# Real Analysis

Math 790

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Exeter, NH
Spring 2022

# Discussion Skills

1. Contribute to the class every day
2. Speak to classmates, not to the instructor
3. Put up a difficult problem, even if not correct
4. Use other students' names
5. Ask questions
6. Answer other students' questions
7. Suggest an alternate solution method
8. Draw a picture
9. Connect to a similar problem
10. Summarize the discussion of a problem

# Real Analysis

## The problems in this text

The method of instruction used with these problems is based on the curriculum at Phillips Exeter Academy, a private high school in Exeter, NH. Most of the beginning of the course (and some of the later) is based on *Real Analysis* by **Frank Morgan** (FM). Most of the end of the course (and some of the earlier) is based on **Aimee Johnson**'s lecture notes and worksheets, which in turn are based on *Introduction to Analysis* by **Maxwell Rosenlicht** (MR). The rest of the problems were written by **Diana Davis** (DD) specifically for this course. If you create your own text using these problems, please give credit as I am doing here, and note who wrote each problem, as I have done in the left margin.

## About the course

This course meets 7 times every two weeks, with one long block (70 minutes) and one short block (40 minutes) each week, plus either one or two regular blocks (50 minutes). The homework for each class is 75 minutes, except that there is no homework for the short blocks. There are occasional tests.

## To the Student

**Contents**: As you work through this book, you will discover that the various topics of real analysis have been integrated into a mathematical whole. There is no Chapter 5, nor is there a section on sequences of functions. The curriculum is problem-centered, rather than topic-centered. Techniques and theorems will become apparent as you work through the problems, and you will need to keep appropriate notes for your records — there are no boxes containing important ideas. Key words are defined in the problems, where they appear italicized.

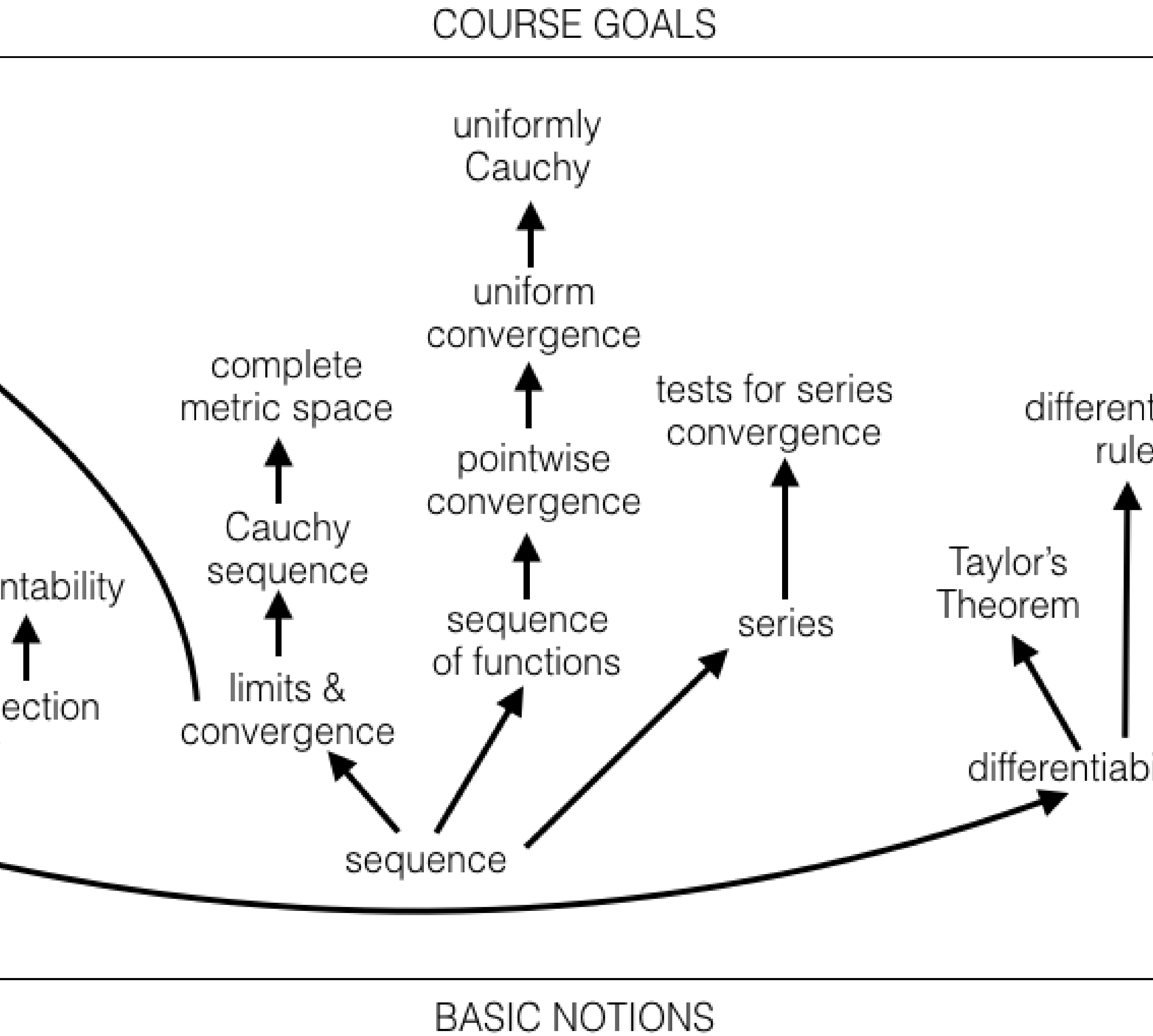
**Your homework**: The first day of class, we will work on the problems on page 1, and your homework is page 2 (possibly 2a and 2b); on the second day of class, we will discuss the problems on page 2, and your homework will be page 3 (possibly 3a and 3b), and so on for each day of the semester. You should plan to spend 75 minutes solving problems for each class meeting.

**Comments on problem-solving**: You should approach each problem as an exploration. Draw a picture whenever appropriate. It is important that you work on each problem when assigned, since the questions you may have about a problem will likely motivate class discussion the next day. Problem-solving requires persistence as much as it requires ingenuity. When you get stuck, or solve a problem incorrectly, back up and start over. Keep in mind that you're probably not the only one who is stuck, and that may even include your teacher. If you have taken the time to think about a problem, you should bring to class a written record of your efforts, not just a blank space in your notebook. The methods that you use to solve a problem, the corrections that you make in your approach, the means by which you test the validity of your solutions, and your ability to communicate ideas are just as important as getting the correct answer.

**Real Analysis**

Below is a map of the ideas in this course, and how they connect, from the basic ideas at the bottom to the course goals at the top. An arrow goes from $A$ to $B$ if we need the ideas from $A$ in order to understand $B$. I made this chart when I was constructing our curriculum.

- Circle topics that you feel you understand well.
- Periodically come back to this chart and circle topics as you master them.



# Real Analysis

in class

sets / FM **Notation.**

- A *set* is a notion that we won't define, because any definition would end up using a word like "collection," which we'd then need to define. We'll just assume that we understand what is meant by a *set*, and let this notion of a set be fundamental.
- We use a capital letter to denote a set, e.g. "Let $S$ be the set of even numbers."
- The symbol $\in$ means "is/be an element of," and $\notin$ means "is not an element of."
- We use a lower-case letter to denote an element of a set, e.g. "Let $s \in S$."
- To describe the elements of a set, use curly braces $\{\}$. For example, $S = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ or $S = \{x : x \text{ is an even number}\}$. The colon ":" means "such that," so that the latter set is read aloud as "$S$ is the set of $x$ such that $x$ is an even number."

Note that the statement $S = \{x : \exists n \in \mathbf{Z} : x = 2n\}$ is equivalent to, but more obfuscating than, both definitions for $S$ given above. Be kind to your reader as much as possible

sets / DD **1**. Let $A = \{1, 2, 3, 4\}$. Which of the following are true statements?

**(a)** $3 \in A$ **(b)** $\{3\} \in A$ **(c)** $5 \in A$ **(d)** $2 \in a$ **(e)** $2 \notin A$

sets / FM **Talking about sets.**

- $X \subset Y$ is read "$X$ is a subset of $Y$," and means that every $x$ in $X$ is also in $Y$:

$$x \in X \implies x \in Y.$$

- An equivalent notation to $X \subset Y$ is $X \subseteq Y$. If one wants to specify that $X \neq Y$, one can write $X \subsetneq Y$. Otherwise, $X \subset Y$ allows for the possibility that $X = Y$.

sets / DD **2**. Let $S$ and $A$ be as above, and let $B = \{1, 2, 3, 4, 5, 6\}$. Which are true? Explain.

**(a)** $A \subset B$ **(b)** $B \subset A$ **(c)** $A \in B$ **(d)** $A \subset S$ **(e)** $S \subset A$

sets / FM **Useful sets.**

- The *empty set* $\varnothing$, the set consisting of no elements.
- The *natural numbers* $\mathbf{N} = \{1, 2, 3, \ldots\}$. In Europe, $\mathbf{N}$ starts with 0.
- The *integers* $\mathbf{Z} = \{-3, -2, -1, -0, 1, 2, 3, \ldots\}$, from the German *zahl* for number.
- The *rationals* $\mathbf{Q} = \{p/q \text{ in lowest terms} : p \in \mathbf{Z}, q \in \mathbf{N}\}$, from *quotient* $= \{\text{repeating or terminating decimals}\}$.
- The *reals* $\mathbf{R} = \{\text{all decimals}\}$, with the understanding that $0.999\ldots = 1$, etc.

Note that these symbols are typeset as $\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}$ and written by hand as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

sets / DD **3**. Let $A$, $B$ and $S$ be as above. Which of the following are true? Explain.

**(a)** $B \subset \mathbf{N}$ **(b)** $S \subset \mathbf{Z}$ **(c)** $\mathbf{Q} \subset \mathbf{R}$ **(d)** $\mathbf{Z} \subset \mathbf{N}$ **(e)** $\varnothing \subset \mathbf{N}$ **(f)** $\varnothing \in A$

# Real Analysis

**The mathematical "or."** In mathematics, "or" means one, or the other, *or both.*

- Shall we meet to do Real Analysis on Monday *or* Thursday? Both!
- To satisfy PEA's math requirement, a student must have passed a course numbered 330 *or above.*

logic / FM **Implication.** There are many ways to say that one statement $A$ implies another statement $B$. The following all mean exactly the same thing:

- If $A$, then $B$.
- $A$ implies $B$ (written $A \implies B$).
- $A$ only if $B$.
- $B$ if $A$ (written $B \impliedby A$).
- not $B$ implies not $A$. (This is the *contrapositive*)

logic / DD **4**. Let statement $A$ be "Max has a valid driver's license in New Hampshire," and let statement $B$ be "Max is over age 16."

**(a)** Write out the five implications above, using these statements.

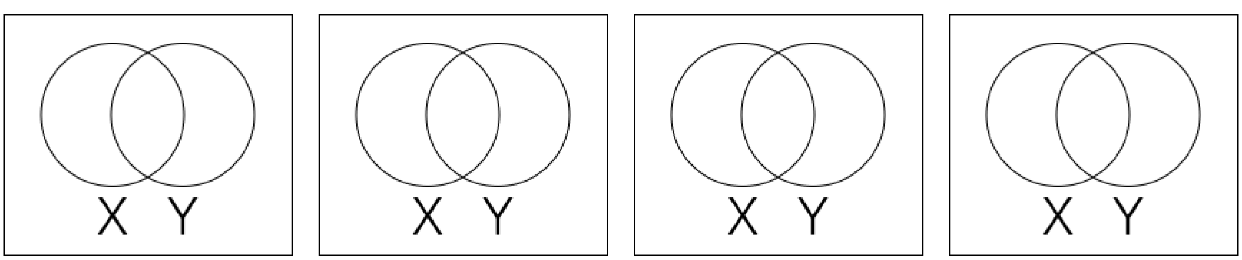
**(b)** Considering this example, do you agree that they are all logically equivalent?

**(c)** The *converse* is $B \implies A$. Is the converse true in this case?

sets / FM **Working with sets.**

- The *intersection* $X \cap Y$ of two sets $X$ and $Y$ is the set of all elements that are in $X$ *and* in $Y$: $X \cap Y = \{x : x \in X \text{ and } x \in Y\}$.
- The *union* $X \cup Y$ of two sets $X$ and $Y$ is the set of all elements that are in $X$ *or* in $Y$: $X \cup Y = \{x : x \in X \text{ or } x \in Y\}$.
- The *complement* $X^C$ of a set $X$ is the set of points *not* in $X$: $X^C = \{x : x \notin X\}$. For this to make sense, the "universal set" that $X$ lives in must be understood.
- The set $X - Y$, or $X \setminus Y$, is the set of all points *in* $X$ that are *not* in $Y$.

sets / DD **5**. Shade the regions corresponding to $X \cap Y$, $X \cup Y$, $X^C$, and $X - Y$, respectively.




sets / FM **6**. Let $X$ be a subset of a universal set $U$, and let $X$ and $Y$ be subsets of $U$. Simplify:

**(a)** $(X \cup Y) \cap (U - X) \cap X$ **(b)** $X \cup (Y \cap X^C)$ **(c)** $(X \cap Y) \cup (X \cap Y^C)$

logic / FM

**Converse and logical equivalence.** The *converse* of the statement "$A$ implies $B$" is the statement "$B$ implies $A$." If a statement and its converse are both true, we say $A$ and $B$ are *logically equivalent*, or in other words $A \iff B$, or in other words "$A$ if and only if $B$," sometimes abbreviated as "$A$ iff $B$."

logic / DD

**1**. Let statement $A$ be "Alex is eligible to vote in the United States" and let statement $B$ be "Alex is a United States citizen." Write out the implication $A \implies B$, its contrapositive, and its converse. Which of these implications are true?

logic / DD

**2**. The lyrics to "This is why I'm hot" by Mims are shown to the right. Does the lyric "I'm hot 'cause I'm fly, you ain't 'cause you not" imply that the notions of "hot" and "fly" are logically equivalent?

[CHORUS]
This is why I'm hot
This is why I'm hot
This is why, this is why,this is why I'm hot
This is why I'm hot
This is why I'm hot
This is why, this is why, this is why I'm hot
I'm hot 'cause I'm fly, you ain't cause you not
This is why, this is why, this is why I'm hot
I'm hot 'cause I'm fly, you ain't cause you not
This is why, this is why, this is why I'm hot

metric / DD

**Metrics.** A *metric* on a set $E$ is a rule that assigns, to each pair $p, q \in E$, a real number $d(p,q)$, called the *distance function*, which is a function $d : E \times E \to \mathbf{R}$, such that:

1. $d(p,q) \geq 0$ for all $p, q \in E$,
2. $d(p,q) = 0$ if and only if $p = q$, NOTE: There are two statements here.
3. $d(p,q) = d(q,p)$, (*symmetry*)
4. $d(p,r) + d(r,q) \geq d(p,q)$ for any $p, q, r \in E$. (the *triangle inequality*)

metric / DD

**3**. Show that each of the following is a metric on $\mathbf{R}^2$:

**(a)** The standard Euclidean metric (i.e. using the Pythagorean theorem),

**(b)** The "Taxicab metric": $d((a,b),(c,d)) = |c-a| + |d-b|$ (also explain the name).

limits / FM

**4**. Consider the following sequences: $1, 1/2, 1/3, 1/4, 1/5, \ldots$ $\quad 3, 1, 4, 1, 5, 9, \ldots$
$1, 0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots$ $\quad 2.1, 2.01, 2.001, 2.0001, \ldots$

**(a)** Which of the sequences converge, and to what limit?

**(b)** Come up with a definition: a sequence *converges* to a *limit* $p$ if...

(Don't look it up; you'll work with the precise definition in your homework. The purpose of this problem is to try to define it, and to see that writing definitions is tricky.)

sets / FM

**5**. (if time) Find infinitely many nonempty sets $S_1, S_2, \ldots$ of natural numbers such that

$$\mathbf{N} \supset S_1 \supset S_2 \supset S_3 \cdots$$

and $\displaystyle\bigcap_{n=1}^{\infty} S_n = \emptyset$. Here the symbol $\displaystyle\bigcap_{n=1}^{\infty} S_n$ means $S_1 \cap S_2 \cap \cdots$, and is used to take the intersection of infinitely many sets.

# Real Analysis

**Metric spaces.** A *metric space* is a set $E$, together with a *metric* $d(p, q)$ that gives the distance between any two points $p, q \in E$.

metric / FM **1**. Show that the following are metric spaces:

**(a)** The set $\mathbf{R}^n$, with the metric

$$d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \max\{|y_1 - x_1|, \ldots, |y_n - x_n|\}.$$

**(b)** Any set $E$, with the *discrete metric* $d(x, y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$ for $x, y \in E$.

**Limits and convergence.** Let $p_1, p_2, \ldots$ be a sequence of points in a metric space $E$.

- A point $p \in E$ is a *limit* of $\{p_i\}_{i=1}^{\infty}$ if, for any $\epsilon > 0$, there exists $N > 0$ such that, whenever $i > N$, $d(p, p_i) < \epsilon$.
- In such a situation, we say that the sequence $p_i$ *converges to* $p$, and write $\lim\limits_{i \to \infty} p_i = p$.

limits / DD **2**. I like to think of sequence convergence as a competition between me and an interlocutor:

**ME:** Consider -1, +1/2, -1/3, +1/4, -1/5, . . . . I claim that this sequence converges to 0.

**INTERLOCUTOR:** Nonsense! Show me that the terms get within 0.1 of 0.

**ME:** Okay, take $k > 10$. After that, the terms are all closer than 0.1.

**INTERLOCUTOR:** Hmm! Now show me that the terms get within 0.0001 of 0.

**ME:** Okay, then take $k > 10000$. After that, the terms are all closer than 0.0001.

**INTERLOCUTOR:** Hmm. Show me that, for any $\epsilon > 0$ that I might ever suggest, the terms get within $\epsilon$ of 0.

**ME:** Okay, take $k >$ ____________, and after that the terms are all closer than $\epsilon$.

limits / DD **3**. Prove, from the definition (i.e. by finding an $N$ that depends on the given $\epsilon$, sometimes called $N(\epsilon)$ to emphasize this dependence), that the sequence $p_n = 1000/n^3$ converges to 0.

**Open balls.** Given a metric space $E$, a point $p_0 \in E$, and a real number $r > 0$, the *open ball* in $E$ with center $p_0$ and radius $r$ is $B_r(p_0) = \{p \in E : d(p_0, p) < r\}$.

open / DD **4**. Sketch the following open balls.

**(a)** In $\mathbf{R}$, the set $B_1(0)$. (Shade the included interval, with open circles for endpoints.)

**(b)** In $\mathbf{R}^2$, the set $B_{1/2}(1, 1)$. (Shade the included region, with a dashed boundary curve.)

open / DD **5**. Express the open interval $(1, 2) \in \mathbf{R}$ as an open ball as above. Then do the same for the general open interval $(a, b)$.

func / FM **Functions.**

- A *function* from $X$ to $Y$ is a rule that assigns, to each $x \in X$, exactly one $y \in Y$.
- We write $f : X \to Y$, and if $f(x) = y$, we write $x \mapsto y$ which is read "$x$ maps to $y$."
- If $f$ maps distinct points to distinct values, then $f$ is called *one-to-one* or *injective*. Equivalently, $f$ is injective if $f(x) = f(y)$ implies that $x = y$.
- $X$ is called the *domain* of $f$, and $Y$ is the *codomain* of $f$.
- The set of all outputs $f(X) = \{f(x) : x \in X\}$ is called the *image* of $f$. If the image is the entire codomain, $f$ is called *onto* or *surjective*. Equivalently, $f$ is surjective if, for each $y \in Y$, there exists an $x \in X$ such that $f(x) = y$.
- A function that is both injective and surjective is called *bijective*.

func / FM **6**. For each of the following, say whether it is injective, surjective, or both (bijective):

**(a)** $f(x) = -x$ **(b)** $f(x) = x^2$ **(c)** $f(x) = \sin x$ **(d)** $f(x) = e^x$ **(e)** $f(x) = x^3 + x^2$.

logic / FM **True and false.** An implication $A \implies B$ is true if $B$ is true, or if $A$ is false (in which case we say that the implication is "vacuously true.") For example, the statement "If 5 is even, then 15 is prime" is vacuously true. An implication is false *only* if $A$ is true and $B$ is false.

logic / FM **7**. Is the statement:

$$\text{If } x \in \mathbf{Q}, \text{ then } x^2 \in \mathbf{N}$$

true or false for the following values of $x$? Justify your answers.

**(a)** $x = 1/2$ **(b)** $x = 2$ **(c)** $x = \sqrt{2}$ **(d)** $x = \sqrt[4]{2}$

logic / DD **8**. Vacuously true statements can be used to make hilarious jokes, with other people who also understand vacuously true statements. Ruin each of the following hilarious jokes by writing each one in the form *if A, then B*:

- Every car I own is a Maserati.
- I've gotten As in all of my Sanskrit courses.
- Swarthmore Football, undefeated since 2000.

# Real Analysis

**Infinite sets.** A set is *countable* if its elements can be listed.
More precisely, a set is *countable* if it is finite, or if its elements can be put in one-to-one correspondence with the natural numbers. Otherwise, the set is called *uncountable*.

count / DD **1**. Show that the set of even natural numbers is countable, by:

**(a)** Showing how to systematically list them;

**(b)** Explicitly constructing a bijective function from $\mathbf{N}$ to the even numbers.

count / DD **2**. Write a proof that the even numbers are countable, using your function from 1(b). The purpose of this problem is to practice constructing a clear, rigorous proof. Do this by filling in the following. In your notebook, write down the entire proof, not just the blanks.

*Proof.* We will show that __________________________________.
We will do this by constructing ______________________________,
and showing that it ____________________________________.
Let $\mathbf{N}$ be the set of natural numbers, and let $S$ be the set of even numbers.
Define $f : \mathbf{N} \to S$ by $f(x) =$ ________ for each $x \in \mathbf{N}$.
First, we will show that $f$ is injective. Suppose that $f(x) = f(y)$. Then ________
______________________________, so $x = y$, as desired.
Now, we will show that $f$ is surjective. Let $x \in S$. Then ________
________________________, so $x = f(n)$ for some $n \in \mathbf{N}$, as desired.
Thus $f$ is injective and surjective, so $f$ is bijective, so there is a bijective function from $\mathbf{N}$ to the even numbers, so ______________________________, as desired.

count / DD **3**. This result seems to be a contradiction: the set of even numbers seems to be a smaller set than $\mathbf{N}$ (half as big!), and yet the two sets have the same size. Explain.

**Open sets.** A set is *open* if there is an open ball around every point.
More precisely, a subset $S$ of a metric space $E$ is *open* if, for each $p \in S$, there exists an $r > 0$ such that $B_r(p) \subset S$.

open / DD **4**. Prove that the empty set is open.

open / DD **5**. Let $S \subset R$ be defined by $S = [0, \infty)$. Show that $S$ is *not* an open set. *Hint*: find a point in $S$ about which there is *no* open ball that is completely contained in $S$.

open / DD **6**. Prove that, for any metric space $E$, the entire space $E$ is an open set.

open / DD **7**. Let $E = [0, \infty)$. Prove that, in $E$, $E$ is open.

## Real Analysis

count / DD **1**. Show that the integers $\mathbf{Z}$ are countable.

seq / DD **2**. For each of the following sequences, say whether it converges or diverges. For those that converge, prove that it converges by finding the limit $p$ and also, given any $\epsilon > 0$, an $N(\epsilon)$.

**(a)** $a_n = \dfrac{\sin n}{n}$ **(b)** $b_n = 1 + (-1)^n$ **(c)** $1, 0, 1/2, 0, 1/4, 0, 1/8, 0, \ldots$

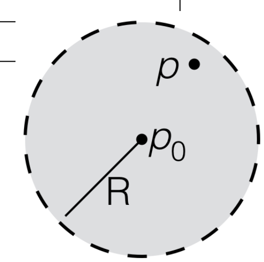
**Precise notions to bound sets.** Let $A$ be a nonempty set of real numbers.

- A real number $u$ is an *upper bound* for $A$ if $a \leq u$ for all $a \in A$.
- A real number $l$ is a *lower bound* for $A$ if $l \leq a$ for all $a \in A$.
- A set is *bounded* if it has both an upper and a lower bound.
- A real number $s$ is the *supremum* ("soo-PREE-mum") or *least upper bound* of $A$ if $s$ is an upper bound for $A$, and $s \leq u$ for any other upper bound $u$ of $A$. The supremum is denoted $\sup(A)$, pronounced "soup $A$," or l.u.b.$(A)$.
- A real number $t$ is the *infimum* ("in-FEE-mum") or *greatest lower bound* of $A$ if $t$ is a lower bound for $A$, and $l \leq t$ for any other upper bound $l$ of $A$. The infimum is denoted $\inf(A)$ or g.l.b.$(A)$.
- A real number $m$ is the *maximum* of $A$ if $m \in A$ and $a \leq m$ for all $a \in A$.
- A real number $n$ is the *minimum* of $A$ if $n \in A$ and $n \leq a$ for all $a \in A$.

Note that, if you are just trying to show that a set is bounded, a super big bound like 1000 works just as well as a bound like 1. There is no need to do extra work to find a tight bound.

bound / AJ **3**. Complete the following table by filling in each box with a number, the letters DNE for "does not exist," or the word "Yes" or "No." Be prepared to justify your answers.

| Set | L.B. | U.B. | min | max | sup | inf | is sup in set? | set bounded? |
|---|---|---|---|---|---|---|---|---|
| $\{x \in \mathbf{R} : 0 \leq x < 1\}$ | | | | | | | | |
| $\{x \in \mathbf{R} : 0 \leq x \leq 1\}$ | | | | | | | | |
| $\{x \in \mathbf{R} : 0 < x < 1\}$ | | | | | | | | |
| $\{1/n : n \in \mathbf{Z} \setminus \{0\}\}$ | | | | | | | | |
| $\{1/n : n \in \mathbf{N}\}$ | | | | | | | | |
| $\{x \in \mathbf{R} : x < \sqrt{2}\}$ | | | | | | | | |
| $\{1, 4, 9, 16, 25\}$ | | | | | | | | |
| $\{(-1)^n(2 - 1/n) : n \in \mathbf{N}\}$ | | | | | | | | |
| $\{\ln(x) : x \in \mathbf{R}, x > 0\}$ | | | | | | | | |
| $\{e^x : x \in \mathbf{R}\}$ | | | | | | | | |




open / DD **4**. Prove that any open ball $B_R(p_0)$ in any metric space $E$ is an open set.
*Hint*: for any point $p$ in the ball, find its distance to the boundary of the ball using $p_0$ and $R$, and choose a smaller number as the radius of the open ball around $p$.

# Real Analysis

bound / AJ **1**. For each of the following statements, either say it is true and explain why, or say it is false and provide a counterexample. *Hint*: Consider the examples from Page 5 # 3.

**(a)** Every set has a maximum.

**(b)** Every set has a minimum.

**(c)** If a set is bounded, then it has a supremum.

**(d)** If a set is bounded, then it has an infimum.

**(e)** If a set has an infimum, then it is bounded below.

**(f)** If a set has a supremum, then it is bounded above.

**(g)** If a set is bounded, then it has both a maximum and a minimum.

**(h)** If a set has a maximum, then it is bounded above.

**(i)** If a set is bounded above, then it has a maximum.

**Negating a statement.** The contrapositive of "$A \implies B$" is "not $B \implies$ not $A$." The statement "not $A$" is the *negation* of statement $A$. I think of this as someone saying "$A$!" and someone replying "No, you're wrong, (negation of $A$)!" For example:

**Person 1:** Everyone in this class is named Max.
**Person 2:** You're wrong! Not everyone in this class is named Max. (true, but not useful)
**Person 1:** How do you know?
**Person 2:** There exists a person in this class not named Max. (checkable! useful!)

logic / DD **2**. Negate the following statements in a checkable, useful manner.

**(a)** All U.S. citizens can vote.

**(b)** Every point of $S$ has a ball around it.

**(c)** Some ball around $p$ contains a point of $S$.

**(d)** One of my classes meets on Saturday.

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**(e)** Notice that the negation of an "all" or "none" statement is an existence statement (parts (a) and (b)), while the negation of an existence statement is an "all" or "none" statement (parts (c) and (d)). Explain why this is the case.

# Real Analysis

**The boundary, interior and closure.** Let $S$ be a subset of a metric space $E$. A point $p \in E$ is a *boundary point* of $S$ if every open ball about $p$ contains points of $S$ and points of $S^C$. The *boundary* of $S$, denoted $\partial S$, is the collection of all of the boundary points of $S$. The *closure* of $S$, denoted $\overline{S}$, is $S \cup \partial S$. The *interior* of $S$, denoted $\overset{\circ}{S}$, is $S \setminus \partial S$.

open / DD **1**. Find $\partial S, \overline{S}$ and $\overset{\circ}{S}$ for each set $S$ that is a subset of the given metric space, with the standard Euclidean metric:

**(a)** $(0,1] \subset \mathbf{R}$ **(b)** $\mathbf{Z} \subset \mathbf{R}$ **(c)** $\mathbf{Q} \subset \mathbf{R}$ **(d)** $B_1(0,0) \subset \mathbf{R}^2$

open / FM **2**. Prove that every point in a set is either a boundary point or an interior point.

The following is an example of a *proof by contradiction*: We begin by supposing the *opposite* of what we want to prove, and we show that it leads to a contradiction (something that is clearly false). This shows that the thing we initially supposed was *false*, which shows that the thing we want to prove is *true*. A proof by contradiction takes the following form:

*Proof.* We will show $A$.
Suppose not $A$.
[Steps of logical reasoning.]
Therefore, not-$A$ is false, so $A$ is true.

seq / FM **3**. **Theorem.** A sequence $\{p_i\}_{i=1}^{\infty}$ of points in a metric space $E$ has at most one limit.

*Proof.* We will show that a sequence of points in a metric space $E$ has at most one limit. We will do this by contradiction, by supposing that it has two different limits, and showing that the two limits must be the same, by showing that the distance between them is 0.

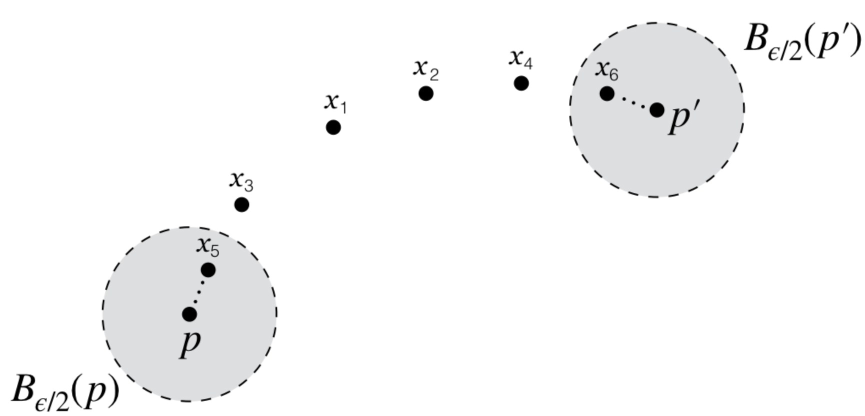
Suppose that the sequence $\{p_i\}_{i=1}^{\infty}$ in metric space $E$ has two different limits, $p$ and $p'$. By definition of $p$ and $p'$ each being a limit point, we know that:

Given any $\epsilon/2 > 0$, there exists $N$ such that $d(p, p_i) < \epsilon/2$ for all $i > N$, and

given any $\epsilon/2 > 0$, there exists $M$ such that $d(p', p_i) < \epsilon/2$ for all $i > M$.

Given $\epsilon > 0$, choose a number $n > \max\{N, M\}$.

(Finish the proof.)
*Hint*: Look at the picture.




logic / DD **4**. We used $\epsilon/2$ to find $N$ and $M$ so that the bound comes out cleanly to $\epsilon$ at the end, but it would also have been fine to use $\epsilon$ and come out with a bound of $2\epsilon$ at the end. Explain why a bound of $2\epsilon$ would also prove that the two limit points coincide.

**Bounded sets.** A set is *bounded* if it is contained in a finite ball.
More precisely, a subset $S$ of a metric space $E$ is *bounded* if there exists $p \in E$ and $r > 0$ such that $S \subset B_r(p)$.

seq / DD

**5**. In $\mathbf{R}$ with the usual Euclidean metric, show that the set $\{1000 - 500/n^2 : n \in \mathbf{N}\}$ is bounded by finding a suitable $p$ and $r$.

open / FM

**6**. **Theorem.** The union of *any* collection of open sets is open.

*Note:* In order to even talk about this, we need a way of indexing an arbitrary collection of sets. We can't use $\bigcup_{n=1}^{k}$ or $\bigcup_{n=1}^{\infty}$, because the collection might be uncountable. The way we get around this is to use the symbol $\alpha$, which represents an arbitrary indexing set: $\bigcup_{\alpha} A_\alpha$.

*Proof.* We will show that ________________________________________.

We will do this by showing that, for any $p$ in the union, there is an open ball around $p$ contained in the union. Let $p \in \bigcup_{\alpha} A_\alpha$. Then for some $\alpha_0$, $p \in A_{\alpha_0}$, because ____________________.

Since $A_{\alpha_0}$ is open, …(complete the proof).

open / FM

**7**. **Theorem.** The intersection of a *finite* number of open sets is open.

*Proof.* We will show that ________________________________________.
We will do this by showing that, for any $p$ in the intersection, there is an open ball around $p$ contained in the intersection. Let $p \in \bigcap_{k=1}^{n} A_k$. Then $p \in A_k$ for all $1 \le k \le n$, because

__________________________________________________.

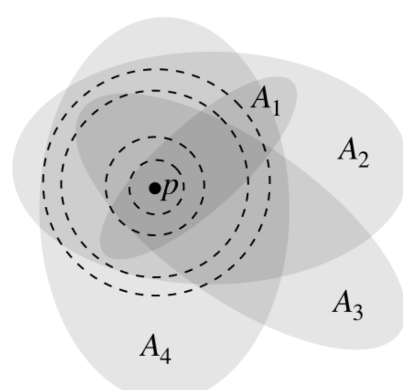
Thus, there exist $r_k \in \mathbf{R}$ such that $B_{r_k}(p) \subset A_k$ for each $k$.
(Complete the proof.) *Hint*: See picture.

open / DD

**8**. Consider the (false!) statement: *The intersection of infinitely many open sets is open.*

**(a)** Explain where the proof above breaks down for infinitely many sets.

**(b)** Give a counterexample to the statement.



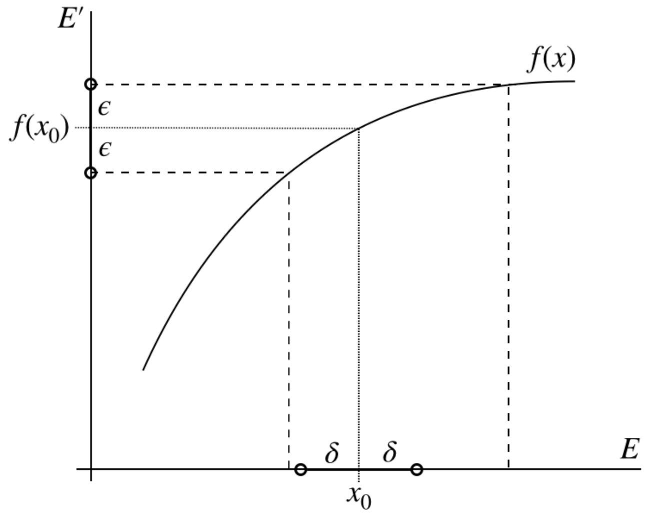
**Continuity.** "Nearby points are sent to nearby points." There are three (!) equivalent definitions of what it means for a function to be *continuous*. We will explore each of them. Then we will prove their equivalence.

*The $\epsilon$–$\delta$ definition of continuity.*

Let $E, E'$ be metric spaces with distance metrics $d, d'$ respectively. Let $f : E \to E'$ be a function, and let $x_0 \in E$. We say that $f$ is *continuous at* $x_0$ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for $x \in E$,

$$d(x, x_0) < \delta \implies d'(f(x), f(x_0)) < \epsilon.$$

We say that $f$ is *continuous* if it is continuous everywhere in $E$.




*Note:* $\delta$ may depend on (be a function of) both $x_0$ and $\epsilon$, because in the definition of continuity, we choose $x_0$ first, then we are given $\epsilon$, and finally we get to choose $\delta$ to make it work.

cont / DD **1**. Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = 1 + 2x$. Explore the epsilon-delta definition of continuity by finding a $\delta$ for each given $x_0$ and $\epsilon$ so that, for any $x \in \mathbf{R}$,

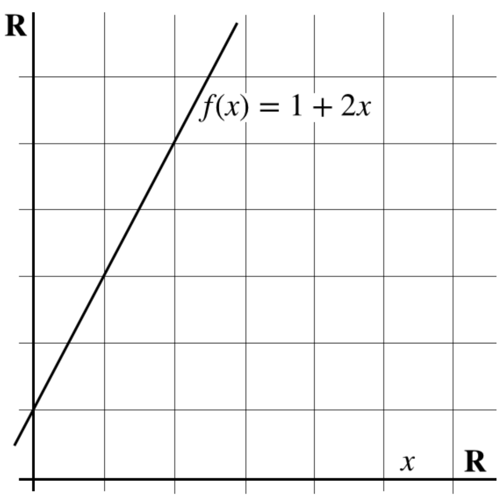
$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon.$$

**(a)** $x_0 = 1, \epsilon = 1$

**(b)** $x_0 = 1, \epsilon = 0.1$

**(c)** $x_0 = 2, \epsilon = 0.001$

*Hint*: Draw in dashed lines as in the figure above.




seq / FM **2**. Prove that "the limit of a sum is the sum of the limits": If $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers, with $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$, then $\lim\limits_{n\to\infty} (a_n + b_n) = a + b$.
*Hint*: At the end, you will want to show that $|(a_n + b_n) - (a + b)| < \epsilon$, so break it into two parts using rules of absolute values, and take $n$ large enough that certain quantities are less than $\epsilon/2$.

seq / DD **3**. Explain why, in the previous problem, we took $\{a_n\}$ and $\{b_n\}$ to be sequences of real numbers, rather than just sequences in an arbitrary metric space.

# Real Analysis

bound / AJ **4**. In each of the following pairs, exactly one of the statements is true. For the one that is true, explain why; for the one that is false, provide a counterexample.

(A1) If a set of real numbers is bounded above, then it has a maximum.

(A2) If a set of real numbers is bounded above, then it has a supremum.

(B1) If a set of real numbers has a supremum, then it has a maximum.

(B2) If a set of real numbers has a maximum, then it has a supremum.

(C1) If a set of real numbers is has an infimum, then the infimum is in the set.

(C2) If a set of real numbers is has a minimum, then the minimum is in the set.

**Closed sets.** A subset $S$ of a metric space $E$ is *closed* if $S^C$ is open.

A *closed ball* in a metric space $E$, with center $p_0$ and radius $r$, is the set $\{p \in E : d(p, p_0) \leq r\}$, or in other words $\overline{B_r(p_0)}$.

closed / DD **5**. Prove that the empty set is closed.

closed / DD **6**. Prove that, for any metric space $E$, the entire space $E$ is closed.

closed / DD **7**. We have now proved that the empty set is both open and closed, and also that any entire space $E$ is both open and closed. Are these contradictions? Explain.

If you have extra time, consider the following problem asked by a former student:

metric / DD **8**. (Rick Muniu) Give an example of a distance function on a metric space that satisfies the triangle inequality, but fails to satisfy one of the other three properties of a metric.

Also remember to write up to hand in your proof of the result in Page 4 # 6.

# Real Analysis

**Bounded sequences.** A sequence of points $\{p_i\}_{i=1}^{\infty}$ in a metric space is *bounded* if it is bounded as a set, i.e. if it is contained in a ball.

seq / DD **1. Theorem.** Every convergent sequence is bounded.

*Proof.* We will show that every convergent sequence is bounded. We will do this by constructing a ball that contains all of the points of the sequence. Take $\epsilon = 1$. Then there exists $N$ such that, for all $n > N$, $d(p_n, p) < 1$, because ____________________________________.
Now take $r = \max\{1, d(p, p_1), d(p, p_2), \ldots, d(p, p_N)\}$.
(Finish the proof).

seq / DD **2.** State the converse (note: converse, not contrapositive) of the theorem in Problem 1. Then either prove it or give a counterexample.

cont / AJ **3.** Let's explore the *epsilon-delta definition of continuity.* For each function $f : \mathbf{R} \to \mathbf{R}$, compute $f(x_0)$ and draw a sketch of $f(x)$ in the vicinity of $x_0$. In the next columns, write out the "allowable output range" $(f(x_0) - \epsilon, f(x_0) + \epsilon)$ for the given $x_0$ and $\epsilon$, and then write out the corresponding "permissible input range" $(x_0 - \delta, x_0 + \delta)$, if it exists, for each value of $\epsilon$ (in the same box). Finally, say if $f$ is continuous at $x_0$.

| function and $x_0$ value | $f(x_0)$ | sketch | $\varepsilon = 1$ | $\varepsilon = 0.1$ | cont? |
|---|---|---|---|---|---|
| $f_1(x) = \|x\|$, $x_0 = 0$ | | | | | |
| $f_2(x) = \begin{cases} x & x \leq 1 \\ 2x - 0.5 & x > 1 \end{cases}$, $x_0 = 1$ | | | | | |
| $f_3(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q} \end{cases}$, $x_0 = 0$ | | | | | |
| $f_4(x) = \begin{cases} 1/x & x \neq 0 \\ 2 & x = 0 \end{cases}$, $x_0 = 0$ | | | | | |
| $f_5(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$, $x_0 = 0$ | | | | | |
| $f_6(x) = \begin{cases} x \cdot \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$, $x_0 = 0$ | | | | | |

cont / AJ **4.** When you *couldn't* find a $\delta$ for a particular $\epsilon$, why not? When you *could* find a $\delta$ for a particular $\epsilon$, was $\delta$ unique? If not, could you find a maximum value for $\delta$? A minimum?

# Real Analysis

**Continuity.** "Nearby points are sent to nearby points." As stated before, there are three equivalent definitions of what it means for a function to be *continuous*, which we will later prove are equivalent. This is the second one.

*The sequence definition of continuity.*

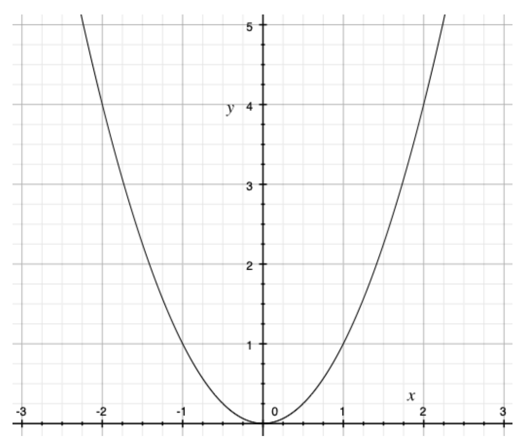
Let $E, E'$ be metric spaces with distance metrics $d, d'$ respectively. Let $f : E \to E'$ be a function, and let $x_0 \in E$.

We say that $f$ is *continuous at* $x_0$ if, for every sequence $\{x_n\} \subset E$ with $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

As before, we say that $f$ is *continuous* if it is continuous everywhere in $E$.

seq / DD **5**. Use the sequence definition of continuity to show that the function $f_2$ from the table earlier in this problem set is not continuous.

**Inverses.**




- If $f : X \to Y$ is one-to-one and onto (bijective), then we define the *inverse* of $f$ to be the function $f^{-1} : Y \to X$, defined such that $f^{-1}(y) = x$ when $f(x) = y$.
- We define the *image* of a set $A \subset X$ as the collection of images of points in $A$, $f(A) = \{f(a) : a \in A\}$.
- No matter if $f$ is bijective or not, we define the *inverse image* of a set $B \subset Y$ as the collection of points in $X$ that map to points in $B$:

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

func / DD **6**. Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x^2$, shown above. Find each of the following, or say why it is not possible:

**(a)** $f\big([-2,-1]\big)$ **(b)** $f^{-1}\big((0,2)\big)$ **(c)** $f^{-1}\big((-2,-1)\big)$ **(d)** $f^{-1}(x)$ for some $x \in \mathbf{R}$

If you have extra time, you can think about these, which will do in class soon:

seq / DD **7**. Let $\{a_i\}$, $\{b_i\}$ be convergent sequences of real numbers under the standard Euclidean metric, with limits $a$ and $b$ respectively. Prove that if $a_i \le b_i$ for all $i$, then $a \le b$.

seq / DD **8**. Prove or give a counterexample for the statement above, with "$\le$" replaced by "$<$."

# Real Analysis

bound / DD **1**. Give an example of each of the following:

**(a)** Sets $A \subset B$ with $\sup(A) < \sup(B)$;

**(b)** Sets $A \subsetneq B$ for which $\sup(A) = \sup(B)$.

**Monotonicity.** We use the following terms to describe sequences of real numbers:

- A sequence $\{a_i\}_{i=1}^{\infty}$ is *increasing* if $a_1 \leq a_2 \leq a_3 \leq \dots$
- A sequence $\{a_i\}_{i=1}^{\infty}$ is *decreasing* if $a_1 \geq a_2 \geq a_3 \geq \dots$
- A sequence is *monotone* if it is either increasing or decreasing.

seq / DD **2**. Prove or give a counterexample: Every convergent sequence of real numbers is monotone.

open / FM **3**. Prove that, for any set $S$, the interior of $S$ is an open set.

open / FM **4**. Prove that, for any set $S$, the interior of $S$ is the largest open set contained in $S$.

closed / DD **5**. Prove that a closed ball is a closed set: for any metric space $E$, any point $p_0 \in E$, and any radius $R > 0$, the closed ball $\overline{B_R(p_0)}$ is closed.

# Real Analysis

*Review for Test 1 – optional problems*

vocab / FM **1**. Write the definition of each term, as a full sentence: **(a)** subset **(b)** countable **(c)** supremum **(d)** minimum **(e)** open **(f)** closed **(g)** metric space **(h)** bounded **(i)** limit **(j)** converge **(k)** lim inf **(l)** monotone **(m)** boundary **(n)** interior **(p)** closure **(q)** continuous **(r)** inverse image

vocab / FM **2**. For each term in problem 1, write down a result (Theorem, etc.) that uses it.

sets / FM **3**. Consider the statement "Every **finite union** of **open** sets is **open**." For each entry in the following table, replace "finite union" and "open" with the other words as indicated, and decide whether the resulting statement is True or False. If it is false, give a counterexample.

| set property | finite $\cup$ | countable $\cup$ | arbitrary $\cup$ | finite $\cap$ | countable $\cap$ | arbitrary $\cap$ |
|---|---|---|---|---|---|---|
| open | | | | | | |
| closed | | | | | | |
| countable | | | | | | |
| uncountable | | | | | | |
| bounded | | | | | | |

sets / FM **4**. Let $A$ be a subset of a metric space $E$.

**(a)** Define the *boundary* of $A$.

**(b)** Define what it means for $A$ to be *open*.

**(c)** Prove that $A$ is open if and only if $A$ contains none of its boundary, i.e. $A \cap \partial A = \emptyset$.

seq / FM **5**. Prove that if $a_n \leq b_n \leq c_n$ for each $n \in \mathbf{N}$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

seq / FM **6**. Is $\mathbf{Q}$ open in $\mathbf{R}$? Justify your answer.

/ DD **7**. In the topic dependence map on Page iii, circle the topics in which you are confident.

# Real Analysis

**Covers.** Given a set $S$, a collection of sets $\{G_\alpha\}$ is a *cover* of $S$ if $S \subset \bigcup_\alpha G_\alpha$.

$\{G_\alpha\}$ is an *open cover* if all of the $G_\alpha$ are open sets.

*Note:* recall (Page 7 # 6) that we use the symbol $\alpha$ to index an arbitrary indexing set, which may not be countable. A picture of an open cover of $\mathbf{R}^2$ by filled ellipses is in the background of the cover (ha!) of *Real Analysis* by Frank Morgan.

cpt / DD **1**. Construct an open cover that:

**(a)** covers $\mathbf{R}$, using unit intervals;

**(b)** covers $\mathbf{R}^2$, using unit balls;

**(c)** covers $\{1/n : n \in \mathbf{N}\}$.

cont / DD **2**. The open set definition of continuity (just below) uses *inverse* images. First, let's think about *images*. Prove or give a counterexample: If $f : E \to E'$ is a continuous function, and $\mathcal{U}$ is open in $E$, then $f(\mathcal{U})$ is open in $E'$.

**Continuity.** Here is the third of the three equivalent definitions of what it means for a function to be *continuous*. We will prove their equivalence soon.

*The open set definition of continuity.*

Let $E, E'$ be metric spaces with distance metrics $d, d'$ respectively, and let $f : E \to E'$.

We say that $f$ is *continuous* if, for every open set $\mathcal{U} \subset E'$, $f^{-1}(\mathcal{U})$ is open in $E$.

cont / DD **3**. Use the open set definition of continuity to show that the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$ is continuous.

**More on open sets.** So far, we have used the "every point is contained in an open ball" characterization of an open set. The following Theorem gives an alternative characterization.

open / DD **4**. **Theorem.** Let $E$ be a metric space, and let $S$ be a subset of $E$, considered as a metric space itself. For a subset $A \subset S$, the following are equivalent:

(1) $A$ is open in $S$.

(2) There exists a set $\widetilde{A}$ that is open in $E$, such that $A = \widetilde{A} \cap S$.

Prove this. *Hint*: To prove that (1) $\implies$ (2), show that $\widetilde{A} \cap S \subset A$ and that $A \subset \widetilde{A} \cap S$.

*Note:* the symbol $\sim$ is "tilde," pronounced "TILL-duh," and $\widetilde{A}$ is read aloud as "$A$ tilde."

# Real Analysis

closed / DD **5**. Prove that the union of a finite number of closed sets is closed. *Hint*: First, argue that $\left(\bigcup_{k=1}^{n} A_i\right)^C = \bigcap_{k=1}^{n} A_i{}^C$. Then argue that, if each $A_i$ is closed, this intersection is open.

closed / DD **6**. Prove that the intersection of any collection of closed sets is closed.
*Hint*: apply a previous result.

An *axiom* is a statement that we take as fact, without proof, generally because it is impossible to prove it from our other axioms and yet it is necessary for the structure of our work. The following statement, which we have previously discussed, is an axiom:

**Completeness axiom.** A nonempty set of real numbers that is bounded from above has a least upper bound.

bound / DD **7**. **Theorem.** Let $S$ be a nonempty, closed subset of $\mathbf{R}$ that is bounded from above. Then $S$ has a maximum element.

*Proof.* We will show that any nonempty, closed subset $S \subset \mathbf{R}$ that is bounded from above has maximum element. We will do this by showing that the least upper bound $a$ of $S$ is contained in $S$. The proof will be by contradiction: we will first suppose that $a \notin S$, and derive a contradiction. (Fill in the reasoning steps in the following proof.)

Let $a$ be the least upper bound of $S$. We know that the least upper bound exists, by the completeness axiom. We want to show that $a \in S$. Suppose, for a contradiction, that $a \notin S$.
$S^C$ is open, because ________________________________________.
Thus there exists some $r > 0$, such that $B_r(a) \subset S^C$, because ________________________.
But then $a - r$ is also an upper bound for $S$, because __________________________.
This contradicts $a$ being the least upper bound for $S$, because ________________________.
Thus $a \in S$, and thus $S$ has a maximum element, as desired.

bound / DD **8**. (Continuation) The statement of the theorem contains the conditions that $S$ is a nonempty, closed subset of $R$. Give a counterexample or explanation for why the conclusion "$S$ has a maximum element" fails to be true if we remove the assumption that:

**(a)** $S$ is nonempty; **(b)** $S$ is closed; **(c)** $S$ is a subset of $\mathbf{R}$.

# Real Analysis

**Accumulation points.** Let $S$ be a subset of a metric space $E$. A point $p \in E$ is an *accumulation point* of $S$ if, for every $\epsilon > 0$, $B_\epsilon(p)$ contains an infinite number of points from $S$. (An accumulation point is also called a *cluster point* or *limit point*.)

sets / DD **1**. For each of the following subsets of $\mathbf{R}$ with the Euclidean metric, describe its set of accumulation points.

**(a)** $\mathbf{Q}$ **(b)** the irrationals **(c)** $(a, b]$ **(d)** $\{1\}$ **(e)** $\{1/n : n \in \mathbf{N}\}$ **(f)** $\mathbf{Z}$

**Isolated points.** Let $S$ be a subset of a metric space $E$. A point $p \in S$ is *isolated* if there exists $r > 0$ such that $p$ is the only point of $S$ in $B_r(p)$.

sets / FM **2**. Let $S$ be a subset of a metric space $E$. Prove that every point of $S$ is either an isolated point in $S$ or an accumulation point of $S$ (but not both).

closed / DD **3**. For a point $p \in \mathbf{R}^n$, consider the set $S = \displaystyle\bigcap_{m=1}^{\infty} \{x \in \mathbf{R}^n : d(x, p) \leq 1/m\}$.

**(a)** Prove that $S$ is closed. *Hint*: Use a previous result.

**(b)** Give a simple description of the set $S$.

**(c)** Prove that any single point $\{p\}$, where $p \in \mathbf{R}^n$, is a closed set.

**A new metric space.**

Let $\mathcal{B} = \{$bounded, real-valued functions on $\mathbf{R}\}$

$= \{f : \mathbf{R} \to \mathbf{R}$ such that there exists $M \in \mathbf{R}$ with $|f(x)| < M$ for all $x \in \mathbf{R}\}$.

For $f, g \in \mathcal{B}$, define $d(f, g) = \sup\{|f(x) - g(x)| : x \in \mathbf{R}\}$.

metric / DD **4**. Let $f(x) = \cos(x)$ and $g(x) = 2$. Explain why $f, g \in \mathcal{B}$, and find $d(f, g)$.

metric / DD **5**. Show that $d$ is a metric on $\mathcal{B}$.

# Real Analysis

cpt / DD

**1**. For each set $S_i \subset \mathbf{R}$, explain why $\mathcal{G}_i$ is an open cover of $S_i$.

**(a)** $S_1 = \{\pi\}$, $\mathcal{G}_1 = \{(1/(n+1), n) : n \in \mathbf{N}\}$

**(b)** $S_2 = [-3, 11]$, $\mathcal{G}_2 = \{(n, n+2) : n \in \mathbf{Z}\}$

**(c)** $S_3 = [2, \infty)$, $\mathcal{G}_3 = \{(n, n+2) : n \in \mathbf{N}\}$

**(d)** $S_4 = (0, 1)$, $\mathcal{G}_4 = \{(1/n, 1 - 1/n) : n \in \mathbf{N}\}$

For some of these, a finite subset of $\mathcal{G}_i$ still covers $S_i$. Which ones?

seq / FM

**Subsequences.** We like sequences to converge, but most don't. Fortunately, most sequences have subsequences that do converge. Given a sequence $a_n$, a *subsequence* $a_{m_n}$ consists of some (infinitely many) of the terms, in the same order.

**The lim sup and lim inf.** Even for sequences that are not convergent, sometimes elements of the sequence do accumulate. The following *always* exist:

- The *lim inf* of a sequence is the smallest limit of any subsequence, or $\pm\infty$.
- The *lim sup* of a sequence is the largest limit of any subsequence, or $\pm\infty$.

*Note:* "lim sup" is pronounced "limm soup."

bound / AJ

**2**. In the following table, write out the first 8 terms of any sequence for which they are not already written out for you. Then find the lim inf and lim sup of each sequence.

| sequence | first 8 terms | lim inf | lim sup |
|---|---|---|---|
| $a_n = 1/n$ | | | |
| $a_n = \sin(n\pi/2)$ | | | |
| $1, -1, 2, -2, 3, -3, 4, -4, \ldots$ | | | |
| $a_n = -n^2$ | | | |
| $a_n = 2 + \dfrac{(-1)^n}{n}$ | | | |
| $a_n = \begin{cases} 3 - e^{-n} & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$ | | | |
| $\frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, \frac{1}{4}, 1\frac{1}{4}, 2\frac{1}{4}, \frac{1}{8}, 1\frac{1}{8}, 2\frac{1}{8}, \ldots$ | | | |

bound / AJ

**3**. Make a conjecture as to what conditions ensure that the lim inf of a sequence equals its lim sup. Then prove your conjecture.

# Real Analysis

**Continuity.** "Nearby points are sent to nearby points." We have seen three definitions; we'll now prove their equivalence.

Let $E, E'$ be metric spaces with distance metrics $d, d'$ respectively. Let $f : E \to E'$ be a function, and let $x_0 \in E$. We say that $f$ is *continuous at* $x_0$ if:

(1) For any $\epsilon > 0$, there exists $\delta > 0$ such that, for $x \in E$,

$$d(x, x_0) < \delta \implies d'(f(x), f(x_0)) < \epsilon.$$

(2) For every sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

(3) For every open set $\mathcal{U} \subset E'$, $f^{-1}(\mathcal{U})$ is open in $E$.
(this one is everywhere, not just at $x_0$)

We say that $f$ is *continuous* if it is continuous everywhere in $E$.

cont / FM **4. Theorem.** The three definitions (1), (2), (3) of continuity are equivalent.

Either prove this using your own method, or follow the proof structure below.

*Proof.* We will prove that the three are equivalent by proving (1) $\iff$ (2) and (1) $\iff$ (3).

**(a)** (1) $\iff$ (2). *Hint*: Prove (1) $\implies$ (2) directly, and (2) $\implies$ (1) using the contrapositive.

**(b)** (1) $\implies$ (3): Let $\mathcal{U}$ be an open set in $E'$. We wish to show that $f^{-1}(\mathcal{U})$ is open, so we need to show that, for any $p \in f^{-1}(\mathcal{U})$, there is an open ball about $p$ in the set ____________.
Let $p \in f^{-1}(\mathcal{U})$. Then $f(p) \in \mathcal{U}$, so there exists $\epsilon > 0$ such that $f(p)$ is contained in ____________ in the set ____________.
Since we assume (1), we can choose $\delta > 0$ such that

$$|x - p| < \delta \implies |f(x) - f(p)| < \epsilon, \text{ and thus}$$
$$|x - p| \leq \delta/2 \implies |f(x) - f(p)| < \epsilon.$$

Here we divided $\delta$ by 2 so that ________________________________________.

Now we have shown that $B(p, \delta/2) \subset f^{-1}(B(f(p), \epsilon)) \subset f^{-1}(\mathcal{U})$, so... (Finish the proof)

**(c)** (3) $\implies$ (1): *Hint*: Since the inverse image of the open ball about $f(p)$ of radius $\epsilon$ is open and contains $p$, it contains some ball $B(p, \delta)$, so $|x - p| < \delta \implies |f(x) - f(p)| < \epsilon$. Fill in the details.

# Real Analysis

**0**. *Hand-in problem.* (essentially Page 13 # 2)

**Theorem.** Let $S$ be a subset of a metric space $E$. Then every point of $S$ is either an isolated point of $S$ or an accumulation point of $S$ (but not both).

*Proof.* ...

If you did not type your first proof in LaTeX, please consider typing this second one. It is a useful (and fun!) skill to learn, and I am happy to help you. (Typing is not required.)

---

closed / DD **1**. **Claim.** For any set $S$,

$$\overline{S} = \bigcap_{C \text{ closed}, \ S \subset C} C.$$

**(a)** Write out the statement in words.

**(b)** Prove it.

Sometimes this is used as the *definition* of the closure of $S$.

We use "Proposition" for a statement that is bigger than a Claim but smaller than a Theorem:

seq / FM **2**. **Proposition.** Every bounded sequence in $\mathbf{R}$ (with the usual Euclidean metric) has a convergent subsequence.

(Prove this using your own method, or follow the structure below.)

We will first show that every bounded sequence of *nonnegative* real numbers in $\mathbf{R}$ has a convergent subsequence. We will do this by explicitly constructing a convergent subsequence. Consider a nonnegative sequence $a_1, a_2, a_3, \ldots$. Each $a_n$ starts off with a nonnegative integer before the decimal point, followed by infinitely many digits (possibly 0) after the decimal point. Since the sequence $a_n$ is bounded, some integer part $D$ before the decimal place occurs infinitely many times, because ________________________________________.
Throw away the rest of the $a_n$. Among the infinitely many remaining $a_n$ that start with $D$, some first decimal place $d_1$ occurs infinitely many times. Throw away the rest of the $a_n$.

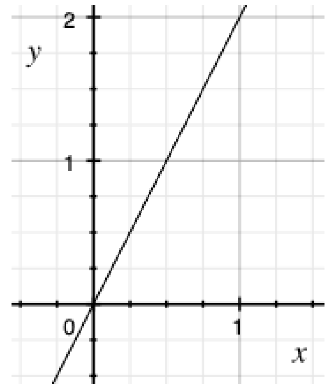
Complete the construction of a number $L = D.d_1 d_2 d_3 \ldots$, and prove that there is a subsequence of $a_n$ converging to $L$.

Finally, show that *every* bounded sequence of real numbers has a convergent subsequence, to complete the proof of the Proposition as stated.

# Real Analysis

cont / DD

**3**. Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = 2x$. Show that $f$ is continuous at $x = 1/2$ (using whatever method you like).

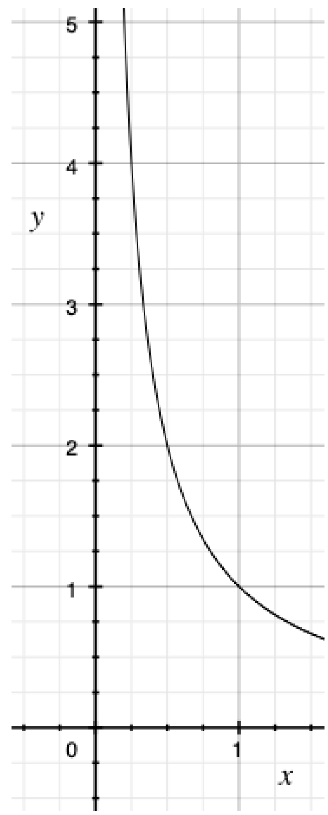



**Uniform continuity.** $f : E \to E'$ is *uniformly continuous* if, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $p, q \in E$,

$$d(p,q) < \delta \implies d'(f(p), f(q)) < \epsilon.$$

uni-con / DD

**4**. Let $E = (0,1)$, and define $f_1, f_2 : E \to \mathbf{R}$ by $f_1(x) = 2x$ and $f_2(x) = 1/x$. These are illustrated to the right.




**(a)** Let $x_0 = 1/2$. Given any $\epsilon > 0$, find $\delta_1$ so that

$$|x - x_0| < \delta_1 \implies |f_1(x) - f_1(x_0)| < \epsilon.$$

**(b)** Let $x_0 = 1/2$. Given any $\epsilon > 0$, find $\delta_2$ so that

$$|x - x_0| < \delta_2 \implies |f_2(x) - f_2(x_0)| < \epsilon.$$

**(c)** Repeat part (a), for $x_0 = 1/10$.

**(d)** Repeat part (b), for $x_0 = 1/10$.

**(e)** Both functions are continuous on $(0,1)$, but only one is uniformly continuous on $(0,1)$. Explain geometrically what causes the difference.

**5**. What is the difference between *continuity* and *uniform continuity*?

---

*Extra problems for if we have time in class*

closed / DD

**6**. Consider the (false!) statement: *The union of infinitely many closed sets is closed.*

**(a)** We proved in Page 12 # 5 that the union of *finitely* many closed sets is closed. Explain where the proof breaks down for infinitely many sets.

**(b)** Give a counterexample to the statement.

metric / DD

**7**. Consider a metric space $E$ with the discrete metric.

**(a)** What do open balls look like in this space? What do closed balls look like in this space?

**(b)** True or False: Any finite set of points, in a metric space $E$ with this metric, is open.

**Sequence definition of a closed set.** So far, to show that a set is closed, we have to show that its complement is open. After proving the following Theorem, we will have another way. In fact, the following Theorem is sometimes used as the *definition* of a closed set.

closed / FM **1**. **Theorem.** Let $S$ be a subset of a metric space $E$. Then $S$ is closed if and only if every convergent sequence of points from $S$ converges to a point in $S$.

Prove this. (Remember that this is an "if and only if" statement, so we need to show both directions of implication.) *Hint*: you proved one direction on your test.

**Cauchy sequences.** A sequence $\{p_n\}_{n=1}^{\infty}$ in a metric space $E$ is *Cauchy* ("COE-she") if, for any $\epsilon > 0$, there exists $N$ such that

$$m, n > N \implies d(p_m, p_n) < \epsilon.$$

You can think of a Cauchy sequence as one that is "trying" to converge, but to a limit that may be outside of its metric space.

Cauchy / DD **2**. Explain the difference between the definition of a *Cauchy* sequence and the definition of a *convergent* sequence.

Cauchy / DD **3**. Consider the sequence $\{a_n = 1/n : n \in \mathbf{N}\}$ in $R^+ = \{x \in \mathbf{R} : x > 0\}$ with the usual Euclidean metric.

**(a)** Show that $\{a_n\}$ is a Cauchy sequence.

**(b)** Show that $\{a_n\}$ does not converge in this metric space.

Cauchy / FM **4**. Prove that a convergent sequence in any metric space is Cauchy.

Cauchy / FM **5**. Prove that a Cauchy sequence in any metric space is bounded. *Hint*: The proof that shows that a convergent sequence is bounded works here, too.

Given a cover $\mathcal{G}$ of a set $S$, a *finite subcover* is a collection consisting of finitely many of the sets in $\mathcal{G}$, that still covers $S$.

cpt / DD **6**. For each of the following open covers $\mathcal{G}_i$ of the set $S_i$, find a finite subcover (a subset of the cover, consisting of finitely many sets) that still covers $S_i$.

**(a)** $S_1 = \{1, 2, 3, 4, 5\}$, $\mathcal{G}_1 = \{(-n, n) : n \in \mathbf{N}\}$

**(b)** $S_2 = [-3, 11]$, $\mathcal{G}_2 = \{(x, x+1) : x \in \mathbf{R}\}$

cpt / DD **7**. Give an example of a set $S$, and an open cover $\mathcal{G}$ of $S$, for which *no* finite subcover of $\mathcal{G}$ covers $S$. Make a conjecture as to what properties of $S$ and $\mathcal{G}$ make this possible.

# Real Analysis

cont / FM

**1**. Let $g : E \to E'$ and $f : E' \to E''$ be continuous functions. Prove that their composition $f \circ g : E \to E''$, i.e. $x \mapsto f(g(x))$, is continuous, using each definition of continuity:

**(a)** epsilon-delta definition **(b)** sequence definition **(c)** open set definition.

**(d)** Which way did you most prefer? Which did you least prefer?

The notion of a *compact* set is very important in analysis; it is why we have been thinking about open covers. In the upcoming Heine-Borel Theorem, we will show that, in $\mathbf{R}^n$, a compact set is just a set that is closed and bounded. The definition itself uses open covers:

**Compactness.** A subset $X$ of a metric space $E$ is *compact* if every open cover of $X$ has a finite subcover.

However, this definition is rather difficult to check. Sure, we can find *an* open cover, but how do you check that *every possible* open cover has a finite subcover?

cpt / DD

**2**. Show that $\mathbf{R}$ is not compact, by finding an open cover that has no finite subcover.

cpt / DD

**3**. Show that $(0, 1]$ is not compact, by finding an open cover that has no finite subcover.

We like *compact* sets because they tend to be the sets that have the properties we want. For example, eventually we will show that a continuous function on a compact set achieves a maximum and minimum, which is very useful in calculus.

open / DD

**4**. $S = [0, \infty)$ is a subset of the metric space $\mathbf{R}$ with the usual Euclidean metric, but $S$ itself is also a metric space, with the inherited metric from $\mathbf{R}$. Which of the following are open sets in $S$?

**(a)** $(0, 1)$ **(b)** $[0, 1)$ **(c)** $(0, 1]$ **(d)** $[0, 1]$

**5**. Prove that for all sequences $\{a_n\} \subset \mathbf{R}$,

$$\liminf \{a_n\} \leq \limsup \{a_n\}.$$

We proved the following for sequences in $\mathbf{R}$. Is it also true for sequences in a general metric space? Either prove it, or give a counterexample:

bdd / DD

**6**. Every bounded sequence has a convergent subsequence.

# Real Analysis

*Optional review problems for Test #2*

vocab / FM **1**. Write the definition of each term, as a full sentence:

**(a)** subsequence **(b)** accumulation point **(c)** isolated point **(d)** Cauchy

**(e)** complete **(f)** cover **(g)** compact **(h)** uniformly continuous

vocab / DD **2**. For each of the terms above, write a result (Theorem, etc.) that uses it.

closed / FM **3**. Prove that any finite set of points in $\mathbf{R}^n$ is closed. *Hint*: use a previous result

sets / DD **4**. Find the set of accumulation points in $\mathbf{R}^2$ for each of the following sets:

**(a)** $\{(p,q) : p,q \in \mathbf{Q}\}$ **(b)** $\{(m/n, 1/n) : m,n \in \mathbf{Z}, n \neq 0\}$

Without looking at your notes, write down the proofs of the following results:

cont / DD **5**. The $\epsilon - \delta$ and sequence definitions of continuity are equivalent.

cont / DD **6**. The $\epsilon - \delta$ and open set definitions of continuity are equivalent.

bdd / DD **6**. Every bounded sequence of real numbers has a convergent subsequence.

bdd / FM **7**. Let $S$ be a nonempty, closed subset of $\mathbf{R}$ that is bounded from above. Then $S$ has a maximum element.

# Real Analysis

uni-con / DD **1**. Let's recall the difference between *continuity* and *uniform continuity.*

**(a)** Explain why $f$ uniformly continuous $\implies$ $f$ continuous.

**(b)** Explain why $f$ continuous $\not\Rightarrow$ $f$ uniformly continuous by providing a counterexample.

In fact, $f$ continuous $\implies$ $f$ uniformly continuous in the special case when the domain of $f$ is *compact.* Let's prove it. This will also give us some experience in applying the fact that every open cover has a finite subcover.

uni-con / FM **2**. **Theorem.** Let $E, E'$ be metric spaces, and let $f : E \to E'$ be a continuous function. If $E$ is compact, then $f$ is uniformly continuous.

*Proof.* Given any $\epsilon > 0$, we will construct a $\delta$ such that, for all $p, q \in E$,

$$d(p,q) < \delta \implies \underline{\hspace{6cm}}.$$

Given $\epsilon > 0$, we know that for each $x_0 \in E$, there is a $\delta_{x_0} > 0$ such that

$$d(p,x_0) < \delta_{x_0} \implies d'(f(p), f(x_0)) < \epsilon/2,$$

because ________________________________________________.

Consider the open ball $\mathcal{U}_{x_0} = \{p : d(p, x_0) < \delta_{x_0}/2\}$. The collection $\{\mathcal{U}_x\}$ of all such open balls covers $E$, because ________________________________________________.

Since $E$ is compact, it has a finite subcover $\{\mathcal{U}_{x_1}, \ldots, \mathcal{U}_{x_n}\}$. Let $\delta = \min\{\delta_{x_i}/2 : i = 1, \ldots, n\}$. We will show that this is the $\delta$ with the desired property.

Suppose that $d(x, x_0) < \delta$. Since $x_0 \in E$, $x_0 \in \mathcal{U}_{x_j}$ for some $j$ in $\{1, \ldots, n\}$, so $d(x_0, x_j) < \delta_{x_j}/2$. Since $d(x, x_0) < \delta \le \delta_{x_j}/2$, we have $d(x, x_j) < \delta_{x_j}$, by ____________________. Therefore,

$$d'(f(x_0), f(x_j)) < \epsilon/2 \text{ and } d'(f(x), f(x_j)) < \epsilon/2, \text{ so}$$

$$d'(f(x), f(x_0)) \le d'(f(x_0), f(x_j)) + d'(f(x), f(x_j)) < \epsilon/2 + \epsilon/2 = \epsilon,$$

as desired.

**Complete metric spaces.** A metric space $E$ is *complete* if every Cauchy sequence of points in $E$ converges to a point in $E$.

Cauchy / DD **3**. Show that each of the following metric spaces (using the usual Euclidean distance metric) is not complete, by finding a Cauchy sequence of points in the space that does not converge to a point of the space:

**(a)** $\mathbf{R}^+ = \{x \in \mathbf{R} : x > 0\}$ **(b) Q**

Cauchy / DD **4**. Prove that a Cauchy sequence that has a convergent subsequence is itself convergent.

cpt / AJ

**5**. **Proposition.** $S = \{0\} \cup \{1/n : n \in \mathbf{N}\} \subset \mathbf{R}$ is a compact set.

**(a)** Draw a picture of this set.

**(b)** *Proof.* We will show that $S$ is compact, by explicitly constructing a finite subcover from any open cover. Consider an arbitrary open cover $\cup\, G_\alpha$ of $S$. 0 must be in some open set $G_\alpha$, because ____________________________________, so call this open set $G_{\alpha_0}$. Then there exists an open ball $B_r(0) \subset G_{\alpha_0}$, because ____________________________________. Then for all $n > 1/r$, we have $1/n \in B_r(0)$, because ____________________________________. So $\{0\} \cup \{1/n : n > 1/r\} \subset G_{\alpha_0}$. (Finish the proof.)

seq-fn / DD

**6**. Graph the following functions, for $n = 1, 2, 3, 4, 5$:

$$f_n(x) = \begin{cases} 1 - nx & \text{if } 0 \le x < 1/n \\ 0 & \text{if } 1/n \le x \end{cases}$$

Is there a "limit function" as $n \to \infty$? If so, say what it is.

# Real Analysis

**0**. *Hand-in problem.* Write up a proof of Page 19 # 4:

**Proposition.** A Cauchy sequence that has a convergent subsequence is itself convergent.

Cauchy / FM **1**. **Theorem. R** is complete. (Prove this)
*Hint*: One method is to use the results of Page 15 # 2 and Page 16 # 5 and argue (using an $\epsilon - N$ proof) that every sequence converges to the same limit as its convergent subsequence.

We use the word "Corollary" for a result that follows directly from an existing Proposition or Theorem:

Cauchy / AJ **2**. **Corollary.** $\mathbf{R}^n$ is complete. (Prove this.)
*Hint*: Repeat the same argument as above, for each coordinate.

sets / AJ **3**. **Proposition.** An infinite subset of a compact metric space has at least one accumulation point.

**(a)** Give an example of the above result: a compact metric space $E$, and an infinite subset $S$ of $E$ that has at least one accumulation point.

**(b)** Prove the Proposition, using your own method or the following outline:

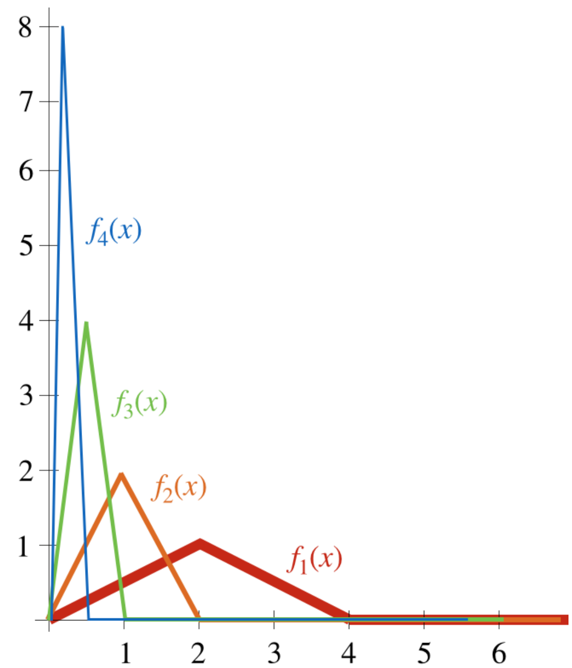
*Proof.* We will prove this by contradiction, by assuming that the subset has no accumulation point, and showing that the subset must be finite. Suppose that $A \subset E$ is an infinite set with no accumulation point. Then for each $p \in E$, there exists an $r_p$ such that $B_{r_p}(p)$ contains only finitely many points of $A$, because ________________________________________.

Then $\bigcup_{p \in E} B_{r_p}(p)$ is an open cover of $E$, because ________________________________________.

Since $E$ is compact, there is a finite subcover $B_{r_1}(p_1) \cup \cdots \cup B_{r_k}(p_k)$ that covers $E$.

(Finish the proof.)

seq-fn / DD **4**. The first four terms of a sequence $f_n$ of functions $R_{\geq 0} \to R_{\geq 0}$ are shown to the right. The piecewise equations for the functions are given below, but the idea of this problem is to think about the picture.




**(a)** Is there some limit function as $n \to \infty$? If so, say what it is.

**(b)** What is the integral over $[0, \infty)$ of $f_n$ for each $n$? Does the integral value have a limit as $n \to \infty$?

$$f_n(x) = \begin{cases} 2^{2n-3}x & \text{if} \quad 0 \leq x \leq 2^{2-n} \\ 2^n - 2^{2n-3}x & \text{if } 2^{2-n} \leq x \leq 2^{3-n} \\ 0 & \text{if } 2^{3-n} \leq x \end{cases}$$

cpt / DD **5**. **Theorem.** Let $K$ and $S$ be subsets of a metric space $E$. Suppose $K \subset S \subset E$. Then $K$ is compact in $S$ if and only if $K$ is compact in $E$.

*Proof.* ($\Longleftarrow$) Suppose that $K$ is compact in $E$. We will show that $K$ is also compact in $S$, by showing that every open cover of $K$ in $S$ has a finite subcover. Take any open cover $\{U_\alpha\}$ of $K$ in $S$. By the Theorem in problem ____________, for each $\alpha$, $U_\alpha = \widetilde{U}_\alpha$ for a set $\widetilde{U}_\alpha \cap E$ that is open in $E$. Then $\{\widetilde{U}_\alpha\}$ gives an open cover of $S$ in $E$. Since $K$ is compact in $E$, there exists a finite subcover of $K$ in $E$, so ...

**(a)** Finish the proof of this direction of implication.

**(b)** Prove the other direction of implication.

# Real Analysis

*Don't just read math; fight it!* – Paul Halmos

**Heine-Borel Theorem.** For a set $S \subset \mathbf{R}^n$, the following are equivalent:
(1) Every sequence in $S$ has a subsequence converging to a point of $S$.
(2) $S$ is closed and bounded.
(3) $S$ is compact: every open cover has a finite subcover.

cpt / FM **1**. *Proof.* We will prove that (3) $\implies$ (2) $\implies$ (1) $\implies$ (3). This will prove that all three criteria are equivalent, because ________________________________________.

cpt / FM **2**. (3) $\implies$ (2). We will prove the contrapositive: if $S$ is not closed or not bounded, then there is some open cover that has no finite subcover.
(Part 1) Suppose that $S$ is not closed. Then some convergent sequence of points from $S$ converges to a point $a$ that is not in $S$, because ______________________. Then $a$ is an accumulation point for $S$, because ______________________. Then the open cover $\{x : |x - a| > 1/n : n \in \mathbf{N}\}$ has no finite subcover, because ______________________.
(Part 2) Suppose that $S$ is not bounded. Then the open cover $\{x : |x| > n : n \in \mathbf{N}\}$ has no finite subcover, because ______________________.

cpt / FM **3**. (2) $\implies$ (1). We will show that, if $S$ is closed and bounded, then every sequence in $S$ has a subsequence converging to a point of $S$. Take any sequence in $S \subset \mathbf{R}^n$. First, just look at the first of the $n$ components of each point. Since $S$ is bounded, the sequence of the first components is bounded, because ______________________________. So for some subsequence, the first components converge, because ______________________________.
Similarly, for a further subsequence, the second components converge. Eventually, for some further subsequence, each of the components converge, because ______________________.
The limit point is in $S$, because ______________________________.

cpt / FM **4**. (1) $\implies$ (3). We will show that, if every sequence in $S$ has a subsequence converging to a point in $S$, then every open cover of $S$ has a finite subcover. First, we will show that every open cover has a *countable* subcover, and then we will show, using a proof by contradiction, that it must actually have a finite subcover. Given an open cover $\{\mathcal{G}_\alpha\}$ of $S$, we will construct a countable subcover. Every point of $S$ lies in a ball of rational radius about a rational point, because ______________________________. Each of these countably many balls is contained in some $\mathcal{G}_{\alpha_0}$, because ______________________________. So a countable open cover $\{V_i\}$ of $S$ is given by ______________________.

cpt / FM **5**. Now suppose that $\{V_i\}$ has no finite subcover. Choose $x_1 \in S \backslash V_1$. Choose $x_2 \in S \setminus (V_1 \cup V_2)$. Continue, choosing $x_n$ in $S \backslash \bigcup_{i=1}^{n} V_i$, which is always possible, because ______________________.
For each value of $i$, there are only finitely many $x_n$, with $n < i$, contained in $V_i$, because ______________________________. Because ______________________________, the sequence $x_n$ has a subsequence converging to some $x \in S$, with $x \in V_i$ for some $i$. Thus infinitely many $x_n$ are contained in $V_i$, which is a contradiction because ______________________.
Thus every open cover of $S$ has a finite subcover, as desired.

# Real Analysis

**1**. *A summary list of previous results.* For each problem listed below, write the statement of the result in plain English (or plain other language of your choice). I have filled in several of them for you, so that you can see what I mean. The purpose of this is to make it easier to remember and reference these results.

**4 # 4**

**4 # 6** The entire metric space is always open.

**7 # 2**

**7 # 3**

**7 # 6**

**7 # 7** A finite intersection of open sets is open.

**8 # 5**

**8# 6**

**9 # 1**

**12 # 5**

**12 # 7**

**15 # 2**

**16# 1**

**16 # 4**

**16 # 5**

**19 # 2** A continuous function on a compact set is uniformly continuous.

**19 # 4**

**20 # 1**

**20 # 2**

**20 # 3**

**20 # 5**

**22 # 2**

**22 # 3**

**22 # 4**

**22 # 5**

Did I miss anything? If so, please add it and let the rest of us know.

cpt / DD **2**. A compact subset of a metric space is bounded. (Prove this.)
*Hint*: use the finite cover.

# Real Analysis

A note on the below: a *sequence* consists of infinitely many terms. Two examples of sequences are $1, 2, 3, \ldots$ and $\pi, \pi, \pi, \ldots$: note that a sequence may have many repeated terms.

On the other hand, in a *set*, each element is unique, so for example $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$ and the "2" only appears once. Thus for the two examples of sequences given above, the *set* of their elements has infinitely many elements $\{1, 2, 3, \ldots\}$ in the first case, and only one element $\{\pi\}$ in the second.

The following three results follow from the Proposition in Page 20 # 3.

cpt / DD **3**. **Corollary 1.** Every sequence of points in a compact metric space has a convergent subsequence.

*Proof.* We will prove the result by explicitly constructing the convergent subsequence. Let $p_n$ be a sequence in a compact metric space $E$. There are two cases, depending on whether the number of different points in the sequence is finite or infinite. If $\{p_1, p_2, p_3, \ldots\}$ is a finite set, then there must be some point $p$ that occurs infinitely many times.

**(a)** Finish the proof of the finite case.

On the other hand, if $\{p_1, p_2, p_3, \ldots\}$ is an infinite set, then it must have at least one accumulation point $p$, because ____________________________________. Choose $n_1$ so that $p_{n_1} \in B_1(p)$. Choose $n_2 > n_1$ so that $p_{n_2} \in B_{1/2}(p)$, and so on. Note that for each $k$, there are infinitely many points of the sequence in $B_{1/k}(p)$, because ____________________________.

(Finish the argument.)

cpt / DD **4**. **Corollary 2.** A compact metric space is complete.

*Proof.* We will show that a compact metric space is complete, by showing that every Cauchy sequence converges to a point in the space. (Do this.)

*Hint*: combine Corollary 1 with another result.

cpt / DD **5**. **Corollary 3.** A compact subset of a metric space is closed.

*Proof.* We will prove this by showing that every convergent sequence of points from a compact subset converges to a point in the compact subset (using the sequence definition of a closed set). (Do this.)

*Hint*: Combine Corollary 2 with other results.

**Connected sets.** A metric space $E$ is *connected* if the only subsets of $E$ that are both open and closed are $E$ and $\emptyset$. A subset $S \subset E$ is connected if it is connected when considered as a metric space.

conn / DD **6**. Write a definition of the word "connected" in the non-mathematical sense (for, say, a subset of $R^2$). Then explain why the above definition is equivalent to that usual meaning.

conn / DD **7**. Give an example of a subset that is connected, and a subset that is not connected, in:

**(a)** $\mathbf{R}^2$ with the Euclidean metric, **(b)** $\mathbf{R}^2$ with the discrete metric.

**Sequences of functions.** Let $f_n : E \to E'$ and let $p \in E$. The sequence $f_1, f_2, \ldots$ *converges at* $p$ if $f_1(p), f_2(p), \ldots$ converges as a sequence of points in $E'$. The sequence $f_1, f_2, \ldots$ *converges* (on $E$) if it converges at every point in $E$. If $f_1, f_2, \ldots$ converges, we define the *limit function* to be $f(p) = \lim_{n\to\infty} f_n(p)$ for each $p \in E$.

seq-fn / DD **1**. For each of the following sequences of functions, explain why the sequence converges, and give the limit function $f$:

**(a)** $f_n : \mathbf{R}^+ \to \mathbf{R}^+$ from Page 19 # 6

**(b)** $f_n : \mathbf{R}^+ \to \mathbf{R}^+$ from Page 20 # 4

The Heine-Borel Theorem says that, in $\mathbf{R}^n$ with the Euclidean metric, *compact* is equivalent to *closed and bounded*. The following two problems further explore the connection between compactness and closed-and-boundedness, for a general metric space.

cpt / DD **2**. Consider the metric space consisting of the integers $\mathbf{Z}$, with the discrete metric. Show that this metric space is closed and bounded, but not compact.

cpt / DD **3**. Prove that any compact subset of a metric space is closed and bounded.
*Hint*: put together several previous results.

**Spaces of functions.** Let $\mathcal{C}(E, E')$ be the set of all continuous functions from the metric space $E$ to the metric space $E'$. Here $\mathcal{C}$ stands for *continuous*. It is a metric space under the distance metric

$$D(f, g) = \max\{d'(f(p), g(p)) : p \in E\},$$

where as usual $d'$ is the distance metric in $E'$. Notice that for this metric to be well defined, the maximum must exist and be finite.

sp-fn / DD **4**. For each of the following, $f, g : \mathbf{R} \to \mathbf{R}$, find $D(f, g)$. *Hint*: draw a picture

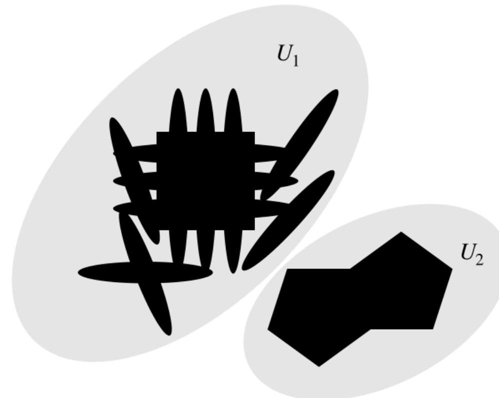
**(a)** $f(x) = \sin(x), g(x) = 0$ **(b)** $f(x) = x + \sin(x), g(x) = x$

metric / DD **5**. Prove that $\mathcal{C}(E, E')$ with the metric $D$ is a metric space.

conn / FM **6**. Give a counterexample to the following statement: If $f : \mathbf{R} \to \mathbf{R}$ is continuous and $S$ is connected, then $f^{-1}(S)$ is connected.

**Other bases.** The *decimal expansion* of a number between 0 and 1 tells, in each decimal place, the number of $1/10^1$s, $1/10^2$s, $1/10^3$s, etc. needed to sum to the number, using digits between 0 and 9. The *binary expansion* and *ternary expansion* do the same, with the number of powers of 1/2 and 1/3, respectively, using digits $\{0, 1\}$ and $\{0, 1, 2\}$, respectively.

Cantor / DD **7**. Write 3/8, 7/16 and 1/3 in binary. Write 5/9, 8/27 and 1/2 in ternary.




**Sets that are connected and not connected.** A subset $S$ of a metric space $E$ is *not connected* if it can be separated by two disjoint open sets $U_1$ and $U_2$ into two nonempty pieces $S \cap U_1$ and $S \cap U_2$, such that $S = (S \cap U_1) \cup (S \cap U_2)$. Otherwise, it is *connected*.

conn / FM **1**. Prove, in two sentences, that any subset of $\mathbf{R}$ that contains two distinct points $a$ and $b$, and does not contain all of the points between $a$ and $b$, is not connected.

conn / DD **2**. Prove that this definition of *connected* is equivalent to the one on Page 22.

conn / FM **3**. Prove that an interval of real numbers is connected (perhaps by contradiction).

Cantor / DD **4**. Show that the set of all possible numbers such as $0.010100011101010\ldots$, with integer part 0 and decimal digits 0 and 1, is uncountable. *Hint*: binary.

**The Cantor set.** Start with the closed unit interval $[0, 1]$. Remove the open middle third $(1/3, 2/3)$, leaving two closed intervals of length $1/3$. Remove the open middle third of each of these, leaving four closed intervals of length $1/9$. Continue. At the $n^{\text{th}}$ step, you have a set $S_n$ consisting of $2^n$ closed intervals each of length $1/3^n$. Let $\mathcal{C} = \bigcap S_n$.

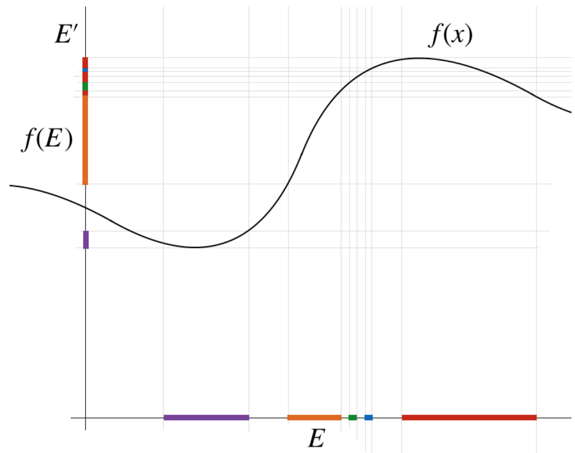
Cantor / DD **5**. Draw $S_n$ for $n = 0, 1, 2, 3, 4$. $S_0$ should take up the entire width of your page.

Cantor / DD **6**. Find the total length of $S_n$ as a function of $n$, and the total length of $\mathcal{C}$.

*Note:* In order to fit a one-semester course into one trimester, a few things needed to be cut. I decided that we should definitely prove the Fundamental Theorem of Calculus, and to make that happen I opted to cut the topic about convergence of sequences of functions, and the related topics about when you can switch the limit and the integral, and when the limit of a sequence of continuous functions is continuous, and the notion of "uniformly Cauchy." Please take a second course in real analysis in college! I have moved this sequence of problems to the Appendix, which we will do if we have extra time.

# Real Analysis

The following result is very important for finding maxima and minima in calculus.

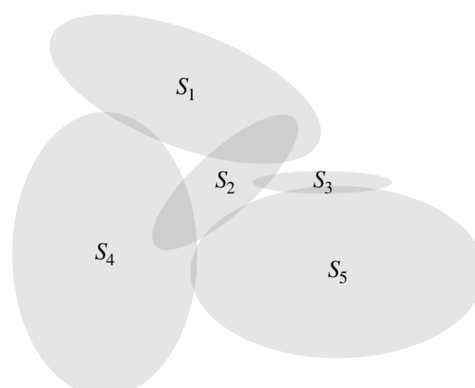



cpt / DD **1**. **Theorem.** The continuous image of a compact set is compact.

*Proof.* Let $f : E \to E'$ be a continuous function, and let $E$ be compact. We will show that its image $f(E)$ is compact by showing that, given any open cover of $f(E)$, we can construct a finite subcover. Let $\{\mathcal{U}_\alpha\}$ be an open cover of $f(E)$. Since each set $\mathcal{U}$ in $\{\mathcal{U}_\alpha\}$ is open in $E'$, each $f^{-1}(\mathcal{U})$ is open in $E$, because ________________________________.

Consider the set $S = \{f^{-1}(\mathcal{U}) : \mathcal{U} \in \mathcal{U}_\alpha\}$. We claim that $S$ is an open cover of $E$. To see this, take any $p \in E$. Then $f(p) \in \mathcal{U}$ for some $\mathcal{U} \in \mathcal{U}_\alpha$, so $p \in f^{-1}(\mathcal{U})$, so $S$ covers $E$. Since $E$ is compact, there is a finite subcover of $S$ covering $E$, i.e. $E \subset f^{-1}(\mathcal{U}_1) \cup \cdots \cup f^{-1}(\mathcal{U}_k)$. Thus $f(E) \subset \mathcal{U}_1 \cup \cdots \cup U_k$.

So for any open cover $\mathcal{U}_\alpha$ of $f(E)$, we have constructed a finite subcover ________________________, so $f(E)$ is compact, as desired.




conn / DD **2**. **Proposition.** Let $\{S_i\}$ be a collection of connected subsets of a metric space $E$. Suppose that there exists $k$ such that, for all $i$, $S_i \cap S_k \neq \emptyset$. Then $\bigcup_i S_i$ is connected.

**(a)** The picture shows an example in the case $E = \mathbf{R}^2$. What is $k$ in this case?

**(b)** *Proof.* We will show that $\cup S_i$ is connected using a proof by contradiction. Suppose that $\cup S_i$ is not connected. Then $\cup S_i$ is contained in the union of two disjoint open sets $A, B$. We will show that $A$ or $B$ must be empty. (Do this.)

*Hint*: write $S_k = (S_k \cap A) \cup (S_k \cap B)$ and use the fact that $S_k$ is connected.

conn / FM **3**. **Theorem.** The continuous image of a connected set is connected.
More formally: Let $f : E \to E'$ be continuous. If $E$ is connected, then $f(E)$ is connected.

(Prove this.)

*Hint*: A one-sentence proof by contradiction is possible.

seq / DD

**4.** **Theorem.** A bounded, monotone sequence of real numbers is convergent.

We will prove this for an increasing sequence; the proof for a decreasing sequence is similar.

*Proof.* Let $a = \sup\{a_k\}$. We will show that $\lim_{k\to\infty} a_k = a$. Given any $\epsilon > 0$, we need to show that there exists $N$ such that, for any $n > N$, $|a - a_n| < \epsilon$. Now $a - \epsilon$ is too small to be a lower bound for $\{a_k\}$, because ________________________________________.
So there exists $N$ such that $a_N > a - \epsilon$, because ________________________________________.
So we know that $a_i > a - \epsilon$ for all $i > N$, because ________________________________________.
Thus $a - \epsilon < a_i < a + \epsilon$ for all $i > N$. (Finish the proof.)

seq / DD

**5.** Consider $\mathbf{R}$ with the metric $d(p,q) = \begin{cases} 0 \text{ if } p = q \\ 1 \text{ if } p \neq q. \end{cases}$. Does the sequence $a_n = 1/n$ have a limit in this metric space? (In other words, does the previous Theorem apply?)

Cantor / DD

**6.** Refer to the construction of the Cantor set $\mathcal{C}$ given on page 26.

**(a)** Explain why the set $S_1$ consists of all numbers between 0 and 1 that have either a 0 or a 2 (but not a 1) as their first digit after the "decimal" point, when expressed in ternary.

**(b)** Explain why $\mathcal{C}$ consists of all of the possible numbers expressed in ternary as infinite strings of 0s and 2s, e.g. $0.020200202202222\ldots$

# Real Analysis

cpt / DD

**1**. **Corollary** to Page 25 # 1. The continuous image of a compact set is bounded.

More formally: If $f : E \to E'$ is continuous, and $E$ is compact, then $f$ is bounded.

(Prove this.)

cpt / DD

**2**. (Continuation) Show that it is necessary for $E$ to be compact, by giving an example of an *unbounded* continuous function on a non-compact metric space.
If your example is a function from $\mathbf{R}$ to $\mathbf{R}$, for extra style points give one example where the space is not closed, and one where the space is not bounded.

**Intermediate Value Theorem:** Let $f : E \to \mathbf{R}$ be continuous, let $E$ be connected, and let $p_1, p_2 \in E$. Then $f$ attains all values between $f(p_1)$ and $f(p_2)$.

IVT / DD

**3**. Prove the Intermediate Value Theorem. *Hint*: put together previous results.

cpt / FM

**4**. **Proposition.** Any closed subset of a compact set is compact.

*Proof.* Let $X$ be a compact subset of a metric space $E$, and let $S$ be a closed subset of $X$. We will show that $S$ is compact, by showing that ______________________________. Let $\{G_\alpha\}$ be an open cover of $S$. We also know that $S^C$ is open, because ________________________.
Then the union $\{G_\alpha\} \cup S^C$ give an open cover of ______________________________.
Since $X$ is compact... (Complete the proof.)

bound / DD

**5**. **Proposition.** A nonempty closed subset of $\mathbf{R}$ has a greatest element if it is bounded from above, and has a least element if it is bounded from below.

*Proof.* We will show that a nonempty closed subset $S$ of $\mathbf{R}$ has a greatest element if it is bounded from above; the proof of the second part is similar.

Let $a = \sup S$. If $a \in S$, we are done, because ______________________________.

If $a \notin S$, then $a \in S^C$. Since $S^C$ is open, there exists $r > 0$ such that $B_r(a) \subset S^C$. Thus, no element of $S$ is greater than $a - r$. (Finish the proof.)

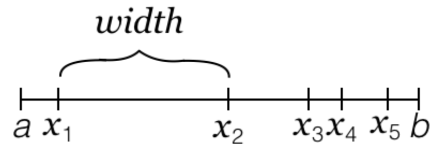
Cantor / DD

**6**. (Rick Parris) Consider again the Cantor set $\mathcal{C}$. Show that $1/4 \in \mathcal{C}$, in two ways:

**(a)** Using the ternary representation of points in $\mathcal{C}$, and

**(b)** using the self-similarity, or *fractal structure*, of $\mathcal{C}$, to show that $1/4$ is never removed.

**Partitions.** Let $a, b \in \mathbf{R}$ with $a < b$. A *partition* of $[a, b]$ is given by a (finite) sequence $x_0, x_1, x_2, \ldots, x_N$ such that $a = x_0 < x_1 < \cdots < x_N = b$. The *width of the partition* is $\max\{x_i - x_{i-1} : i = 1, \ldots, N\}$.



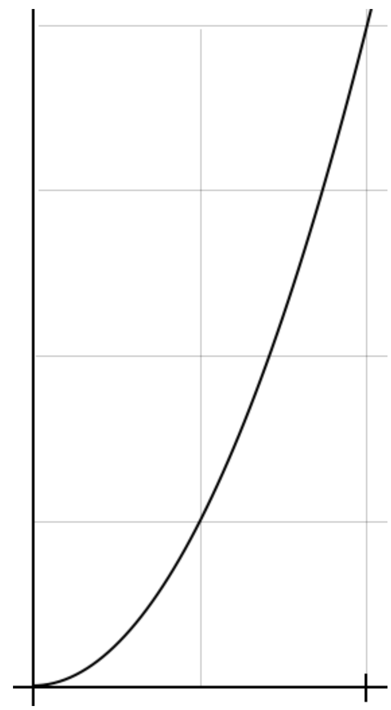

part / DD **7**. If you partition $[a, b]$ using the partition $x_0, x_1, x_2, \ldots, x_N$, how many subintervals of the form $[x_{i-1}, x_i]$ do you get? If they are equally spaced, what is the length of each?

**Riemann sums.** A *Riemann sum* for $f$ on $[a, b]$, corresponding to the partition $x_0, x_1, x_2, \ldots, x_N$ of $[a, b]$, is

$$\sum_{i=1}^{N} f(x_i') \cdot (x_i - x_{i-1}),$$

where $x_i' \in [x_{i-1}, x_i]$ is a representative point in each subinterval.

Riem / DD **8**. Compute the Riemann sum for $f(x) = x^2$ on the interval $[0, 2]$, with $N = 4$. Use a partition, and a representative point in each interval, *that no one else in the class will think of.* You are welcome to use a calculator for the computations. In the picture, draw in your partition, including the rectangles whose areas sum up to your Riemann sum value.



**Differentiability.** In our continuing quest to put all of calculus on a rigorous mathematical basis, we will now study differentiability. Let $\mathcal{U} \subset \mathbf{R}$ be an open set, let $f : \mathcal{U} \to \mathbf{R}$, and let $x_0 \in \mathcal{U}$. We say that $f$ is *differentiable at* $x_0$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{exists.}$$

We denote this limit by $f'(x_0)$ and call it *the derivative of* $f$ *at* $x_0$.

diff / DD

**1**. Answer these questions about the definition of differentiability.

**(a)** Why must the domain and range of $f$ be real numbers?

**(b)** Why must the domain $\mathcal{U}$ be open?

**Integrability.** Let $a, b \in \mathbf{R}$ with $a < b$, and let $f : [a, b] \to \mathbf{R}$. Then $f$ is *Riemann integrable* on $[a, b]$ if there exists a number $A$ such that, for any $\epsilon > 0$, there exists $\delta$ such that, if we take *any* partition of width $\delta$, and if we take $S$ to be *any* Riemann sum value associated to such a partition, then $|A - S| < \epsilon$. In this case, we say that

$$A = \int_a^b f(x)\, dx$$

is the *Riemann integral* of $f$.

You can think of integrable functions from $\mathbf{R} \to \mathbf{R}$ as those that are bounded, and are not discontinuous everywhere on an interval.

Riem / DD

**2**. Draw a picture of the function defined on $[0, 3]$ by: $f(x) = \begin{cases} 0 \text{ if } x \in [0,1] \cup [2,3] \\ 1 \text{ if } x \in (1,2) \end{cases}$.

**(a)** Find a partition of $[0, 3]$ of width $\leq 1/4$, and draw it on the interval $[0, 3]$.

Find the value of each of the following Riemann sums, when the representative point $x_i'$:

**(b)** is the left endpoint of each interval;

**(c)** yields the maximum value of $f$ on its interval;

**(d)** yields the minimum value of $f$ on its interval.

Refer to the definition of Riemann integrability, given above.

**(e)** For $\epsilon = 1/10$, can you find an $A$ and a $\delta$ that satisfy the definition?

**(f)** What are the maximum and minimum values of a Riemann sum associated to a partition of width $1/4$? And what are the maximum and minimum values for width $1/10$?

**(g)** Is $f(x)$ Riemann integrable?

**(h)** Explain why, if the difference between the maximum and minimum Riemann sum values for a function $f$ approaches 0 as the partition width approaches 0, then $f$ is integrable.

Cantor / FM

**3**. Prove that there is a bijection between elements of the Cantor set (*Hint*: ternary) and elements of the set $[0, 1]$ (*Hint*: binary). Are you surprised?!

MVT / FM **1. Proposition (finding critical points).** *If a real-valued function is differentiable at an interior minimum or maximum point, then its derivative is* 0 *there.*

This fact is the basis for finding maxima and minima in calculus. We will also need it to prove the Mean Value Theorem, which is likewise essential for calculus.

*Proof.* We will show that, for an open set $\mathcal{U} \subset \mathbf{R}$ and a function $f : \mathcal{U} \to \mathbf{R}$, if $f$ attains a maximum or minimum at $x_0 \in \mathcal{U}$, and if $f$ is differentiable at $x_0$, then $f'(x_0) = 0$. We will prove the result directly, showing that at a maximum or minimum, the limit is 0.

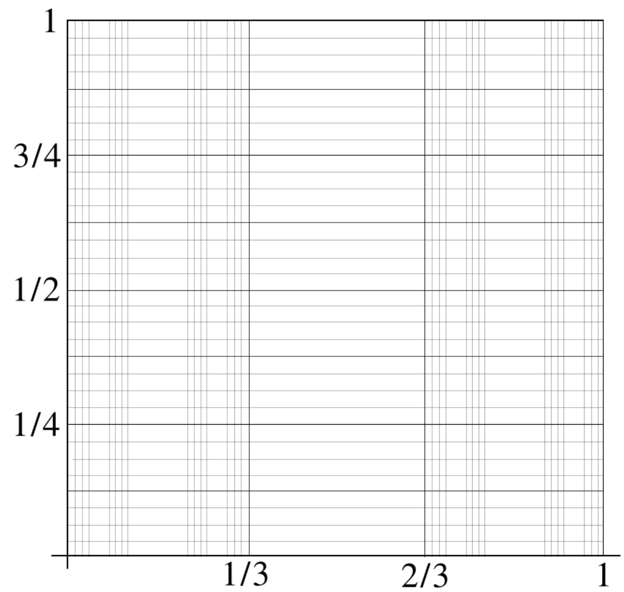
Since $f$ is differentiable, we know that $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists.

Suppose that $x_0$ is a local minimum. For $x$ near $x_0$, the numerator in the definition is non-negative, because ________________________________. If $x_0 > x$, the denominator is positive, and thus $f'(x)$ is a limit of non-negative numbers, because ________________________. For $x_0 < x$, the denominator is negative, and thus similarly $f'(x)$ is a limit of non-positive numbers. Therefore, the only possibility is $f'(x) = 0$, because ____________________________. The proof when $x_0$ is a local maximum is similar.

Cantor / DD **2. The Cantor function.** Define $f_{\mathcal{C}} : [0,1] \to [0,1]$ as follows:

First, on the *complement* of the Cantor set, $\mathcal{C}^C$, define $f_{\mathcal{C}}$ as follows: On the open middle third of the interval, $f$ is $1/2$. On the open middle thirds of the two remaining intervals, $f$ is $1/4$ and $3/4$, respectively. On the open middle thirds of the remaining intervals, $f$ is $1/8$, $3/8$, $5/8$ and $7/8$, respectively. Continue in this manner to assign a value to every point in $\mathcal{C}^C$.

Then, *on* $\mathcal{C}$, define $f_{\mathcal{C}}$ as follows: For any point $p \in \mathcal{C}$, express $p$ in ternary as a decimal point followed by an infinite string of 0s and 2s. Divide this number by 2 to yield a decimal point followed by an infinite string of 0s and 1s, and interpret it in binary; this is $f_{\mathcal{C}}(p)$.




**(a)** Make a sketch of the graph of $f_{\mathcal{C}}$ on the axes to the right.

**(b)** Consider the function $f_{\mathcal{C}} : \mathcal{C} \to [0,1]$, which is just the Cantor function restricted to the Cantor set. Is it onto? Is it continuous?

<sub></sub>

bound / DD **3**. A nonempty bounded subset of $\mathbf{R}$ has an infimum and a supremum. (Prove this)

Riem / DD **4**. Prove, from the definition, that $f(x) = 3$ is Riemann integrable on $[0,1]$.

Riem / DD **5**. Prove, using the definition of the Riemann integral, that

$$\int_a^b \Big(f(x) + g(x)\Big)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx.$$

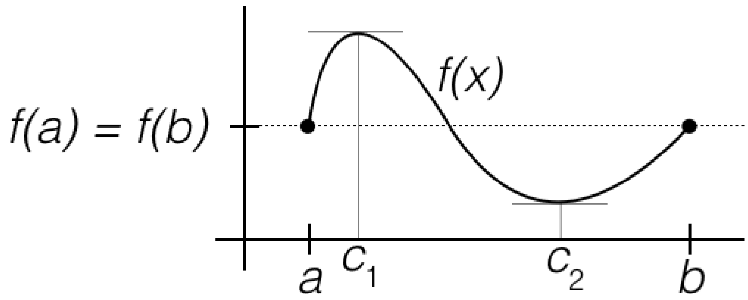
*Hint*: Such a proof will require $\epsilon$ (probably $\epsilon/2$), $\delta$ (probably $\delta = \min\{\delta_1, \delta_2\}$), and a chain of inequalities with summations. Use the sum $\int_a^b (f+g)\ dx$ as the number $A$ in the definition.

MVT / DD **6**. **Corollary** to Page 25 # 1. *A continuous real-valued function on a nonempty compact metric space attains a maximum and minimum.* (This is essential for calculus!)

*Proof.* Let $f : E \to \mathbf{R}$ be a continuous function on a nonempty compact metric space $E$. Then $f(E)$ is closed and bounded because ____________________________________ and nonempty because ____________________________________. A nonempty bounded set has an infimum $a$ and a supremum $b$, by Page __ # __. Furthermore, $a$ and $b$ are accumulation points of $f(E)$, so there are sequences in $f(E)$ converging to $a$ and $b$. Since $f(E)$ is closed, the limits of these sequences are in $f(E)$, because ____________________________________. Thus $a, b \in f(E)$, so $f(E)$ attains a maximum and minimum.

MVT / DD

**1. Rolle's Theorem.** *Suppose that $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$, and that $f(a) = f(b)$. Then for some $c \in (a, b)$, $f'(c) = 0$.*




Rolle's Theorem is the basis for the Mean Value Theorem, which we will see is very important in calculus.

*Proof.* We know that $f$ attains a maximum and a minimum on $[a, b]$, by ________________________________________. If the maximum occurs at some point $p$ on the interior, then by ________________________________________ $f'(p) = 0$, so let $c = p$ and we are done. The same argument holds for the minimum. If neither the maximum nor the minimum occurs on the interior, then they both occur at the endpoints, so... (Finish the proof).

**2. Proposition.** *If $f$ is differentiable at $x_0$, then $f$ is continuous at $x_0$.*

*Proof.* We will show that, if $\mathcal{U}$ is an open subset of $\mathbf{R}$, and $f : \mathcal{U} \to \mathbf{R}$ is differentiable at $x_0 \in \mathcal{U}$, then $f$ is continuous at $x_0$.

Since $f$ is differentiable at $x_0$, we know from Page __ # __ that for any $\bar{\epsilon} > 0$, we can choose $\bar{\delta} > 0$ so that

$$|x - x_0| < \bar{\delta} \implies |f(x) - f(x_0) - f'(x_0)(x - x_0)| < \bar{\epsilon}|x - x_0|.$$

Thus, $|x - x_0| < \bar{\delta}$ implies

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f(x_0) - f'(x_0)(x - x_0)| + |f'(x_0)(x - x_0)| \qquad (1) \\ &\leq (\bar{\epsilon} + |f'(x_0)|) \cdot |x - x_0|. \qquad (2) \end{aligned}$$

**(a)** Justify each of the inequalities (1) and (2).

Now choose $\delta = \min\left\{\bar{\delta}, \dfrac{\epsilon}{\bar{\epsilon} + |f'(x_0)|}\right\}$.

**(b)** Use the above to show that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$, as desired.

**(c)** Explain why we used the symbols $\bar{\delta}$ and $\bar{\epsilon}$ at the beginning instead of $\delta$ and $\epsilon$.

Riem / FM

**3.** Compute directly from the definition that $\int_0^1 x^2\, dx = 1/3$, as follows:

**(a)** Divide $[0, 1]$ into $n$ subintervals of width $1/n$. Show that, if we evaluate the following Riemann sum at the right endpoint of each interval, the result is

$$\sum_{k=1}^{n} f(x)\frac{1}{n} = \frac{1}{n^3}\sum_{k=1}^{n} k^2.$$

**(b)** Use the formula $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(2n + 1)(n + 1)}{6}$ and take the limit as $n \to \infty$.

Riem / MR

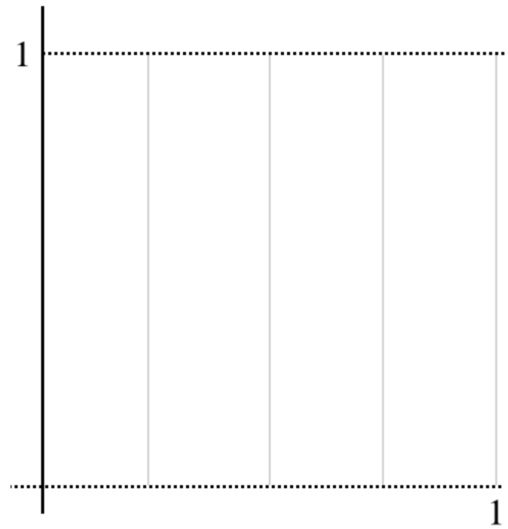
**4**. Show that, if $f : [a, b] \to \mathbf{R}$ is integrable, and $f(x) \geq 0$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\, dx \geq 0.$$

Riem / DD

**5**. **The characteristic function of the rationals.** Consider the function defined on $[0, 1]$ by:

$$\chi_{\mathbf{Q}}(x) = \begin{cases} 1 \text{ if } x \in \mathbf{Q} \\ 0 \text{ if } x \notin \mathbf{Q} \end{cases} .$$




For a partition of $[0, 1]$ of width $1/4$, find the value of each of the following Riemann sums, when the representative point $x_i'$:

**(a)** is the left endpoint of each interval;

**(b)** yields the maximum value of $\chi_{\mathbf{Q}}$ on its interval;

**(c)** yields the minimum value of $\chi_{\mathbf{Q}}$ on its interval.

**(d)** Find a partition of $[0, 1]$ of width $\leq 1/10$, and repeat parts (b)-(d).

**(e)** For $\epsilon = 1/2$, can you find an $A$ and a $\delta$ that satisfy the definition of Riemann integrability? How about for $\epsilon = 1/10$?

**(f)** What are the maximum and minimum values of a Riemann sum associated to a partition of width $1/4$? How about for width $1/10$?

**(g)** Is it possible to get a value of 0.123456789 for a Riemann sum of $\chi_{\mathbf{Q}}(x)$ on $[0, 1]$? Describe all possible values that you can get as a Riemann sum.

**(h)** Is $\chi_{\mathbf{Q}}(x)$ Riemann integrable on $[0, 1]$?

# Real Analysis

Riem / MR **1**. Show that, if $f, g : [a, b] \to \mathbf{R}$ are integrable, and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

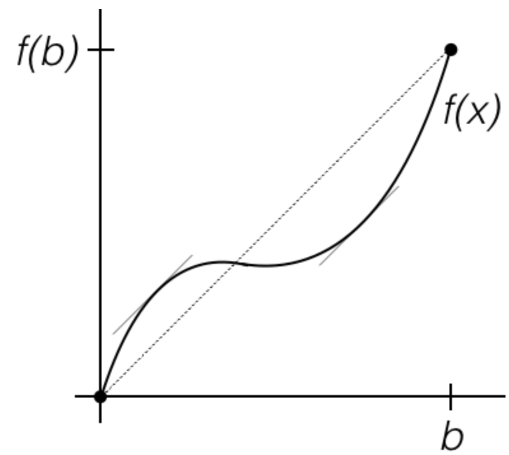
$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$$

*Hint*: You can prove this from scratch, but it is easier to apply a previous result.

MVT / DD **2**. The hypothesis for the preceding Rolle's Theorem, and for the upcoming Mean Value Theorem, is that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

**(a)** Why did we need $f$ to be continuous on $[a, b]$ instead of just $(a, b)$?

**(b)** Why don't we ask for $f$ to be differentiable on $[a, b]$ instead of just on $(a, b)$?




MVT / DD **3**. The picture on the right is meant to illustrate the Mean Value Theorem. On the same axes, sketch the function $g(x) = f(x) - x$, under the assumption that $f(b) = b$.

MVT / DD **4**. **Mean Value Theorem.** Suppose that $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Then for some $c \in (a, b)$,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The MVT is essentially the same as Rolle's Theorem, just "tilted," or perhaps we could call it "vertically sheared."

*Proof.* By horizontal scaling and translation, we may assume that $[a, b] = [0, 1]$, because ______________________________________. If $f(0) = f(1)$, then we are done, because ______________________________________. If not, then by vertical scaling and translation, we may assume that $f(0) = 0$ and $f(1) = 1$, because ______________________________________. Let $g(x) = f(x) - x$. (Finish the proof)

MVT / DD **5**. Check the Mean Value Theorem for the function $f(x) = x^3$ on $[0, 1]$. (This means: determine $a$, $b$, $f(a)$, and $f(b)$ in this case, and find the $c$ that satisfies the equation above.)

FTC / FM **6**. Show that, if $|f(x)| \leq M$ for all $x \in [a, b]$, then $\left| \int_a^b f(x)\, dx \right| \leq M(b - a)$.

*Hint*: You can do this from scratch, but it is easier to use a previous result.

**7**. Read the following story. Ponder its wisdom.

## Law & Order: MVT, by Evelyn Lamb, in *Scientific American*, Oct 13, 2019:

*In the criminal justice system, velocity-based offenses are considered especially unimportant. In New York, the dedicated detectives who investigate these minor misdemeanors are members of an elite squad known as the Moving Violation Team. These are their stories.*

[Open with aerial shot of the New York State Thruway. It is a beautiful fall day. Traffic is on the heavy side but moving freely. Zero in on a car passing below a set of cameras on the road.]

*DUN DUN*

[One week later, in Syracuse, NY. MICHELLE ROLLINS is bringing the mail inside. Their wife CARLA GOFF is sitting on the couch.]

MICHELLE ROLLINS: What's this? [opens envelope] Really? A ticket? But I didn't see any cops when I was driving last week.

CARLA GOFF: They have those cameras mounted above the roads now.

MICHELLE ROLLINS: I saw those. They were right near the toll plazas. I was never speeding when I was near one of the cameras. This is garbage! They can't prove I was speeding.

[A few days later, in the Moving Violation Team office. Detective DOROTHY BERNSTEIN is going through papers, filing some and tossing others. Her colleague EDDIE WILLIAMS looks on.]

DOROTHY BERNSTEIN: We've got another driver contesting the ticket.

EDDIE WILLIAMS: They just don't stop, do they? They have no idea what they're in for.

*DUN DUN*

[Inside the courtroom. Judge CHARLOTTE SCOTT presiding. Another *DUN DUN* for good measure.]

BAILIFF: Please rise.

CHARLOTTE SCOTT: You may be seated. What do we have today? Ah, a contested speeding ticket. Plaintiff, opening statement, please.

MICHELLE ROLLINS: Your honor, I received a speeding ticket, but I was never pulled over.

DOROTHY BERNSTEIN: Are you familiar with the cameras we have to record license plates for tolls?

MICHELLE ROLLINS: Sure.

DOROTHY BERNSTEIN: They also record your location and time.

MICHELLE ROLLINS: Of course, but I don't see how that's relevant.

DOROTHY BERNSTEIN: There are multiple cameras. We recorded you driving here, at mile marker 192, [holds up blurry photo of a car passing under a camera on the road] at 12:47 pm on October 12. Then we took this photograph of you at mile marker 148. Can you read the timestamp on that photograph for me?

MICHELLE ROLLINS: Are these theatrics really necessary?

CHARLOTTE SCOTT: Just answer the question.

MICHELLE ROLLINS: It says [squints] 1:21 pm.

DOROTHY BERNSTEIN: In 34 minutes, you traveled 44 miles. Is that correct?

MICHELLE ROLLINS: Yes.

DOROTHY BERNSTEIN: The speed limit for this entire portion of the highway is 65 miles per hour. Would you agree that your average speed was above 65 miles per hour?

MICHELLE ROLLINS: [muttering] 68 minutes, 88 miles, 60 minutes, 65 miles, plus 8 is 73, the extra is less than a mile... [regular voice] Yes, it was.

DOROTHY BERNSTEIN: As a matter of fact, it was 77.65 miles per hour.

MICHELLE ROLLINS: But that doesn't prove anything. The speed limit is not an average speed limit. You have to show I was traveling above 65 miles per hour at some point.

DOROTHY BERNSTEIN: Mx. Rollins, are you familiar with the Mean Value Theorem?

*DUN DUN*

[But no scene change]

MICHELLE ROLLINS: Yeah, I took calculus. That's the theorem that says that if your average rate of change between two endpoints is M, then your instantaneous rate of change at some point between two endpoints must have been M, if –

CHARLOTTE SCOTT: [bangs gavel] Case closed!

MICHELLE ROLLINS: Wait a minute, I didn't finish! That's *if* the function is a continuous function on the whole closed interval and differentiable on the open interval!

CHARLOTTE SCOTT: Are you saying the function describing your position was somewhere discontinuous or non-differentiable?

MICHELLE ROLLINS: I didn't say that, but, with all due respect, it's not my responsibility to prove they weren't but Detective Bernstein's to prove they were. Detective Bernstein, can you show that time and position are continuous, rather than discrete, quantities?

DOROTHY BERNSTEIN: Oh, please! Your honor, all widely-used modern and classical physical theories that are used to make predictions about real-world behavior use the assumption of continuous time. If time is not continuous, it is close enough on a practical level to assume such.

MICHELLE ROLLINS: By the same token, though, all numbers can be practically represented – to any degree of accuracy we desire – by rational numbers, can they not?

DOROTHY BERNSTEIN: Objection, your honor, irrelevant.

CHARLOTTE SCOTT: Mx. Rollins, where are you going with this?

MICHELLE ROLLINS: I promise it is highly relevant. At all points of my journey, we can assume the time and my position were rational numbers, using Detective Bernstein's "close enough on a practical level" criterion. Therefore my position was a function of time defined on the rational numbers. The mean value theorem does not hold for functions defined over the rationals! Take, for example, the function that is 0 for all rational numbers $q$ such that $q^2$ is less than 2 and 1 for all rational numbers whose squares are larger than 2. The average value of this continuous function on the $[0, 2]$ interval is strictly between 0 and 1, but the function only takes the values 0 and 1.

[A gasp ripples through the courtroom, which somehow is full of an audience of people despite the fact that this is a very boring traffic case.]

CHARLOTTE SCOTT: [bangs gavel] Order! Order! Detective Bernstein?

DOROTHY BERNSTEIN: [Stammering] Wait – I – what – You can't be serious!

CHARLOTTE SCOTT: If the detective cannot counter Mx. Rollins' argument, I have no choice but to dismiss the ticket.

[DOROTHY BERNSTEIN sinks into her chair. EDDIE WILLIAMS brings her a cup of coffee. MICHELLE ROLLINS leaves the courtroom to a flock of reporters outside.]

DOROTHY BERNSTEIN: Thanks, Eddie. I can't believe they're getting away with it.

EDDIE WILLIAMS: We can only hope the next one hasn't thought so deeply about the mean value property.

*DUN DUN*

Law & Order: MVT, by Evelyn Lamb, in *Scientific American*, Oct 13, 2019.

# Real Analysis

**Fundamental Theorem of Calculus.** Let $f$ be a continuous function on $[a, b]$.

I. $\frac{d}{db}\int_a^b f(x)\ dx = f(b)$.

II. If there exists $F$ such that $f(x) = F'(x)$, then $\int_a^b f(x)\ dx = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a)$.

FTC / FM **1**. *Proof of I.* We have

$$\frac{d}{db}\int_a^b f(x)\ dx = \lim_{h\to 0}\frac{\int_a^{b+h} f(x)\ dx - \int_a^b f(x)\ dx}{h} \tag{1}$$

$$= \frac{\int_b^{b+h} f(x)\ dx}{h}. \tag{2}$$

**(a)** Justify equation (1) .

For (2), we use the fact that if $f$ is integrable on $[a, c]$ and $a < b < c$, then $\int_a^c f = \int_a^b f + \int_b^c f$, whose proof is straightforward; we omit it here. Now if $h > 0$, we have

$$\min_{|x-b|\le|h|} f(x) \le \frac{\int_b^{b+h} f(x)\ dx}{h} \le \max_{|x-b|\le|h|} f(x), \tag{3}$$

by ______________________________________.

**(c)** Justify why equation (3) also holds when $h < 0$.

**(d)** Explain why, as $h \to 0$, the left and right sides of (3) both approach $f(b)$.

**(e)** Finish the proof, that

$$\frac{d}{db}\int_a^b f(x)\ dx = \frac{\int_b^{b+h} f(x)\ dx}{h} = f(b).$$

FTC / FM **2**. *Proof of II.* By (I), we have

$$\frac{d}{db}\left(F(b) - \int_a^b f(x)\ dx\right) = F'(b) - f(b) = f(b) - f(b) = 0.$$

**(a)** Justify the equalities above.

By ______________________________________, there is a constant $C$ such that

$$\frac{d}{db}\left(F(b) - \int_a^b f(x)\ dx\right) = 0 \implies F(b) - \int_a^b f(x)\ dx = C.$$

**(c)** Finish the proof by setting $b = a$ and deducing the desired statement.

FTC / FM **3**. Let $F(x) = \int_0^x e^{-t^2}\ dt$. Compute $F'(x)$ and $F'(0)$.

Riem / AJ

**4. Theorem.** Every continuous, real-valued function $f$ is integrable on $[a, b]$.

*Proof.* We will show that any sequence of Riemann sums whose partition widths converge to 0 is Cauchy. This will prove the result, because the sequence of Riemann sums is a Cauchy sequence of real numbers, which converges because ______________________________.
The limit it converges to is then the Riemann integral $\int_a^b f(x)\, dx$.

We know that $f$ is uniformly continuous, because ______________________________.
So, given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon. \tag{1}$$

Consider two Riemann sums, each with width less than $\delta/2$. Their subintervals intersect (if we break up $[a, b]$ at all the places where either of the partitions has a subinterval break) in smaller subintervals of width also less than $\delta/2$, because ______________________.
On each subinterval, the values $f(x)$ from the two Riemann sums $S_1$ and $S_2$ come from points at distance at most $\delta/2$ from a point in the intersection, and hence at distance at most $\delta$ from each other, because ______________________________. By (1), these values differ by at most $\epsilon$. Let $x_i^1$ and $x_i^2$ be the representative values chosen for $S_1$ and $S_2$, respectively. Summing over the smaller subintervals, we see that the Riemann sums can differ by at most

$$\begin{aligned}
\left|\sum \left(f(x_i^1) - f(x_i^2)\right)(x_i - x_{i-1})\right| &\le \sum \left|f(x_i^1) - f(x_i^2)\right|(x_i - x_{i-1}) && (2)\\
&\le \sum \epsilon(x_i - x_{i-1}) && (3)\\
&= \epsilon \sum (x_i - x_{i-1}) && (4)\\
&= \epsilon(b - a). && (5)
\end{aligned}$$

Justify the lines (2) (3) (4) (5).

Since $b - a$ is finite, we can make $\epsilon(b - a)$ as small as we like, so the sequence is Cauchy, proving the result.

# Real Analysis

*Choose some of the following 10 problems to complete: those that interest you*

The following result is equivalent to the Mean Value Theorem (MVT). In words, it says that "how a function varies on an interval depends on the length of the interval, multiplied by some bound on the derivative in that interval."

FTC / MR **1**. **Corollary 1 to the MVT.** Suppose that $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Then for some $c \in (a, b)$, $f(b) - f(a) = f'(c) \cdot (b - a)$. (Prove this.)

The following result is the reason why we need the Mean Value Theorem to do calculus.

FTC / FM **2**. **Corollary 2 to the MVT.** On an open interval where $f'$ is always 0, $f$ is constant. (Prove this.)

---

Prove the following:

FTC / MR **3**. **Corollary 1 to the FTC.** A continuous function has an antiderivative.

More precisely: If $f$ is a continuous, real-valued function on an open interval $\mathcal{U} \subset \mathbf{R}$, then there exists a real-valued function $F$ on $\mathcal{U}$ such that $F'(x) = f(x)$.

FTC / MR **4**. **Corollary 2 to the FTC.** Antiderivatives differ by a constant.

More precisely:If $F$ and $G$ are both antiderivatives of $f$, then $F - G = C$ for some constant $C$.

FTC / MR **5**. **Corollary 3 to the FTC.** If $F$ is the antiderivative for $f$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

More precisely: If $\mathcal{U} \subset \mathbf{R}$ is open and $F : \mathcal{U} \to \mathbf{R}$ has continuous derivative $f$, then for $a, b \in \mathcal{U}$, $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

---

Fun with the Cantor function.

**6**. Recall the Cantor function $f_{\mathcal{C}}$, defined on Page 28. It is a non-constant, continuous function on $[0, 1]$, with derivative 0 everywhere except on the Cantor set, a set that has measure (total length) 0. Does $f_{\mathcal{C}}$ violate Corollary 2 to the MVT? Explain why or why not.

Cantor / DD **7**. (Brian Jenike) Where is $f_{\mathcal{C}}$ continuous? Prove your answer correct.

Cantor / DD **8**. (Hari Srinivasulu) Where is $f_{\mathcal{C}}$ differentiable? Prove your answer correct.

Cantor / DD **9**. Let $\chi_{\mathcal{C}}$ be the characteristic function of the Cantor set: It is 1 on $\mathcal{C}$, and 0 otherwise. Compute $\displaystyle\int_0^1 \chi_{\mathcal{C}}\,dx$.

*Hint*: Given $\epsilon > 0$, choose $n$ such that $(2/3)^n < \epsilon$ and $\delta < 1/3^n$, and use the partition of $[0, 1]$ determined by the intervals of the Cantor set in the $n^{\text{th}}$ step of construction.

Cantor / DD **10**. Recall that you sketched the Cantor function $f_{\mathcal{C}}(x)$ on Page 28. Find $\displaystyle\int_0^1 f_{\mathcal{C}}(x)\,dx$.

*Hint*: refer to your picture

# Real Analysis

## Appendix: Sequences of functions

I omitted these problems so that we would be able to prove the Fundamental Theorem of Calculus. If you would like to do them, please do! Note that these problems may use results proved on earlier pages.

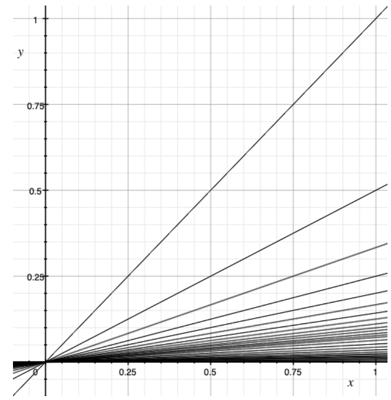
seq-fn / DD **1**. Say whether the following sequences (also used in the following problems) converge, and if so, to what limit function:

**(a)** $f_n : \mathbf{R} \to \mathbf{R}$ given by $f_n(x) = x/n$.

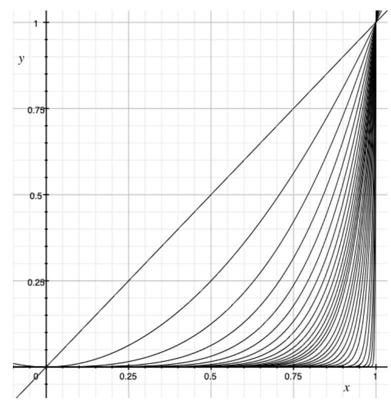
**(b)** $g_n : [0,1] \to [0,1]$ given by $g_n(x) = x^n$.

**(c)** $h_n : \mathbf{R} \to \mathbf{R}$ given by

$$h_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{n-1}\, x^{2n-1}}{(2n-1)!}.$$




It is a bit disturbing that, in some cases that we have seen, a limit of continuous functions is not continuous. The problem is that although each $f_n \to f$ at each point, for some functions, some points take longer to converge than others. We can express this difference precisely, as the difference between *pointwise convergence* (now) and *uniform convergence* (later):




**Pointwise convergence.** A sequence of functions $f_n : E \to E'$ *converges pointwise* to $f$ on $E$ if, for every $x \in E$, and given any $\epsilon > 0$, there exists an $N$ such that

$$n > N \implies d'(f_n(x), f(x)) < \epsilon.$$

Notice that in this definition, $N$ depends on both $x$ and $\epsilon$.

seq-fn / DD **2**. For the given $x$ and $\epsilon$, find $N$ such that $n > N \implies |f_n(x) - f(x)| < \epsilon$.

**(a)** $x = 1/2, \epsilon = 0.01$ **(b)** $x = 0.9, \epsilon = 0.01$

seq-fn / DD **3**. Repeat the above problem for $g_n(x)$ with $g(x)$.

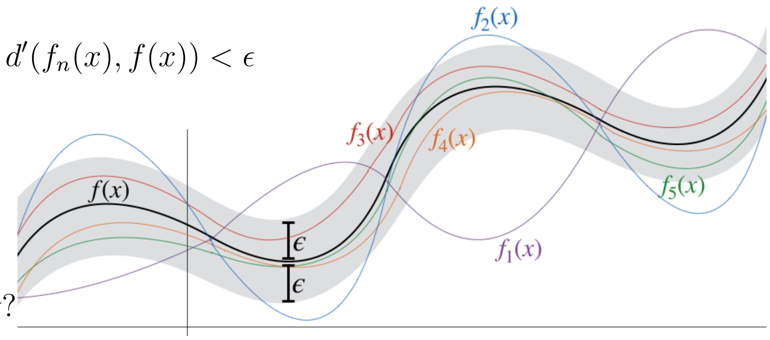
# Real Analysis

**Uniform convergence.** A sequence of functions $f_n : E \to E'$ converges *uniformly* to $f$ on $E$ if, given any $\epsilon > 0$, there exists an $N$ such that

$$n > N \implies d'(f_n(x), f(x)) < \epsilon$$

for all $x \in E$.

seq-fn / DD **4.** Explain the difference between the definition of pointwise convergence and the definition of uniform convergence. In Page A1 # 1, which functions converge uniformly?




We have previously seen that some limits of continuous functions are not continuous. For the limit to be continuous, *uniform* convergence is exactly what we need.

seq-fn / FM **5.** **Theorem.** *A uniform limit of continuous functions is continuous.*

*Proof.* We will show that, for a uniformly convergent sequence of functions $f_1, f_2, \ldots$, with $f_n : E \to E'$ for each $n$, $\lim\limits_{n\to\infty} f_n(x) = f(x)$ is continuous. More precisely, we will show that, given any $\epsilon > 0$, we can find $\delta$ such that **(a)** ________________. The idea is that because the sequence of functions converges uniformly, we can handle all points $x$ near any particular point $p$ by looking at one $f_n$ (specifically $f_{N+1}$) that is uniformly near the limit function $f$.

Given any $\epsilon > 0$, choose $N$ such that for all $x \in E$,

$$n > N \implies d(f_n(x), f(x)) < \epsilon/3. \tag{6}$$

We can find such an $N$ because **(b)** ________________________.

We know that $f_{N+1}$ is continuous because **(c)** ________________________. Since $f_{N+1}$ is continuous, given any $p \in E$, we can choose $\delta > 0$ such that

$$d(x,p) < \delta \implies d'(f_{N+1}(x) - f_{N+1}(p)) < \epsilon/3. \tag{7}$$

We can find such a $\delta$ because **(d)** ________________________. So if $d(x,p) < \delta$, we have

$$\begin{aligned} d'(f(x), f(p)) \;&\le\; d'(f(x), f_{N+1}(x)) \;+\; d'(f_{N+1}(x), f_{N+1}(p)) \;+\; d'(f_{N+1}(p), f(p)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon, \end{aligned}$$

as desired. Here the inequality in the first line is true by **(e)** ________________________. On the right hand side, the first term is less than $\epsilon/3$ by equation **(f)** ______________, the second term is less than $\epsilon/3$ by equation **(g)** ______________, and the third term is less than $\epsilon/3$ by equation **(h)** ______________.

# Real Analysis

For a sequence of points, we wanted a way to discuss convergence without knowing what the limit is. This notion is "Cauchy" − for any $\epsilon > 0$, there exists $N$ such that $n, m > N \implies d(a_n, a_m) < \epsilon$. We would like to have an analogous way to talk about a sequence of *functions* converging, without knowing what the limiting function is. This notion is *uniformly Cauchy,* combining the notion of the sequence of functions converging uniformly, with the notion of the function values being a Cauchy sequence at each point. The definition of *uniformly Cauchy* is in the beginning of the proof below.

Cauchy / DD **6. Theorem.** *A sequence of functions mapping to a complete metric space is uniformly convergent if and only if the sequence of functions is uniformly Cauchy.*

*Proof.* We will prove that, for a sequence of functions $f_n : E \to E'$, if $E'$ is complete, then $f_n$ is uniformly convergent if and only if, for any $\epsilon > 0$, there exists $N$ such that

$$n, m > N \implies d'(f_n(p), f_m(p)) < \epsilon \text{ for all } p \in E,$$

or in other words, the sequence of $f_n$ is uniformly Cauchy.

*Proof.* ($\Rightarrow$) We will assume that $f_n \to f$ uniformly, and show that $f_n$ is uniformly Cauchy.

Suppose that $f_n$ converges uniformly to $f$. Then, given $\epsilon > 0$, there exists $N > 0$ such that $d'(f(x), f_n(x)) < \epsilon/2$ for all $n > N$ and all $x \in E$, because **(a)** ____________________.
So for all $n, m > N$, we have

$$\begin{aligned} d'(f_n(x), f_m(x)) &\leq d'(f_n(x), f(x)) + d'(f(x), f_m(x)) \\ &< \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

as desired. Here the inequality in the first line is true by **(b)** ____________________.
Each of the terms on the right hand side of the first line are less than $\epsilon/2$ because **(c)** ____________.

($\Leftarrow$) We will assume that $f_n$ is uniformly Cauchy, and show that $f_n \to f$ uniformly.

Suppose that the sequence $f_n$ is uniformly Cauchy. Then for every $x \in E$, and for every $n$, $\{f_n(x)\}$ is a Cauchy sequence in $E'$, because **(d)** ____________. Since $E'$ is complete, $\{f_n(x)\}$ converges to a point in $E'$ because **(e)** ____________________, and we will call this point $f(x)$. This shows that $f_n \to f$ pointwise. Now we need to show that $f_n \to f$ uniformly.

Given any $\epsilon > 0$, choose $N$ such that

$$n, m > N \implies d'(f_n(x), f_m(x)) < \epsilon/2 \text{ for all } x \in E. \tag{1}$$

We can find such an $N$ because **(f)** ____________________. We want to show that, for any $n > N$ and any $x \in E$, $d'(f_n(x), f(x)) < \epsilon$.

(continued on next page)

(continued)

Fix a particular such choice of $n$ and $x$. Now look at the ball $B_{\epsilon/2}(f_n(x))$. The sequence $f_1(x), f_2(x), \ldots$ eventually enters this ball and stays within it, by **(g)** ________________. So $f(x) \in \overline{B_{\epsilon/2}(f_n(x))}$, because **(h)** ____________________________. Here we use the closure of the ball because **(i)** ____________________________. Thus,

$$d'(f_n(x), f(x)) \ \le \ \epsilon/2 \ < \ \epsilon,$$

as desired. Here the first inequality is true because **(j)** ____________________________.

seq-fn / FM **7**. Let $f_n(x) = x/n$.

**(a)** Prove that $f_n(x) \to f(x) = 0$ uniformly on $[0, 1]$.

**(b)** Does $f_n(x) \to 0$ uniformly on $\mathbf{R}$?

**Derivatives and integrals with sequences of functions.**

switch / DD **8**. Let $f_n(x) = x^n$ on $[0, 1]$ (we have seen this before).

**(a)** Find $f(x) = \lim\limits_{n\to\infty} f_n(x)$. Does $f_n \to f$ uniformly on $[0, 1]$?

**(b)** Find $\int_0^1 f_n(x)\, dx$, as a function of $n$, and $\int_0^1 f(x)\, dx$.

**(c)** Is $\displaystyle \lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \int_0^1 \lim_{n\to\infty} f_n(x)\, dx$ true in this case?

uni-con / FM **9**. (For fun.) Consider the statement: If $f_n \to f$ uniformly, then ${f_n}^2 \to f^2$ uniformly.

**(a)** Give a counterexample to the statement.

**(b)** Add a simple hypothesis, and prove the revised statement.

switch / DD **10**. Let $g_n(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ n & \text{if} \quad 0 \le x \le 1/n \\ 0 & \text{if } 1/n < x < 1 \end{cases}$ on $[0, 1]$. Repeat Page A4 # 8 for this function.

switch / DD **11**. Make a conjecture: For a sequence of functions $f_n$ on $[a, b]$, when can you switch the limit and the integral? We will prove a Theorem about this on the next page.

We have seen that for $f_n(x) = x^n$ on $[0, 1]$, $\lim \int f_n = \int \lim f_n$, and just above we saw an example of a sequence of functions where the two are *not* equal. You may wonder, based on the above examples, *when* you can switch the limit and the integral. The following Theorem tells you that, as usual, *uniform* convergence is the key! See the next Theorem:

# Real Analysis

switch / AJ **12. Theorem.** For a uniformly convergent sequence of continuous functions $f_n$ on $[a, b]$, you can switch the limit and the integral:

$$\lim_{n\to\infty} \int_a^b f_n(x)\ dx = \int_a^b \lim_{n\to\infty} f_n(x)\ dx.$$

*Proof.* Given $\epsilon > 0$, choose $N$ such that, for all $x \in [a, b]$, $n > N \implies |f_n(x) - f(x)| < \epsilon$. We can do this because ____________________________________. Then

$$\left| \int_a^b f_n(x)\ dx - \int_a^b f(x)\ dx \right| = \left| \int_a^b (f_n(x) - f(x))\ dx \right| \tag{1}$$

$$\leq \left| \int_a^b |f_n(x) - f(x)|\ dx \right| \tag{2}$$

$$< \int_a^b \epsilon\ dx = \epsilon(b-a). \tag{3}$$

Justify each of the three equalities/inequalities above. Then complete the proof.

switch / AJ **13.** Give a counterexample to the above Theorem when each of the following hypotheses is omitted: **(a)** $f_n$ are continuous, **(b)** $f_n$ converge uniformly.

switch / AJ **14.** Write the converse of the above Theorem. Then prove it or give a counterexample.

switch / DD **15.** The purpose of the "fill-in" theorems was to teach you how to read proofs: by fighting with every single statement and justifying to yourself why it is true. Practice this skill by reading the following theorem copied from *Introduction to Analysis* by Maxwell Rosenlicht (p. 140). Work through the proof, figuring out why each part is true and how it all works.

**Theorem.** Let $f_n : \mathcal{U} \to \mathbf{R}$, where $\mathcal{U} \subset \mathbf{R}$ is open. Suppose that $f_n'$ is continuous for all $n$, $f_n'$ converges uniformly on $\mathcal{U}$, and for some $a \in \mathcal{U}$, $\{f_n(a)\}$ converges. Then

- $\lim f_n$ exists,
- $\lim f_n$ is differentiable, and
- $(\lim f_n)' = \lim f_n'$.

*Proof.* By the Fundamental Theorem of Calculus, we have that, for all $x \in \mathcal{U}$ and all $n \in \mathbf{N}$,

$$\int_a^x f_n'(t)\ dt = f_n(x) - f_n(a).$$

Let $\lim\limits_{n\to\infty} f_n' = g$. By the Theorem in Page 28 # 1, $\lim\limits_{n\to\infty} (f_n(x) - f_n(a))$ exists for any $x \in U$, and equals $\int_a^x g(t)\ dt$. Since $\lim\limits_{n\to\infty} f_n(a)$ exists, so does $\lim\limits_{n\to\infty} f_n(x)$. Setting $\lim\limits_{n\to\infty} f_n(x) = f(x)$ we have

$$f(x) - f(a) = \int_a^x g(t)\ dt$$

for each $x \in U$. A second use of the Fundamental Theorem of Calculus gives $f' = g$, which is what was to be proved.

**16. Theorem (Differentiating series of functions).** *For a sequence of functions that converges somewhere, whose derivatives are continuous, and whose sequence of partial sums is uniformly convergent, you can switch the derivative and the integral.*

*Proof.* We'll show that, if $f_k'$ is continuous for all $k$, and $\sum_{k=1}^{n} f_k'$ is uniformly convergent, and $\sum_{k=1}^{n} f_k(a)$ converges for some $a$, then $\lim_{n\to\infty} s_n' = \left(\sum_{k=1}^{n} f_k\right)' = f'$.

**(a)** Use previous work to show that, for a sequence of partial sums $s_n$, if $s_n'$ is continuous, and $s_n'$ is uniformly convergent, and $\{s_n(a)\}$ converges for some $a$, then $\left(\lim_{n\to\infty} s_n\right)' = \lim_{n\to\infty} s_n'$.

Now we have

$$\begin{aligned}
\lim_{n\to\infty} s_n' &= \lim_{n\to\infty} (f_1 + f_2 + \cdots + f_n)' && (1)\\
&= \lim_{n\to\infty} (f_1' + f_2' + \cdots + f_n') && (2)\\
&= \lim_{n\to\infty} \sum_{k=1}^{n} f_k' && (3)\\
&= \sum_{k=1}^{\infty} f_k' && (4)\\
&= \left(\lim_{n\to\infty} s_n\right)' && (5)\\
&= \left(\sum_{k=1}^{n} f_k\right)' = f', && (6)
\end{aligned}$$

as desired.

(Justify each of the six above equalities.)

**17. Corollary.** *For a uniformly convergent series, you can differentiate term by term.*

(Prove this.)

uni-con / DD **4.** We have studied:

- continuous and *uniformly* continuous functions,
- pointwise convergent sequences of functions and *uniformly* convergent sequences of functions, and
- *uniformly* Cauchy sequences of functions.

Look at all the definitions, and explain what "uniformly" means in general.