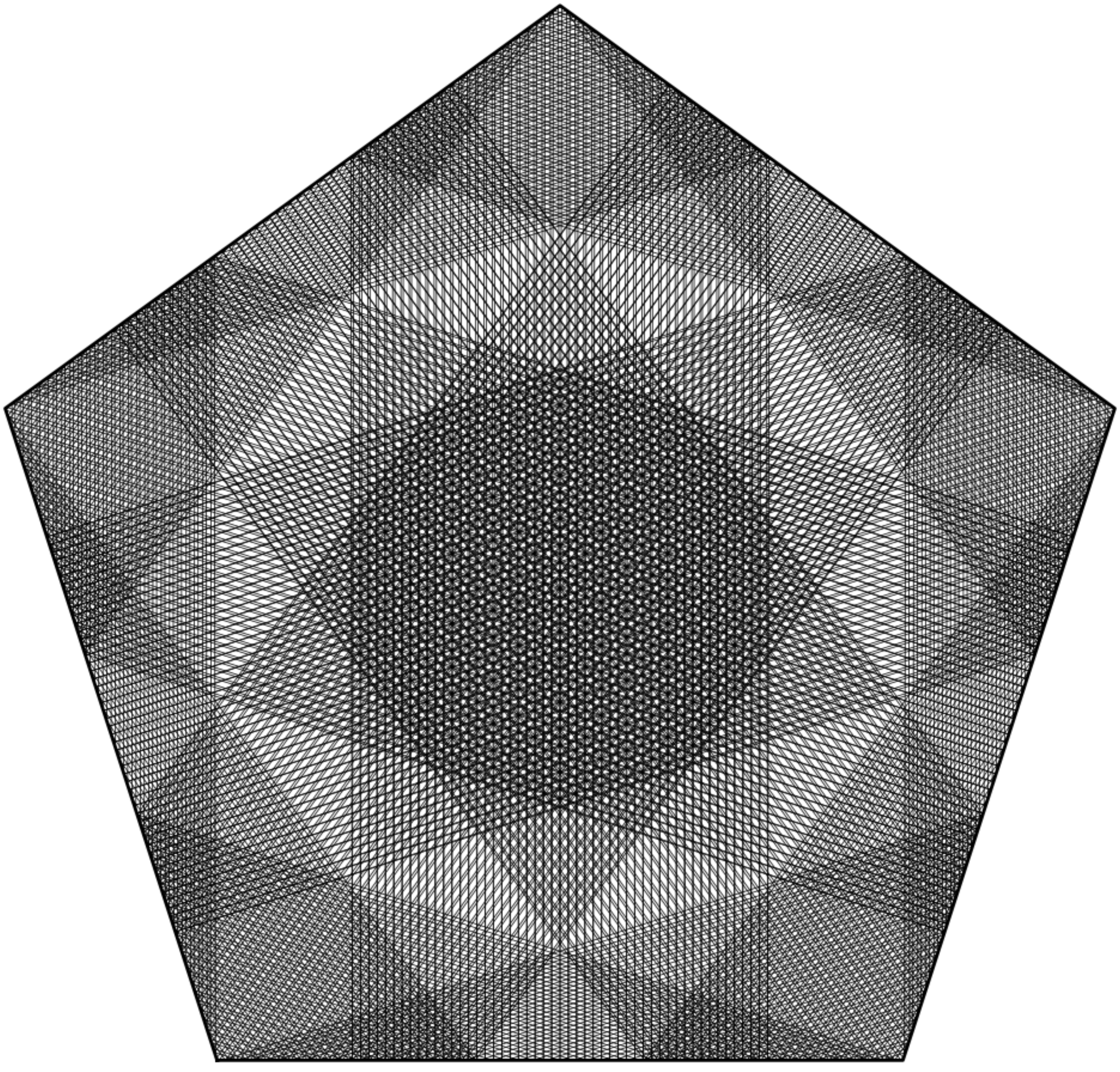


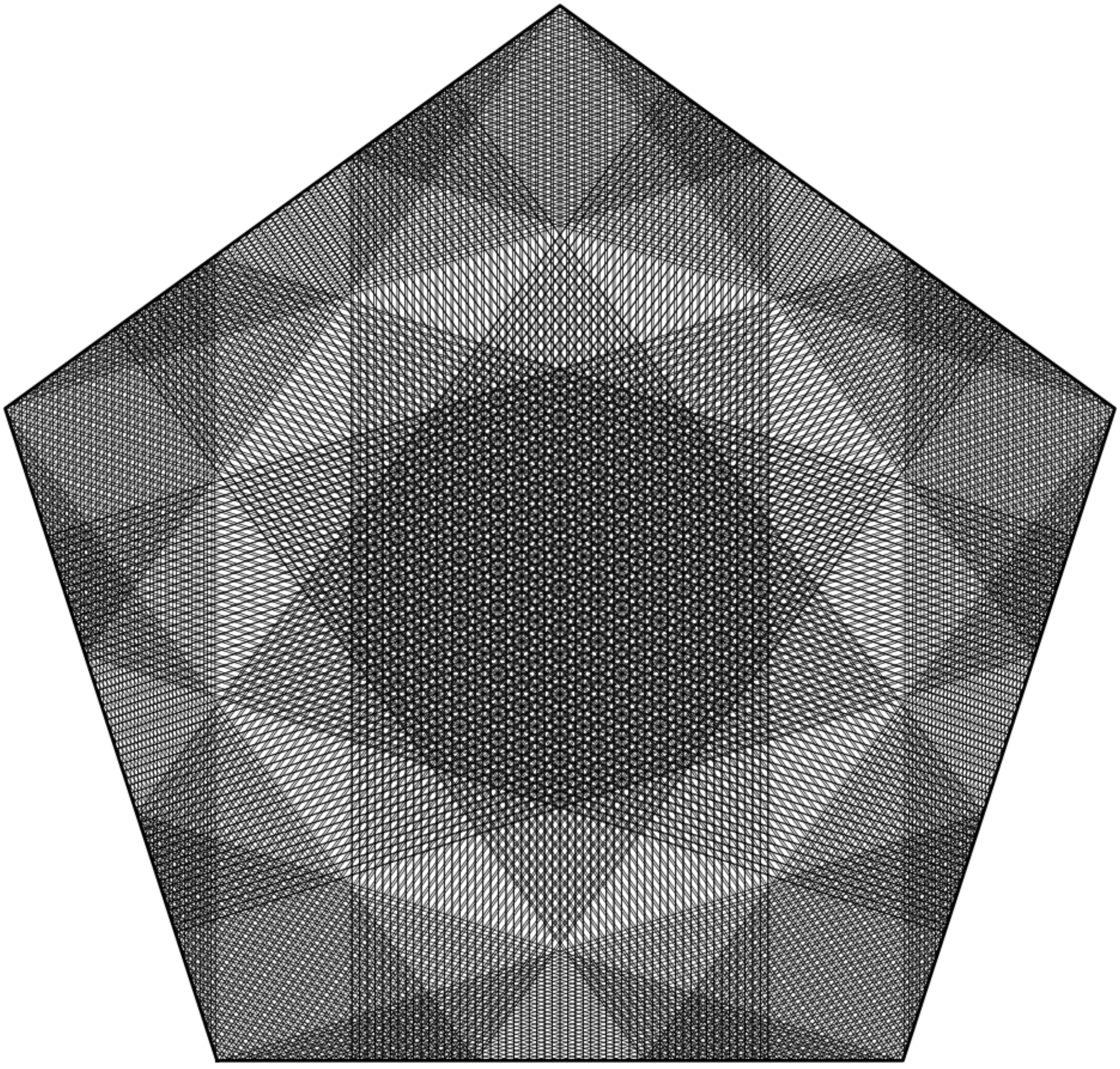


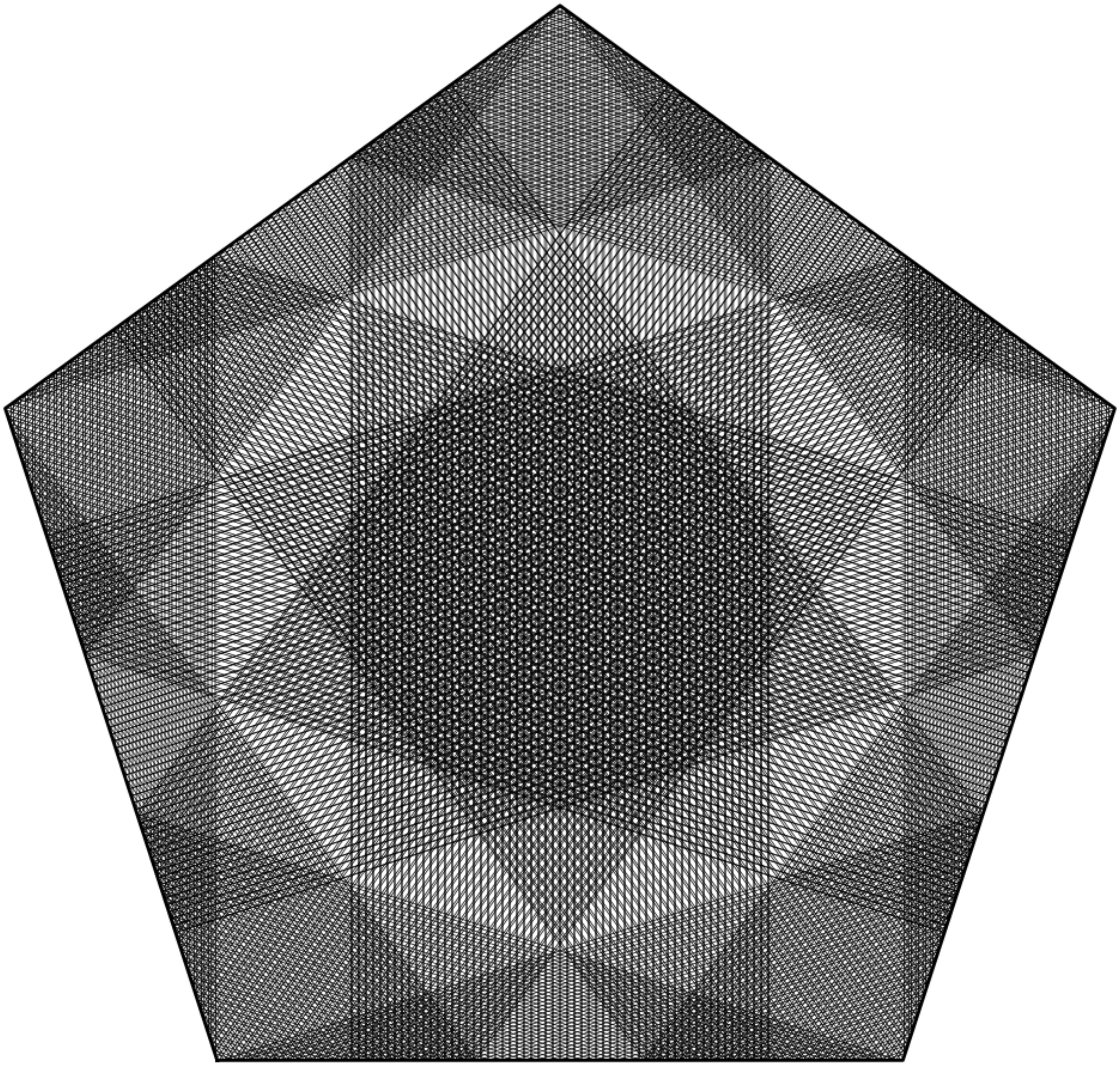


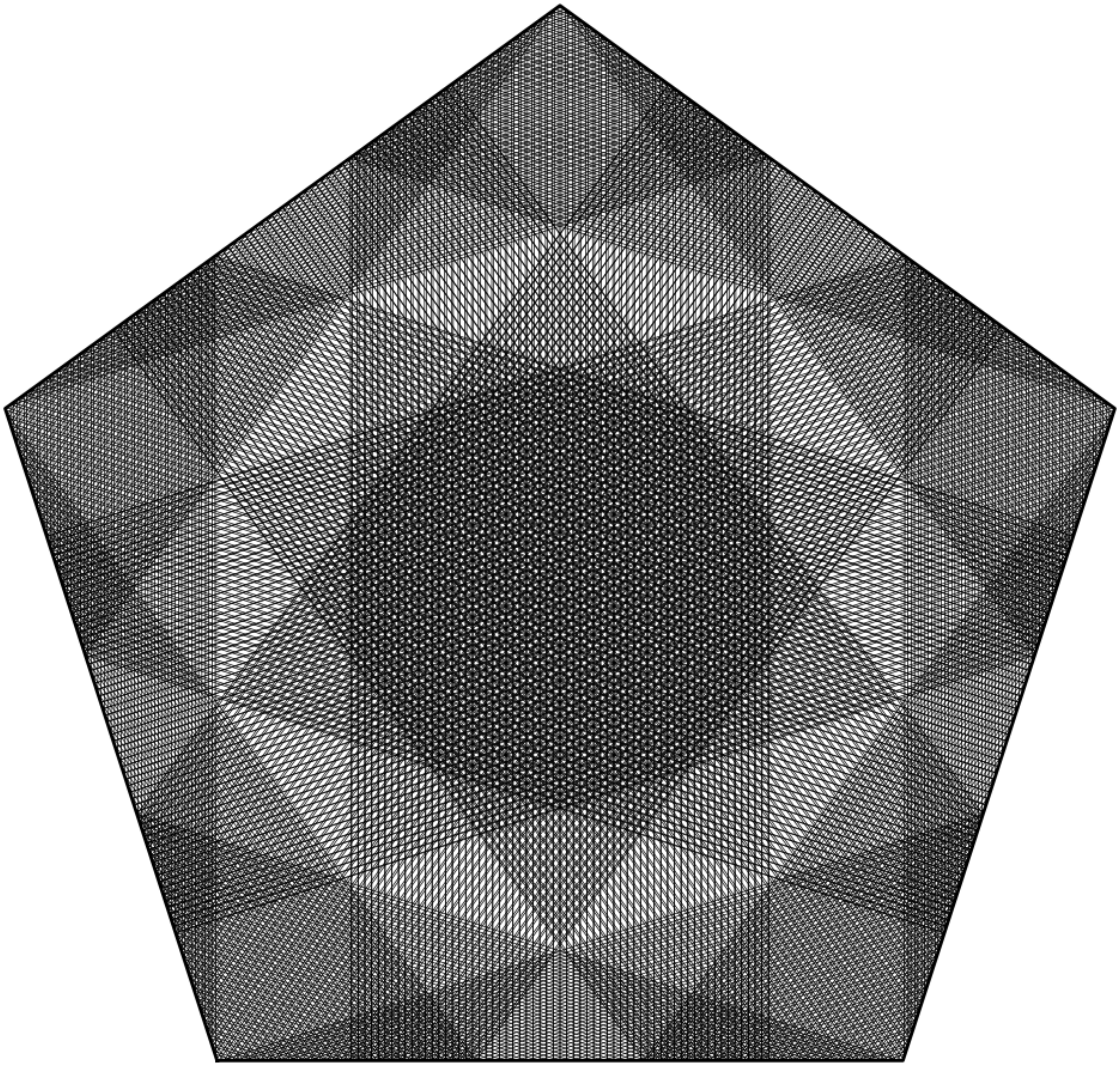


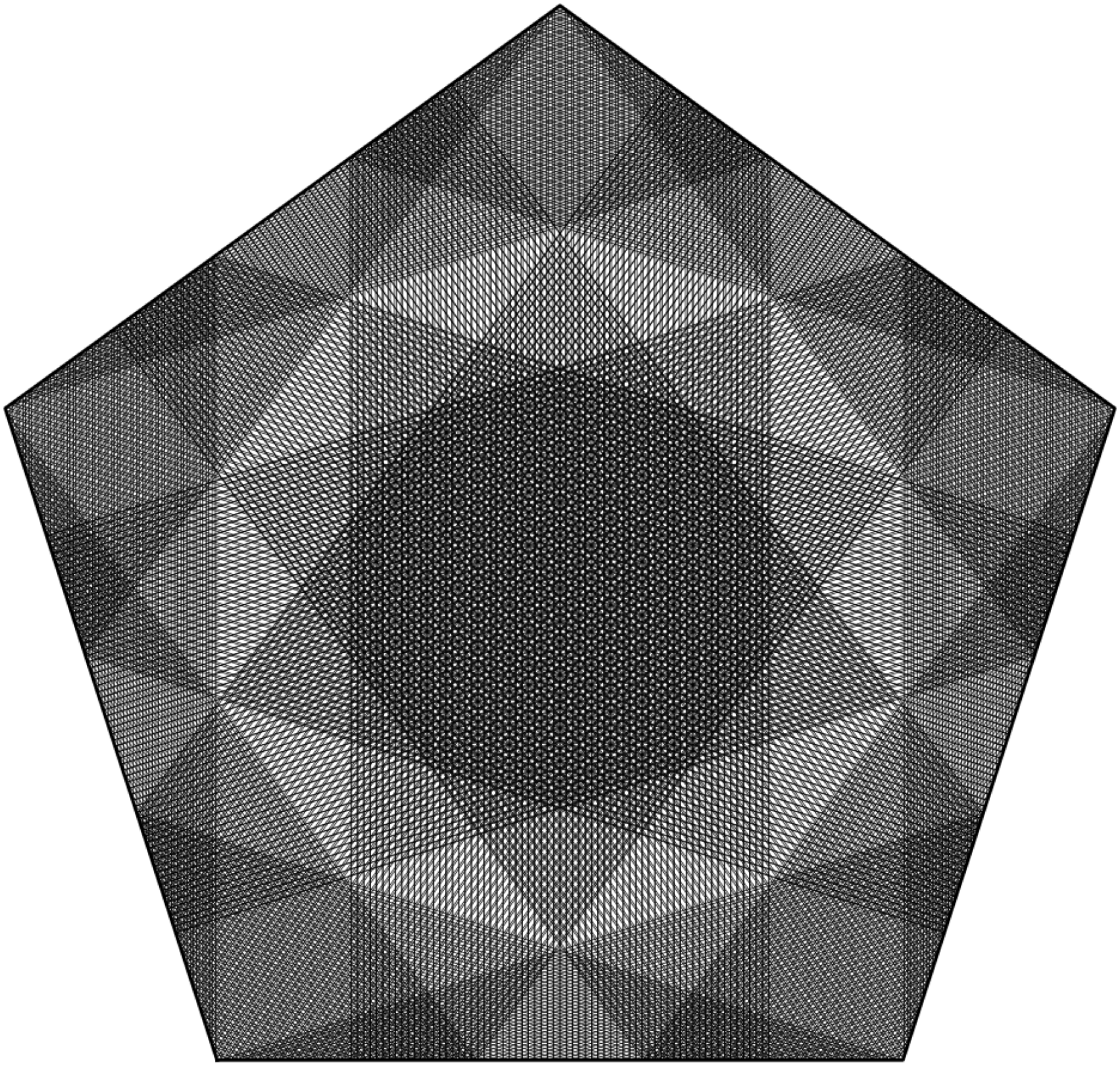


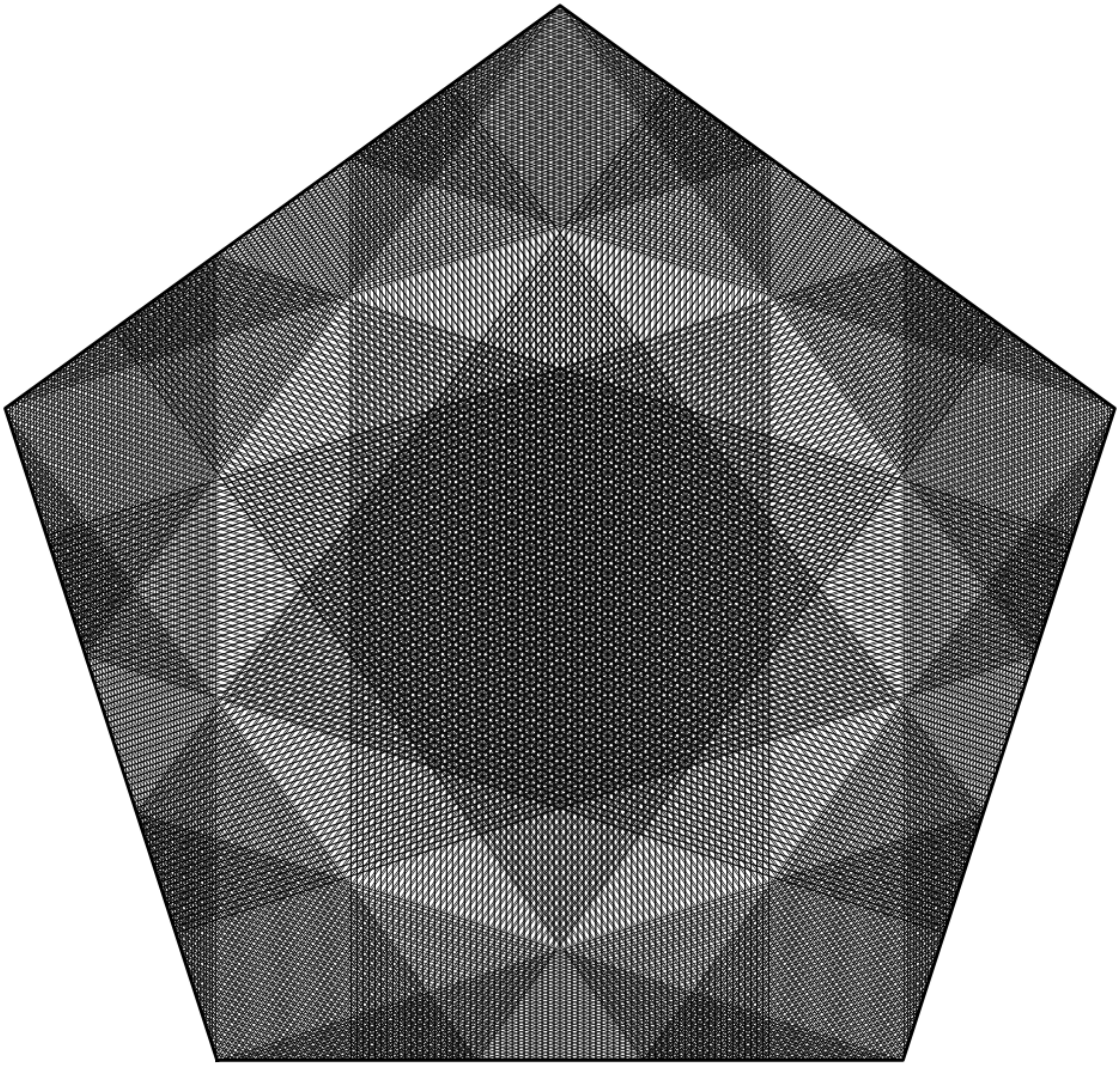


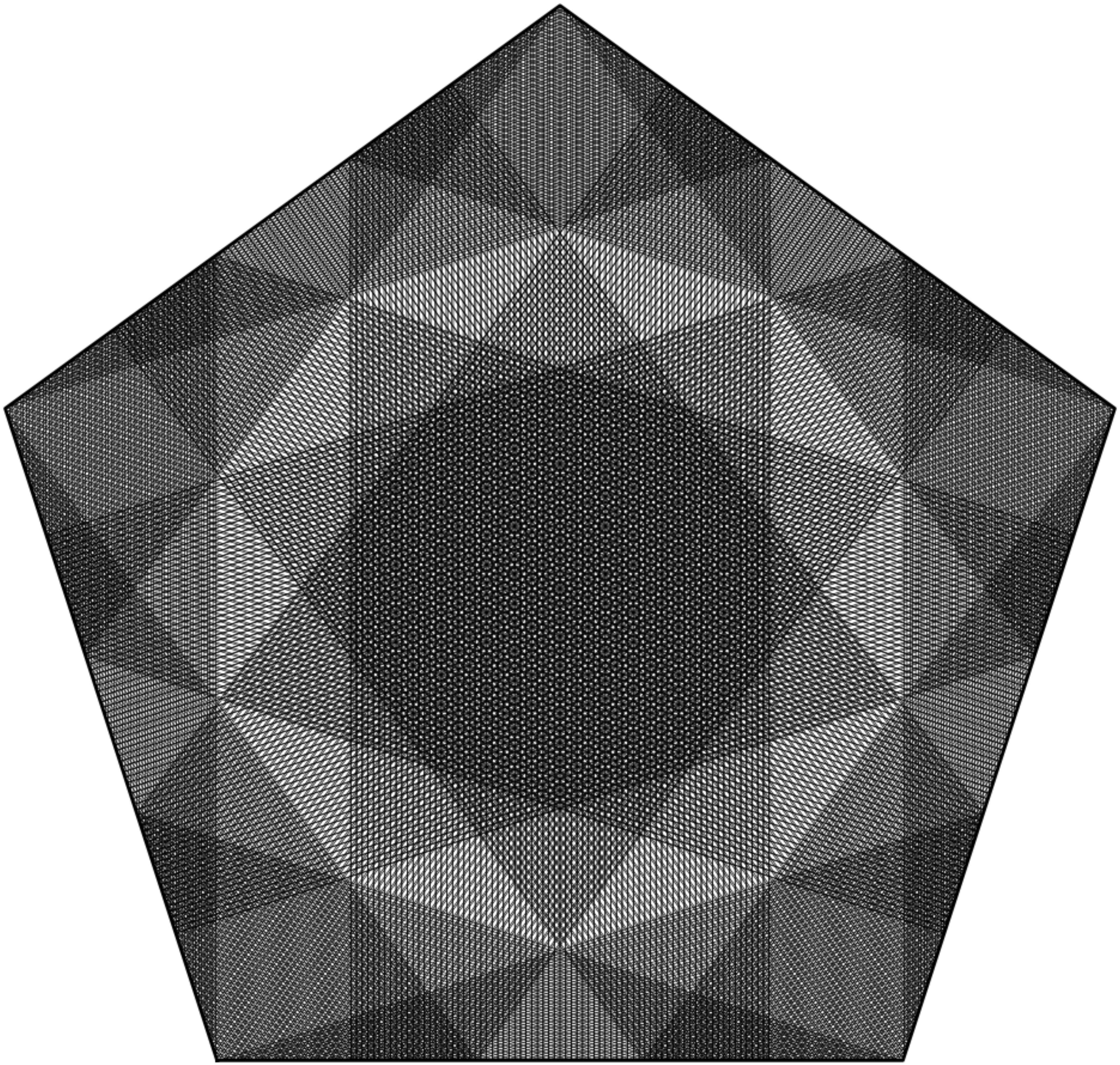


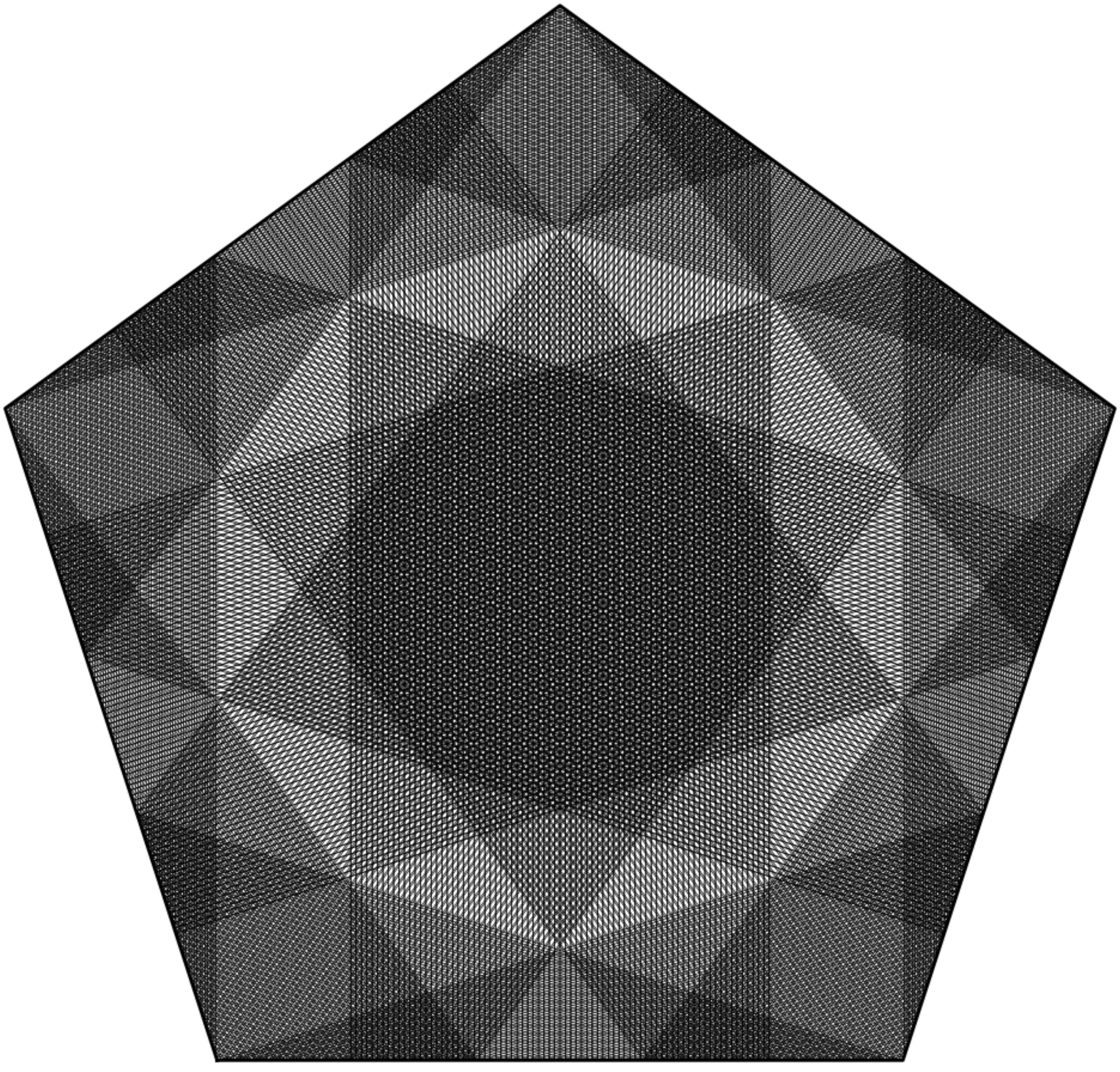


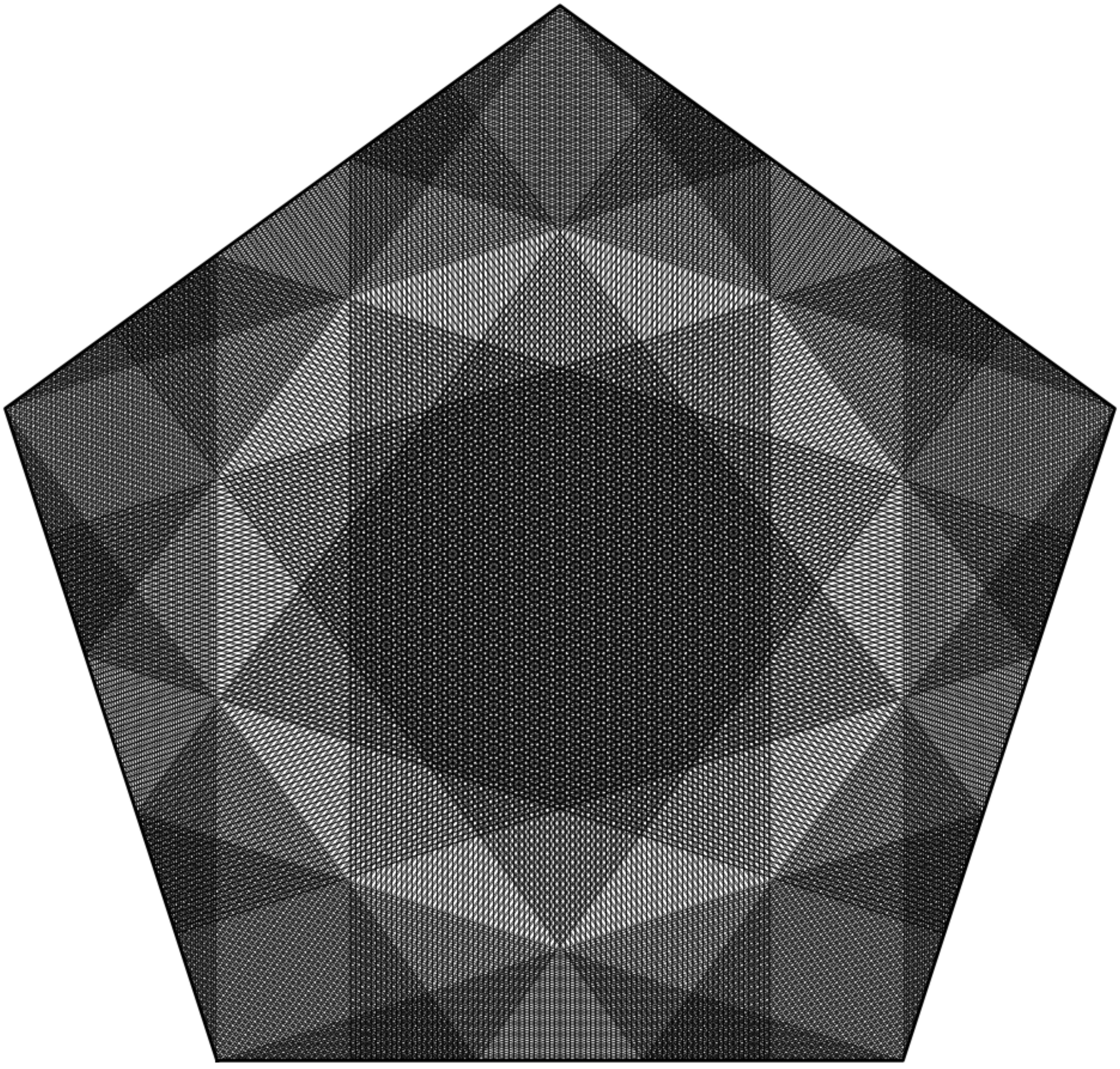


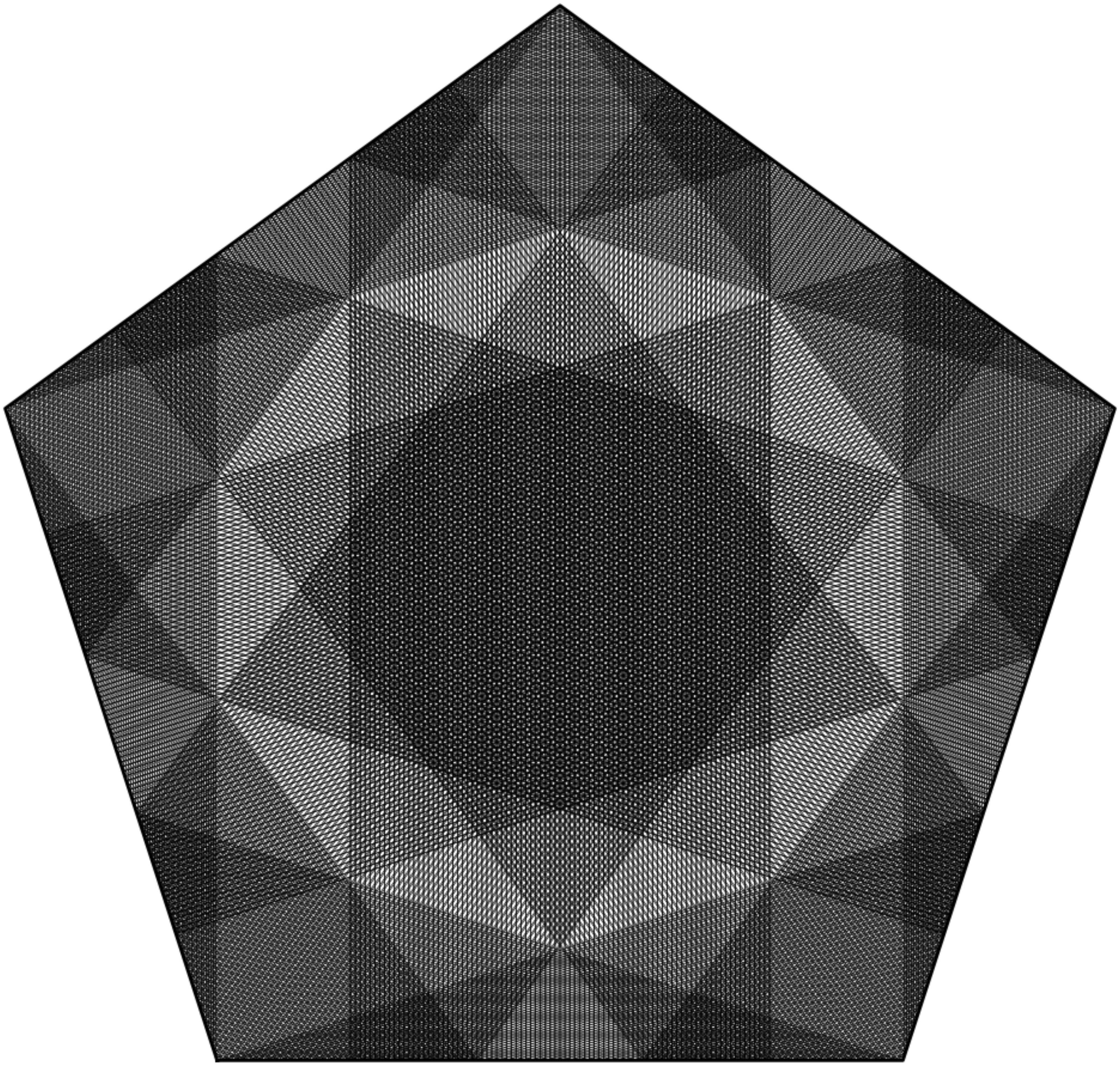


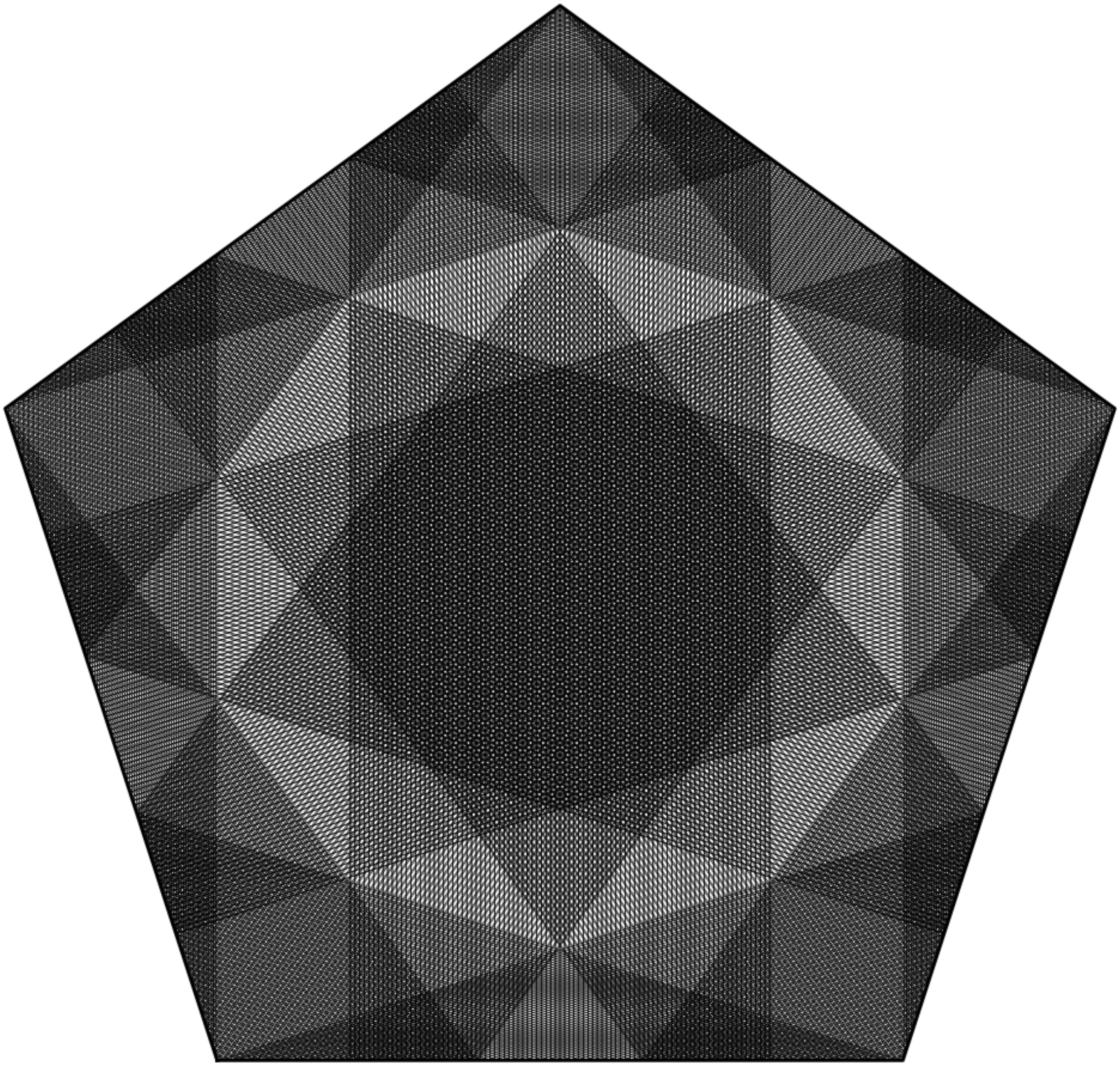


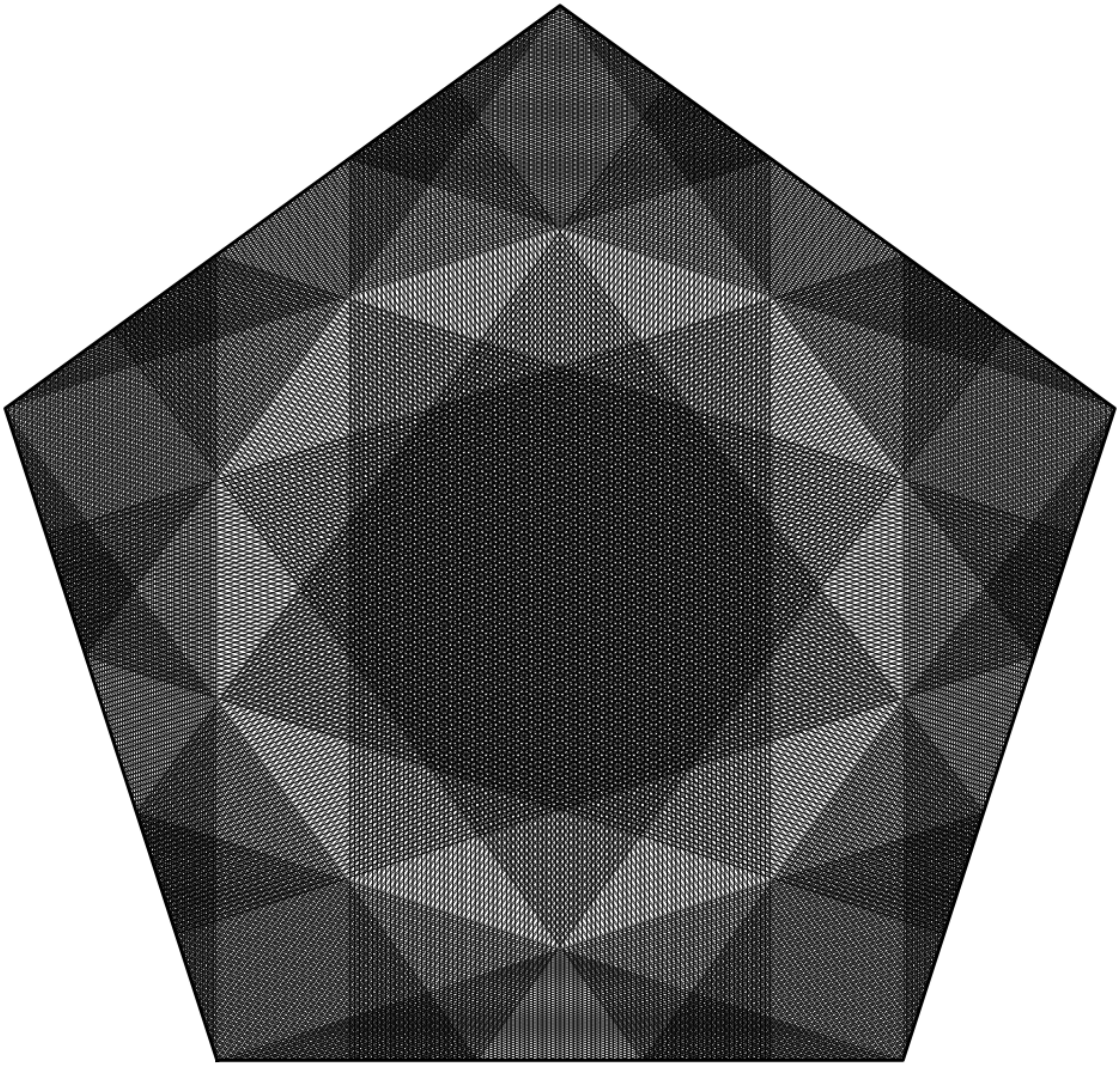


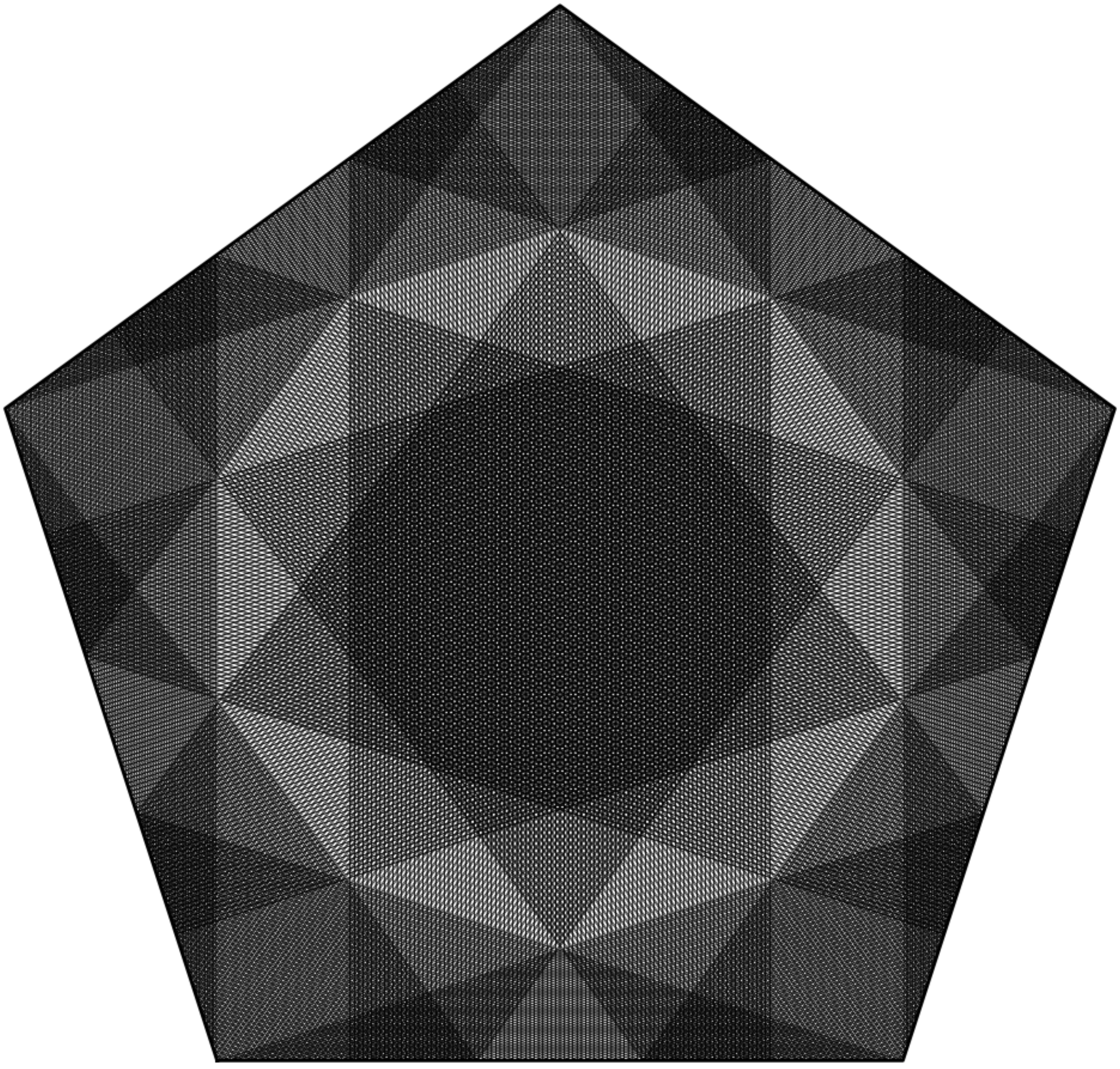


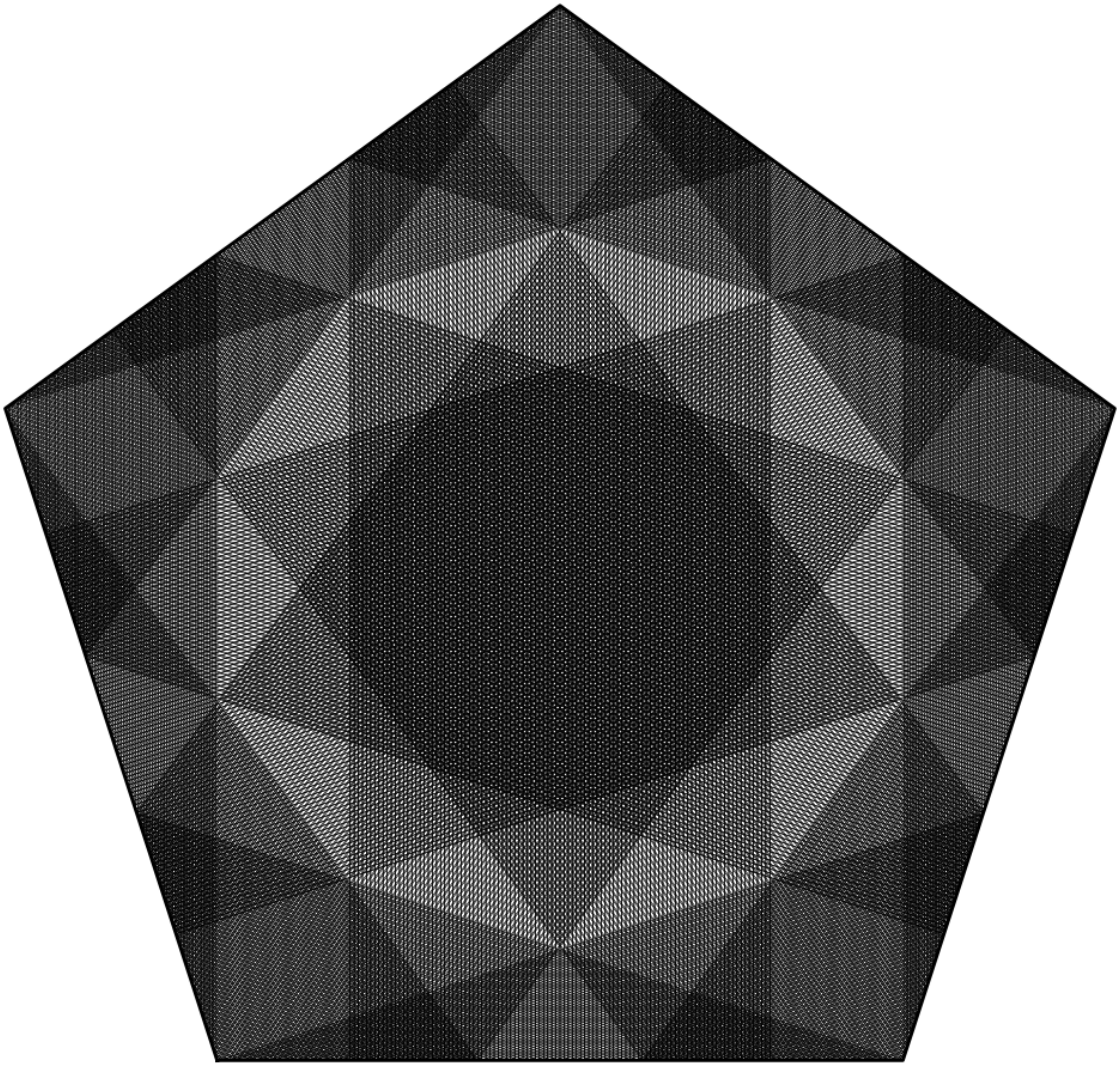




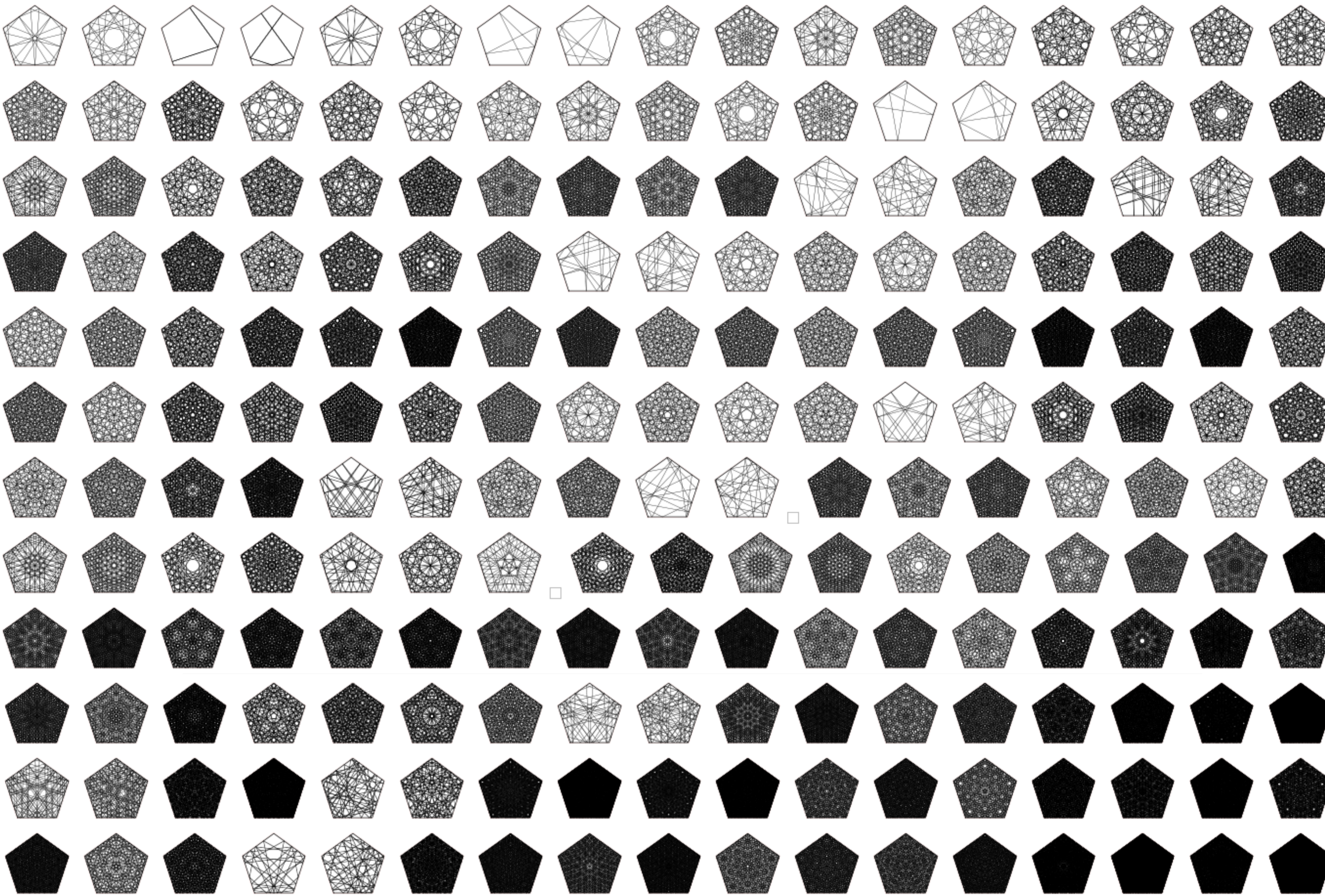








Library of pentagon billiard trajectories



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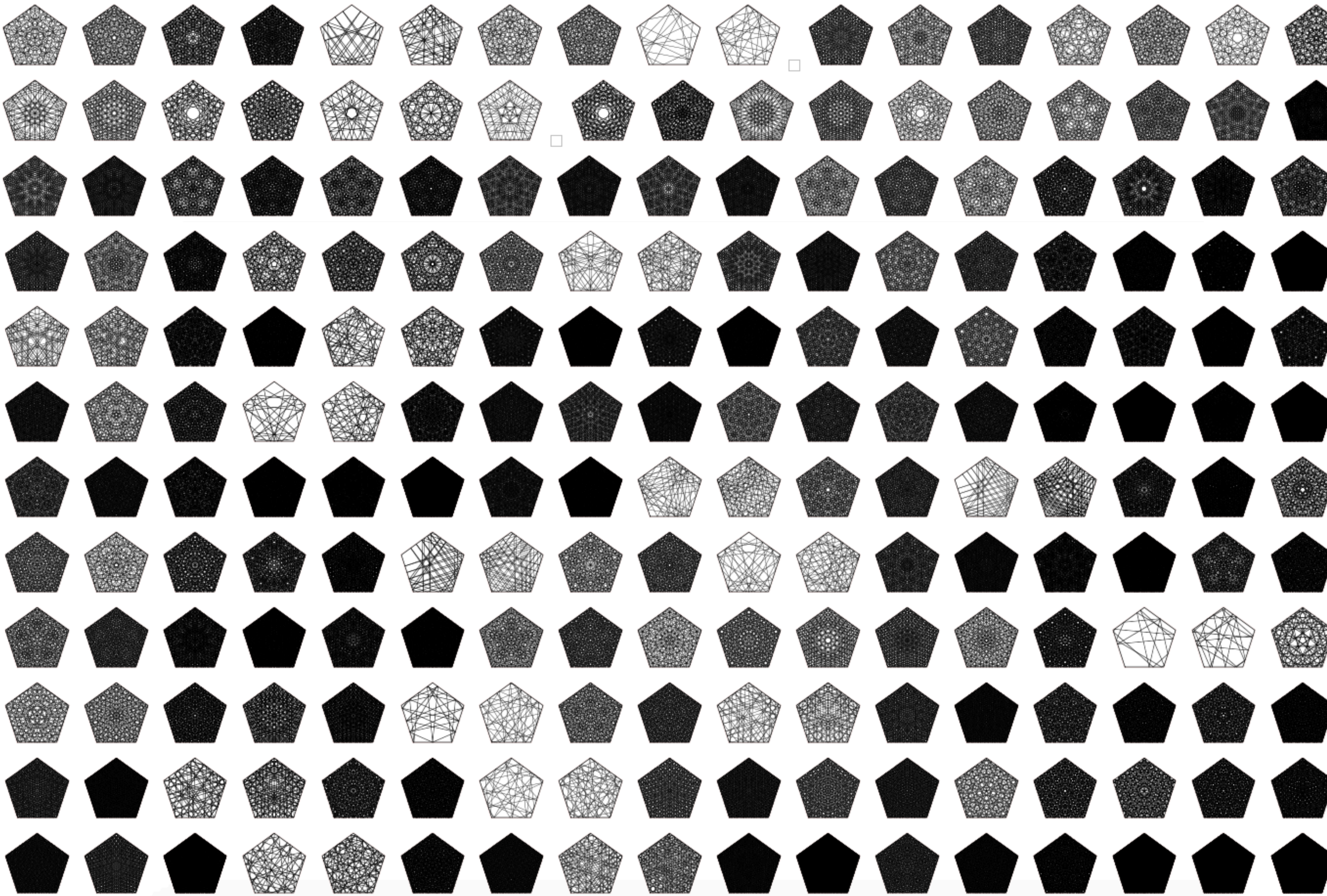
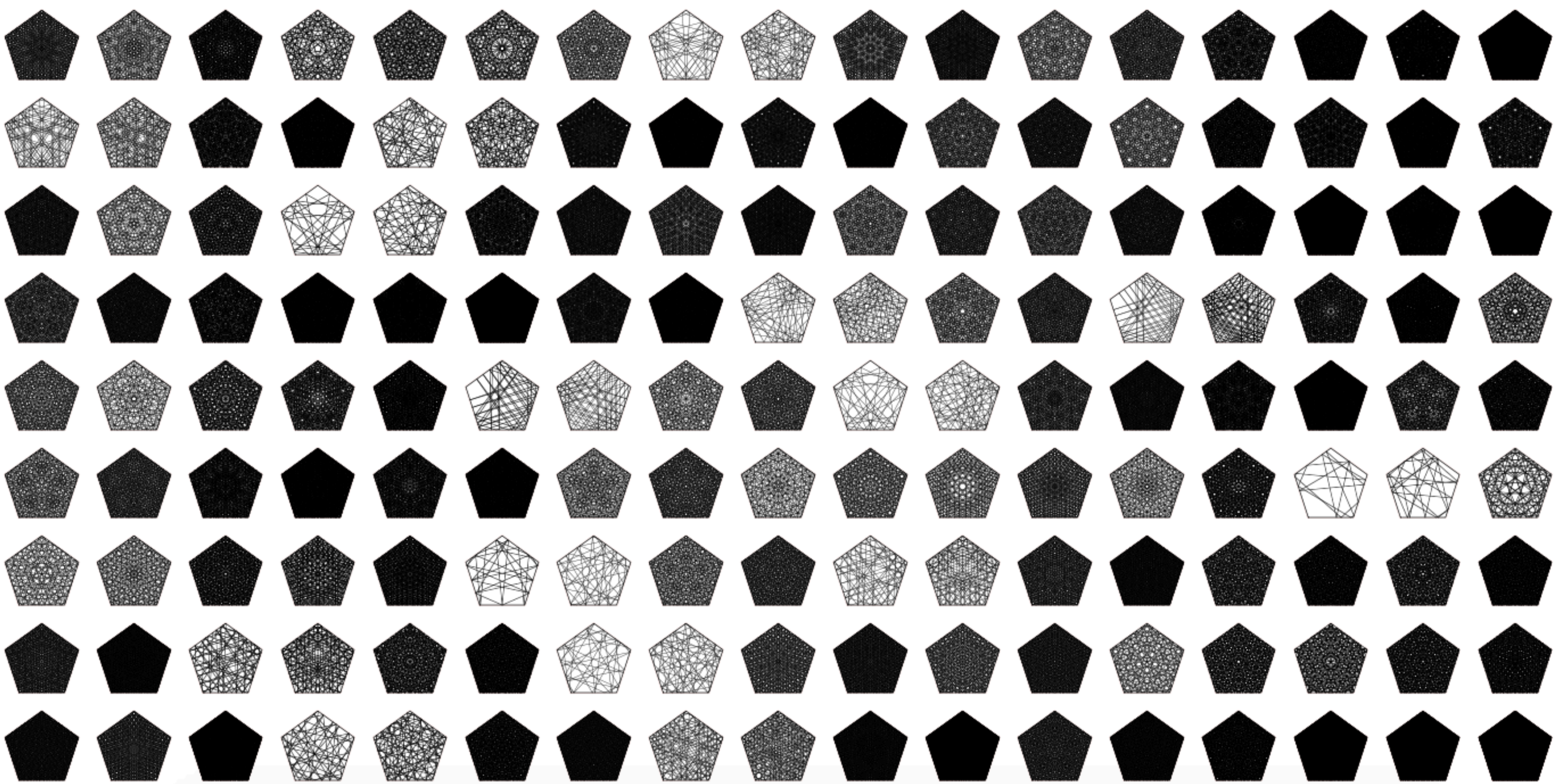


Library of pentagon billiard trajectories

Conjecture from observing library:

As the length of the path increases, periodic billiard trajectories "equidistribute" in the pentagon.



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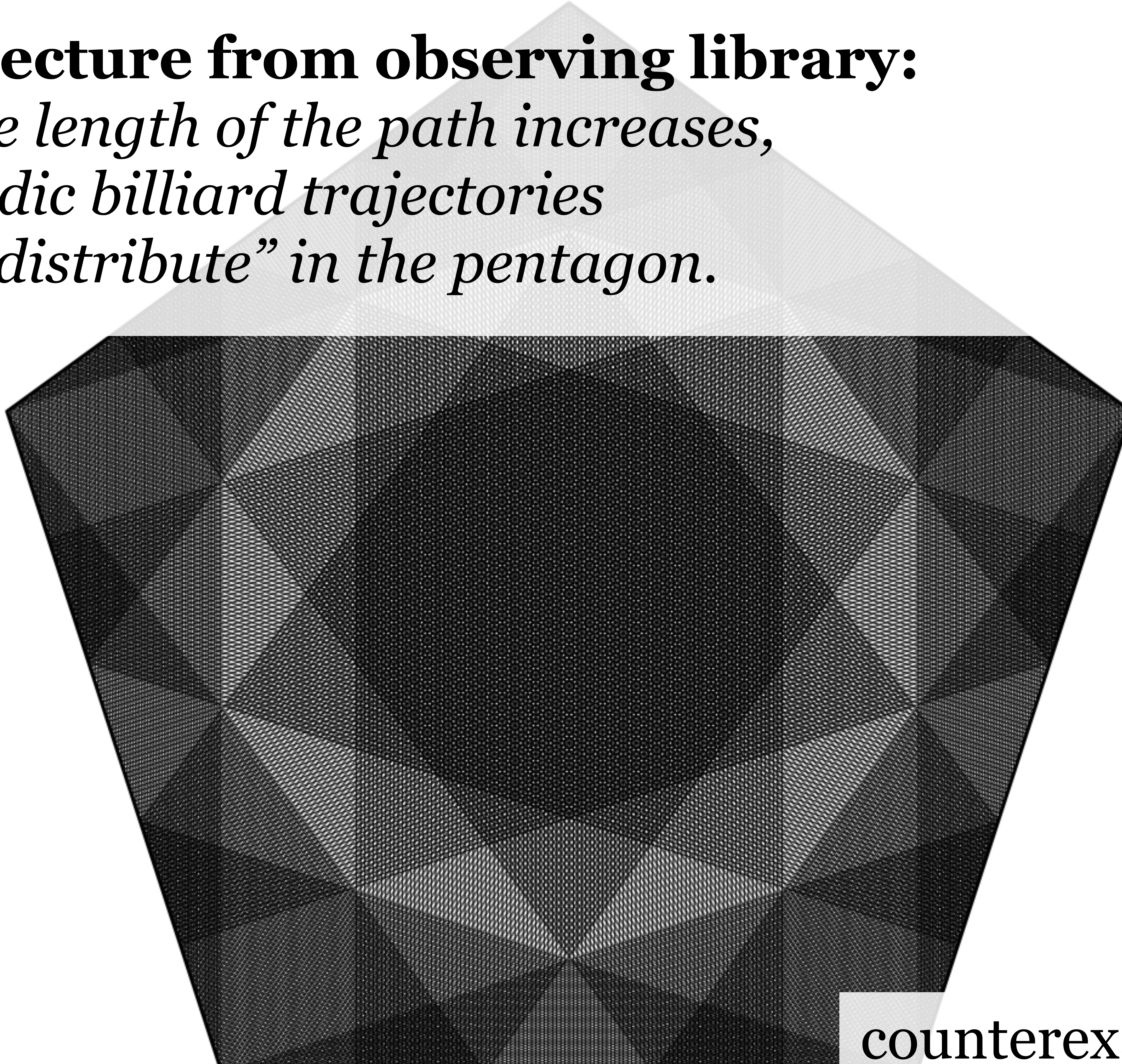
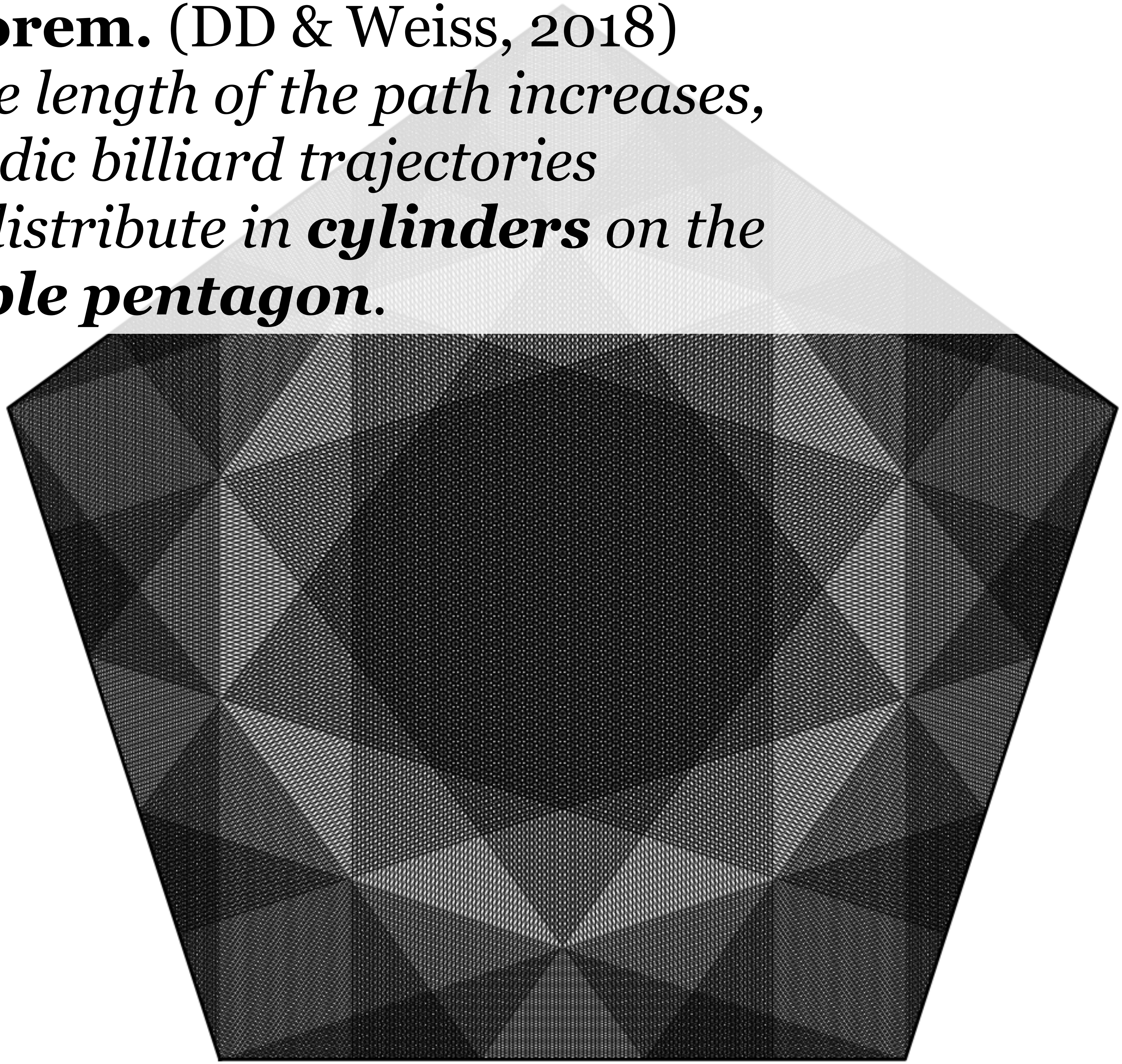


counterexample.

Theorem. (DD & Weiss, 2018)
As the length of the path increases, periodic billiard trajectories equidistribute in ***cylinders*** *on the* ***double pentagon***.



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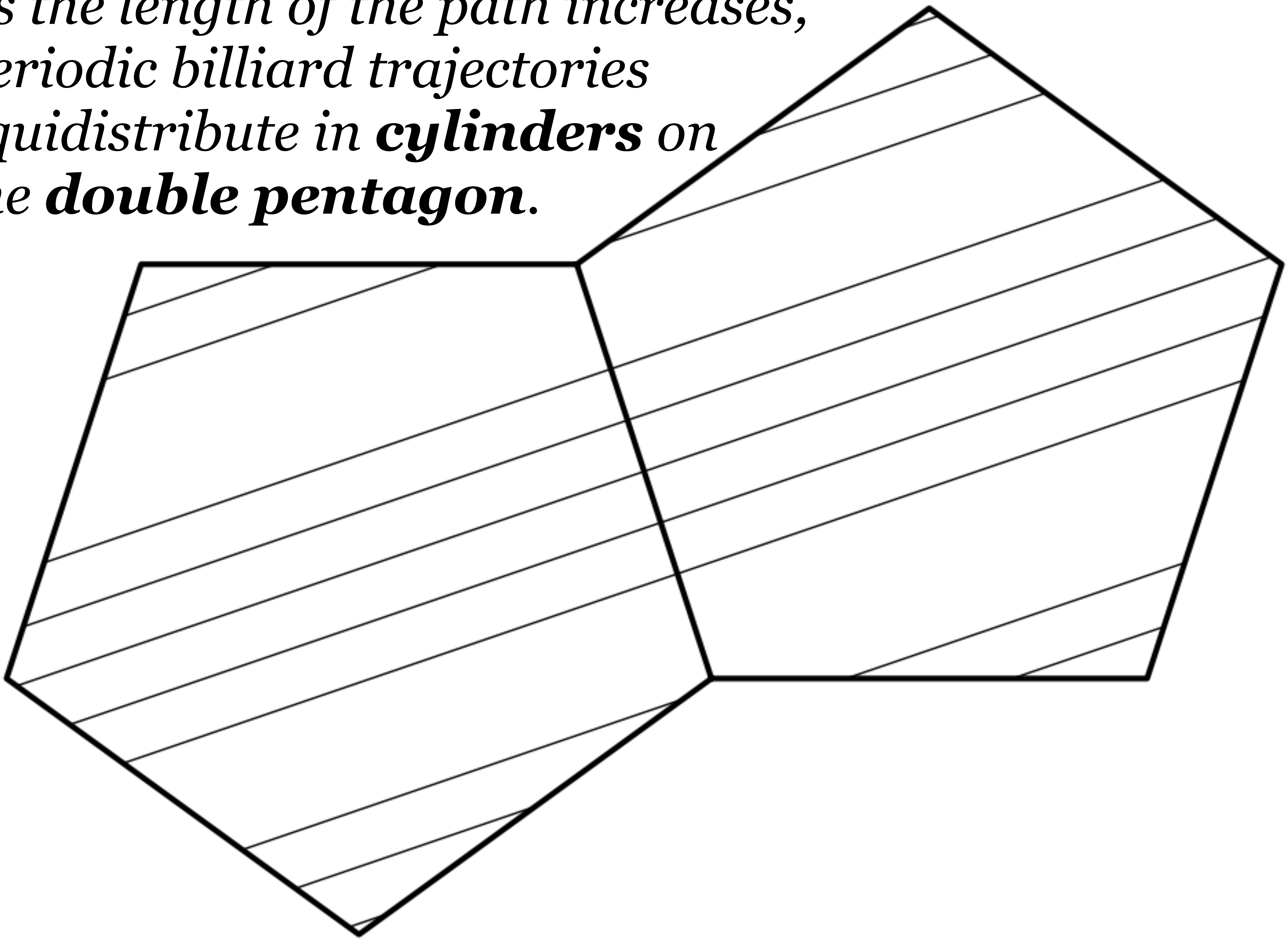

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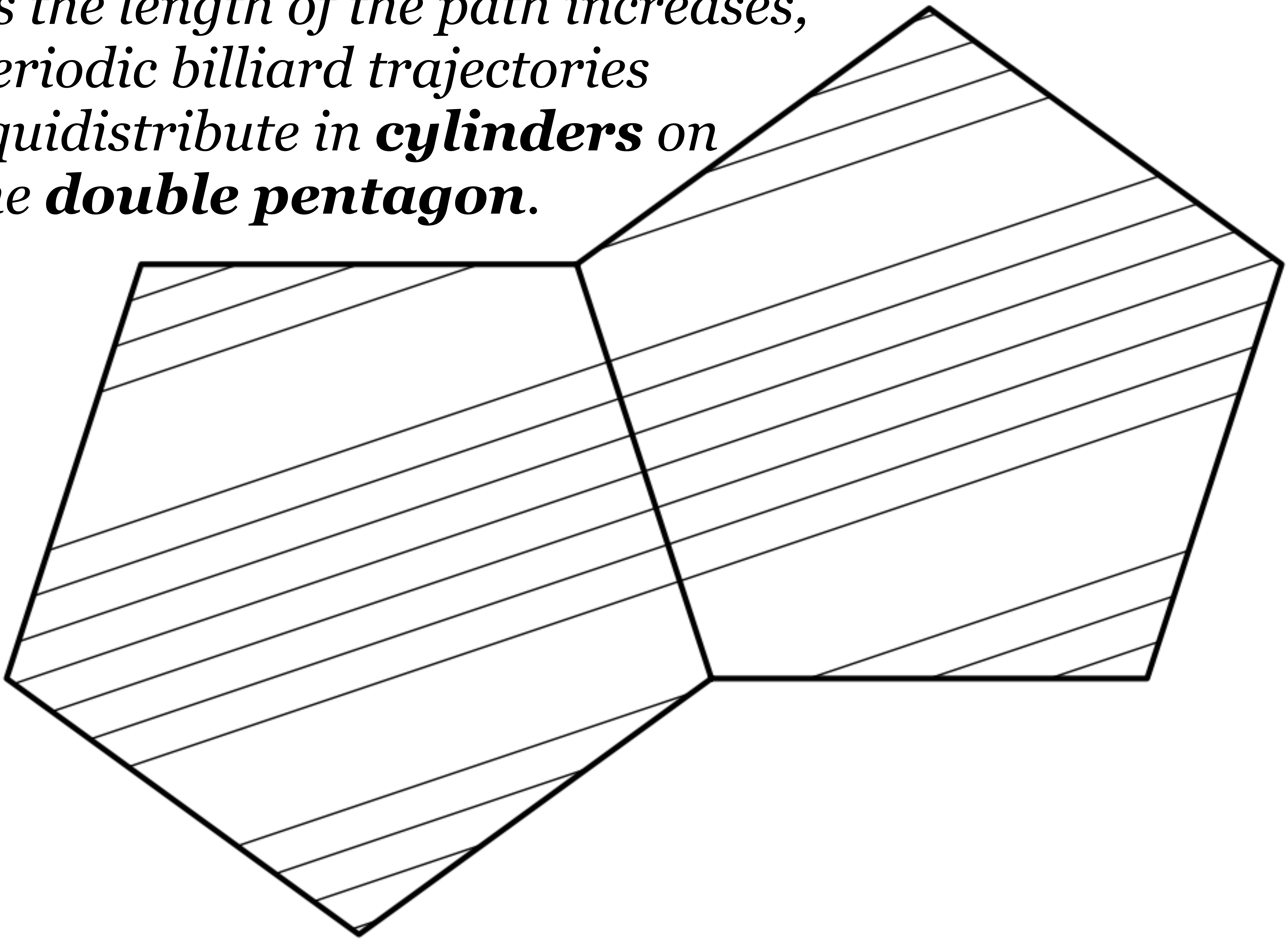

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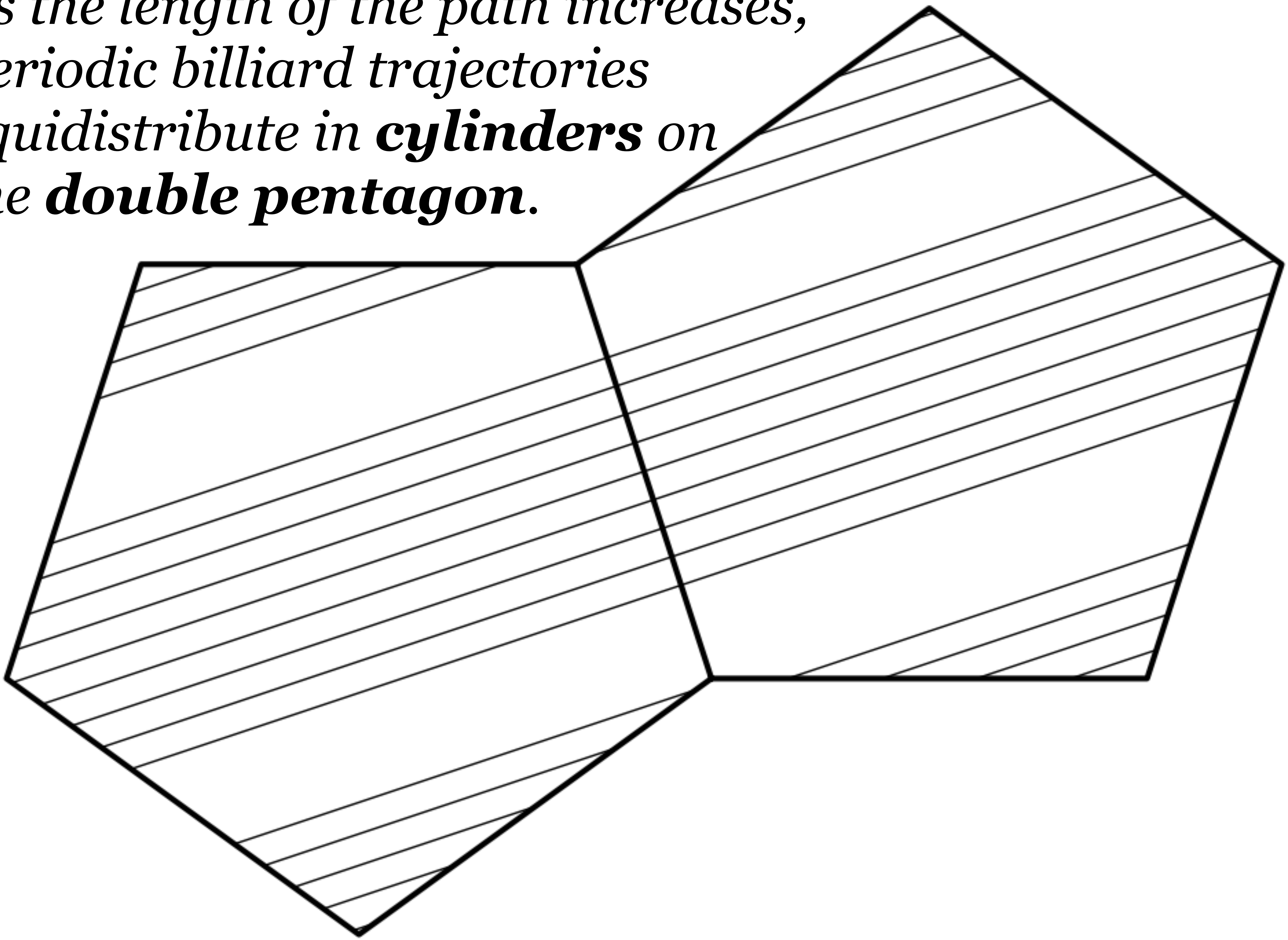

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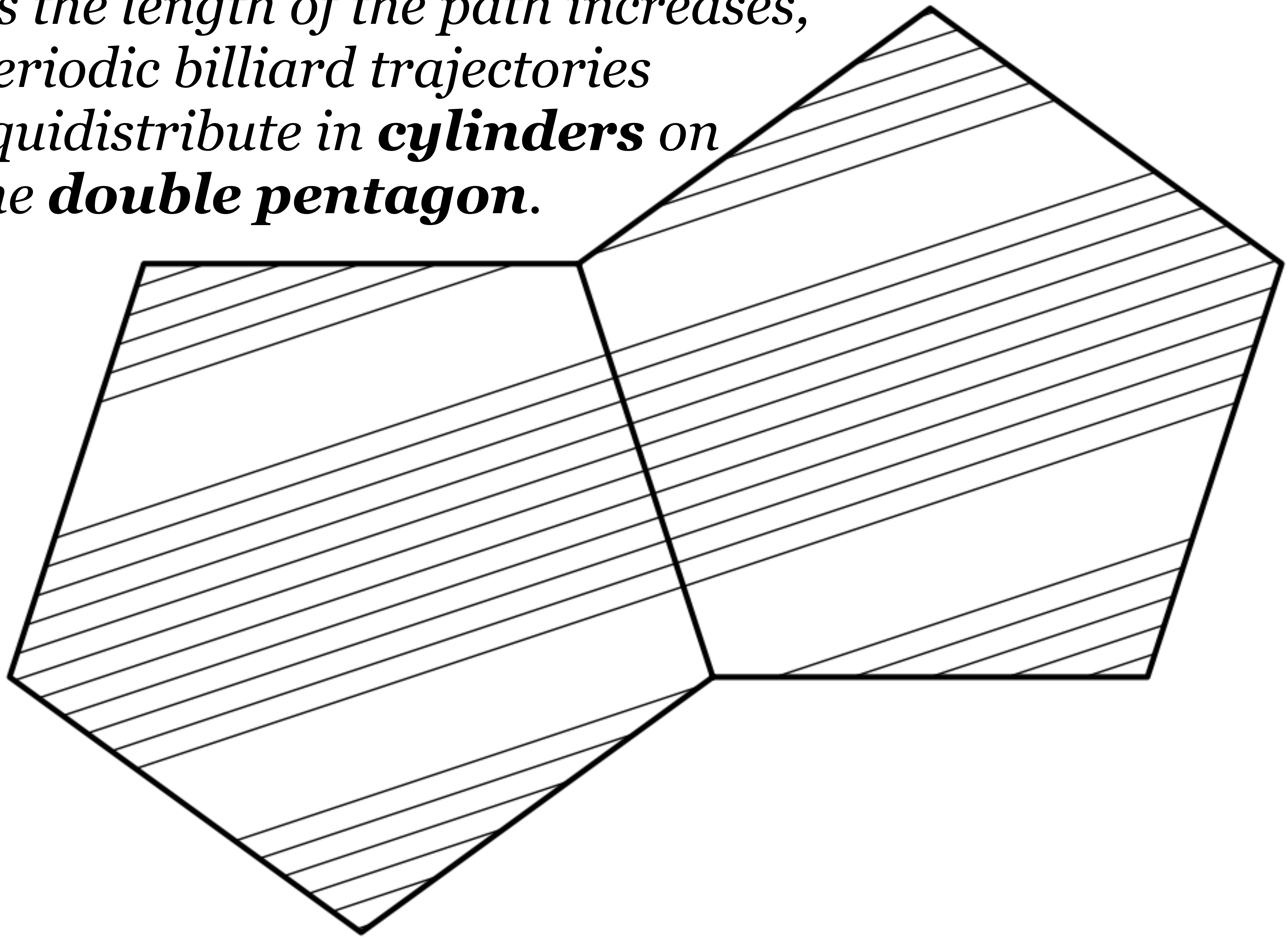

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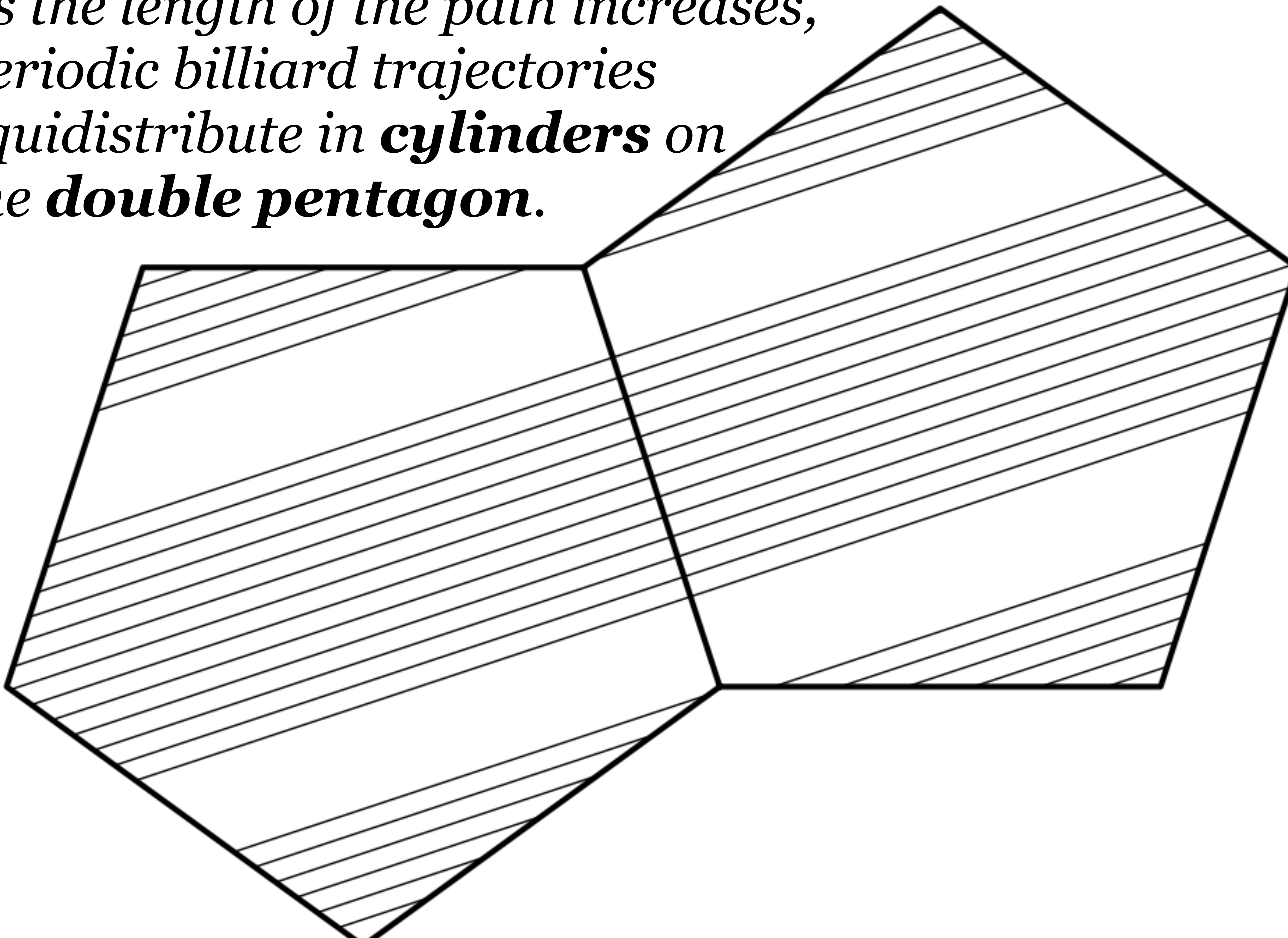

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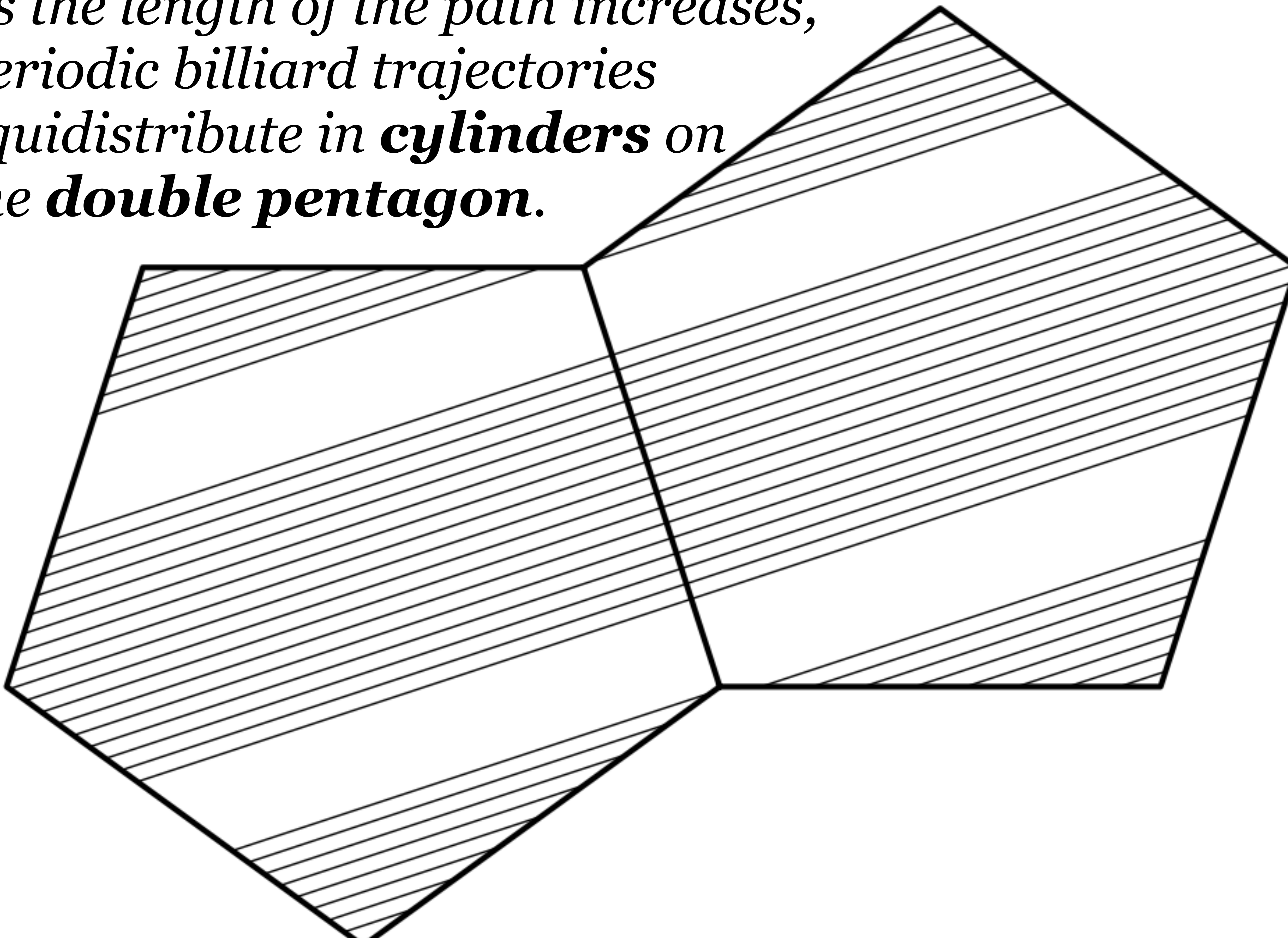

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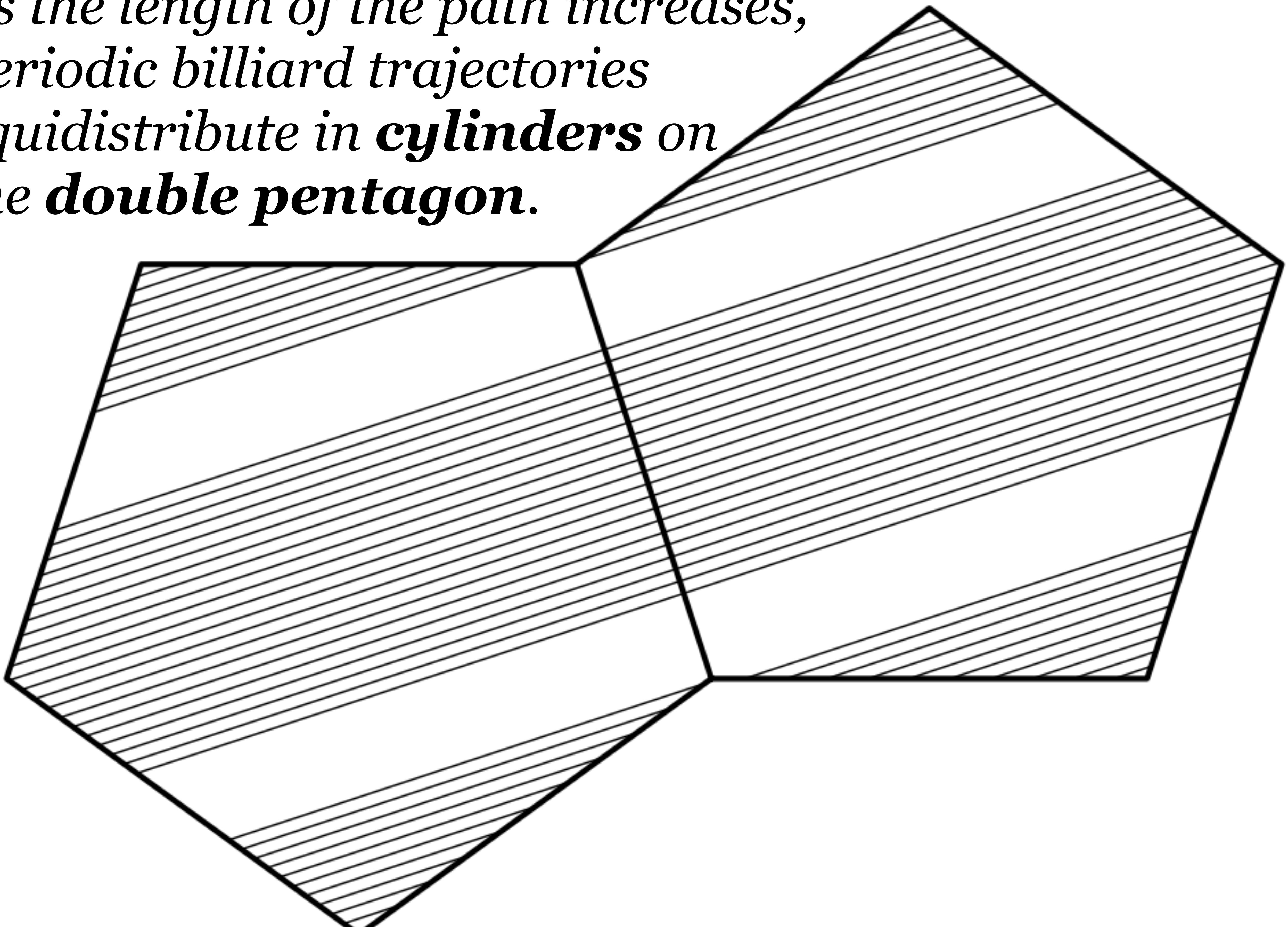

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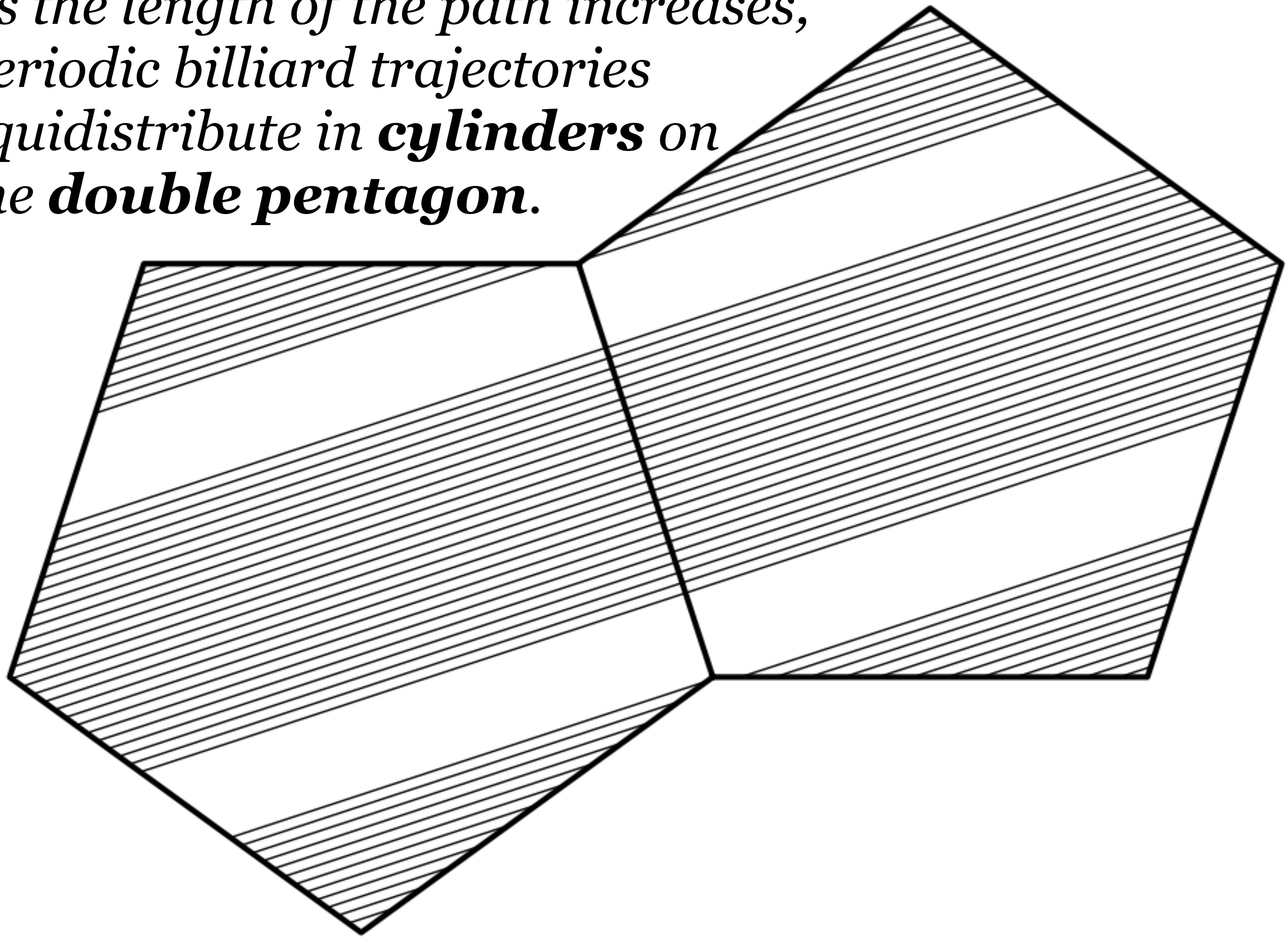

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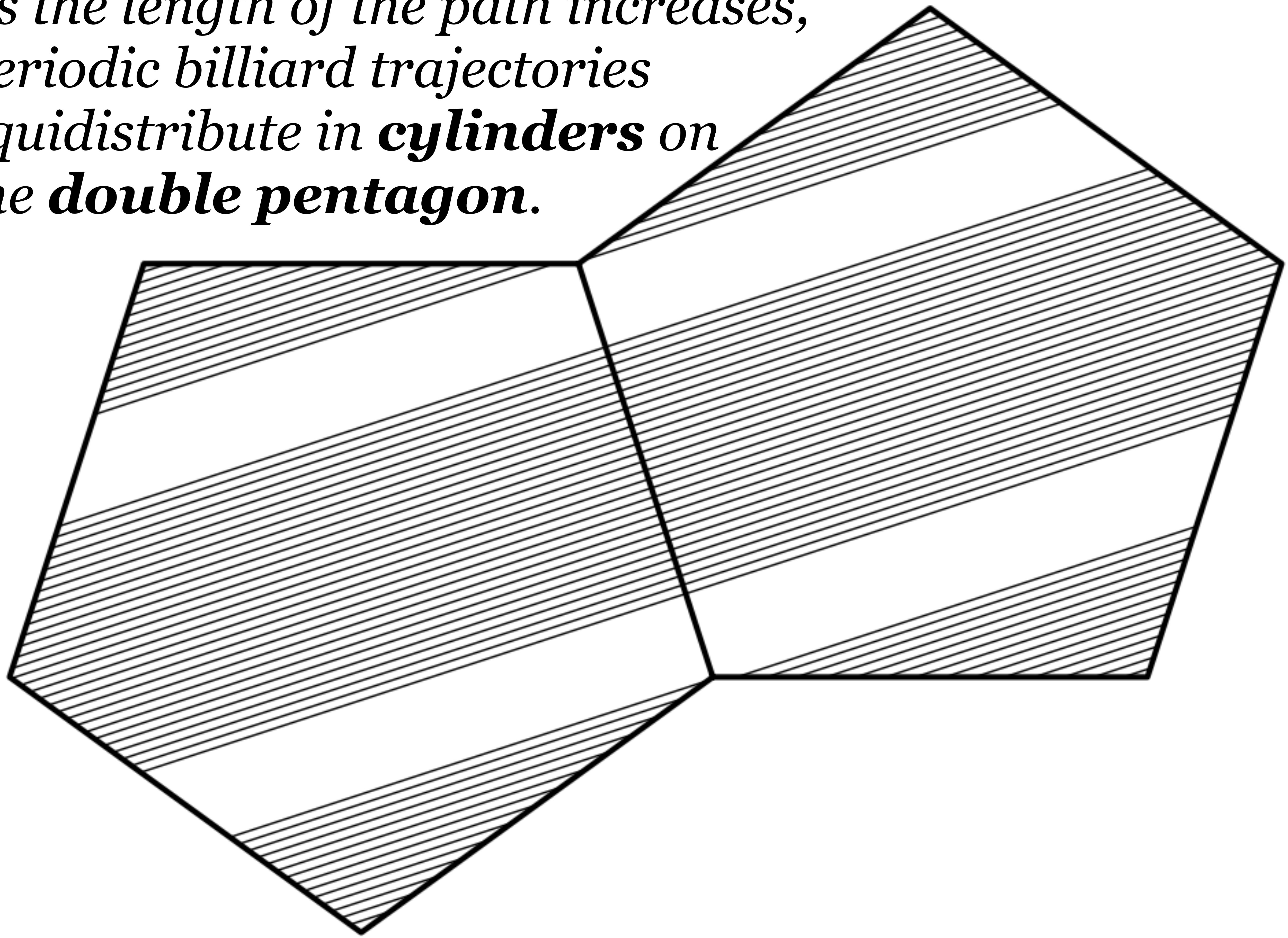

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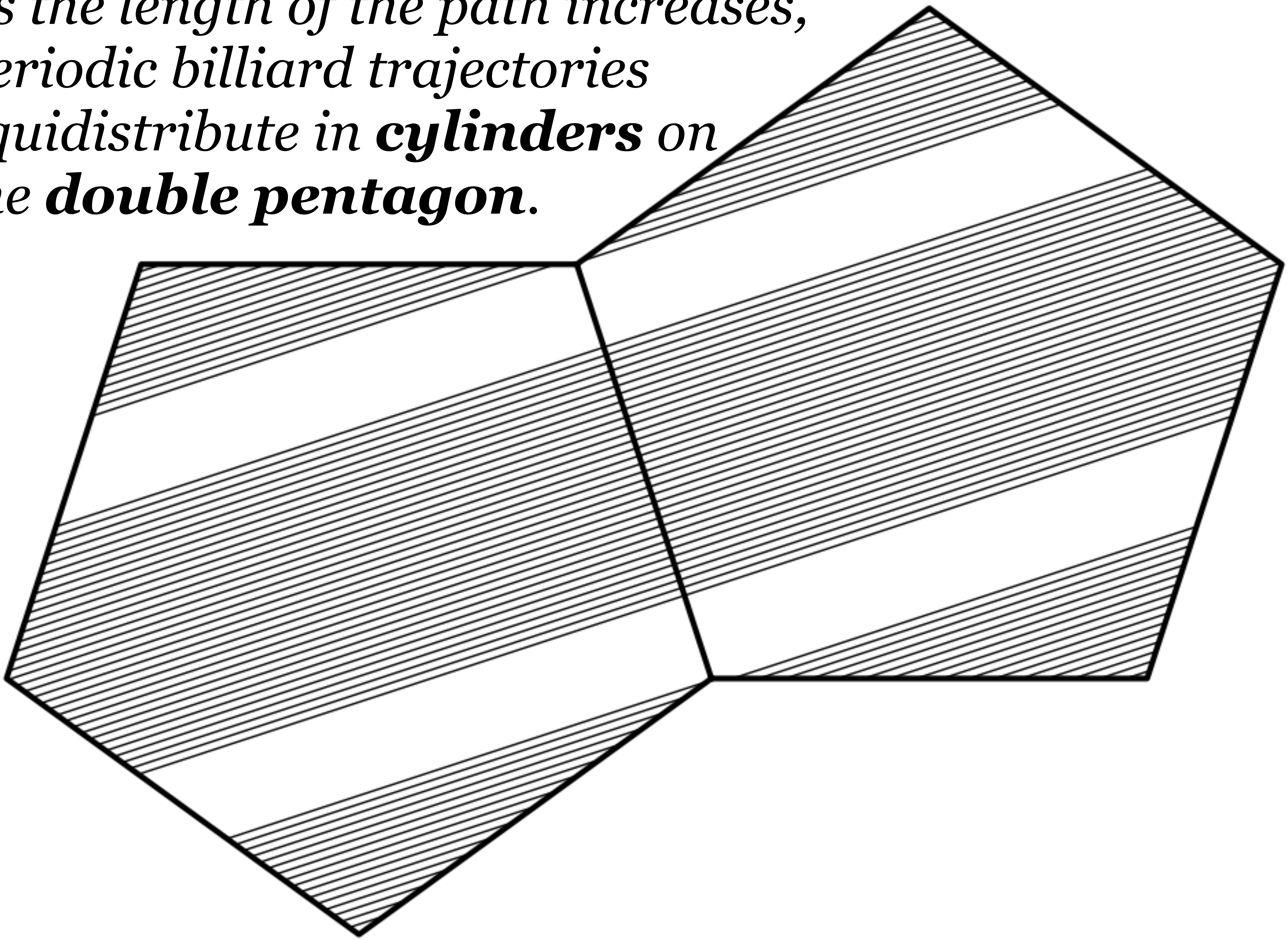

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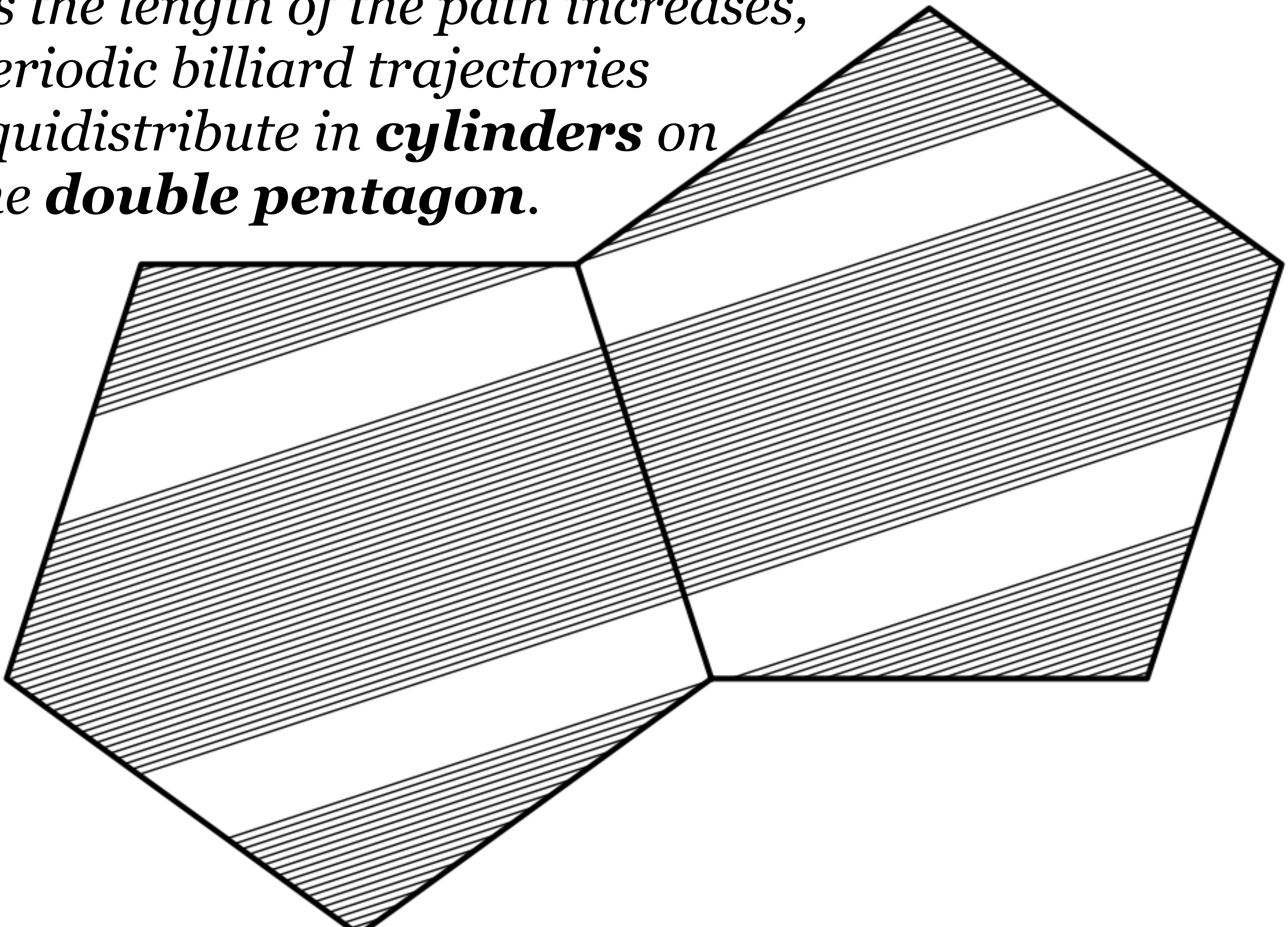

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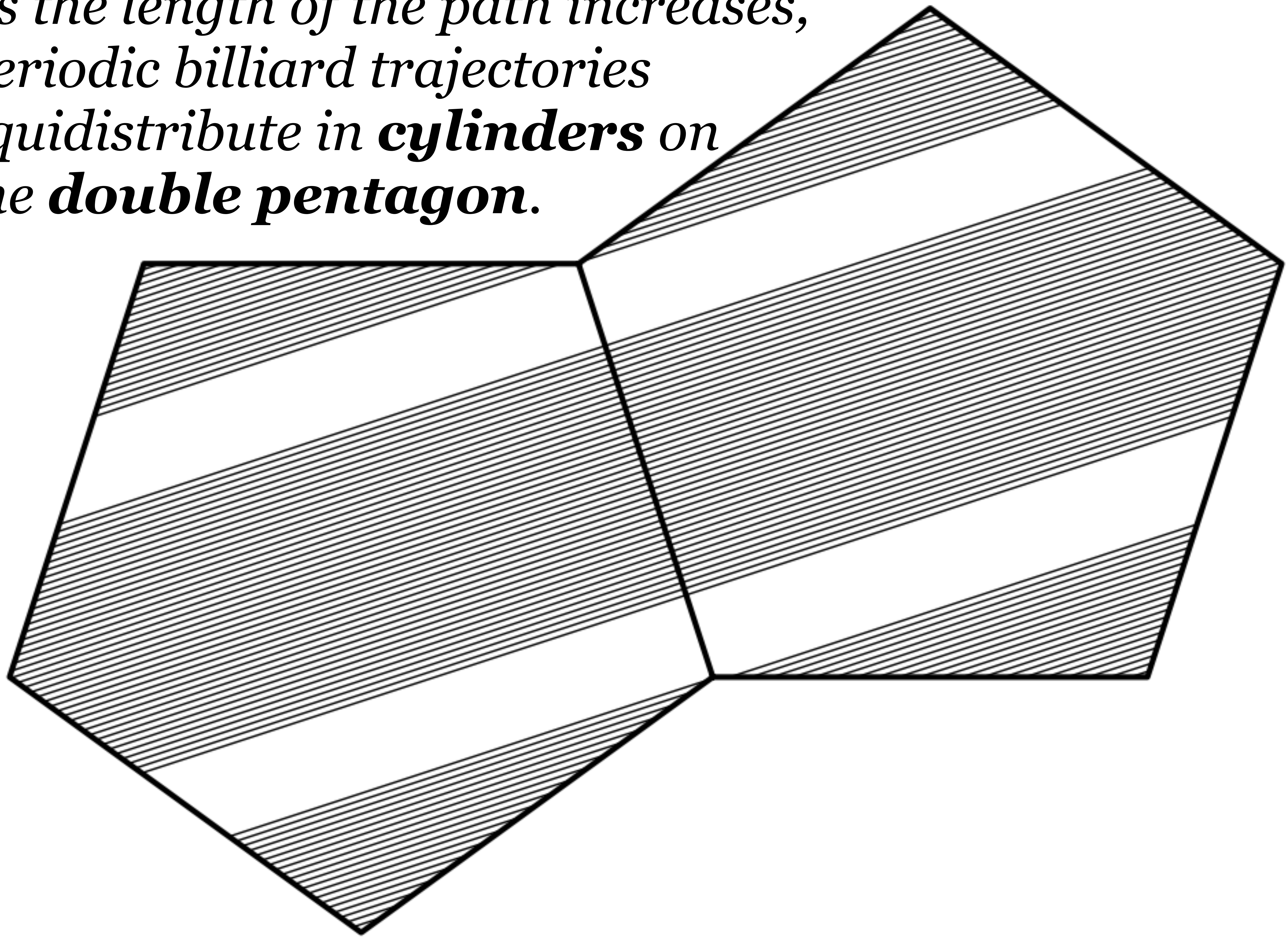

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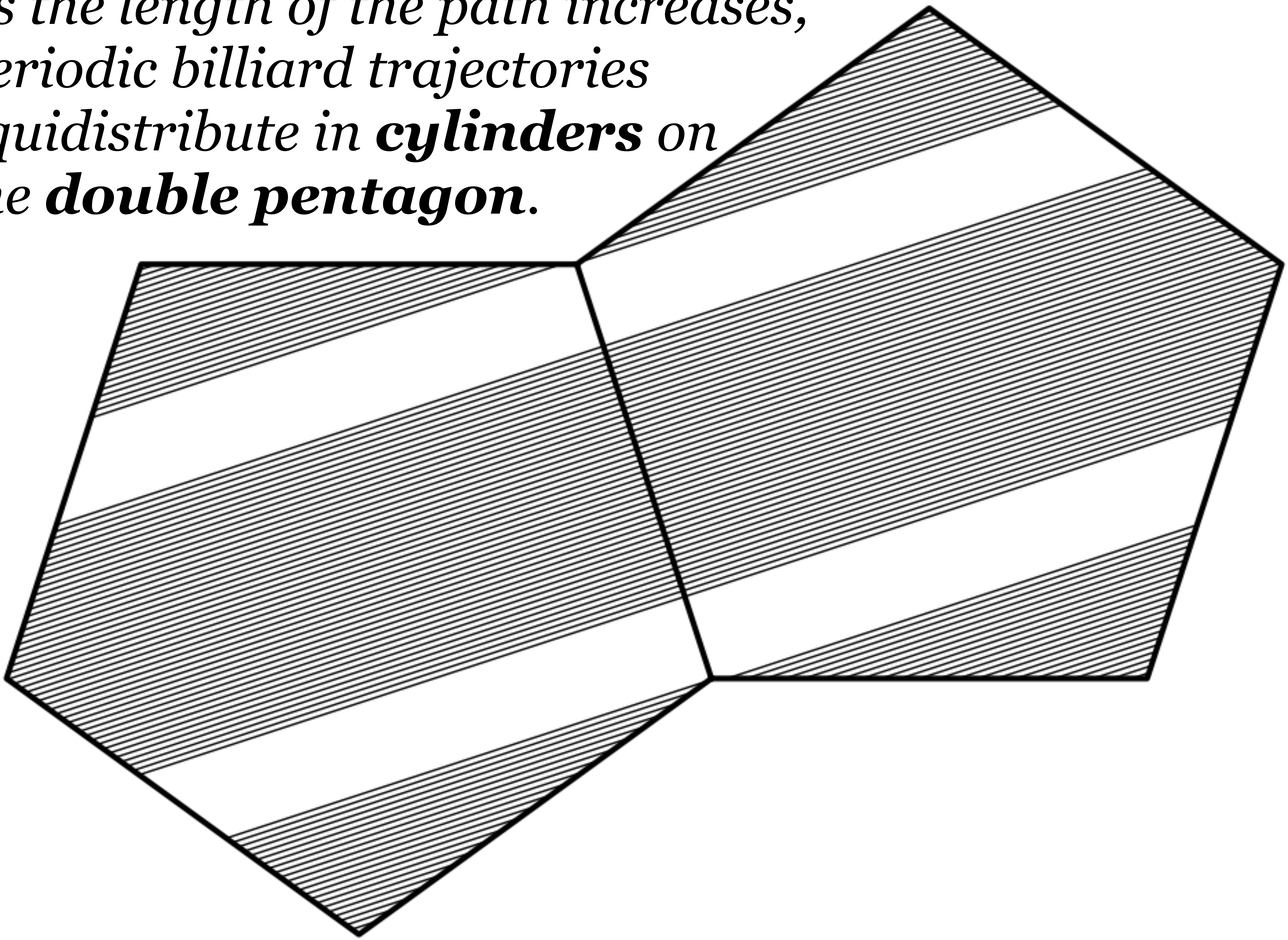

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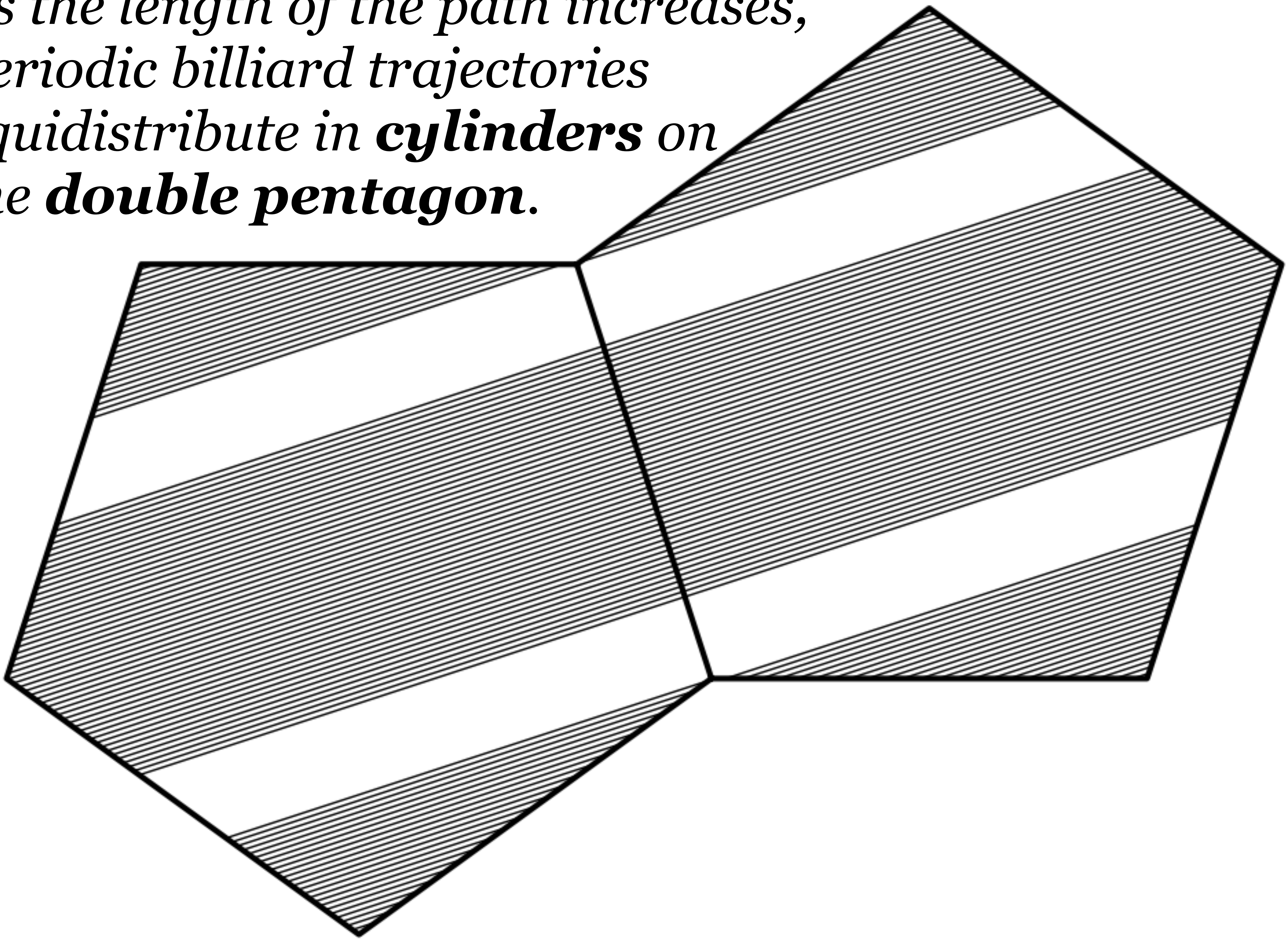

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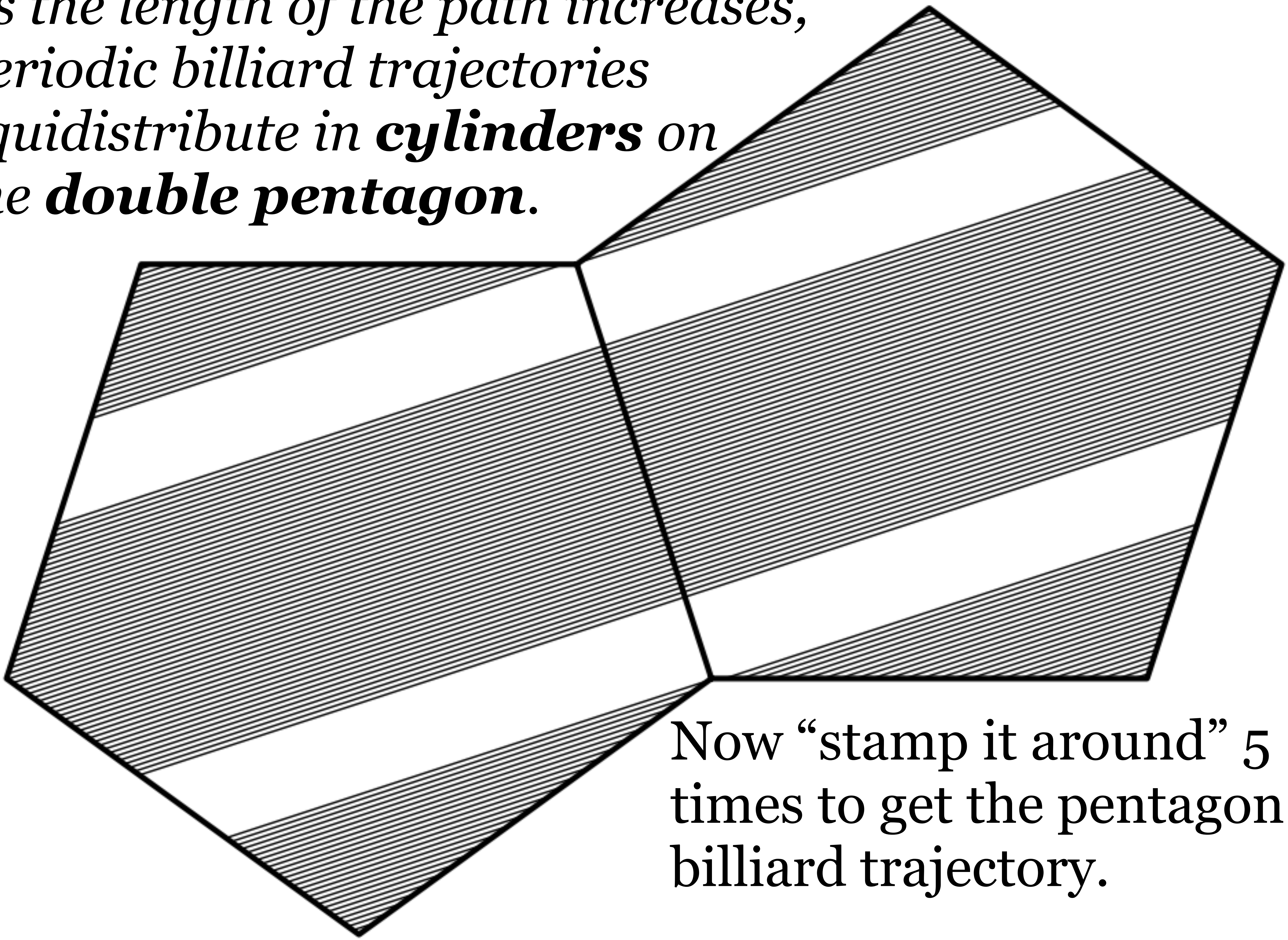

Now "stamp it around" 5 times to get the pentagon billiard trajectory.

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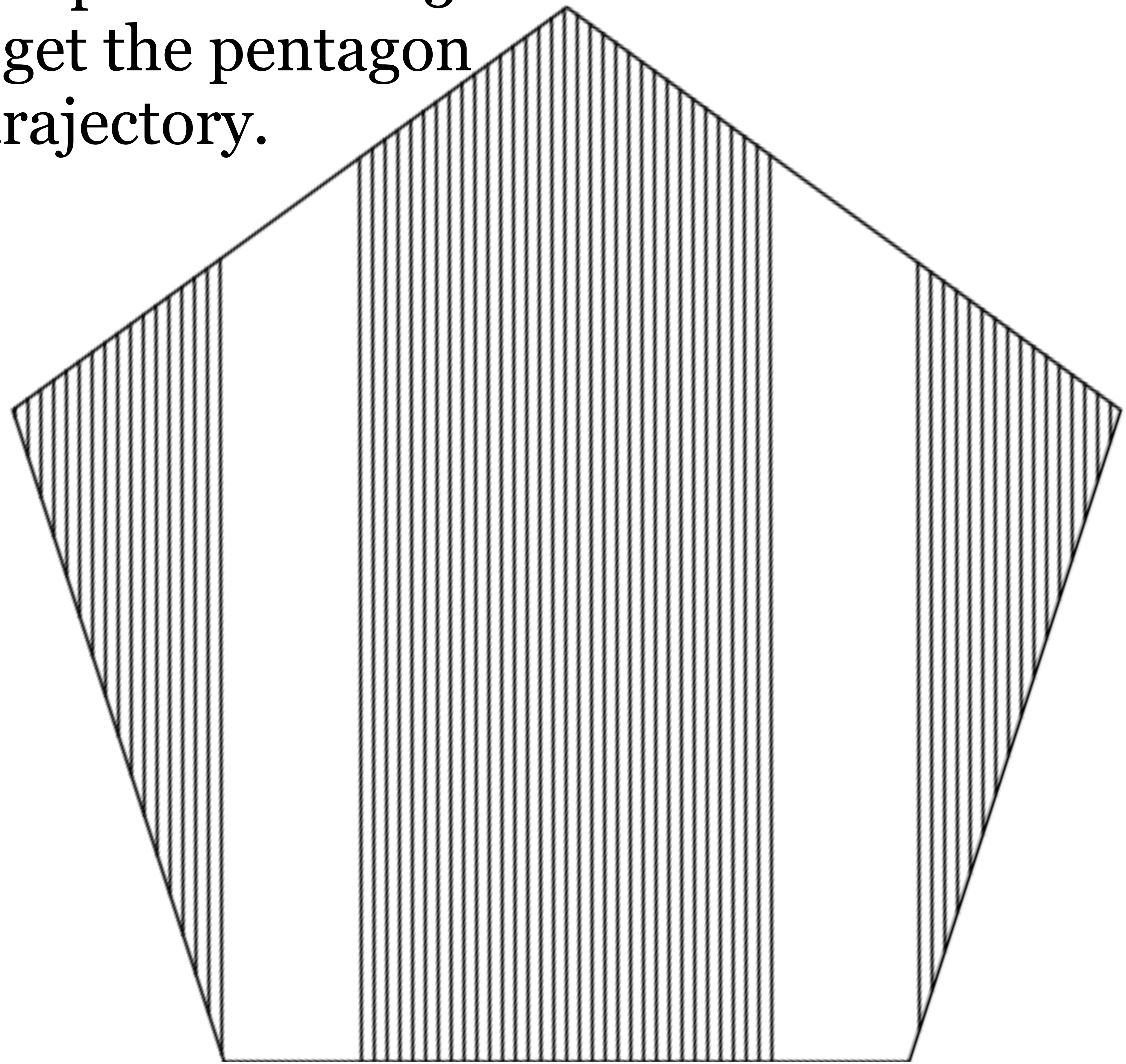

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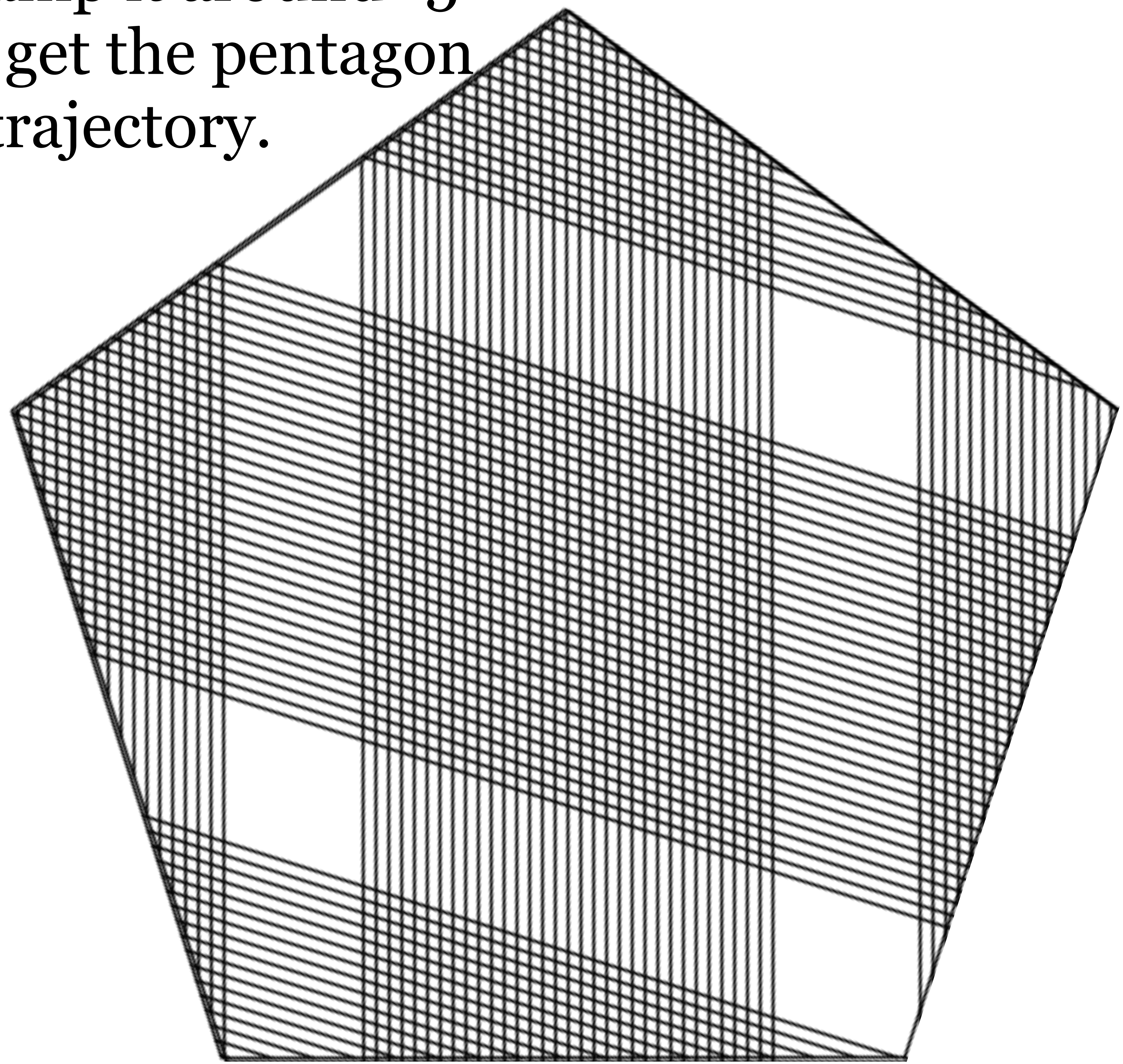

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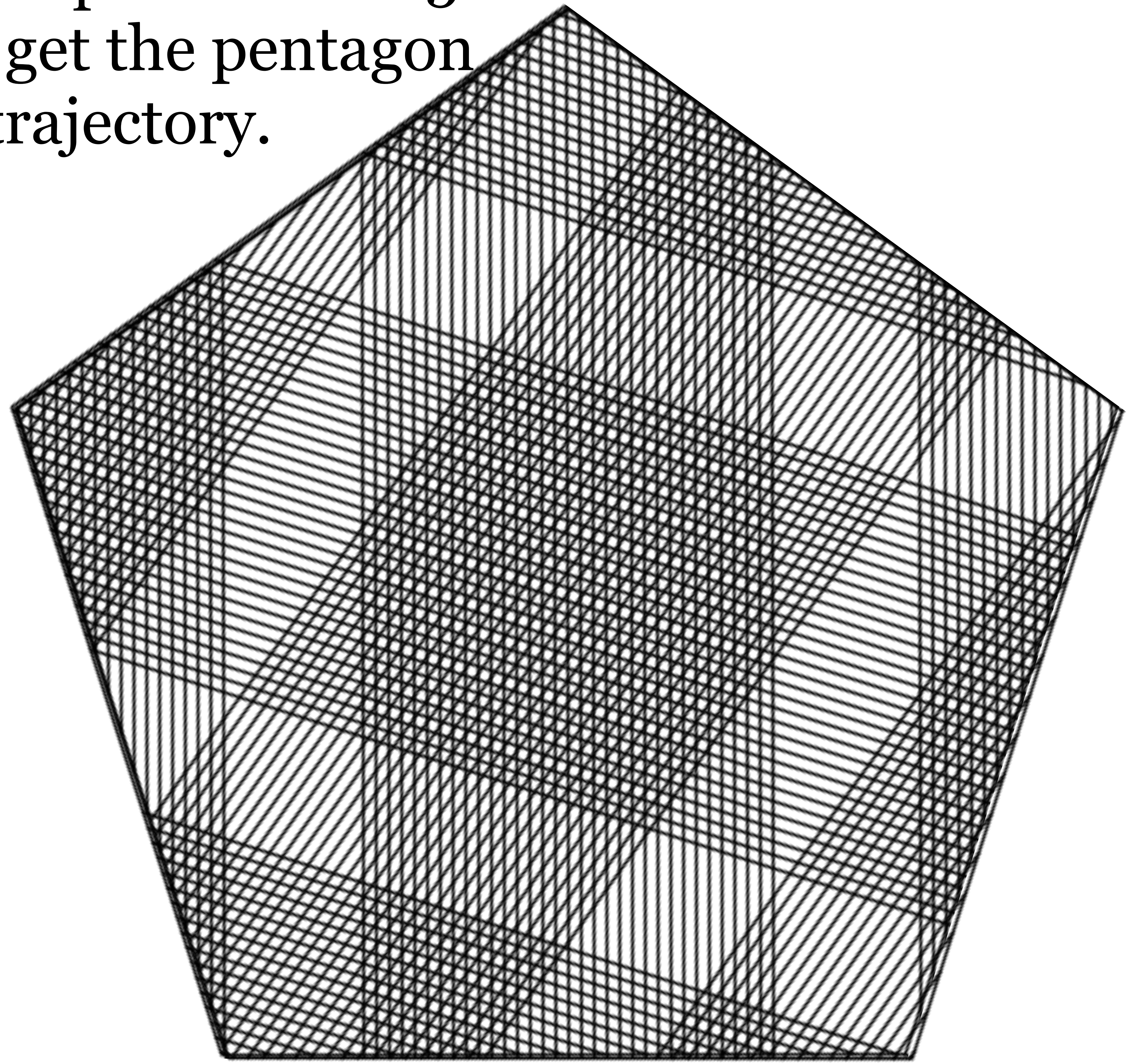

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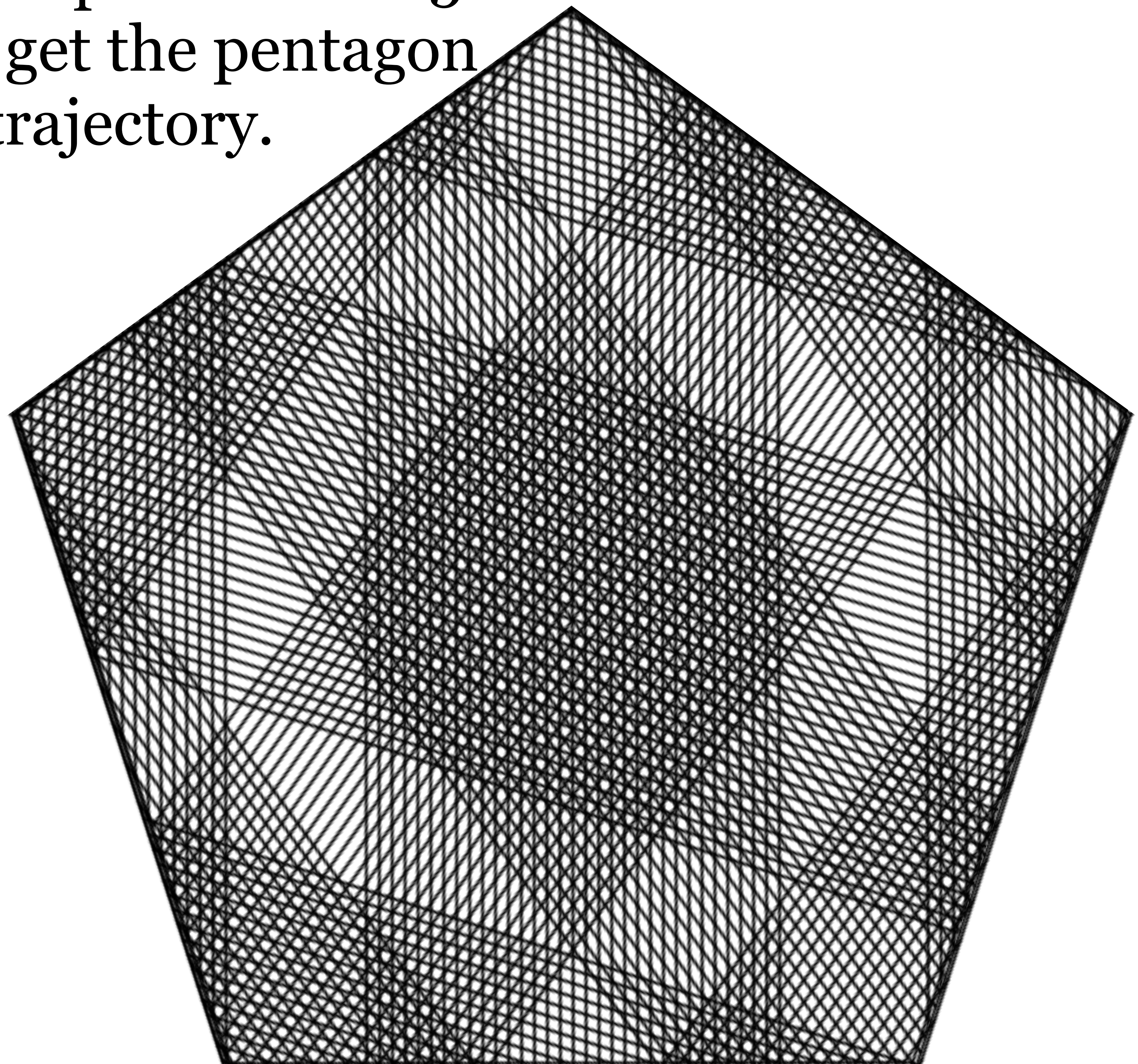
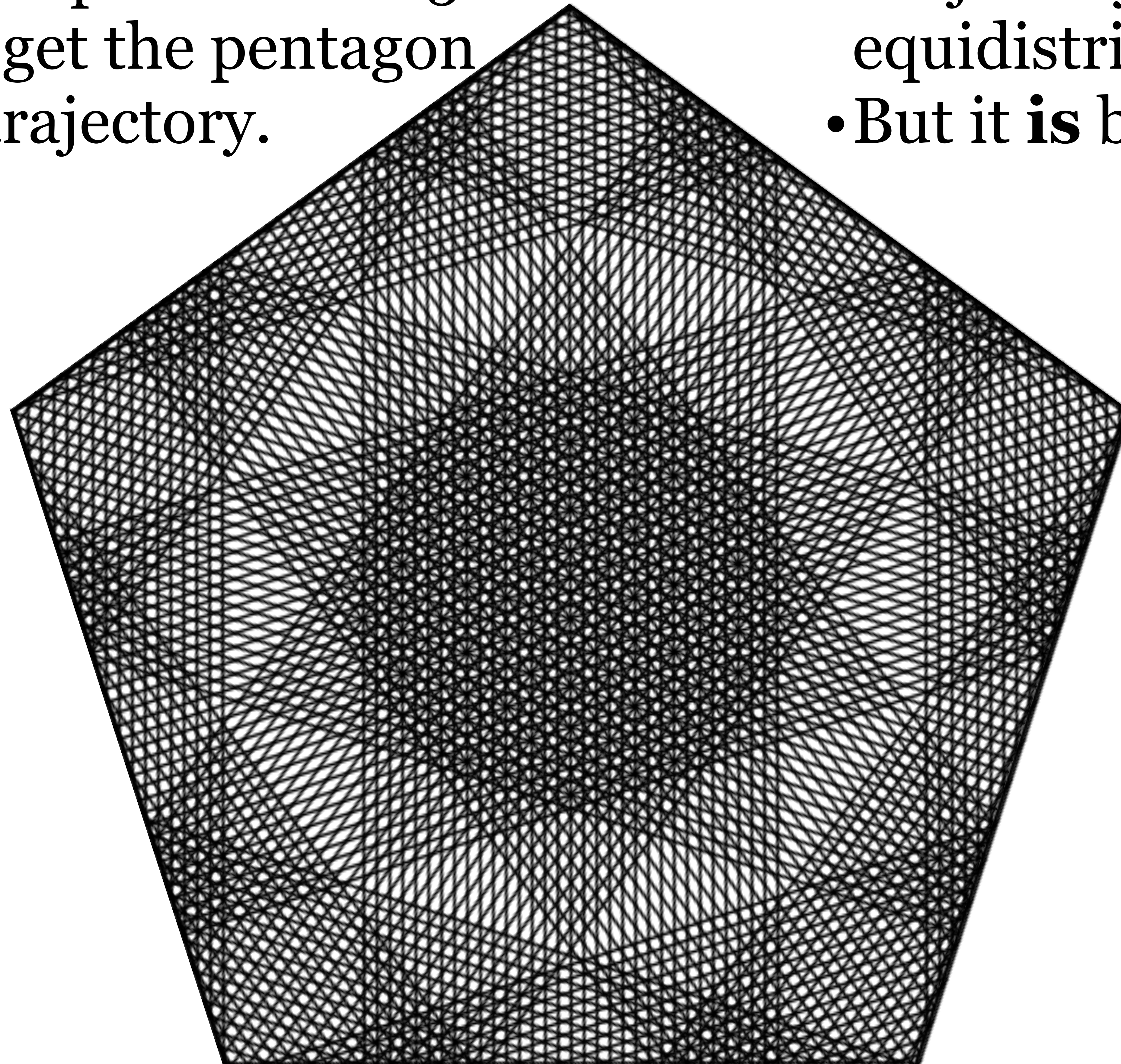
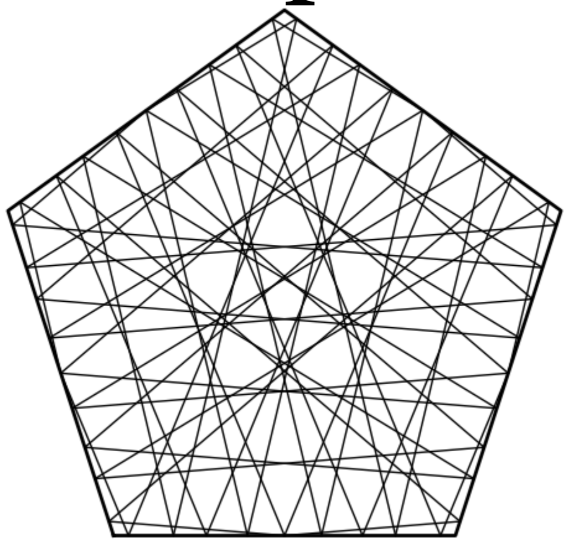


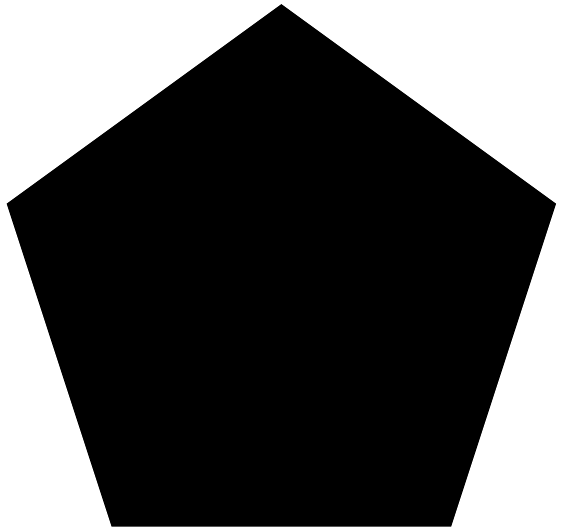
Now "stamp it around" 5 times to get the pentagon billiard trajectory.

- Trajectory is **not** equidistributed,
- But it **is** beautiful.



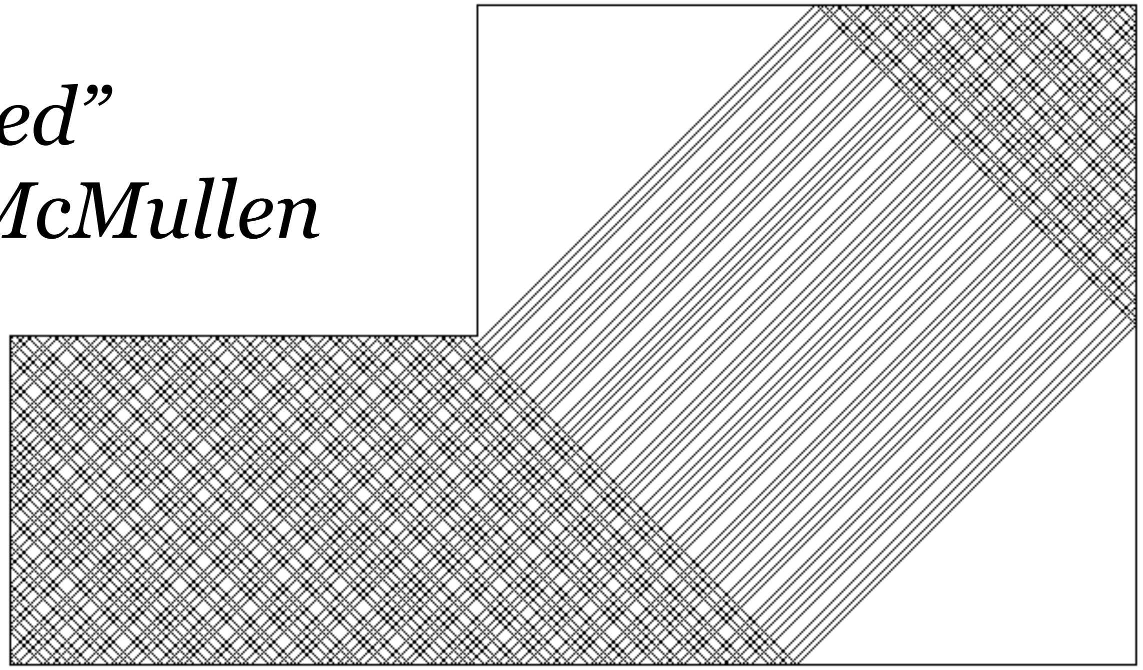
Recall: Regular polygons exhibit **optimal dynamics:**
Every path is either periodic or equidistributed.





Other tables can have **non-optimal dynamics:**
For example, dense in one region, misses another.

"Trapped"
by Curtis T. McMullen



(Moon Duchin showed me this)

Can this happen on the regular pentagon?

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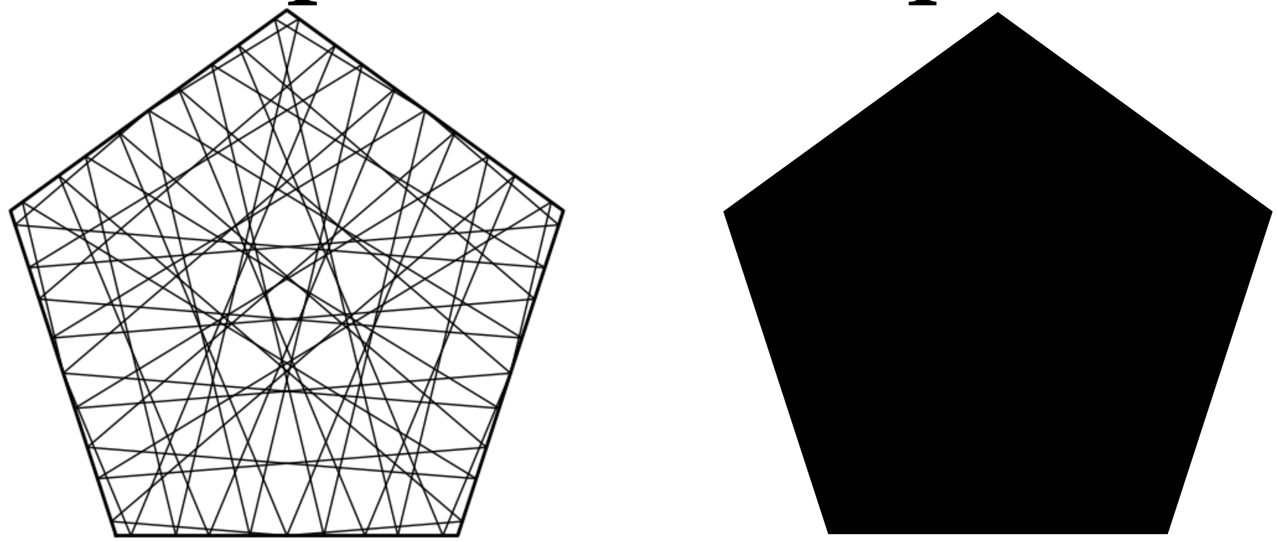

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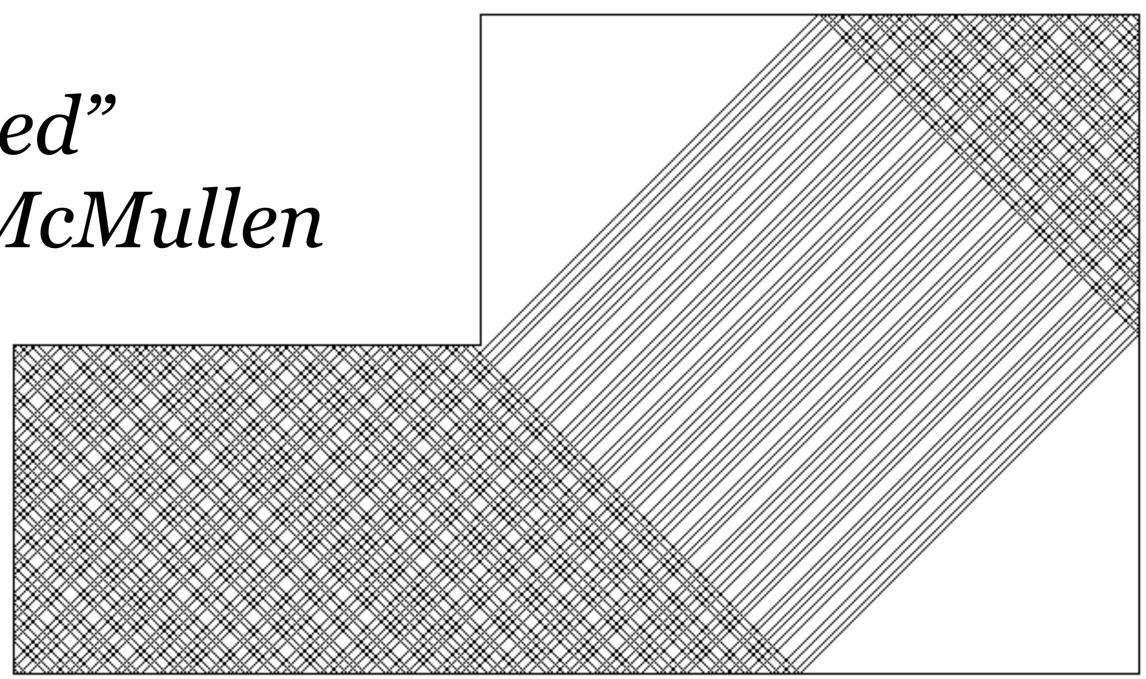

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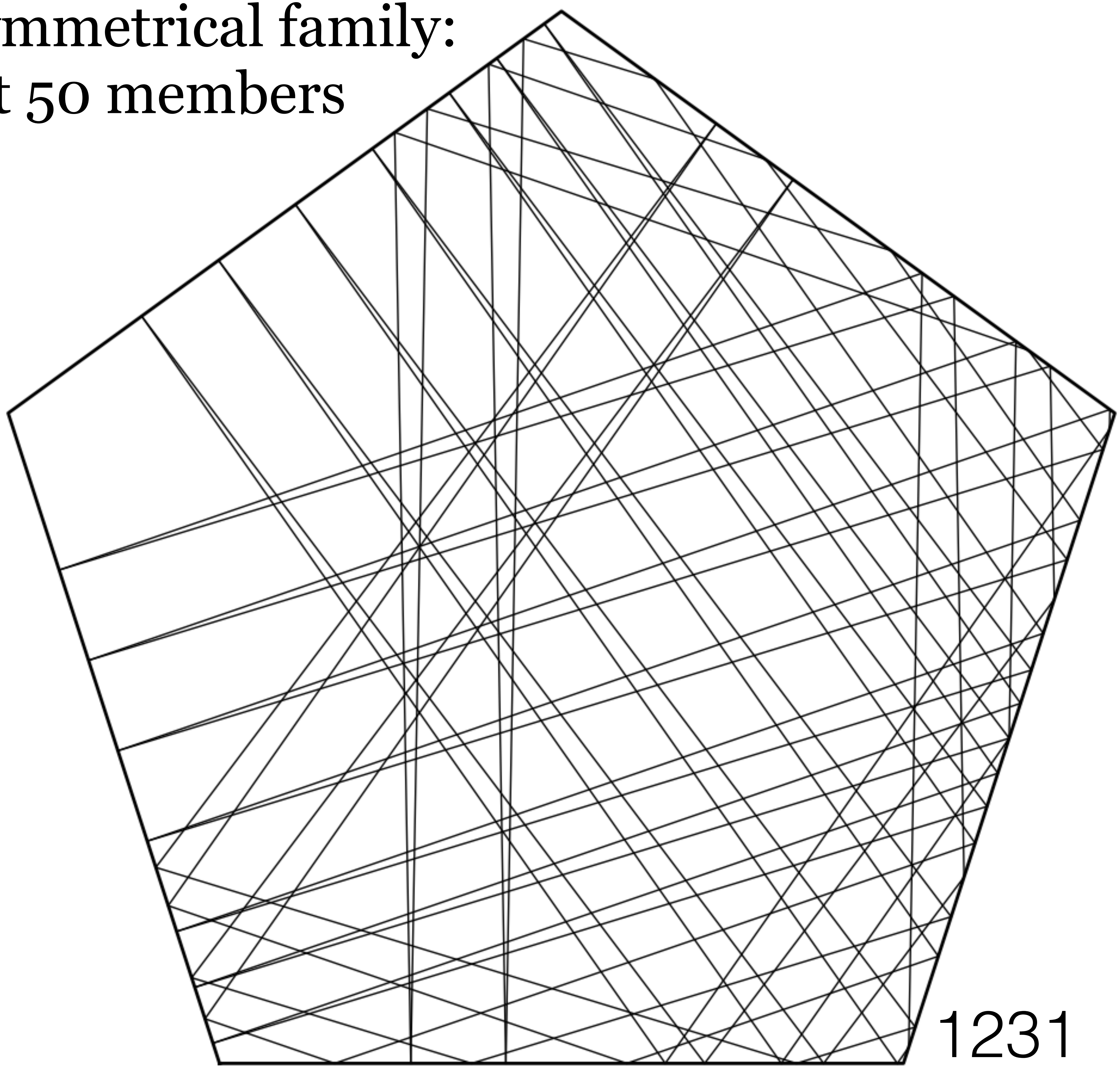
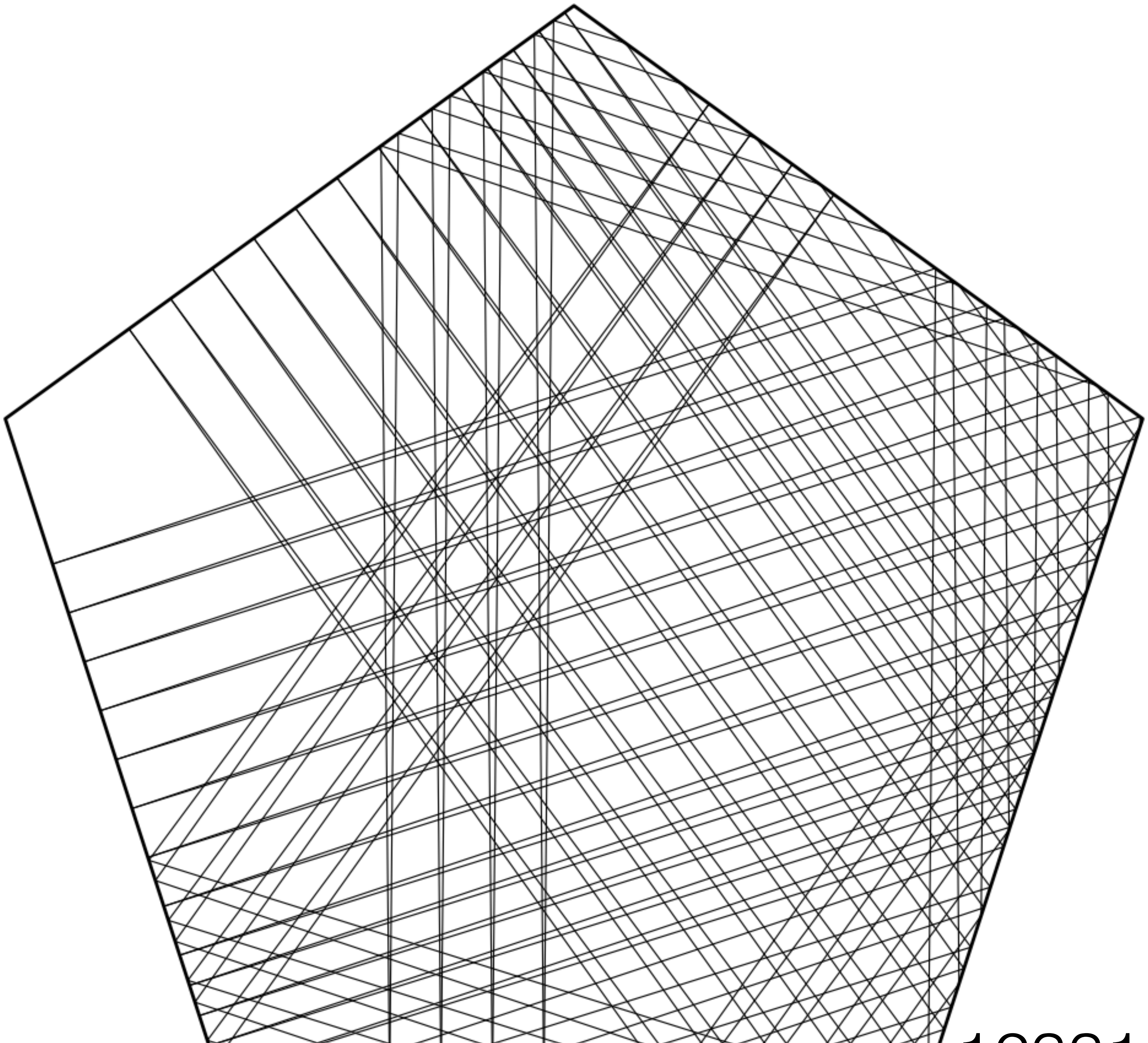
Can this happen on the regular pentagon? **Yes!**

An asymmetrical family:
its first 50 members





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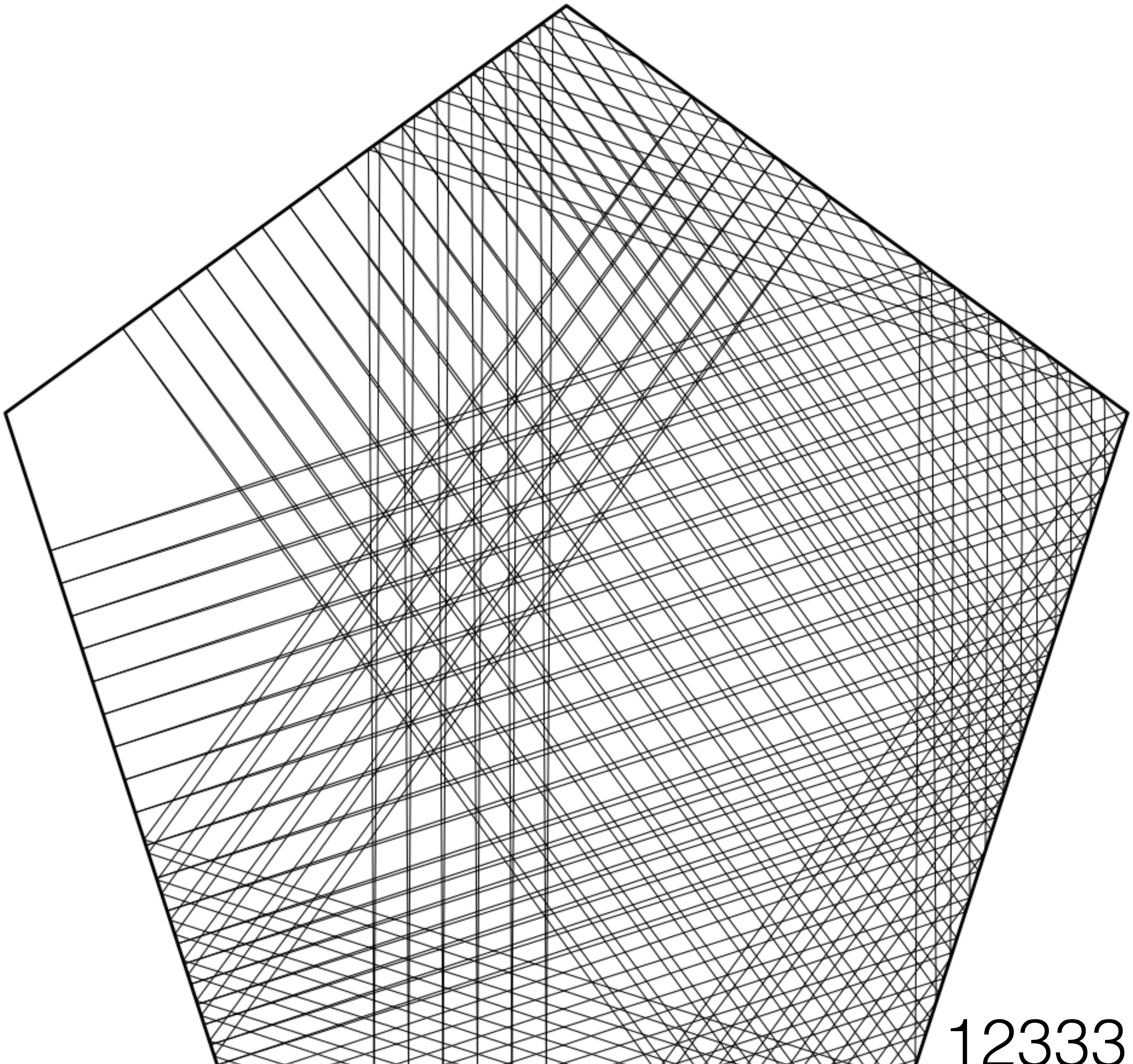

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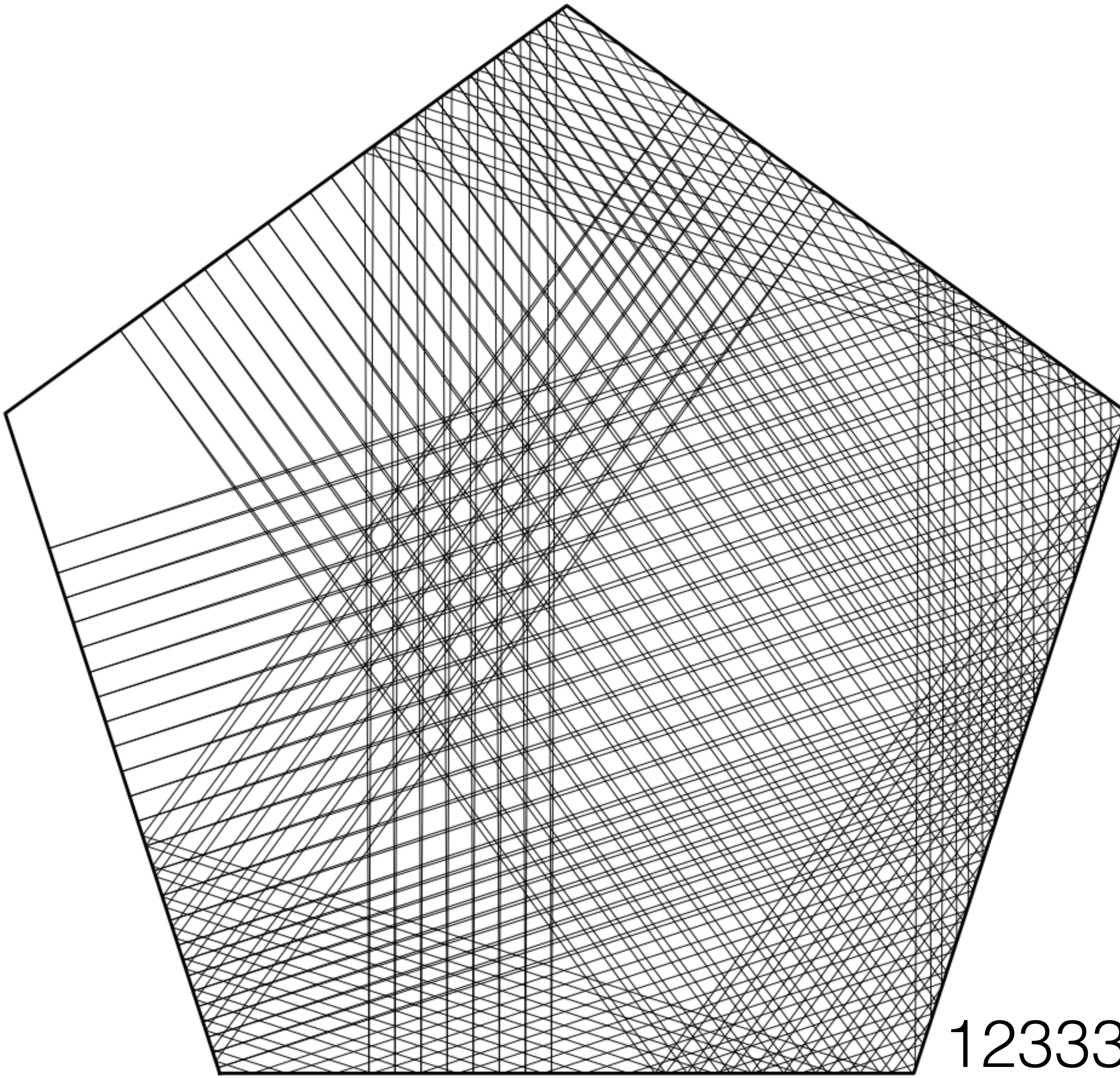

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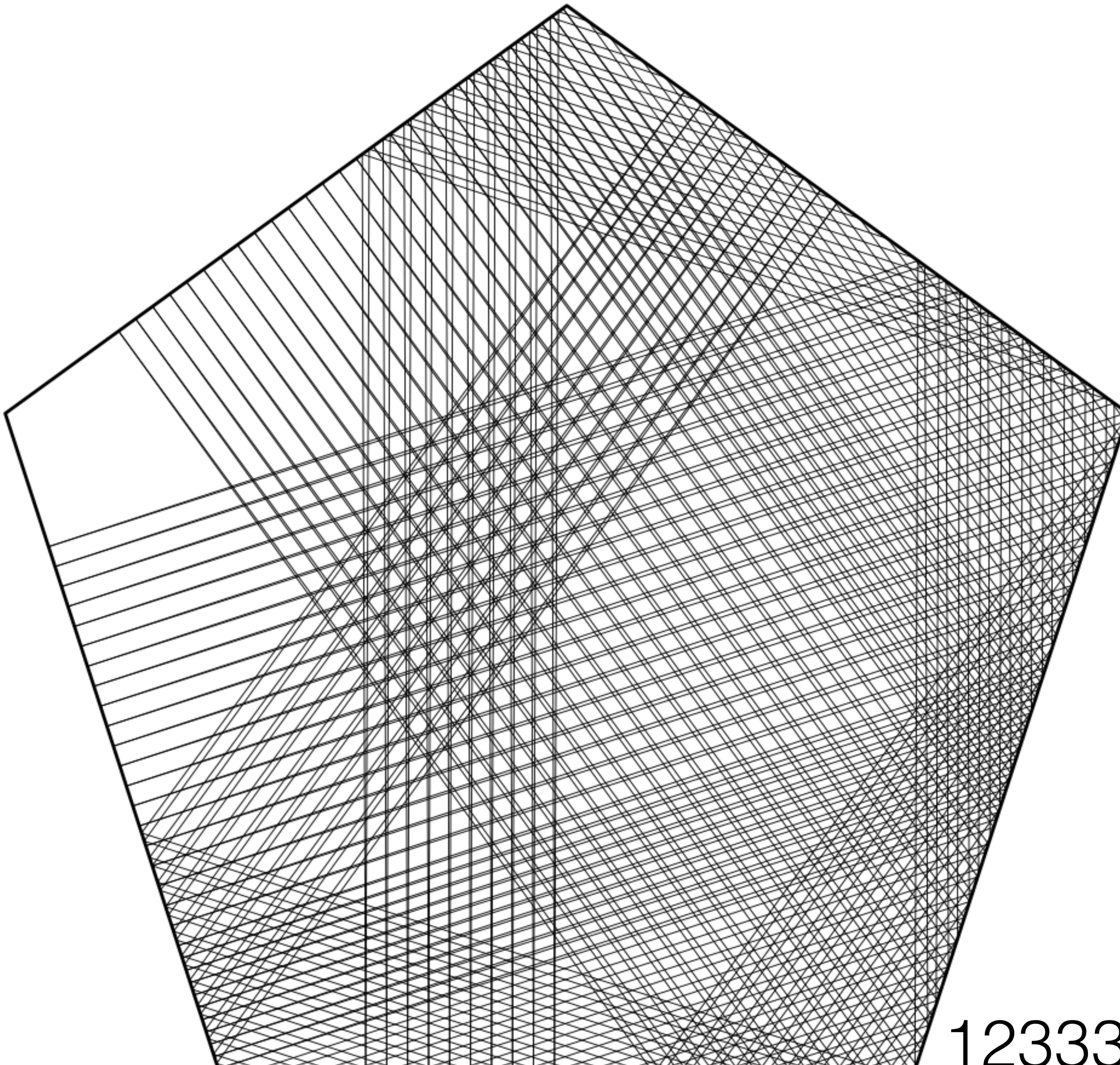

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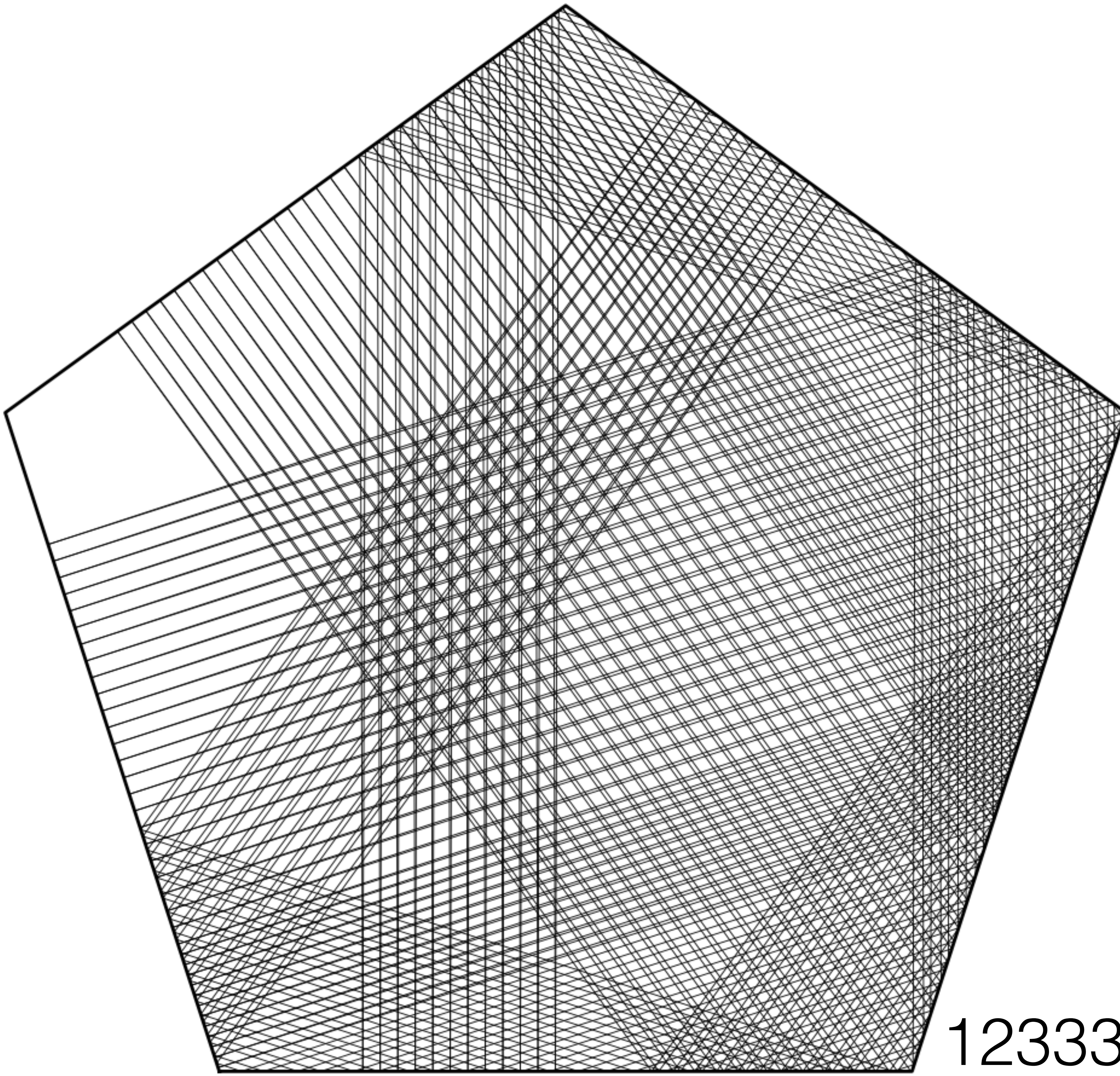

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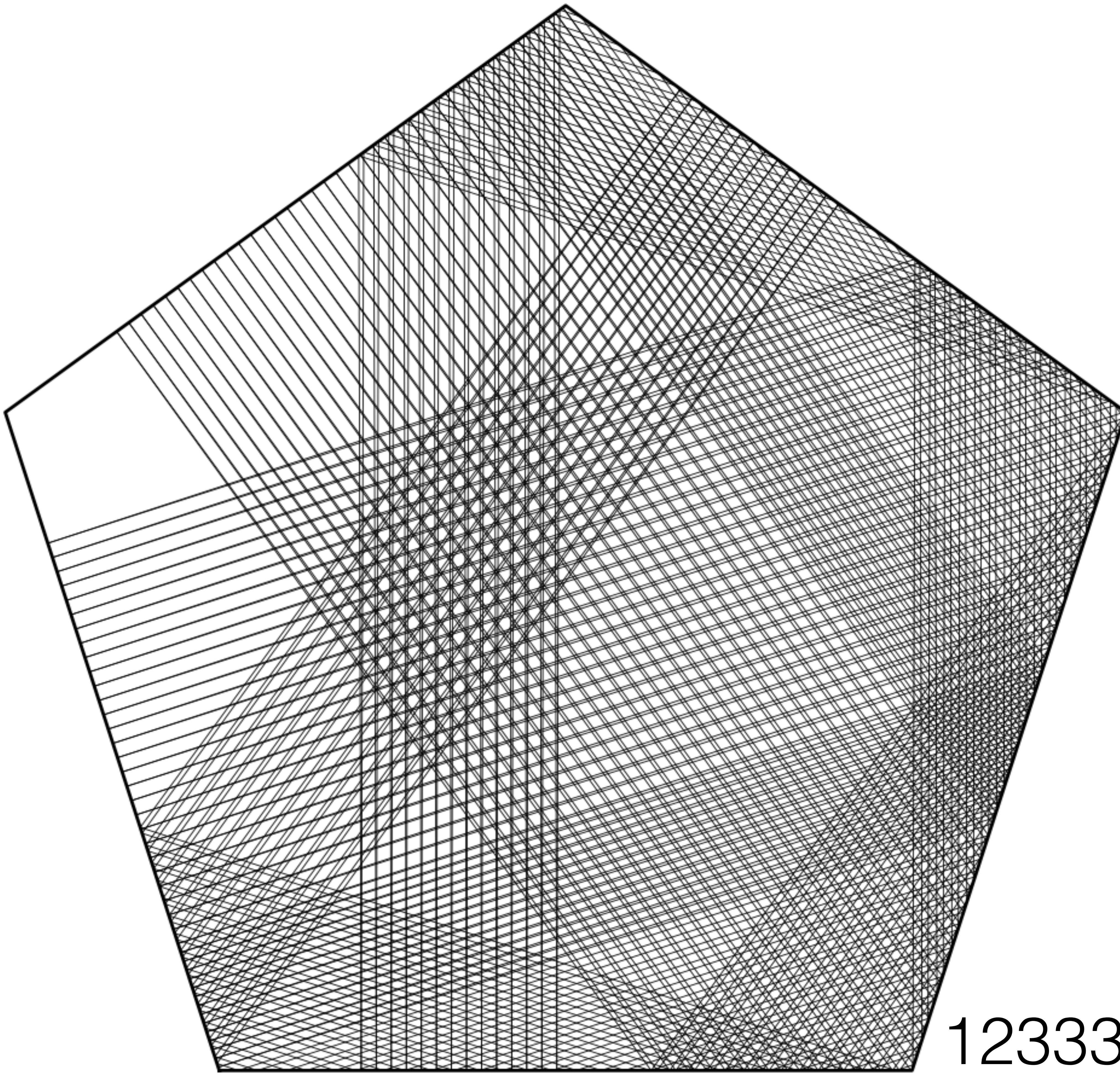

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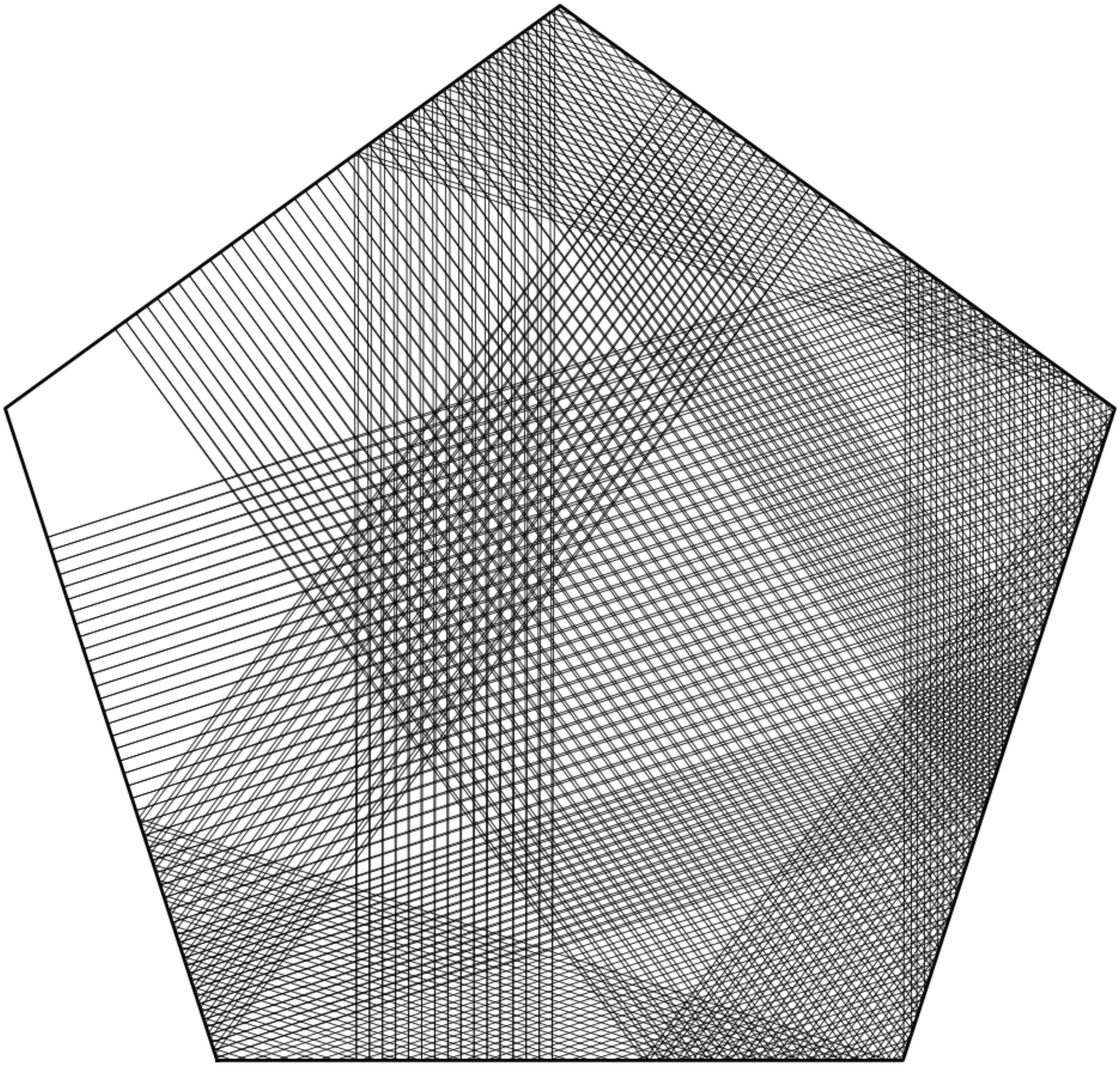
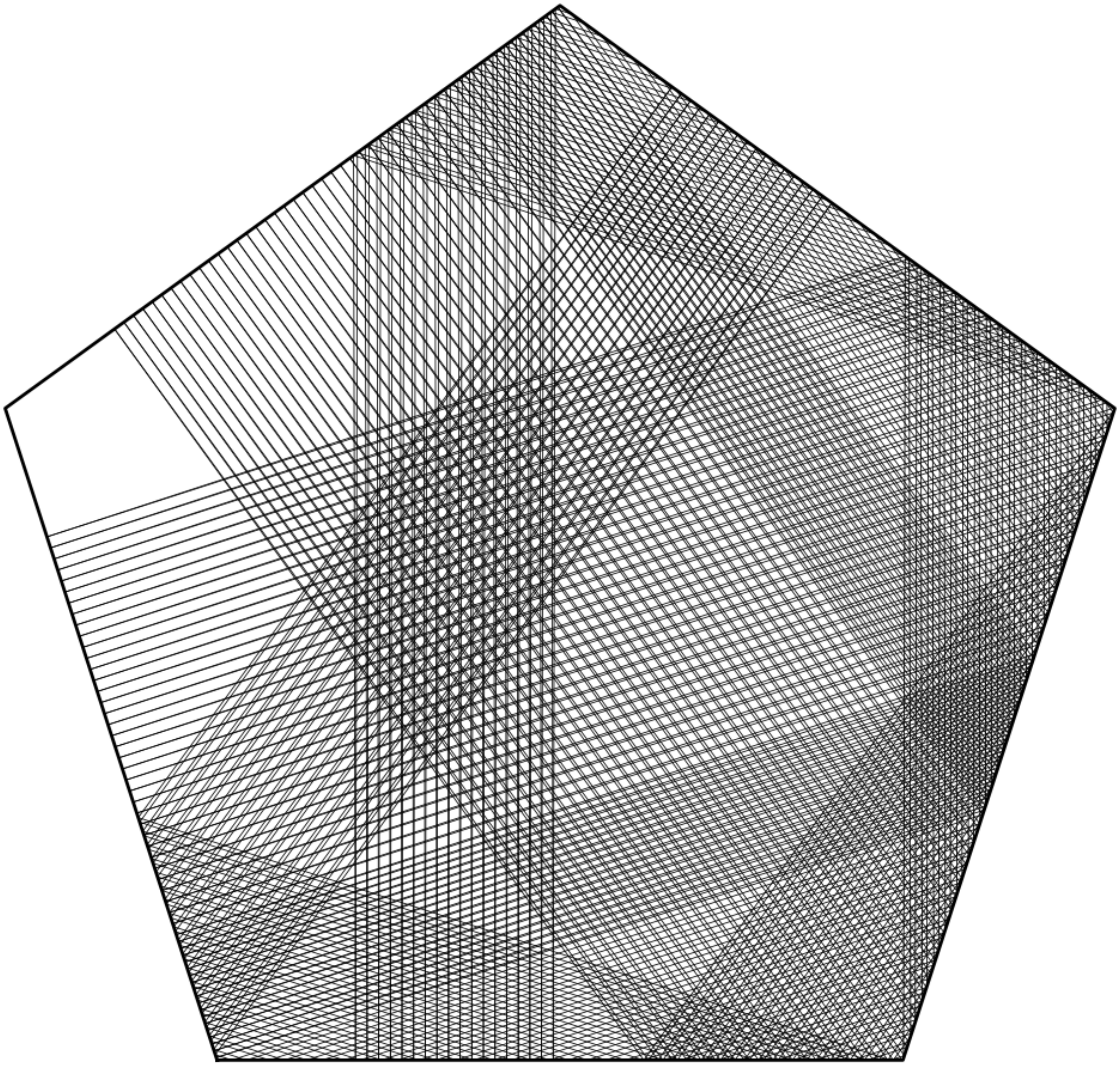
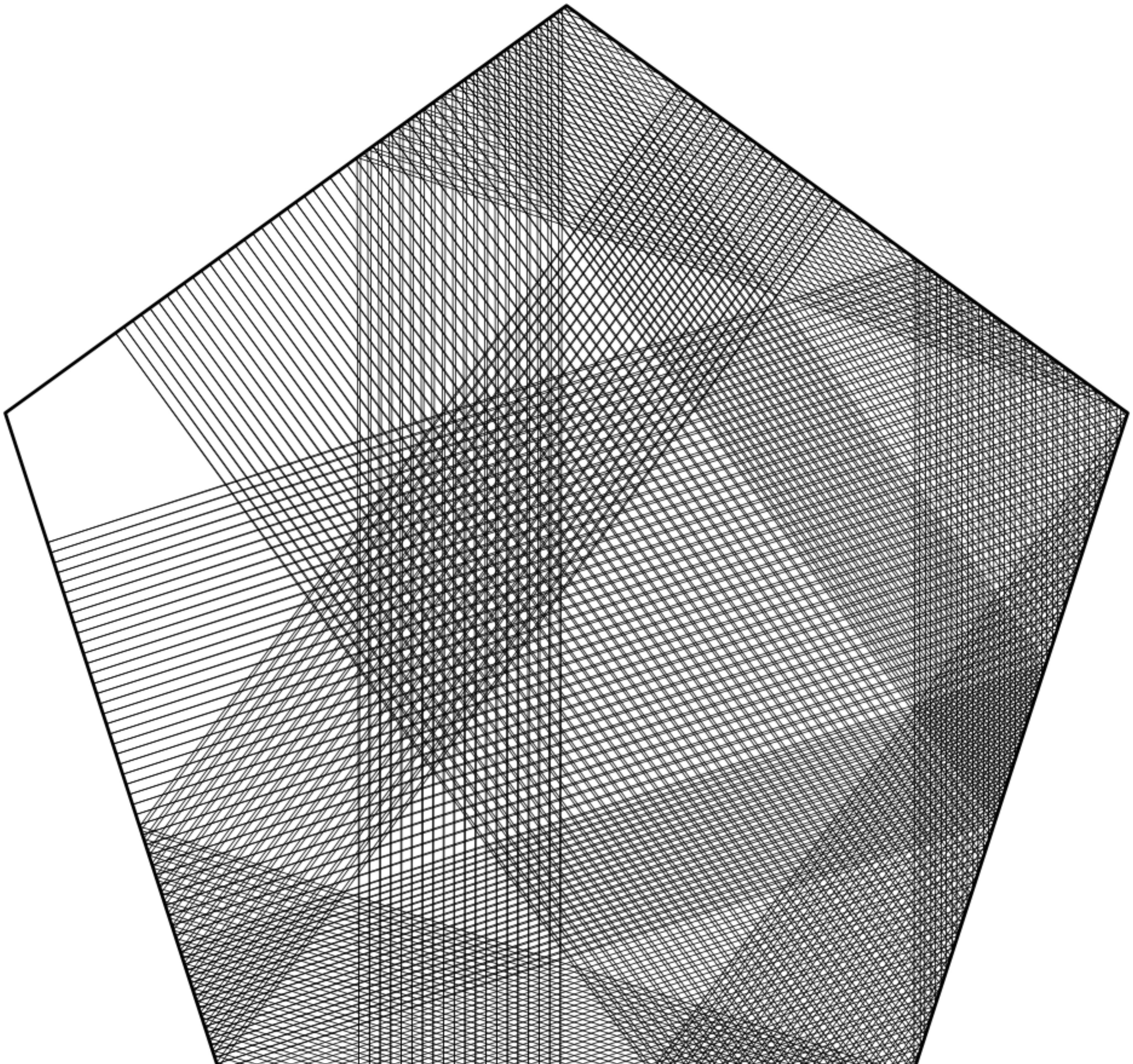
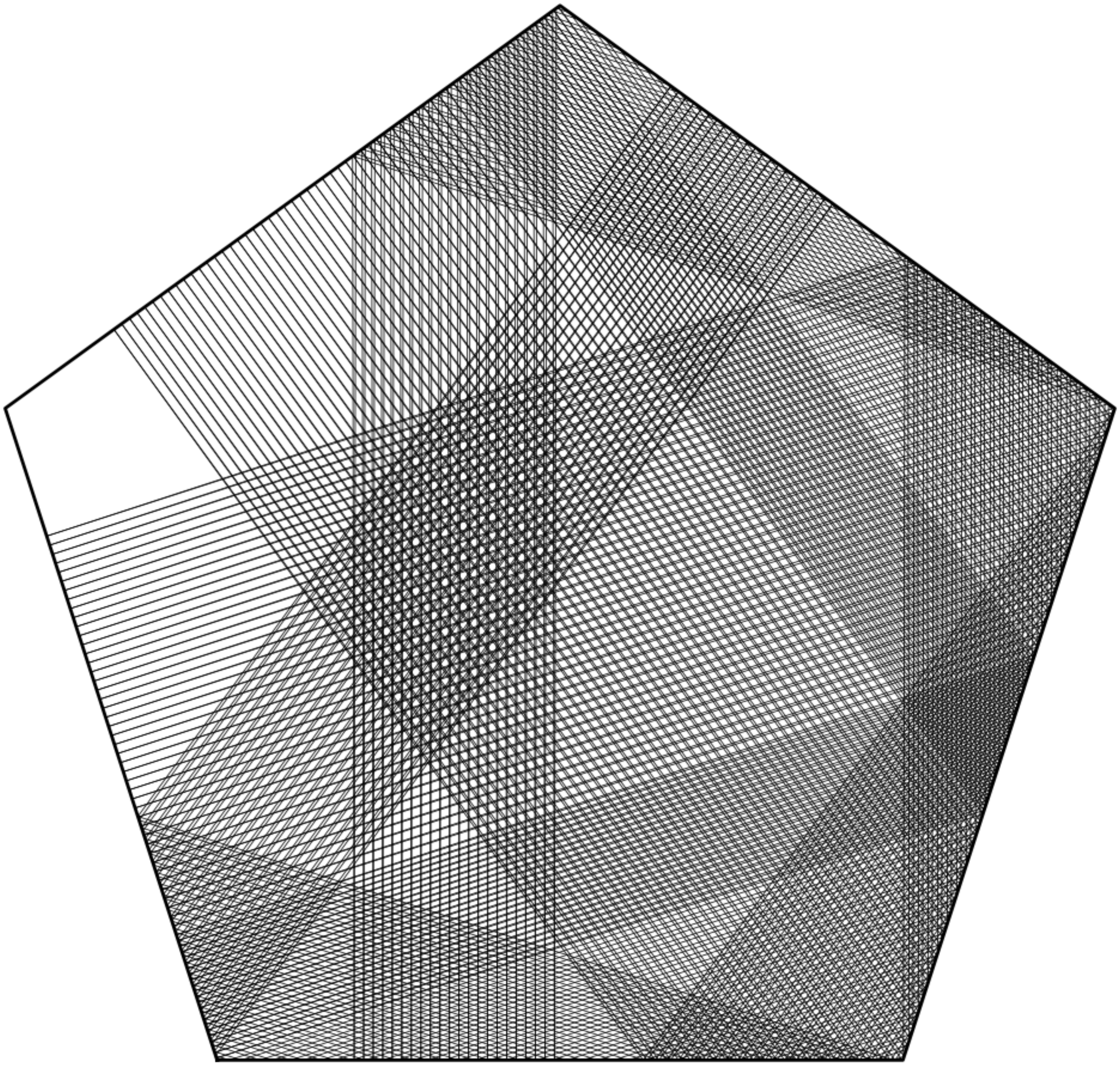
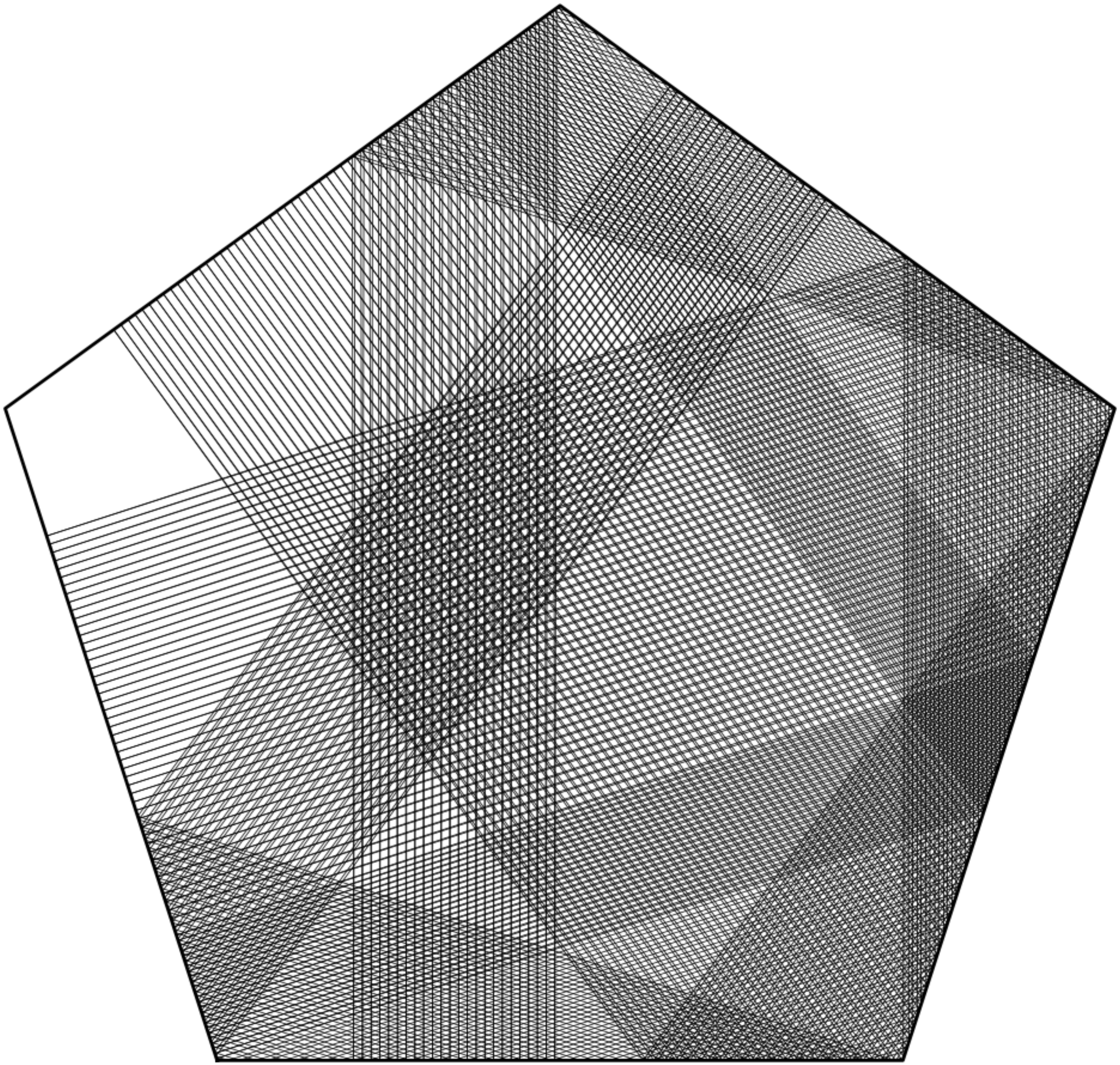
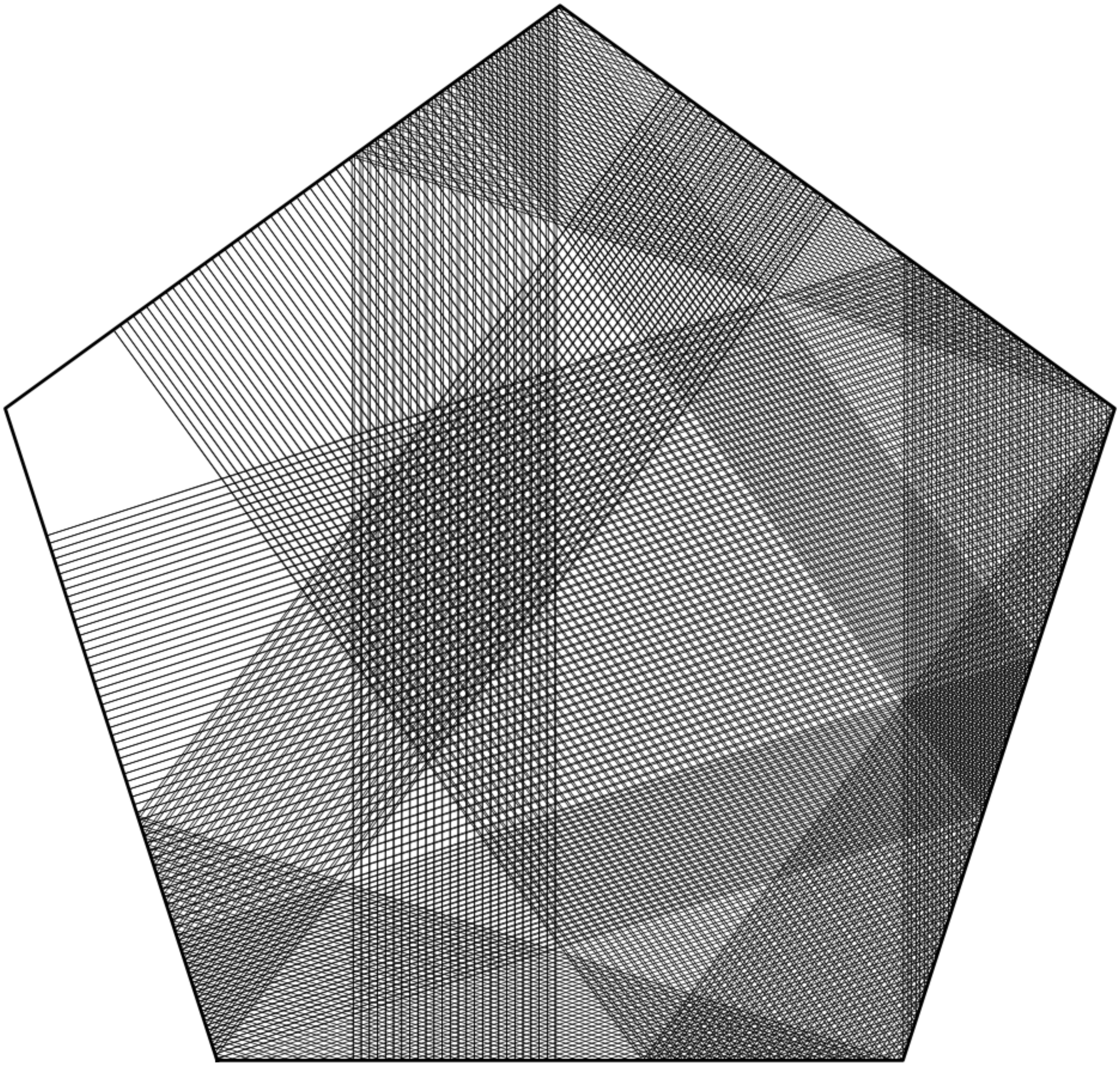


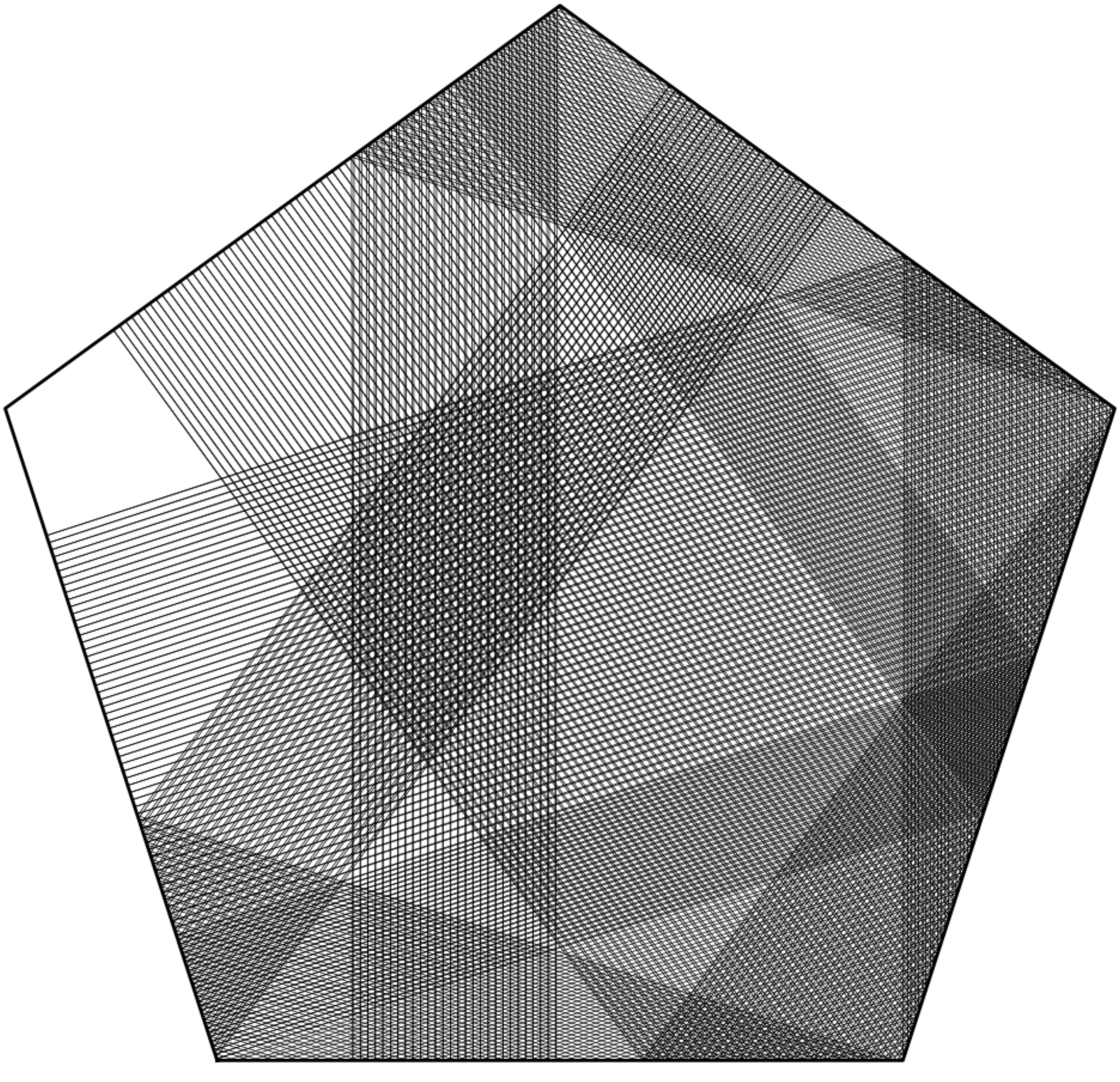


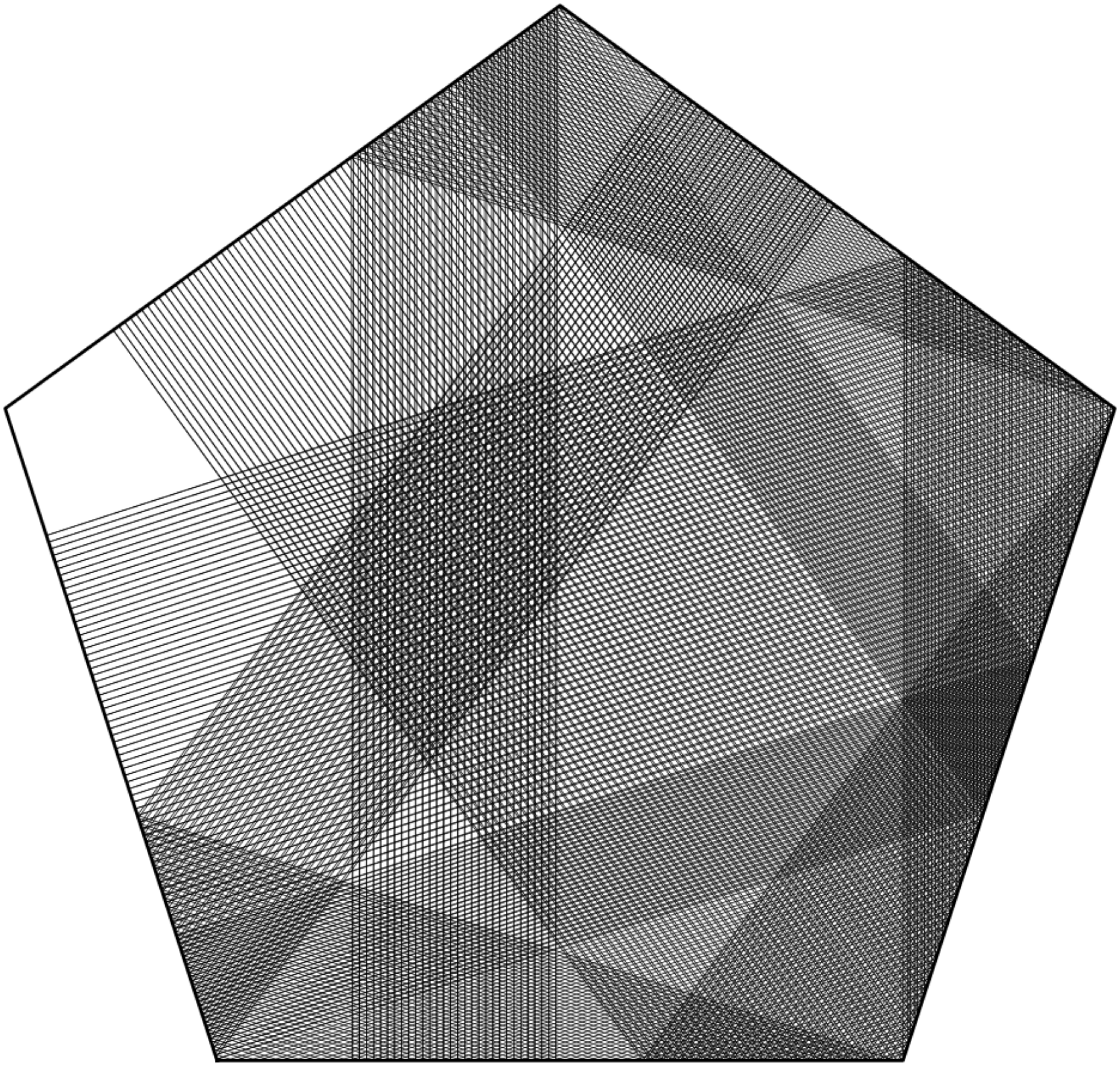


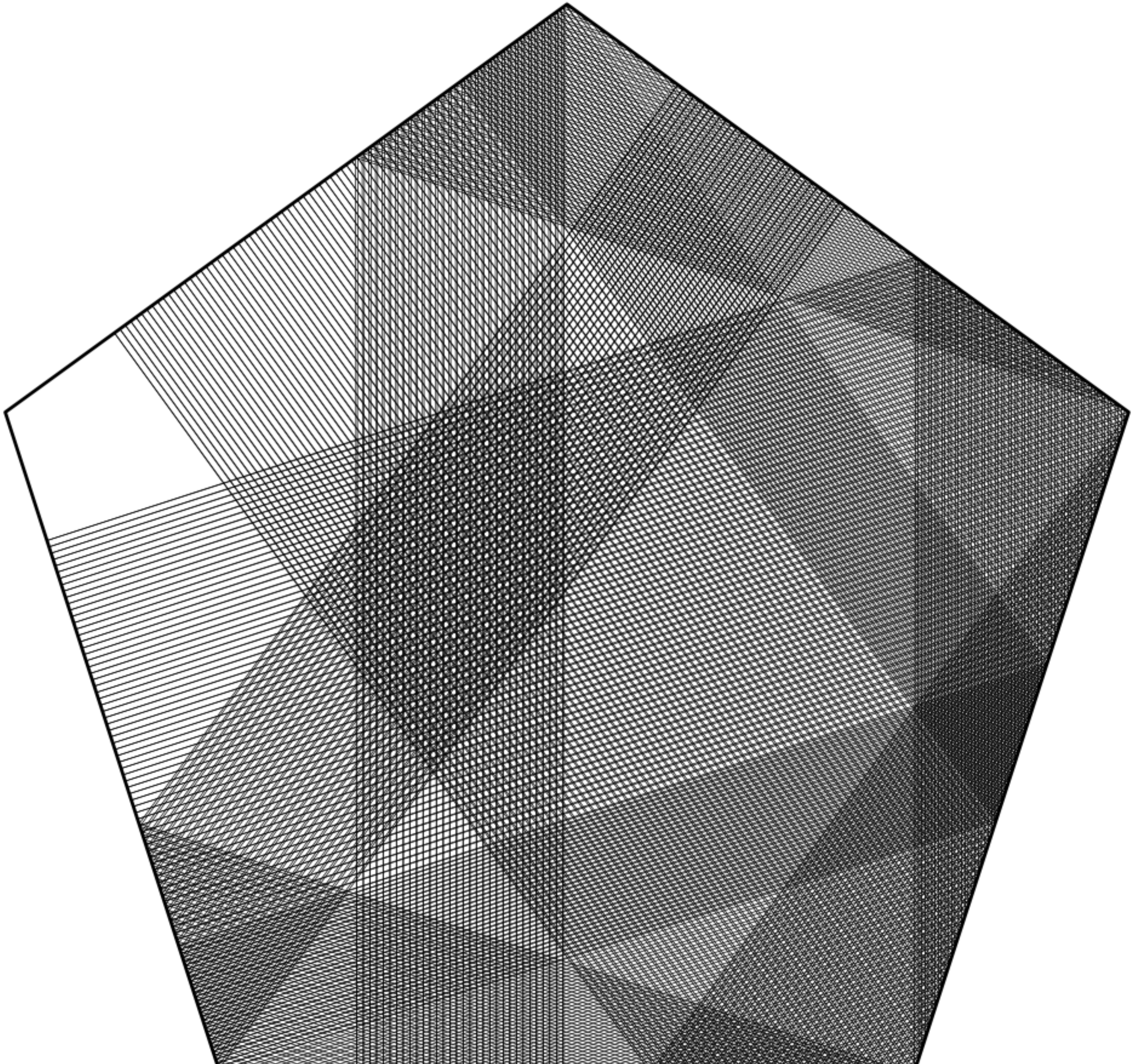


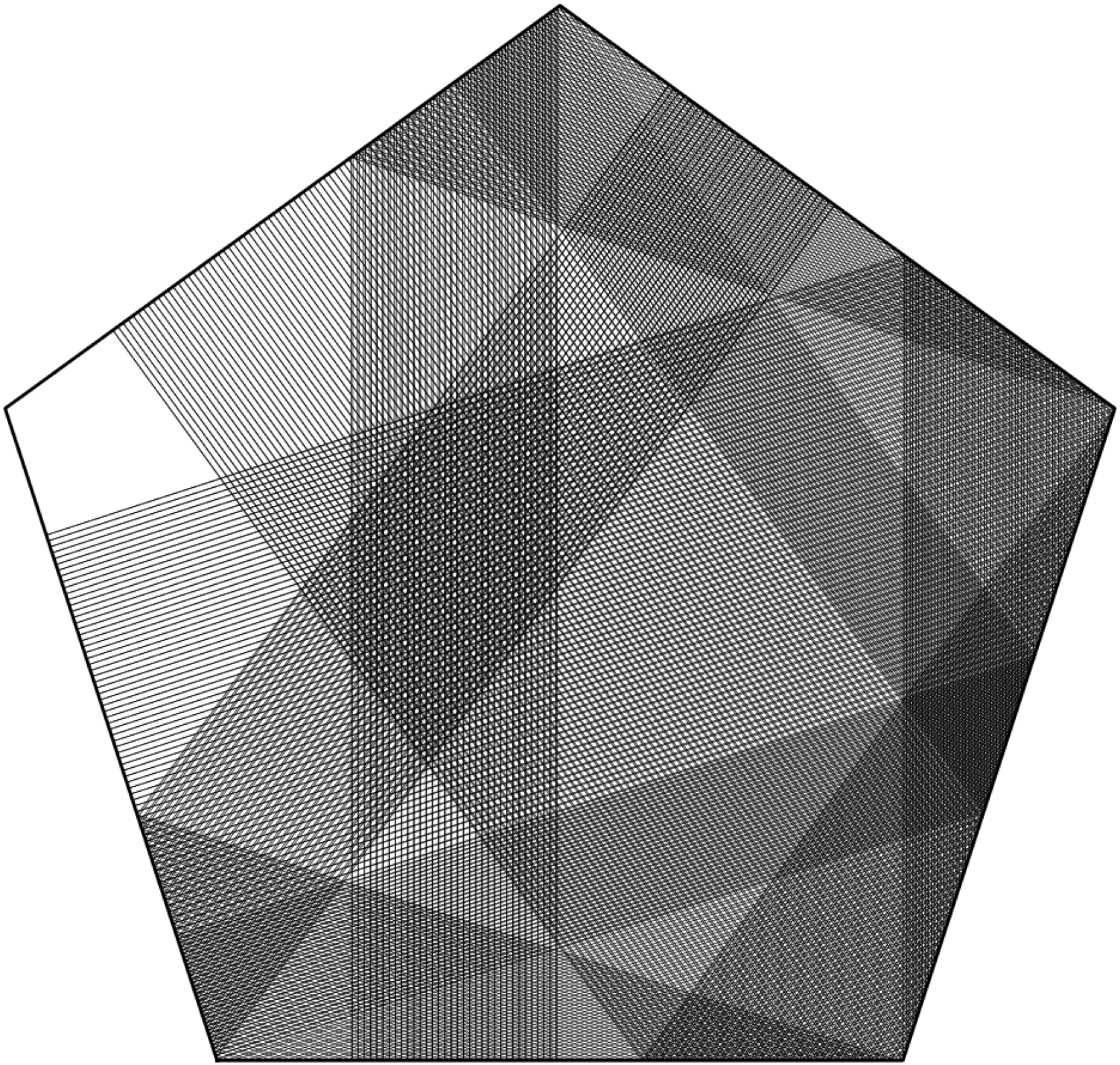


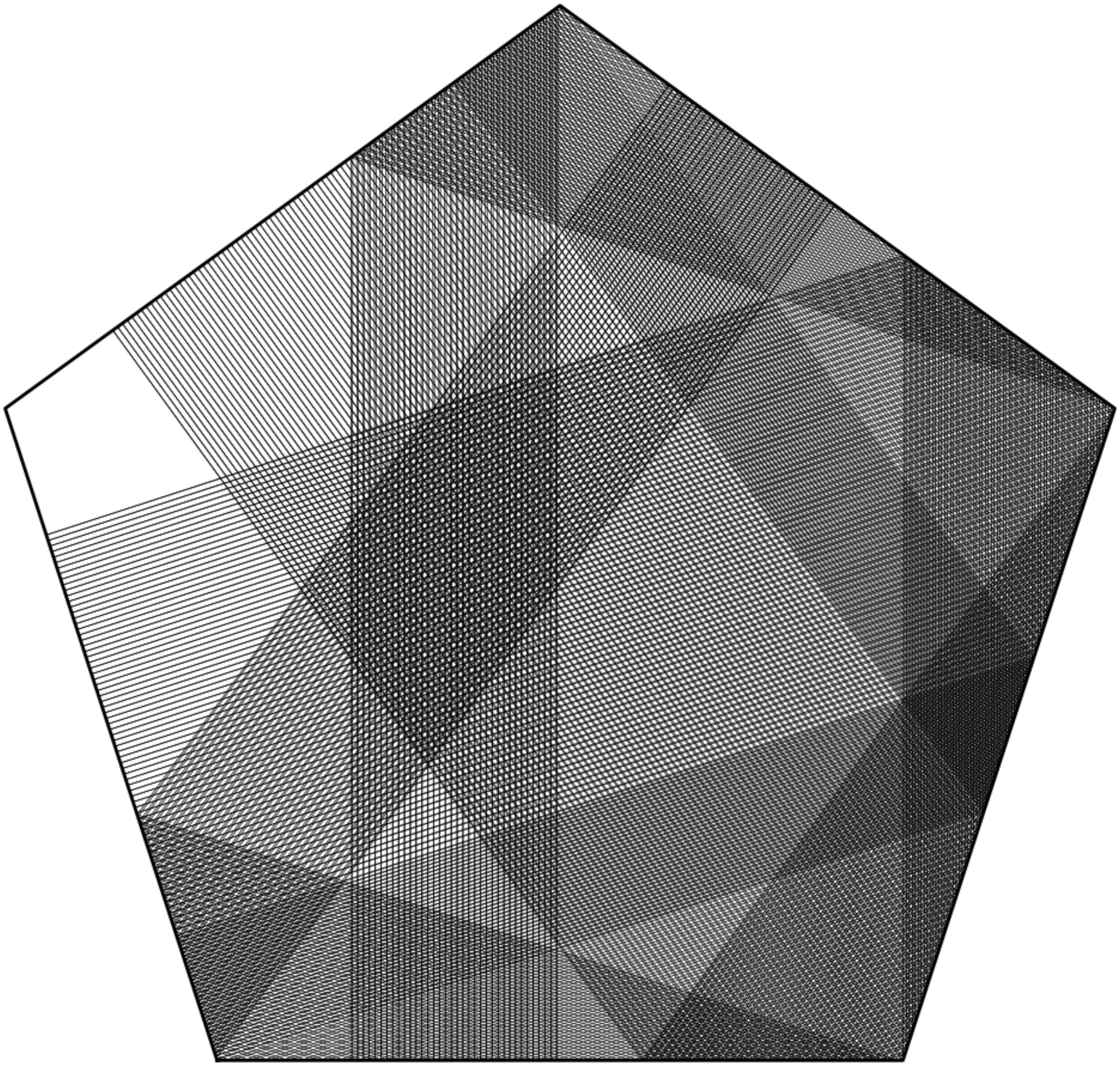


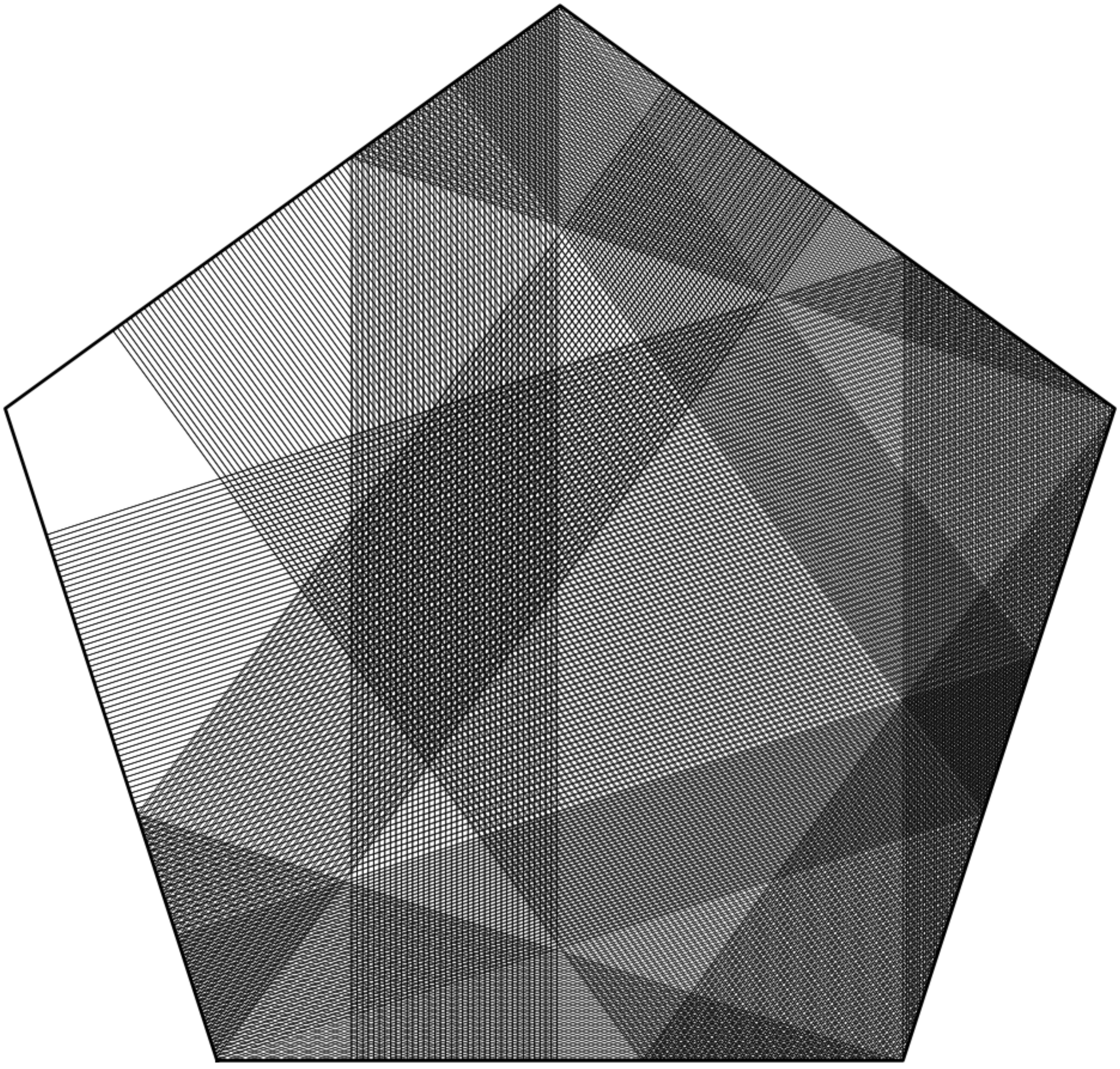


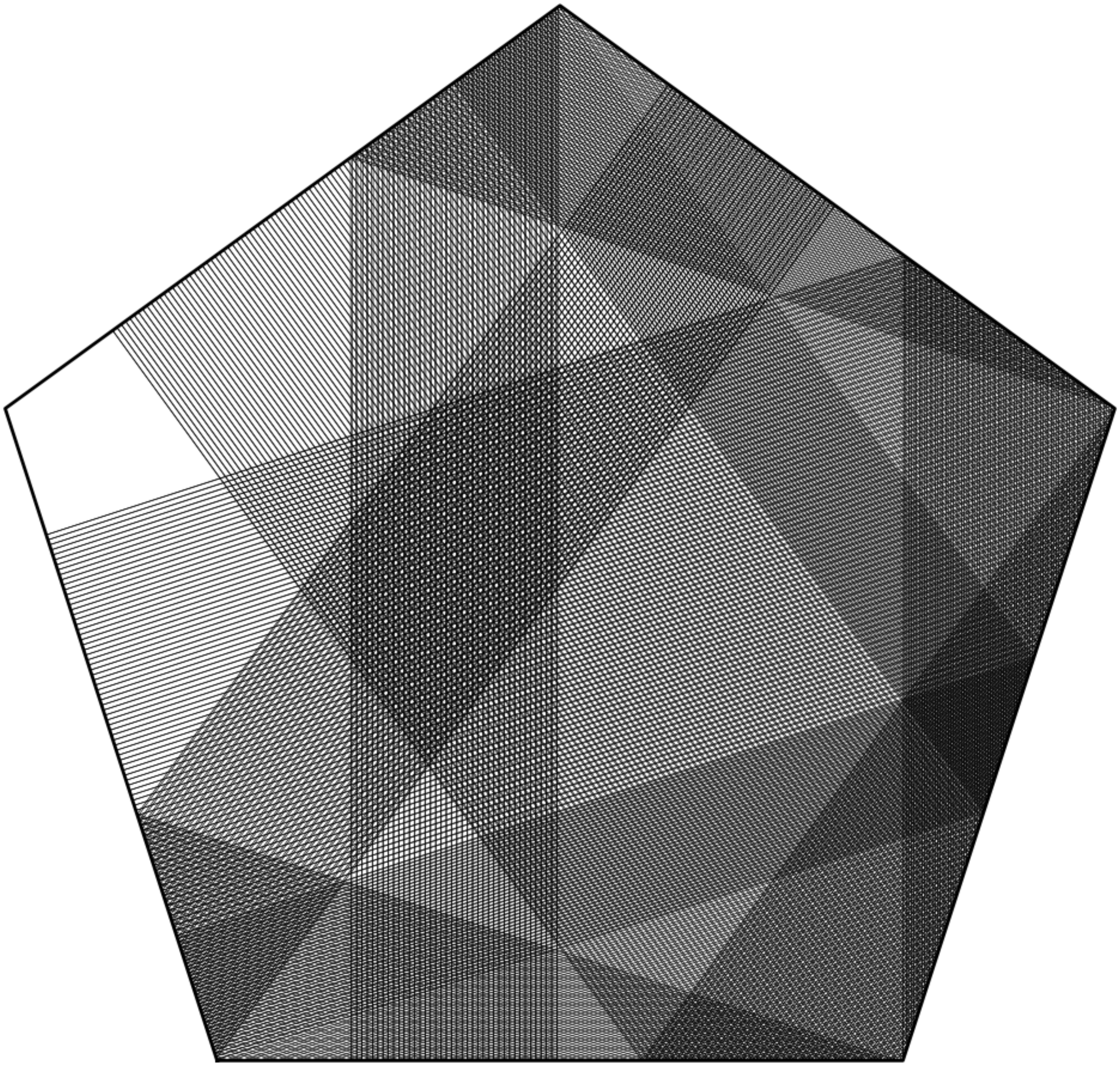


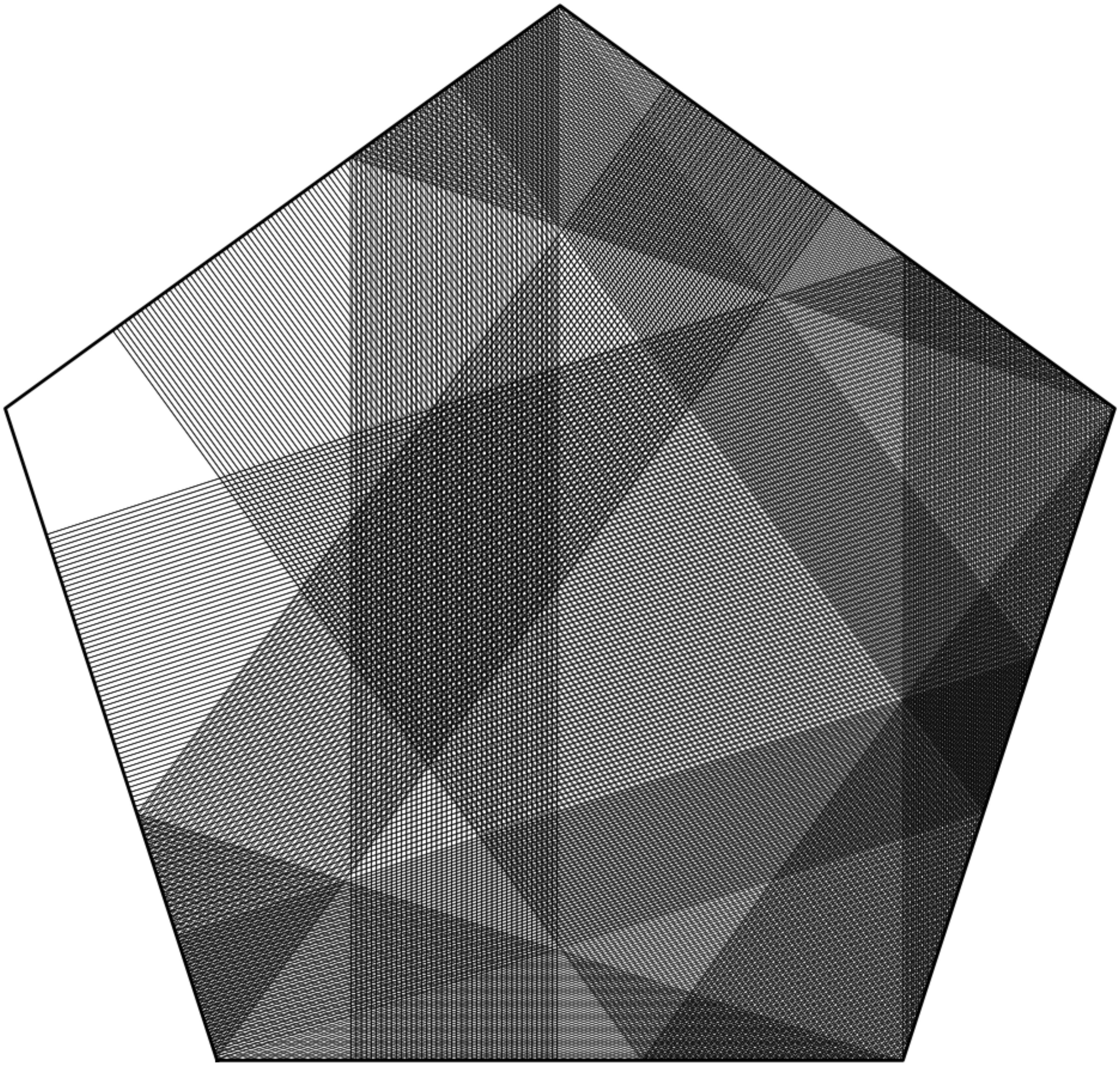


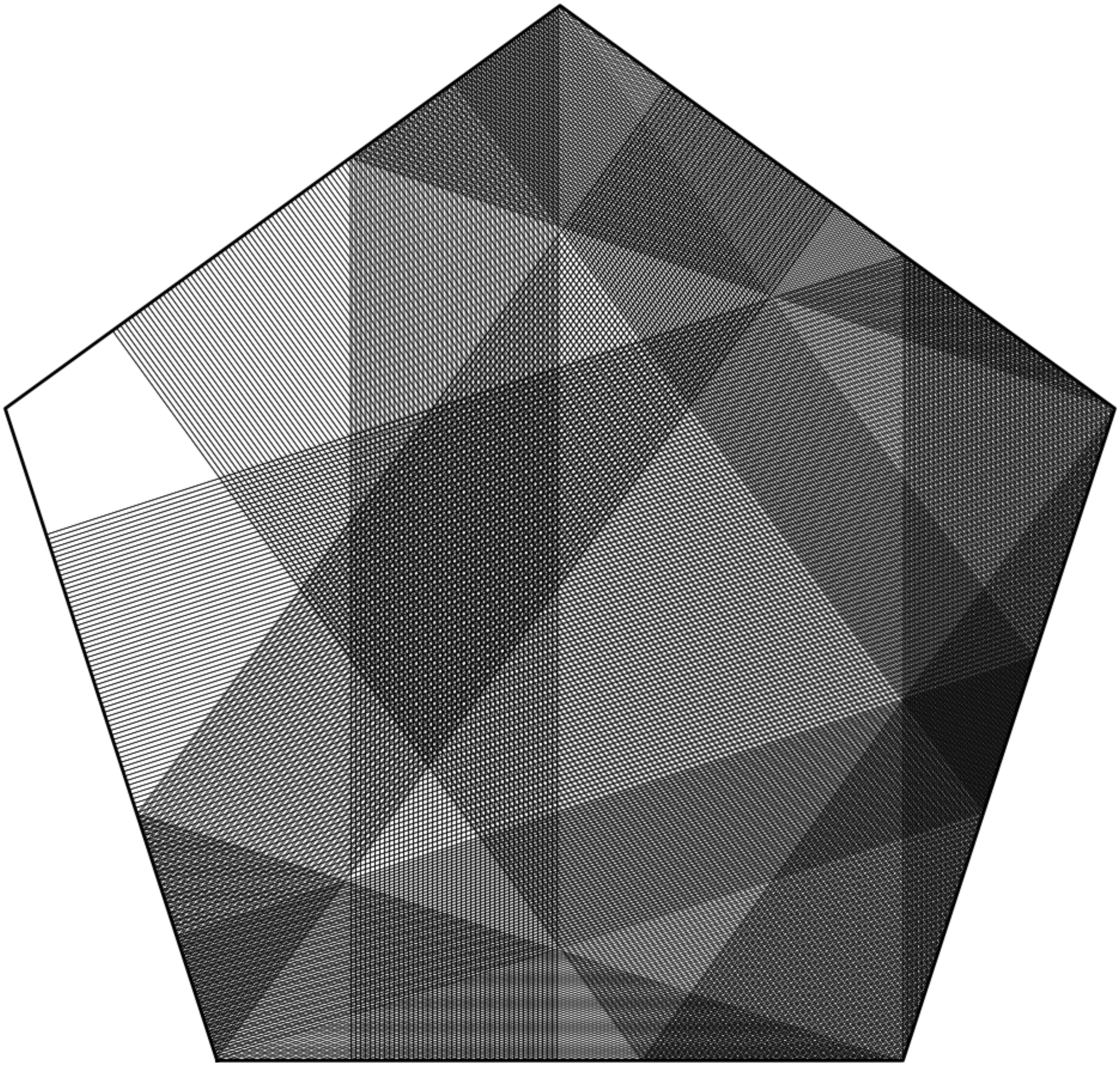


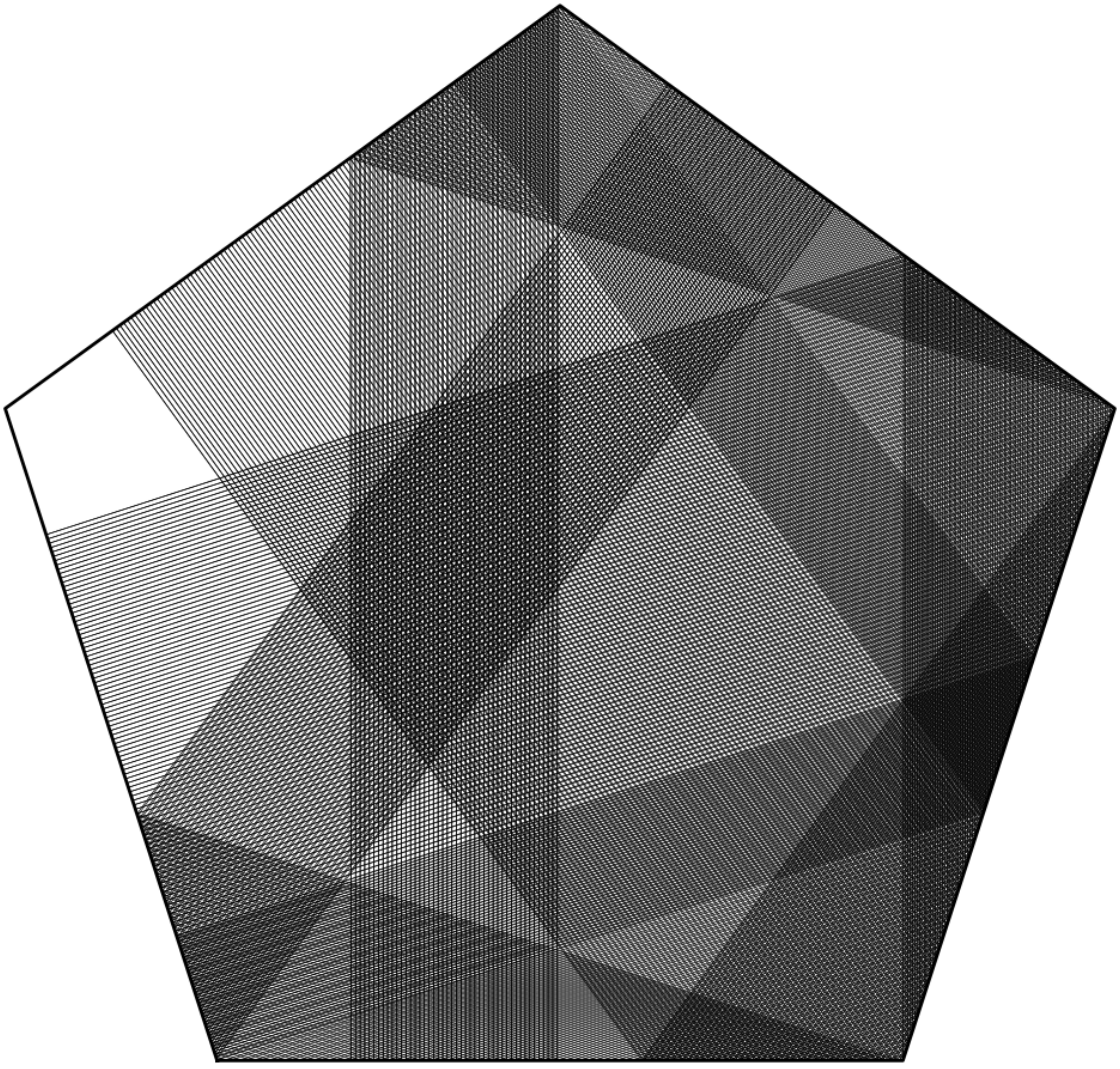


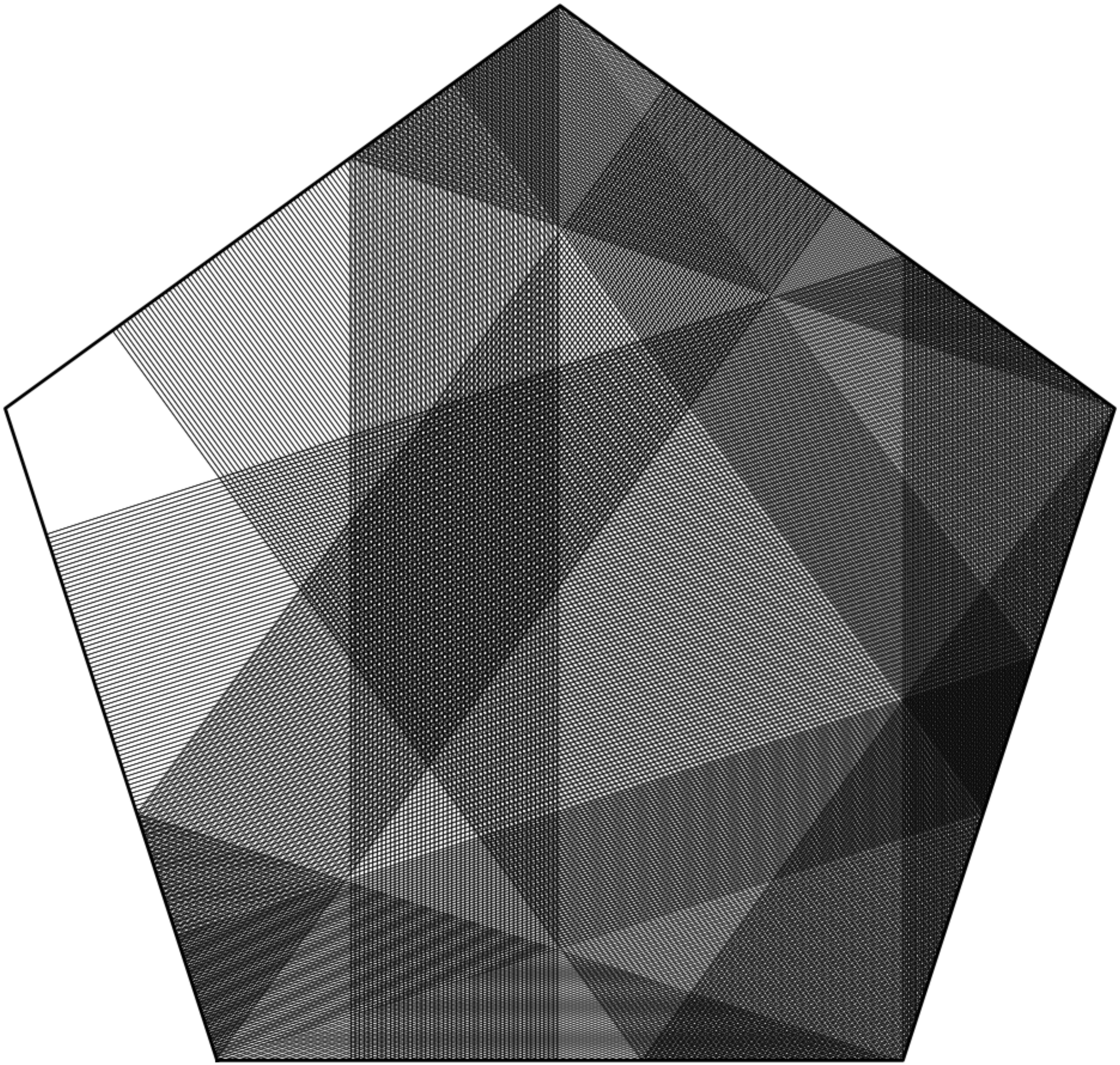




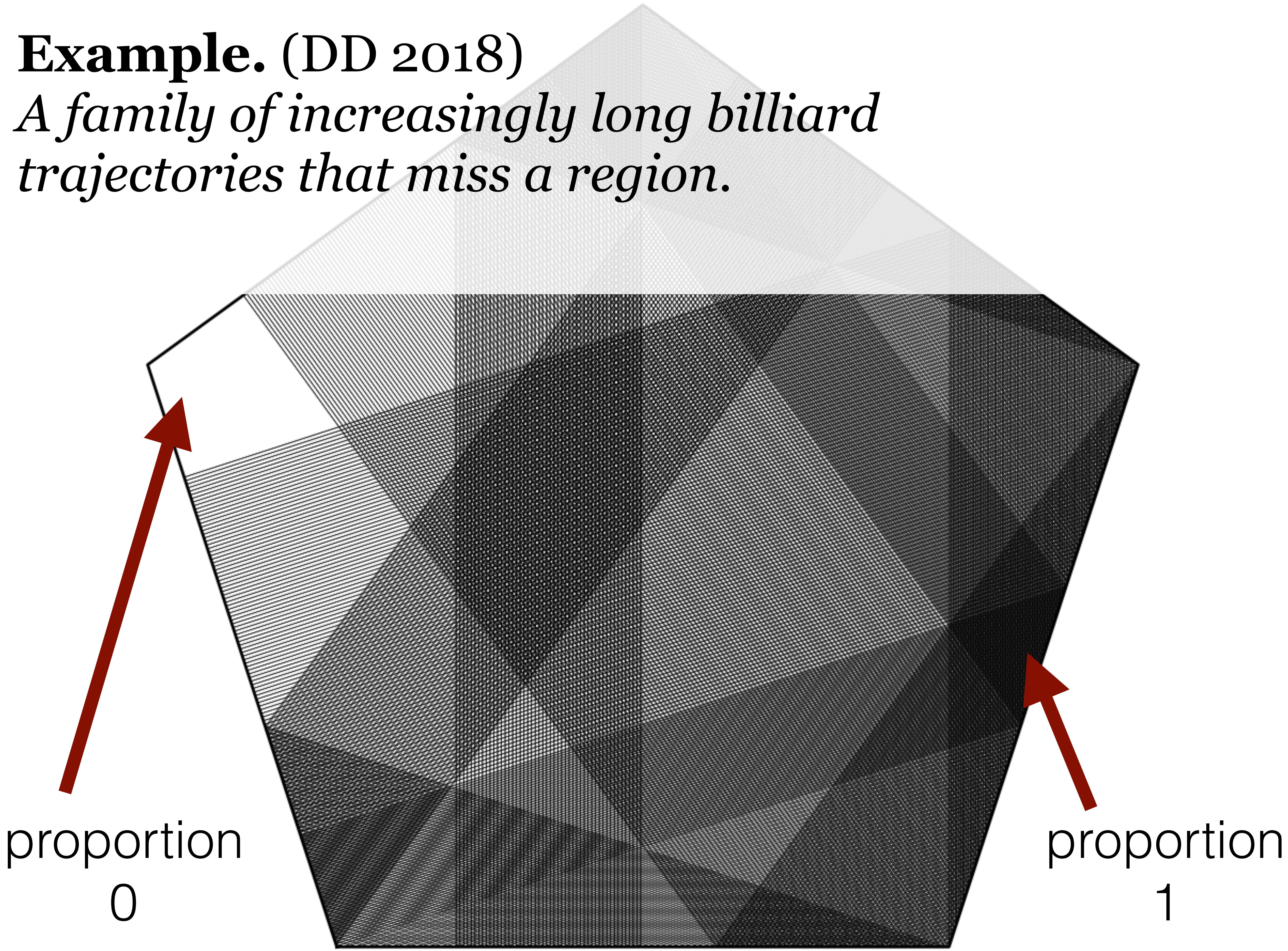




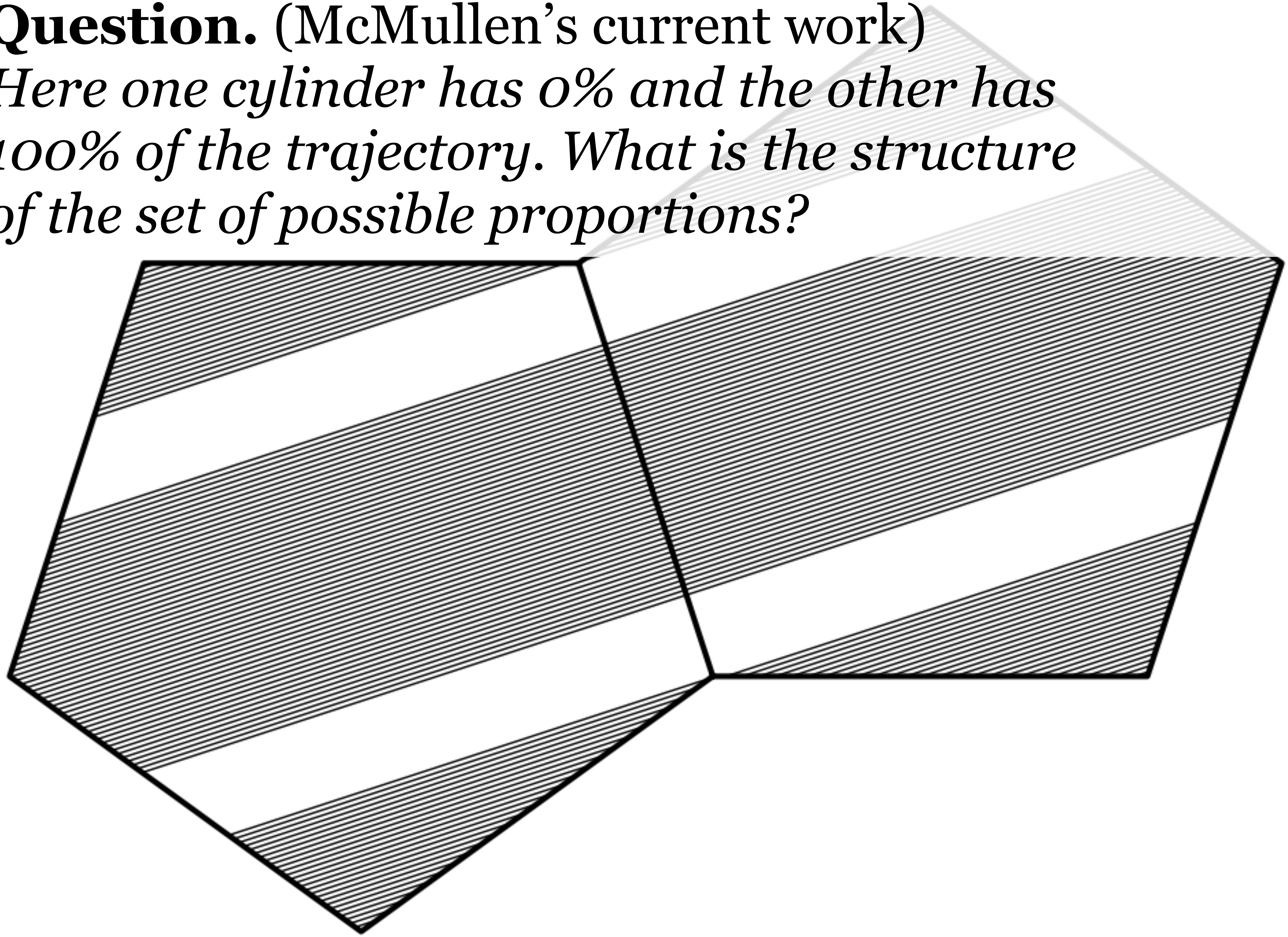




Example. (DD 2018)
A family of increasingly long billiard trajectories that miss a region.



Question. (McMullen's current work)
Here one cylinder has 0% and the other has 100% of the trajectory. What is the structure of the set of possible proportions?



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Theorem. *(McMullen 2018)*
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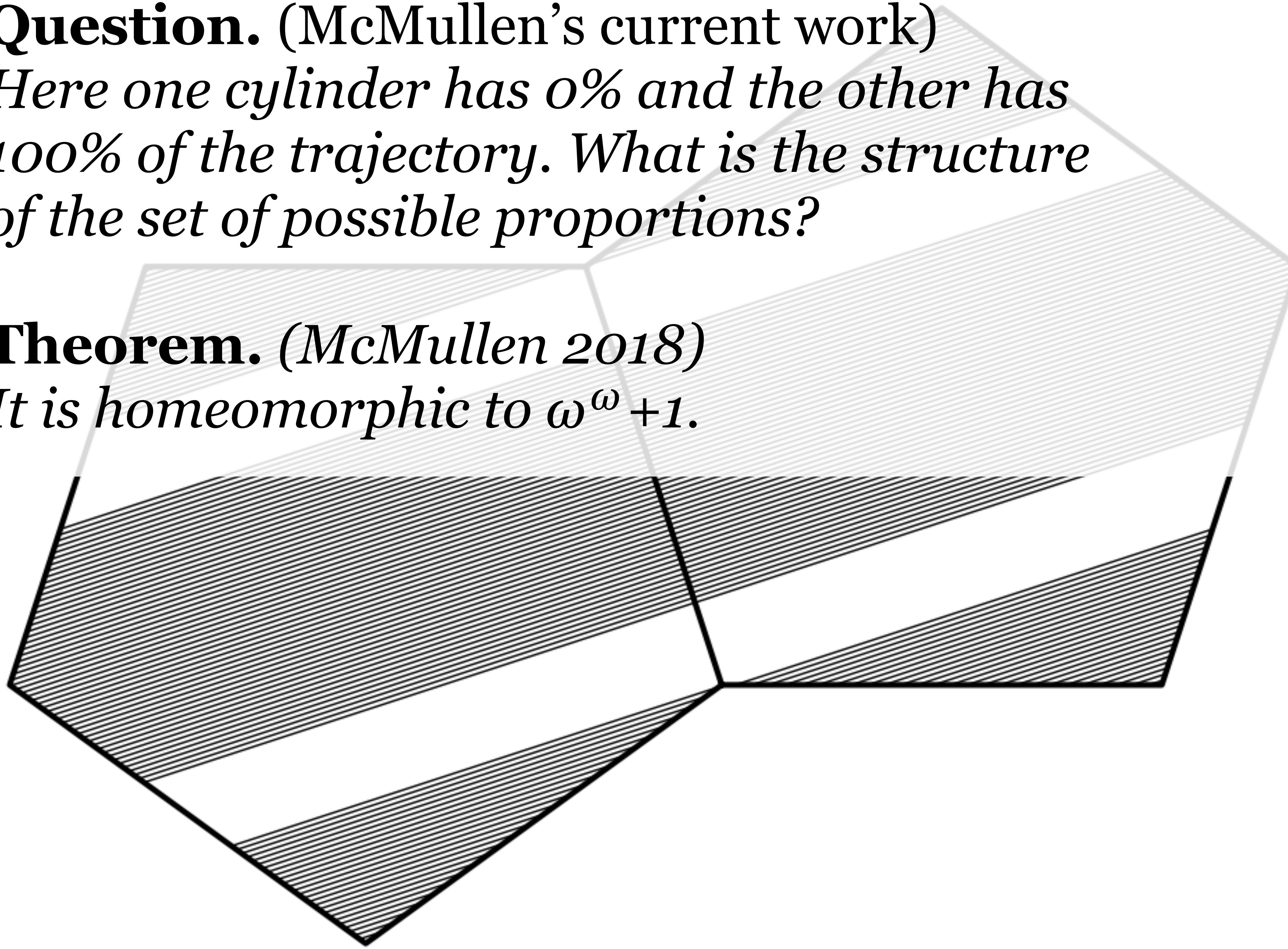

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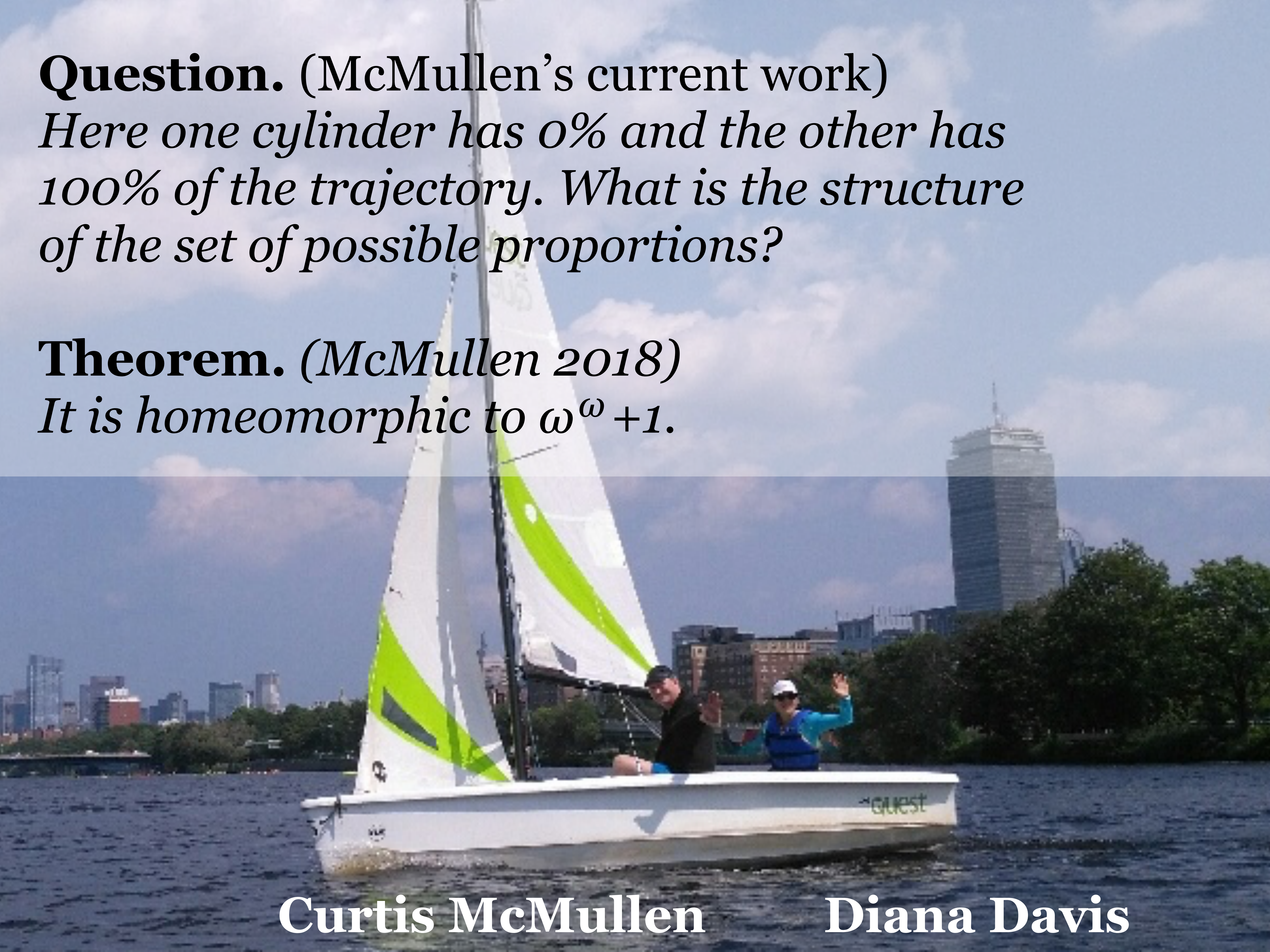
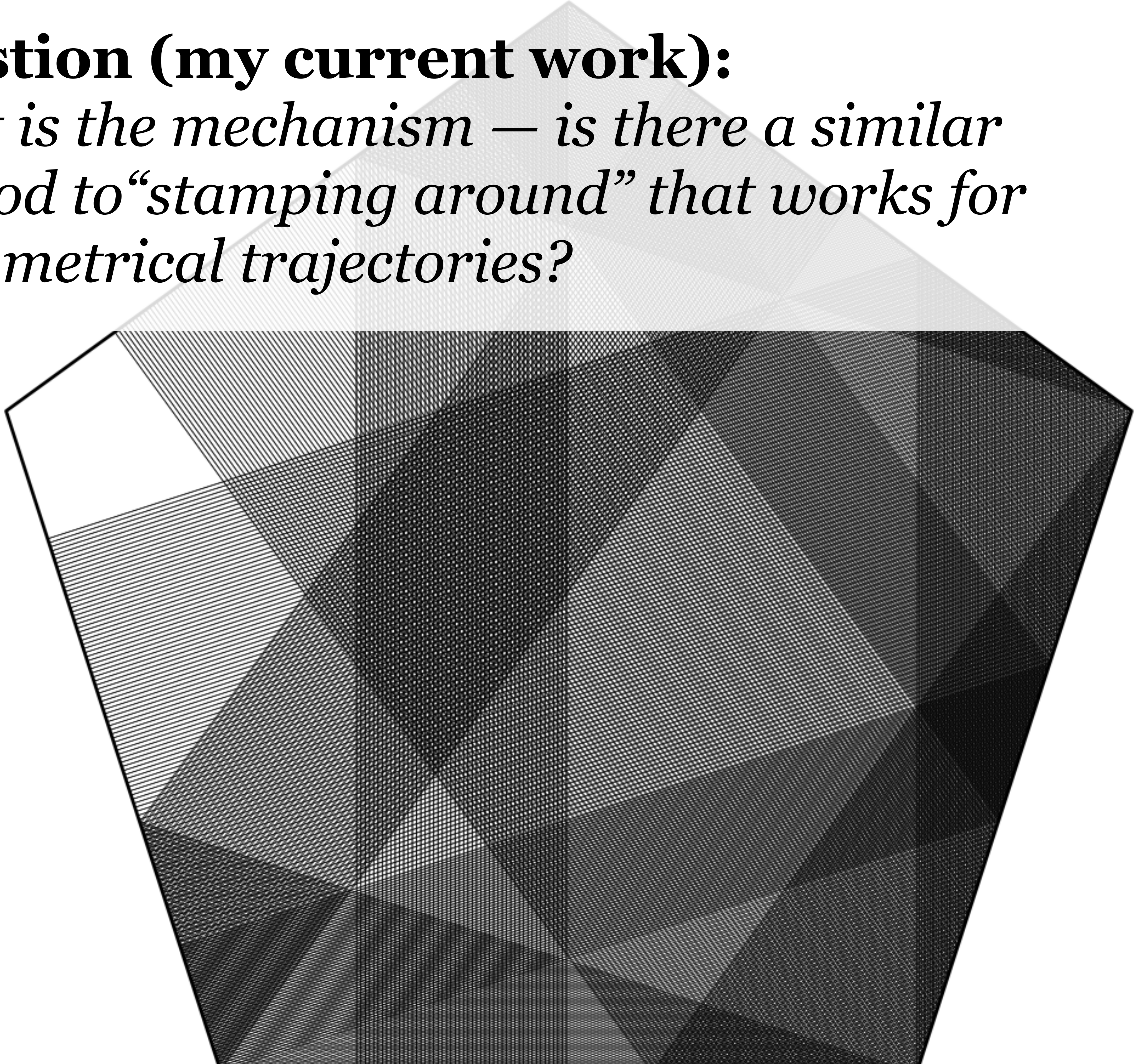
Theorem. *(McMullen 2018)*
It is homeomorphic to $\omega^{\omega}+1$.



Question (my current work):

What is the mechanism — is there a similar method to"stamping around" that works for asymmetrical trajectories?



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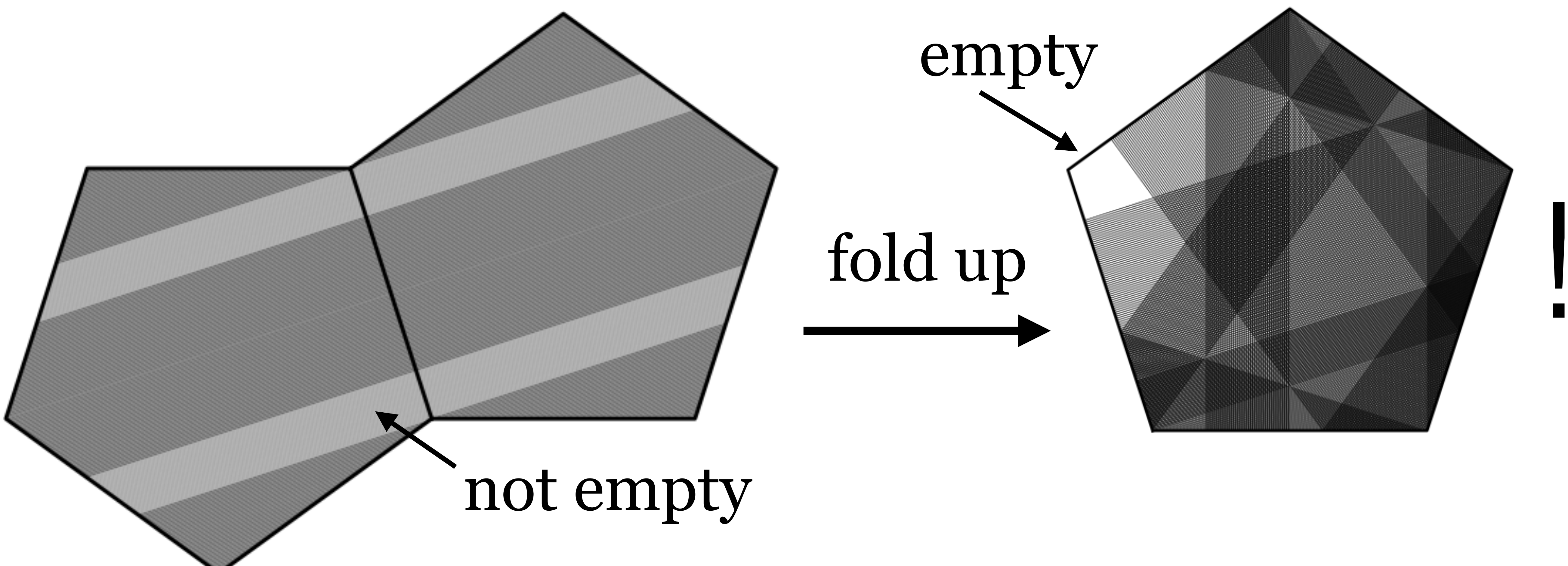
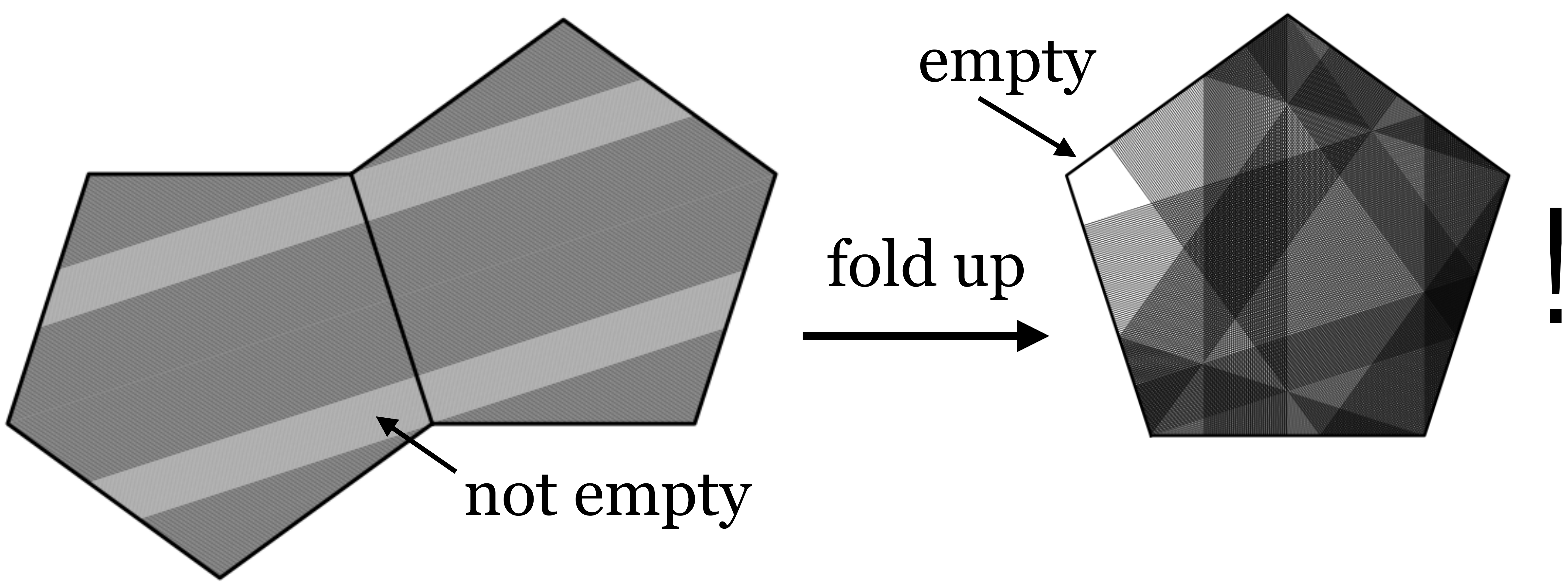


Future work:

- **Understand the mechanism from surface trajectories to asymmetrical billiard paths**
- Understand billiard trajectory behavior
- Prove conjectures about symbolic dynamics
- Extend to other shapes



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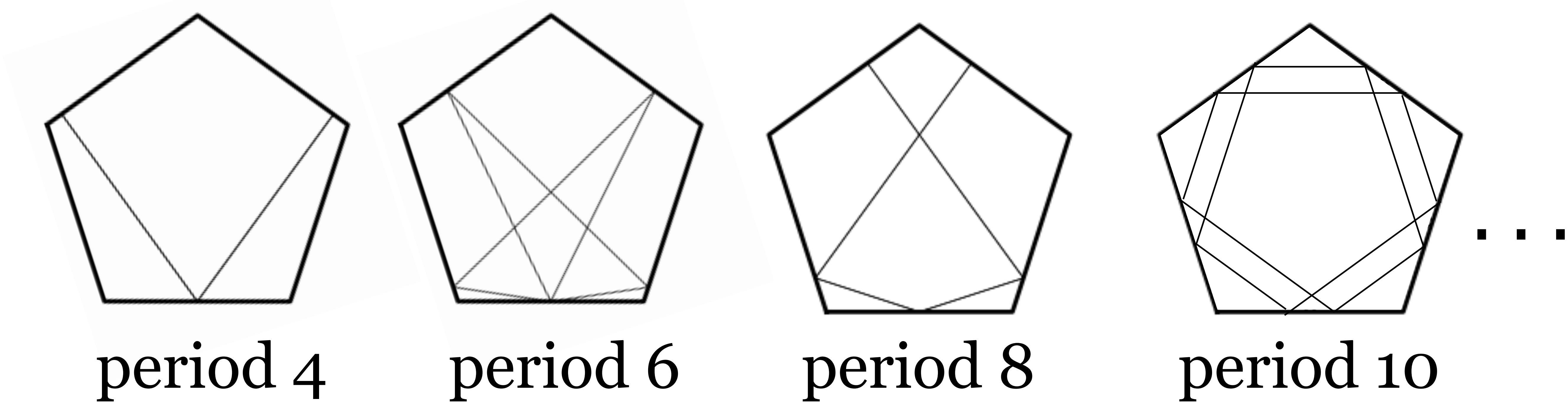
Does every even number arise as a billiard period?



Experimentally: all except 2, 12, 14 and 18 (up to 4000).

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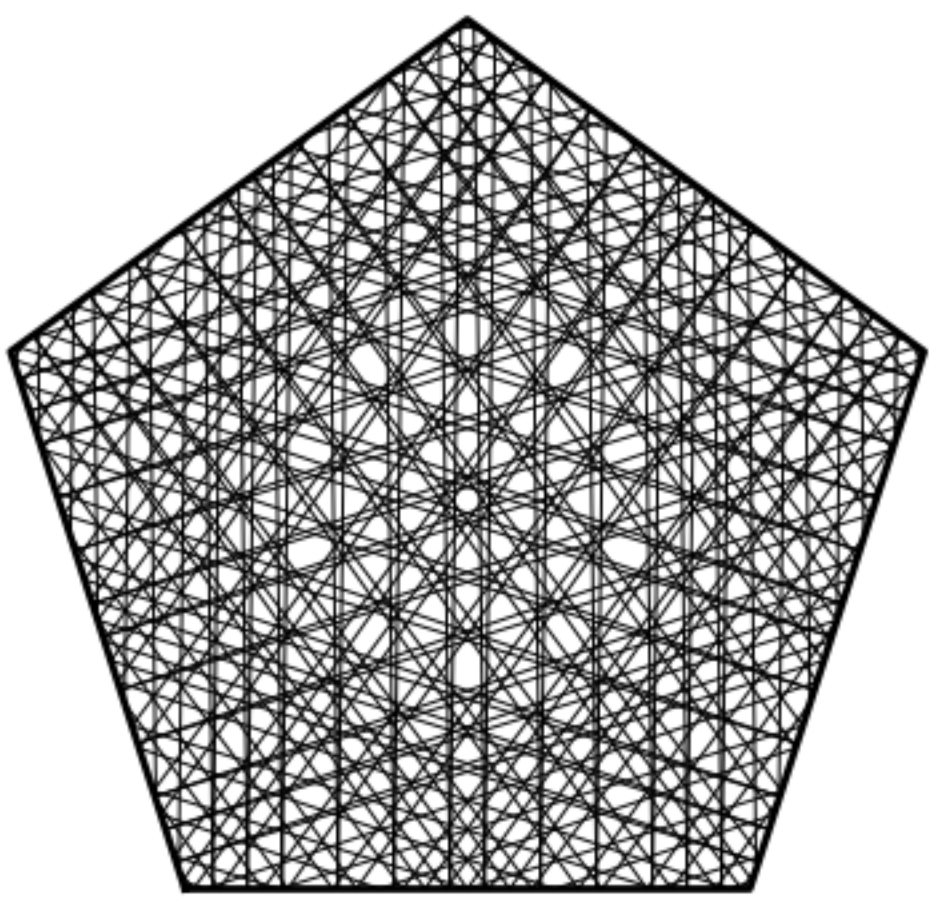
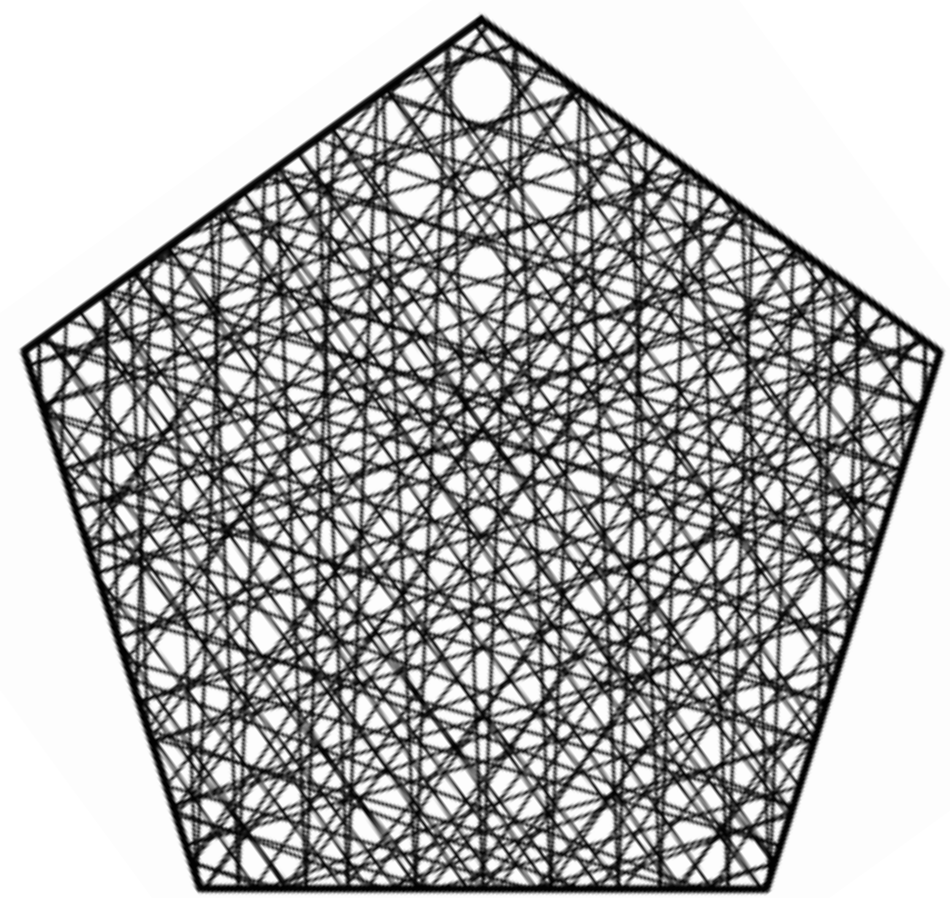


Why are there “holes”?



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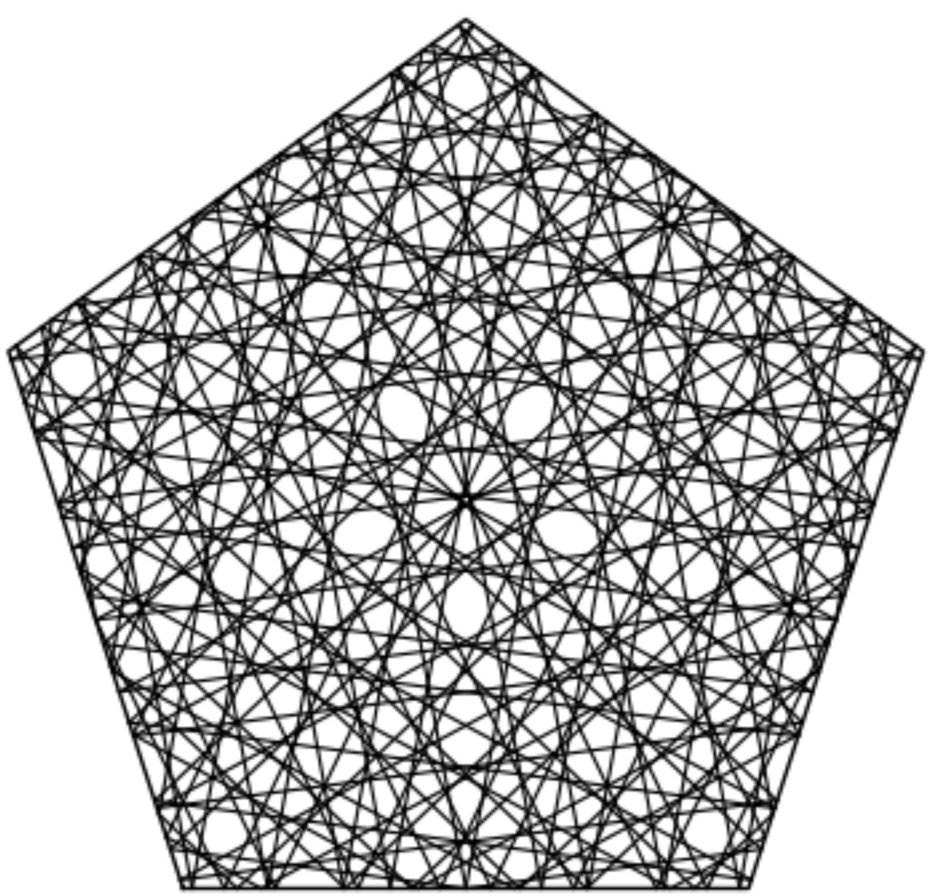

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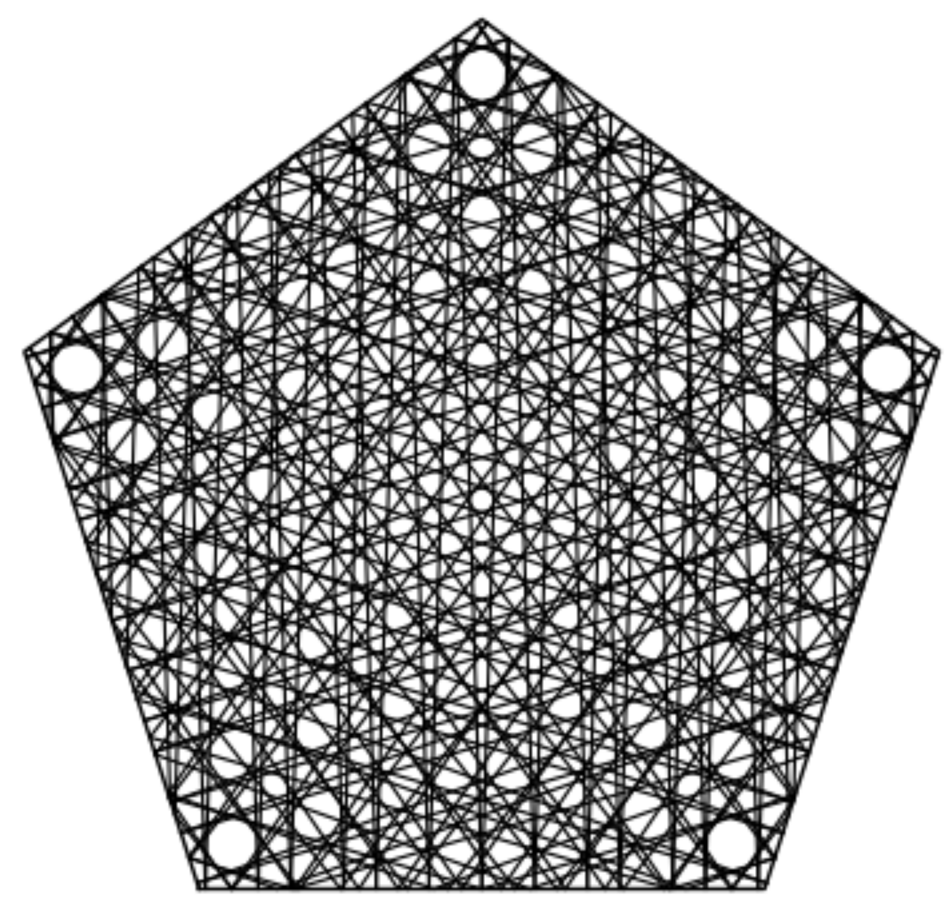

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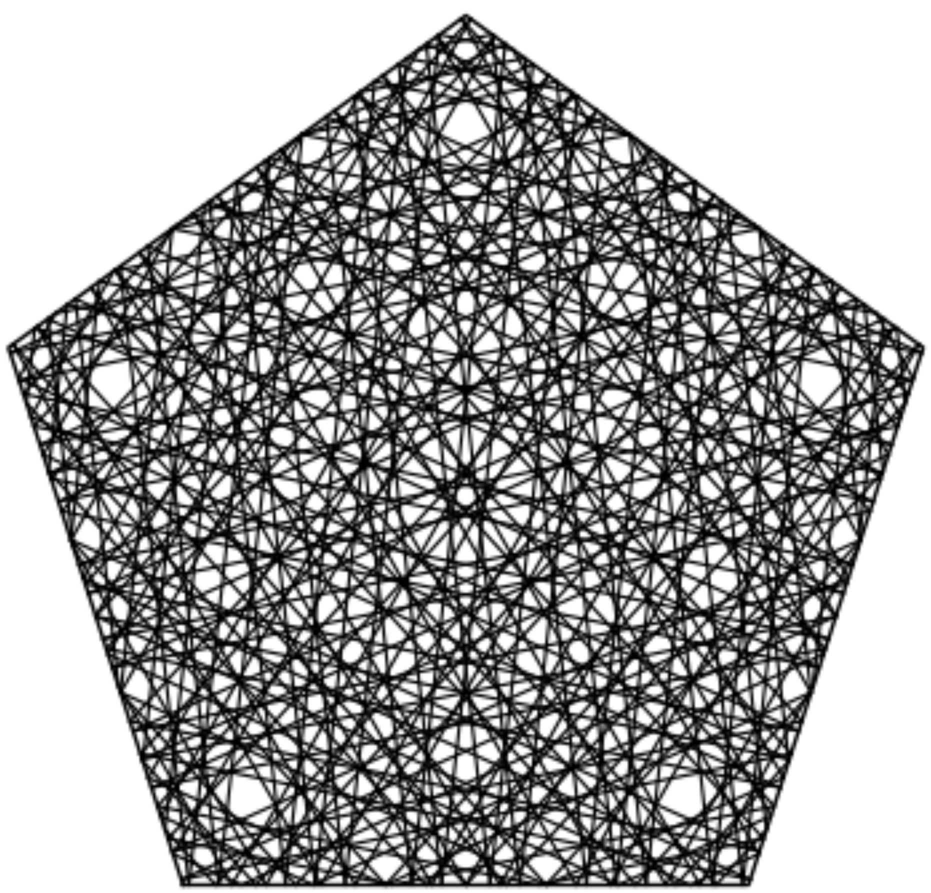

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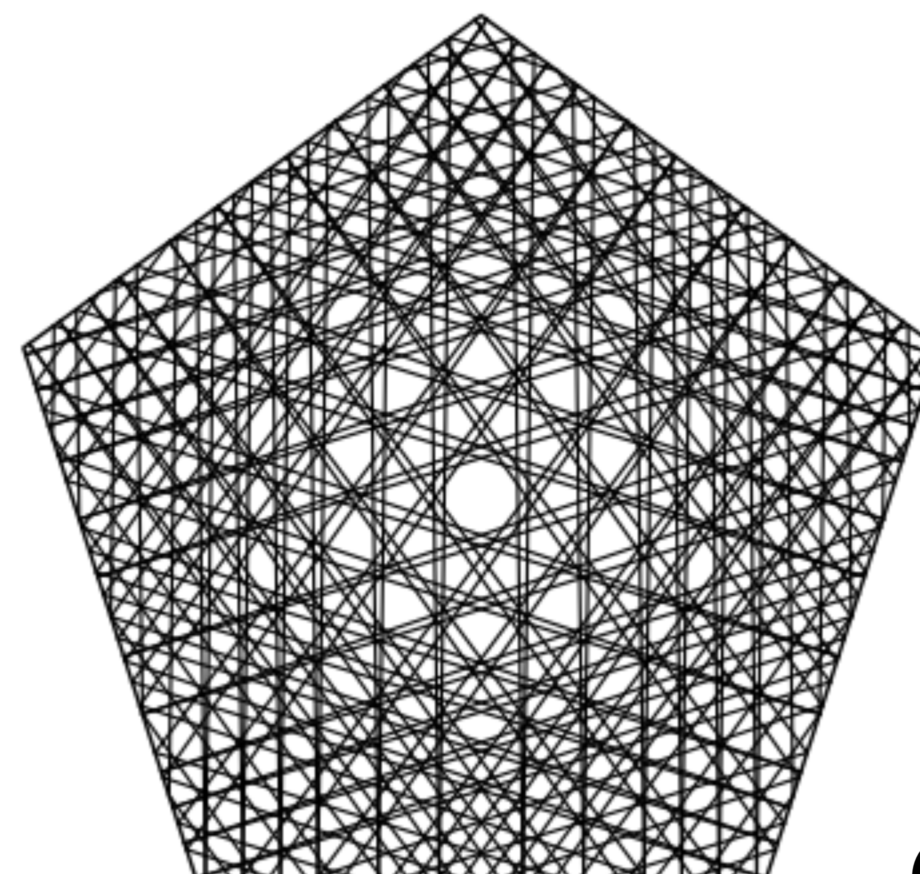

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Conjecture. (DD & Lelièvre)
For the subword length n sufficiently large, the complexity of each individual *aperiodic billiard sequence on the regular pentagon is* $15n + 10$.

Conjecture. (DD & Lelièvre)
The complexity p(n) of the billiard language *of the regular pentagon asymptotically approaches* $\frac{10}{\pi^2}n^3$.

Future work:

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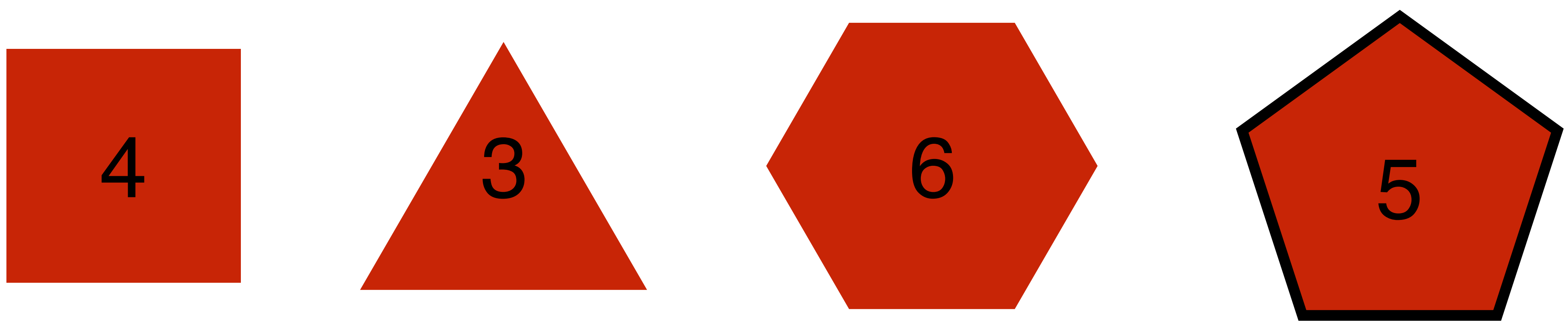
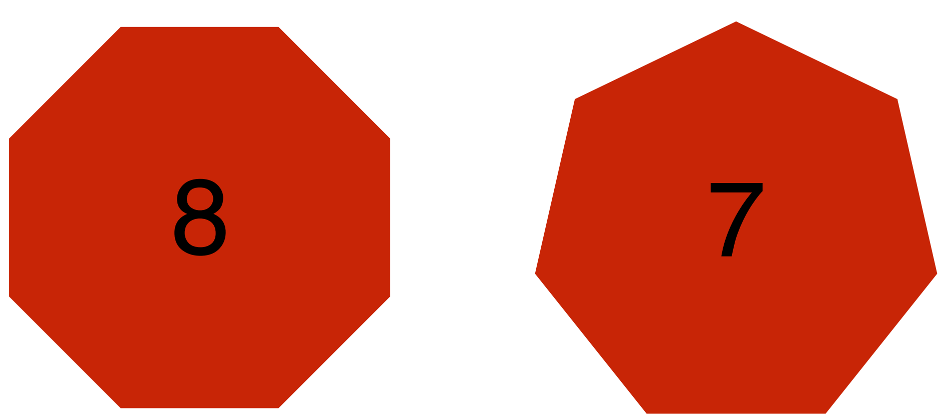
- **Extend to other shapes**

We understand periodic billiard paths on:



Next steps:



…

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- Diana Davis and Samuel Lelievre, *Periodic paths on the pentagon, double pentagon and golden L*, arXiv:1810.11310.

- Curtis T. McMullen, *Billiards and the arithmetic of non-arithmetic groups,* Coxeter Lecture Series, Fields Institute, 2 November 2018; slides at `http://math.harvard.edu/~ctm/expositions/home/text/talks/toronto/2018/III/slides.pdf`

- Curtis T. McMullen, *Gallery,* `http://math.harvard.edu/~ctm/gallery/`