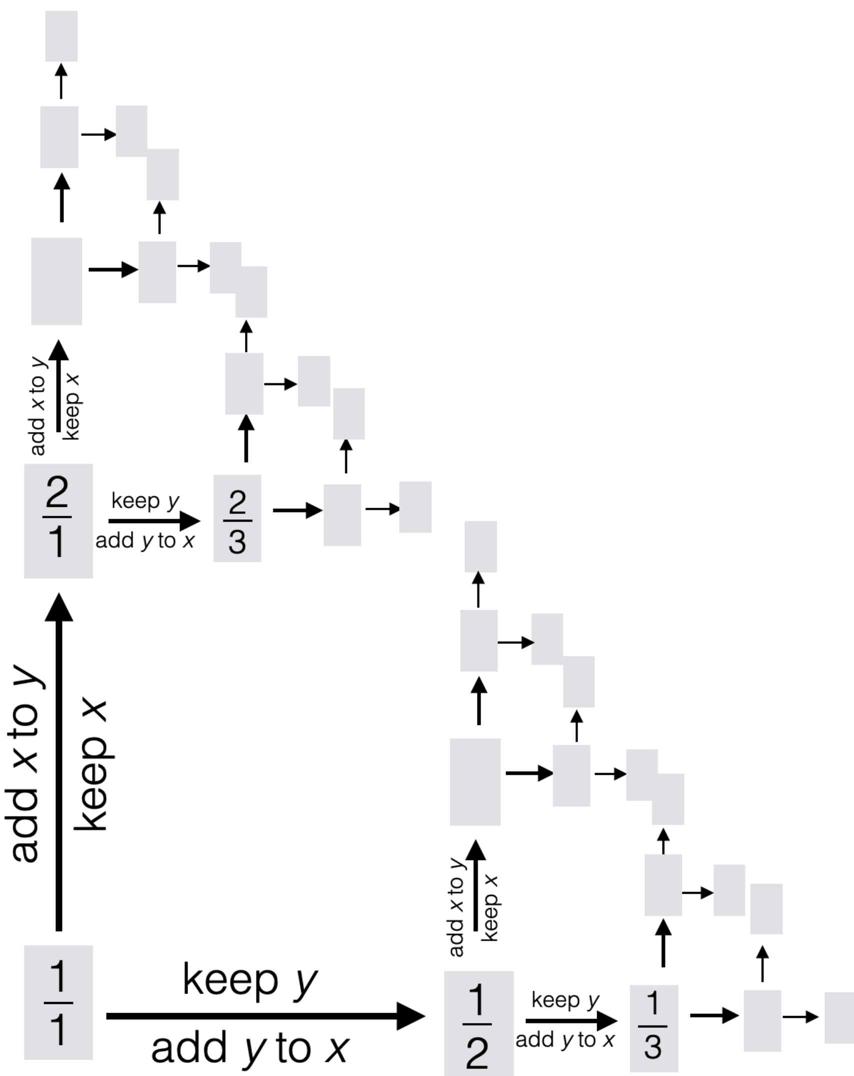
Billiards, Surfaces and Continued Fractions

1. You know what's beautiful? The structure of the rational numbers is beautiful. What does this mean, "the structure of the rational numbers"? Well, it turns out that the rational numbers have a *binary tree* structure (called the *Farey tree*):

1. Start with the fraction $\frac{1}{1}$ as the root.
2. For each node of the tree, create two "children": One by keeping the numerator ("y") the same and replacing the denominator ("x") with $x + y$, and one by keeping the denominator ("x") and replacing the numerator ("y") with $x + y$.
3. Continue forever.

Fill in the first five levels of the tree below.



2. **(a)** Explain why *every* fraction that shows up in the tree is in lowest terms: in other words, x and y never have any common factors.

(b) Explain why *every* rational number appears in this (infinite) tree.

A *continued fraction* gives an expanded expression of a given number. To obtain the continued fraction expansion for a number, say 15/11, we do the following:

$$\frac{15}{11} = \mathbf{1}+\frac{4}{11} = 1+\frac{\mathbf{1}}{11/4} = 1+\frac{1}{\mathbf{1}+7/4} = 1+\frac{1}{1+\mathbf{1}+3/4} = 1+\frac{1}{2+\frac{1}{4/3}} = 1+\frac{1}{2+\frac{1}{\mathbf{1}+\frac{1}{3}}}.$$

The idea is to pull off 1s until the number is less than 1, and then take the reciprocal of what is left, and repeat until the reciprocal is a whole number.

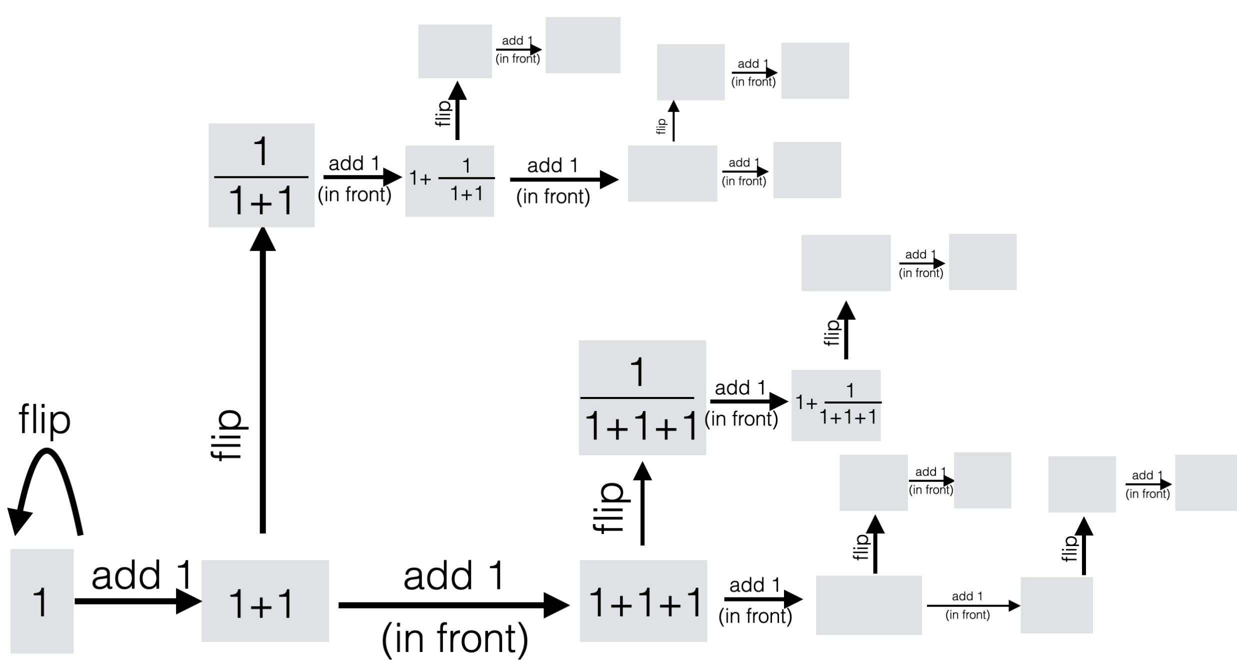
We can denote this compactly by recording only the initial numbers, so $15/11 = [1; 2, 1, 3]$. The semicolon indicates that the initial 1 is outside of the fraction.

3. Find the continued fraction expansion of 12/7, showing all your steps as above.

4. I prefer to build my continued fractions from the inside, out:
At each step, you can either:

1. *flip*, which means to take the inverse of the number you have, or
2. *add* 1 to the number you have.

Starting with the number 1 in the lower left corner, fill out the first few levels of this tree.



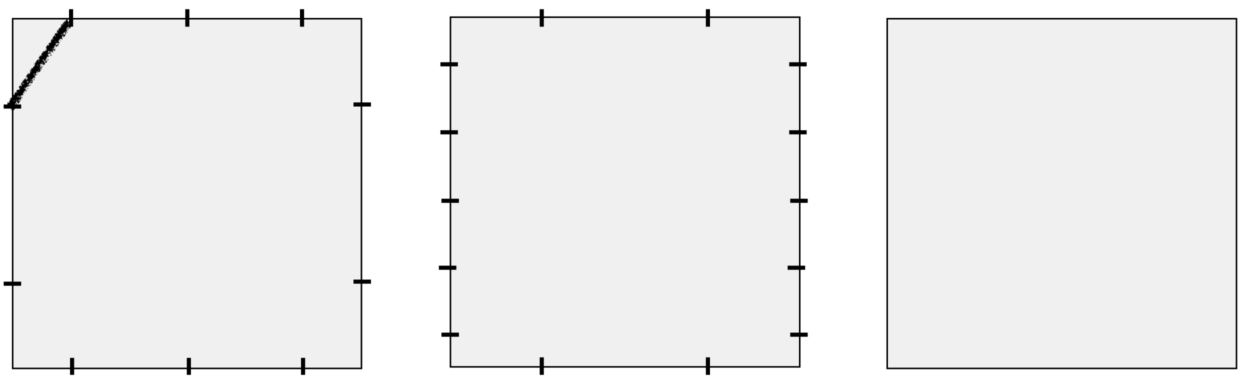
5. Explain why:

(a) *every* rational number can be expressed as a continued fraction, and

(b) *every* rational number will eventually show up in the tree shown above.

6. Billiards! The pictures below show some scratchwork for drawing periodic billiard trajectories on the square billiard table.

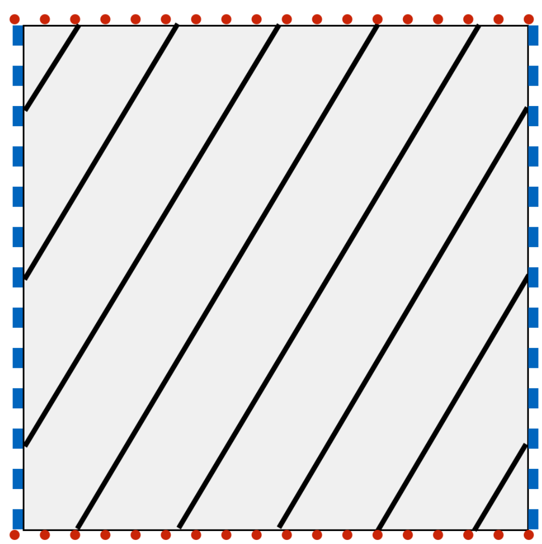
- Starting at the top-left corner, connect the top mark on the left edge to the left-most mark on the top edge with a line segment, as shown. Then connect the other five pairs with parallel segments (of slope $3/2$, in this case), down to the bottom-right corner.
- Do the same with symmetric line segments of the opposite slope ($-3/2$ in this case).
- Repeat for the middle square. What are the slopes of the segments in that case?



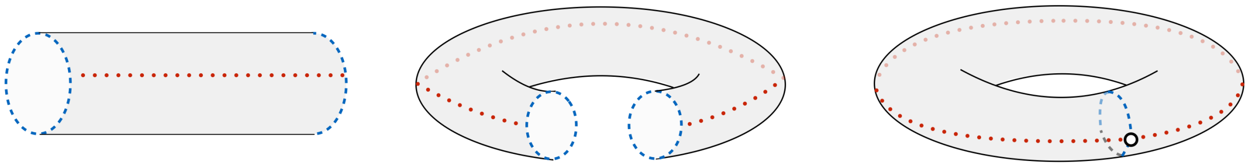
(a) Explain why this strategy gives a *billiard path*: a path of a bouncing ball where the angle of incidence equals the angle of reflection at each bounce.

(b) In the third square, draw a billiard trajectory of slope $\pm 3/5$. Exactly where should you place the tick marks (bounce points) to make this happen?

7. The left picture below shows a top view of a room with magic walls: when you cross one of them, you reappear out the opposite wall. So really, the top and bottom walls are *identified*, and the left and right walls are *identified*. (This fact is represented by the dots and dashes, respectively.) If you cut out the square, you can match up those edges and see it as a continuous room. The right picture shows the same room, with one continuous straight-line path passing through it. Explain.



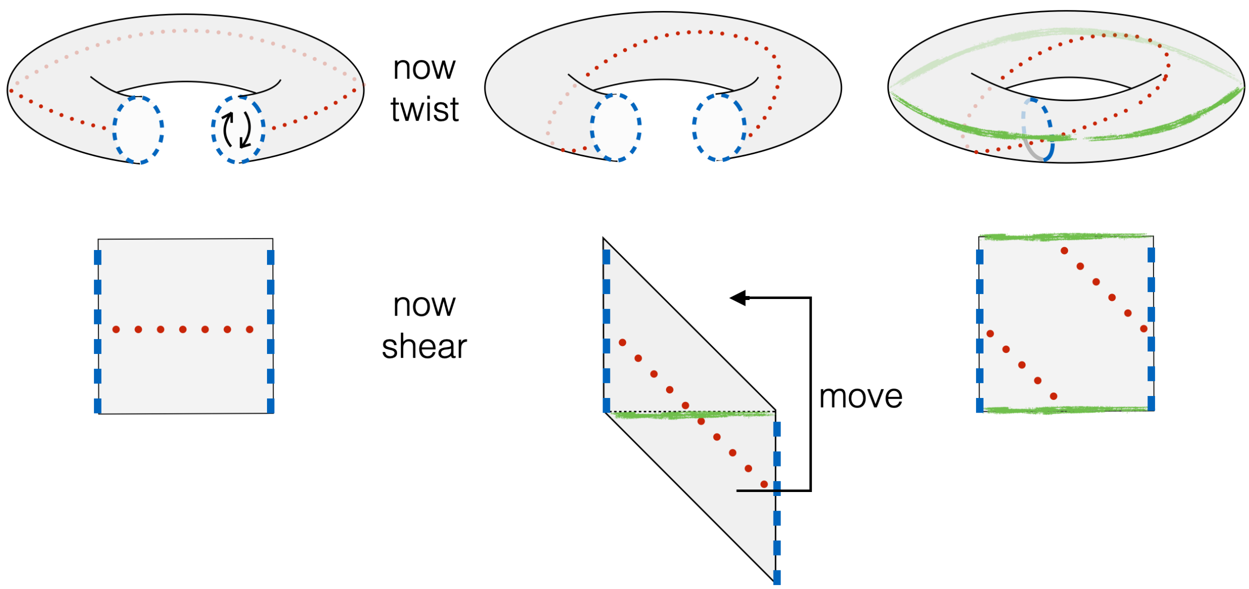
8. To understand billiards on the square, we actually use the "room with magic walls" *surface* instead. **Explain why**, if a piece of paper were very stretchy, gluing the top edge to the bottom edge and *simultaneously* gluing the left edge to the right edge would result in the steps shown below. The result is called a *torus*, the surface of a donut.



Another powerful tool in mathematics is:

1. Understand something simple (like a really basic, short path).
2. Understand what happens when you change the thing (stretch the path).
3. Profit: Now you understand more complicated things (longer paths)!

Consider a simple path: the equator of a torus (dotted, top left picture below). If you *twist* the torus as shown, that path becomes a bit more complicated: it goes around the torus once, as before, and now also passes once through the hole.
On the second row below, you can see the same action on the square picture.

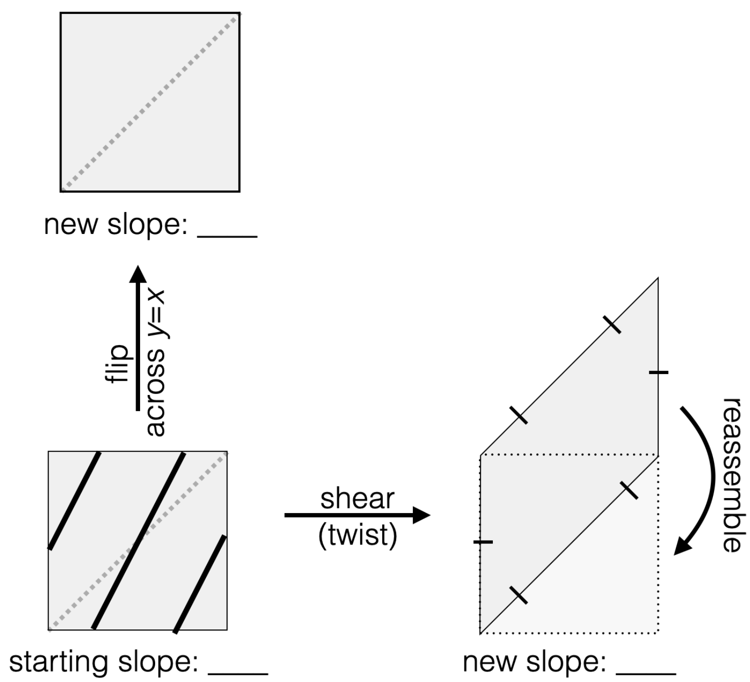


Our burning question is:

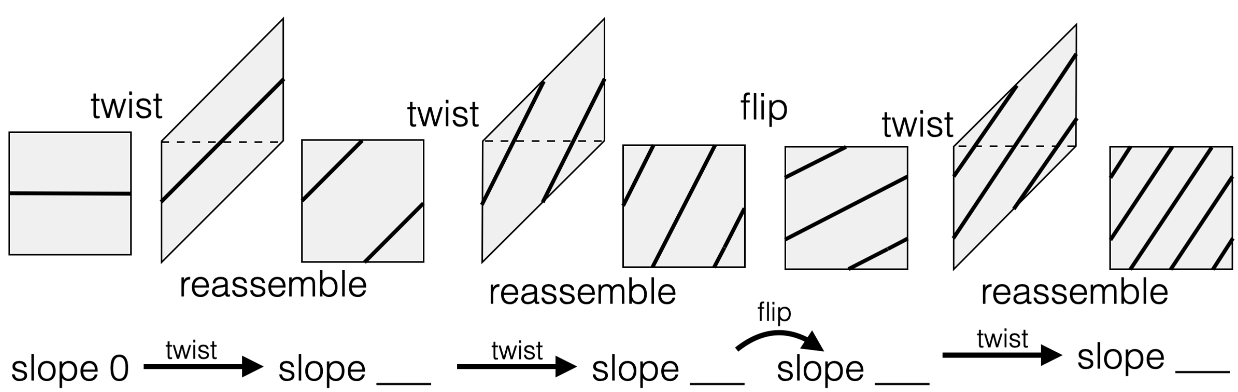
What happens to a trajectory on the square torus when you twist it?

It turns out that we can build all trajectories from the simplest trajectory using twists, so if we can understand twists, we can understand everything!

9. Given a trajectory on the square torus, we want to know what happens to that trajectory if we apply a symmetry of the surface. To do this, we can sketch the trajectory before and after applying the symmetry. Sketch the image of the given trajectory under the *flip* and the *shear (twist)*, and say what happens to the slope.



10. The pictures below show how to start with a simple trajectory of slope 0, and apply twists and flips to end up with increasingly complicated trajectory. Fill in the slopes of each trajectory along the way.



11. What effect does a twist have on the slope? What effect does a flip have on the slope?

12. Explain how twists and flips are related to continued fractions.

13. Prove that we can get *every* periodic trajectory on the square torus via a sequence of twists and flips.

Okay, so that's the happy ending:

- To study billiards on the square, we instead study paths on the square torus.
- You can get *every* periodic path on the square torus by starting with the simplest path (horizontal path, or "equator") and applying a sequence of twists and flips.
- So if you can understand what happens to a path when you twist, and what happens when you flip, by induction you understand every periodic path on the torus.
- Good news: We can understand what happens when you twist or flip!
- Thus we can also understand every periodic billiard path on the square.

Some additional problems for fun

14. Explain why

(a) a number is rational if and only if its continued fraction expansion is finite;

(b) a number is *algebraic*, i.e. the root of a polynomial, such as $\sqrt{5}$ or $\sqrt[3]{2+\sqrt[5]{4}-\sqrt{7}}$, if and only if its continued fraction expansion is eventually periodic (repeating).

A number is called *transcendental*, like π and e, if its continued fraction is infinite and non-repeating. People look for patterns in these, as they do for the decimal representation.

15. Suppose 100 ants are on a log 1 meter long, each moving either to the left or right with unit speed. Assume the ants collide elastically (when they hit each other, each ant immediately turns around and goes the other way), and that when they reach the end of the log, they fall off. What is the longest possible waiting time until all the ants are off the log?