Billiards, Surfaces and Geometry

Math 690

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The problems in this text

This style of problems is based on the curriculum at Phillips Exeter Academy, a private high school in Exeter, NH. Most of the problems were written by Diana Davis, some based on her previous book *Lines in positive genus: An introduction to flat surfaces* and some specifically for this course; these are labeled in the margin as **DD**. Some problems are taken from *Geometry and Billiards* by Serge Tabachnikov, labeled in the margin as **ST**. The dissection problems are taken from *Mostly Surfaces* by Rich Schwartz, labeled in the margin as **RS**. Problems from Phillips Exeter Academy's materials are labeled **PEA**. Anyone is welcome to use this text, and these problems, so long as you do not sell the result for profit. If you create your own text using these problems, please give appropriate attribution, as I am doing here.

About the course

This course met three or four mornings a week for 50 minutes. The nine students in the class did one page of homework in preparation for class, and posted their solutions online. Class time consisted of discussing solutions. A proof course was the prerequisite.

To the Student

Contents: As you work through this book, you will discover that various topics about geometry, surfaces and billiards have been integrated into a mathematical whole. There is no Chapter 5, nor is there a section on ellipses. The curriculum is problem-centered, rather than topic-centered. Techniques, definitions and theorems will become apparent as you work through the problems, and you will need to keep appropriate notes for your records.

Your homework: Each page of this book contains the homework assignment for one night. The first day of class, we will work on the problems on page 1, and your homework is page 2; on the second day of class, we will discuss the problems on page 2, and your homework will be page 3, and so on for each of the 35 class days of the semester. You should plan to spend one hour each night solving problems for this class.

Comments on problem-solving: Please approach each problem as an exploration. Reading each question carefully is essential, especially since definitions, highlighted in italics, are routinely inserted into the problem texts. It is important to make large, clear, accurate diagrams, and paper models, whenever appropriate. Useful strategies to keep in mind are: create an easier problem, work backwards, and recall a similar problem. It is important that you work on each problem when assigned, since the questions you may have about a problem will likely motivate class discussion the next day.

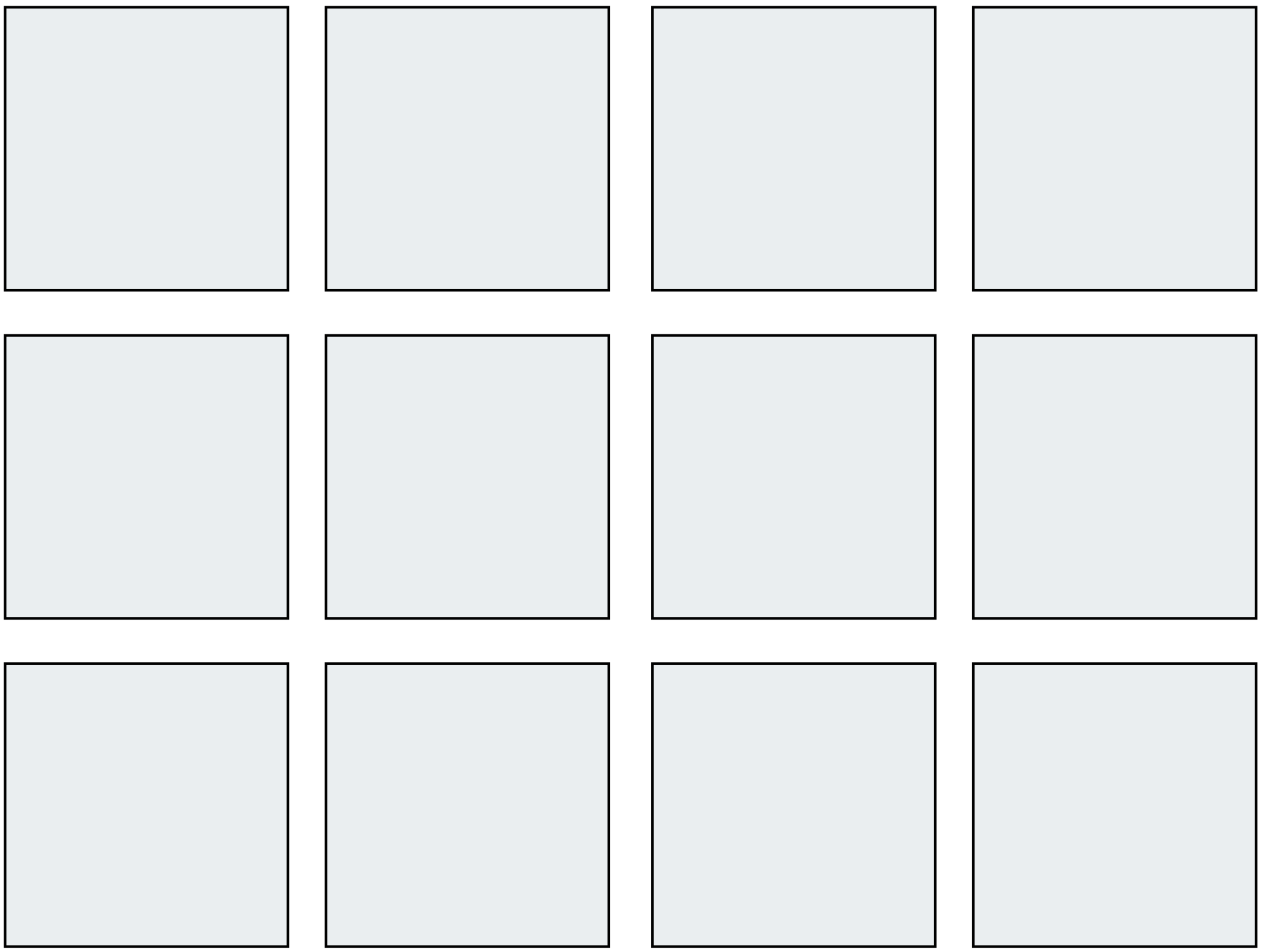
Problem-solving requires persistence as much as it requires ingenuity. When you get stuck, or solve a problem incorrectly, back up and start over. Keep in mind that you're probably not the only one who is stuck, and that may even include your teacher. If you have taken the time to think about a problem, you should bring to class a written record of your efforts, not just a blank space in your notebook. The methods that you use to solve a problem, the corrections that you make in your approach, the means by which you test the validity of your solutions, and your ability to communicate ideas are just as important as getting the correct answer.

Billiards, Surfaces and Geometry

DD **1.** Consider a ball bouncing around inside a square billiard table. We'll assume that the table has no "pockets" (it's a billiard table, not a pool table!), that the ball is just a point, and that when it hits a wall, it reflects off and the angle of incidence equals the angle of reflection, as in real life.

(a) A billiard path is called *periodic* if it repeats, and the *period* is the number of bounces before repeating. Construct a periodic billiard path of period 2.

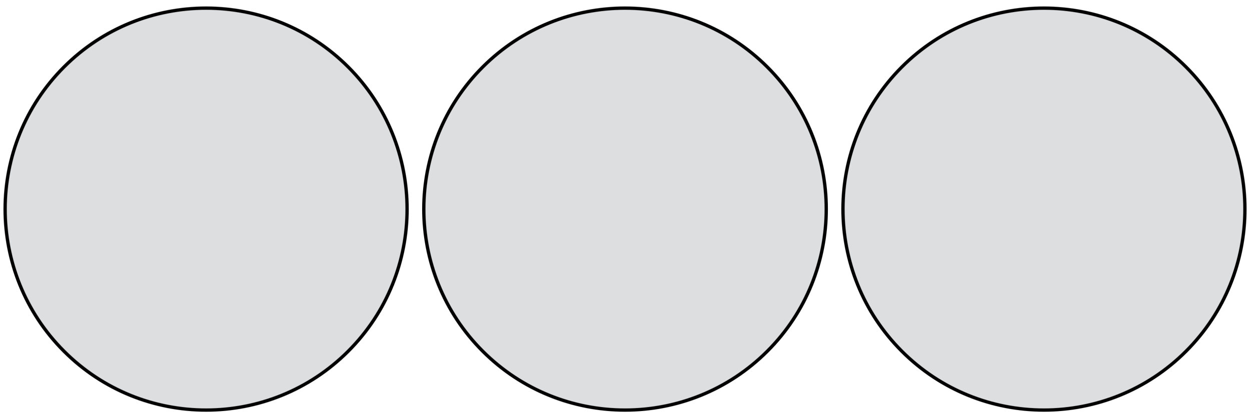
(b) For which other periods can you construct periodic paths?



→ *more problems on the other side!* ←

DD **2.** Now consider a *circular* billiard table. Again assume that the ball is just a point, and that when it bounces off, the angle of incidence equals the angle of reflection. Note that in a billiard table with curved edges, the ball reflects off of the *tangent line* to the point of impact.

Draw several examples of billiard trajectories in a circular billiard table. Describe the behavior in general.



ST 1.8 **3.** Suppose 100 ants are on a log 1 meter long, each moving either to the left or right with unit speed. Assume the ants collide elastically (when they hit each other, each ant immediately turns around and goes the other way), and that when they reach the end of the log, they fall off. What is the longest possible waiting time until all the ants are off the log?

From class:

Proposition. For periodic billiard paths on the square billiard table:

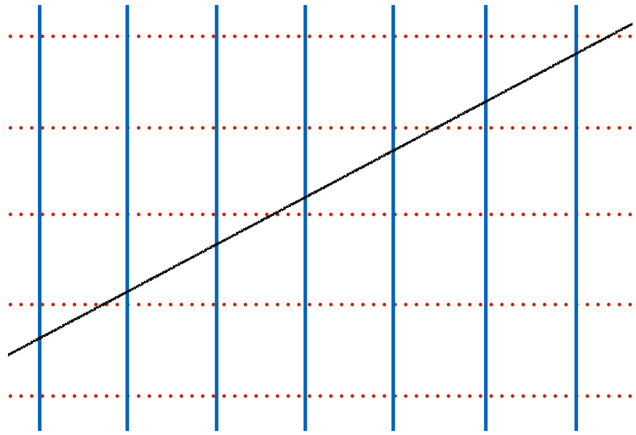
1. Every path has an even period, and
2. Every even number is the period of some billiard path.

Proof.

1. For a path in the direction of angle θ, bouncing against a vertical side transforms the direction to angle $180 - \theta$, and bouncing against a horizontal side transforms the direction to angle $-\theta$. A subsequent bounce transforms the angle back to θ, or to $180 + \theta$. Thus after an even number of bounces, the angle is in the set $\{\theta, 180 + \theta\}$, and after an odd number of bounces, the angle is in the set $\{-\theta, 180 - \theta\}$. For a path to be periodic, it must return to its original direction, which can only occur after an even number of bounces.
2. To construct a path of period $2(n + 1)$: it meets the midpoints of the two vertical edges, and meets the top and bottom edges at points separated by distances of $\frac{1}{2n}, \frac{1}{n}, \dots \frac{1}{n}, \frac{1}{2n}$, with the path forming n diamonds. (Picture needed for this part.)

Billiards, Surfaces and Geometry

DD **1.** Draw a line on an infinite square grid, and record each time the line crosses a horizontal or vertical edge. We will assume that the direction of travel along a line is always left to right. We could record the line to the right with the sequence …●○●●○●●○●●○●●○…, or we could assign A to horizontal and B to vertical edges, and record it as $\ldots BABBABBABBA\ldots$.

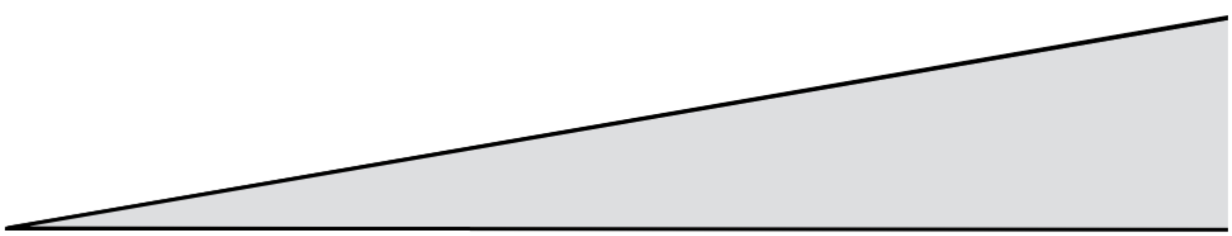


(a) What is the slope of the line in the picture?

(b) Record this *cutting sequence* of colors, or of As and Bs, for several different lines. Describe any patterns you notice. What can you predict about the cutting sequence, from the line?

(c) What should you do if the line hits a vertex?

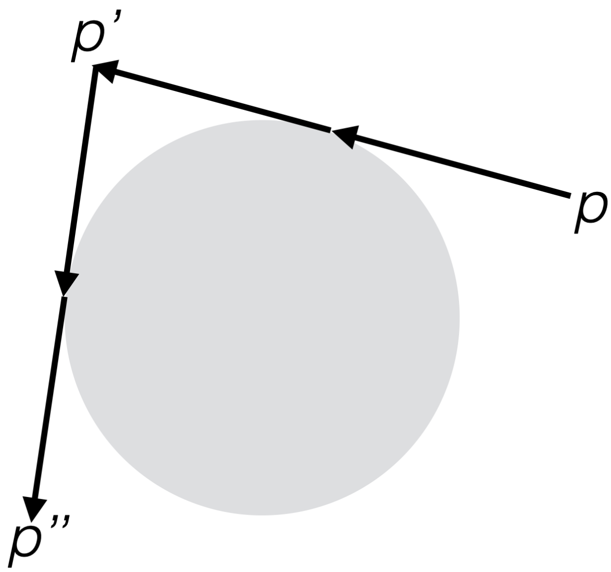
Contextual note. The sequences of symbols in problem 1 are called *cutting sequences*. Dr. Caroline Series (pictured to the left), a British mathematician, wrote a series of papers exploring these sequences and linking them to other areas of mathematics, in the 1980s. We will see that cutting sequences are related to group theory and continued fractions; Series also explained their relationship with hyperbolic geometry.



DD **2.** Consider a billiard "table" in the shape of an infinite sector with a small vertex angle, say 10°. Draw several examples of billiard trajectories in this sector (calculate the angles at each bounce so that your sketch is accurate). Is it possible for the trajectory to go in toward the vertex and get "stuck"? Find an example of a trajectory that does this, or explain why it cannot happen.

$\rightarrow$ *more problems on the other side!* $\leftarrow$

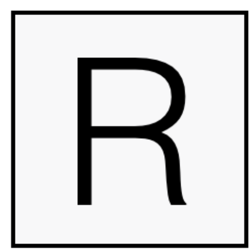
DD **3.** *Outer billiards.* Though it may seem strange to call it "billiards," we can also define a billiard map on the *outside* of a billiard table. We choose a starting point p, and a direction, either clockwise or counter-clockwise. Then, draw the tangent line from p to the table in that direction to find the point of tangency. Double the vector from p to the point of tangency, and add this to p to get p', as in the picture. We repeat the construction to find p'', and so on.



Draw several examples of outer billiards on a circular table, and describe the behavior in general.

By the way, outer billiards are sometimes called "dual billiards." When talking about several kinds of billiards, you can use the term "inner billiards" for regular billiards.

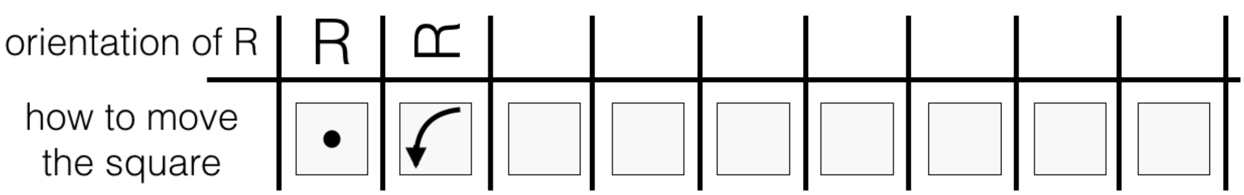
4. *Symmetries of the square.* If you turn a square 90° counter-clockwise, it looks the same as before. We call a 90° counter-clockwise rotation a *symmetry* of the square, because after you do it, you have a square just like the original. Let's find all the symmetries of the square.



(a) Cut out a square and draw an **R** on one side, as shown, and also hold it up to the light and trace through a backwards **R** on the back.

(b) How many different symmetries of the square can you find? Record in the first line of the table below the appearance of the **R** for each one.

(c) In the second line of the table, indicate how to move the square to achieve that position.



From class:

Proposition. For a trajectory on the square grid that passes through a lattice point:

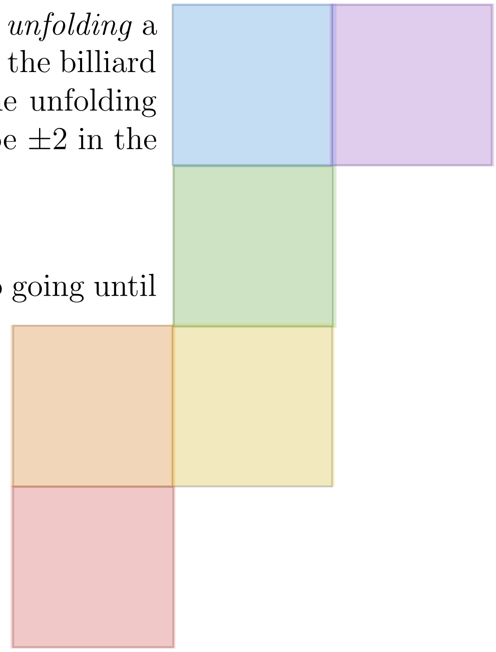
1. If its slope is rational, it passes through another lattice point, and
2. if its slope is irrational, it does not.

Proof.

1. If the slope is a rational number y/x in lowest terms, then another lattice point is found by adding the vector $[x, y]$ to the original point.
2. Since the slope is irrational, if you move horizontally a whole number of units, you move vertically an irrational number of units, so you cannot land on a lattice point.

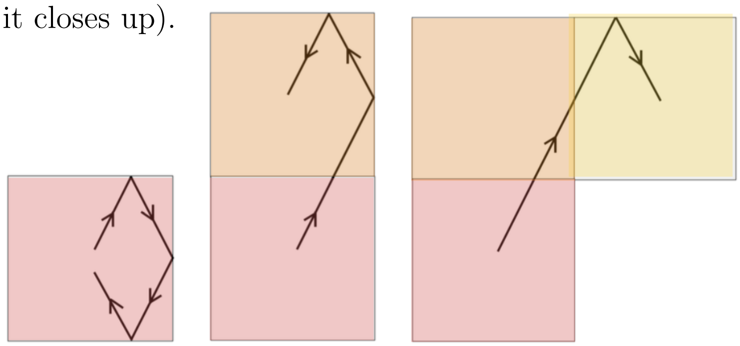
Billiards, Surfaces and Geometry

DD **1.** A powerful tool for understanding inner billiards is *unfolding* a trajectory into a straight line, by creating a new copy of the billiard table each time the ball hits an edge. Two steps of the unfolding process are shown for a small piece of trajectory of slope ± 2 in the square.



(a) Draw some more steps of the unfolding.

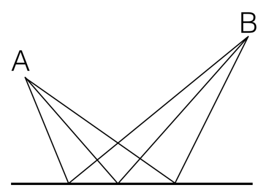
(b) Draw the complete billiard path in the square (keep going until it closes up).



(c) Use the unfolding to explain why a trajectory with slope 2 yields a periodic billiard trajectory on the square. (We always assume that one edge is horizontal.)

(d) Which other slopes yield a periodic billiard trajectory?

ST **2.** *A proof of the billiard reflection law, part I.* The *Fermat principle* says that light propagates from point A to point B along the path that takes the least possible time. Since our paths are in the Euclidean plane, this is just the shortest path. Consider a single reflection in a flat mirror ℓ (the horizontal line in the picture), and find the point X along the line that minimizes the distance $AX + XB$. Explain how to obtain the billiard reflection law (angle of incidence equals angle of reflection) as a consequence.



3. The *continued fraction expansion* gives an expanded expression of a given number. To obtain the continued fraction expansion for a number, say 15/11, we do the following:

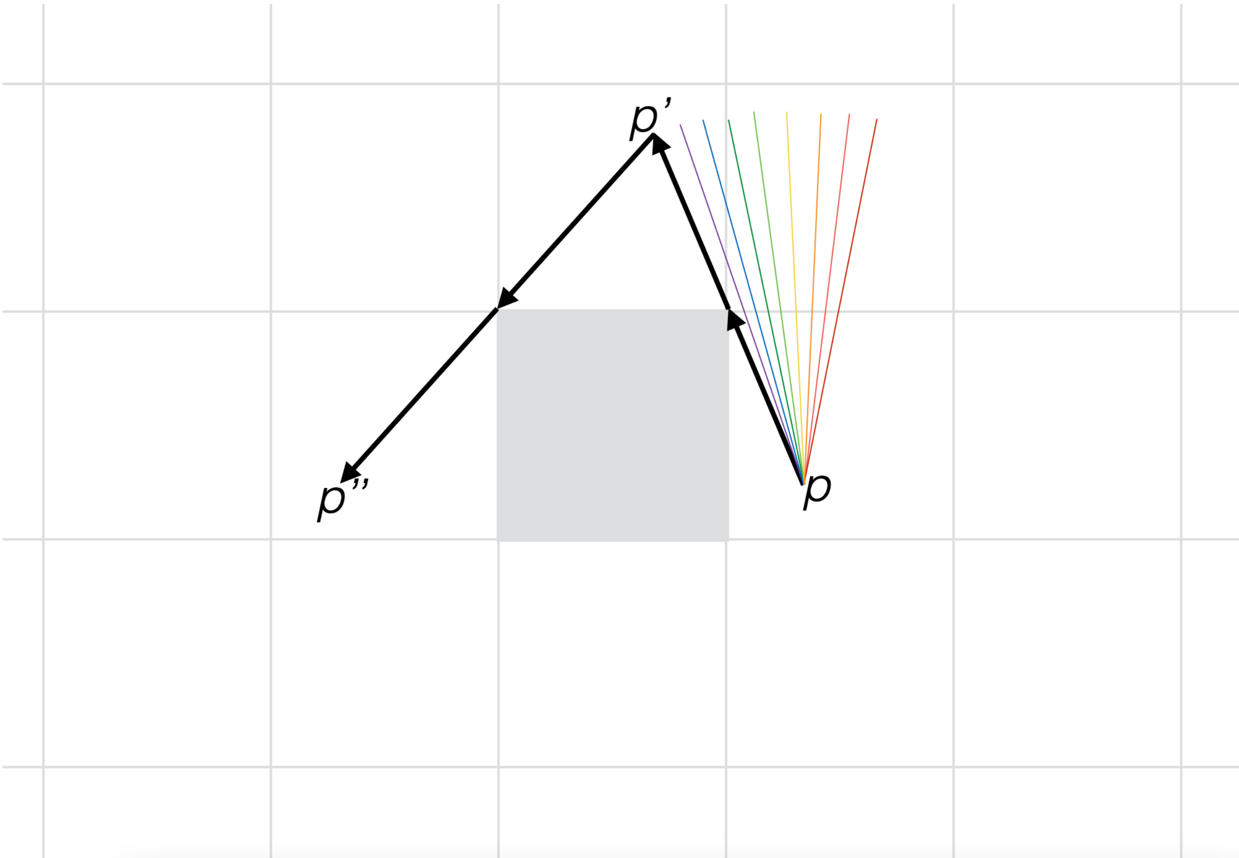
$$\frac{15}{11} = 1 + \frac{4}{11} = 1 + \frac{1}{11/4} = 1 + \frac{1}{1 + 7/4} = 1 + \frac{1}{2 + 3/4} = 1 + \frac{1}{2 + \cfrac{1}{4/3}} = \mathbf{1} + \cfrac{1}{\mathbf{2} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{3}}}}.$$

The idea is to pull off 1s until the number is less than 1, take the reciprocal of what is left, and repeat until the reciprocal is a whole number. Since all the numerators are 1, we can denote the continued fraction expansion compactly by recording only the bolded numbers: $15/11 = [1; 2, 1, 3]$. The semicolon indicates that the initial 1 is outside the fraction.

Find the first few steps of the continued fraction expansion of π. Explain why the common approximation 22/7 is a good choice. Find the best fraction to use, if you want a fractional approximation for π using integers of three digits or fewer.

DD **4.** We can also play outer billiards on polygonal tables. Here, the "tangent line" is always through a vertex – you can think of sweeping a line counter-clockwise until it hits a vertex, as shown.

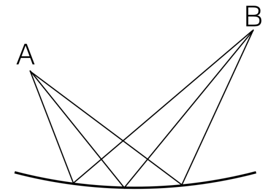
Draw several examples of outer billiards on a square table. Can you find any periodic trajectories? *Hint:* be accurate. Use a ruler.



Contextual note. The outer billiards system was proposed as a toy model for planetary motion: the table is the sun, and the point is the planet bouncing around it. It is easier to analyze a "discrete" dynamical system, where a planet jumps from place to place, than a "continuous" dynamical system where planets move smoothly, continuously interacting with each other.

It is a problem of great importance to know whether our solar system is stable or whether Earth will spin out, away from the sun. Related to this, it was for a long time an open problem whether there exists a shape of table, and a starting point, where the point eventually bounces off to infinity. Richard Schwartz (left) gave one example of such a table, and Dmitry Dolgopyat and Bassam Fayad gave another, both in 2009.

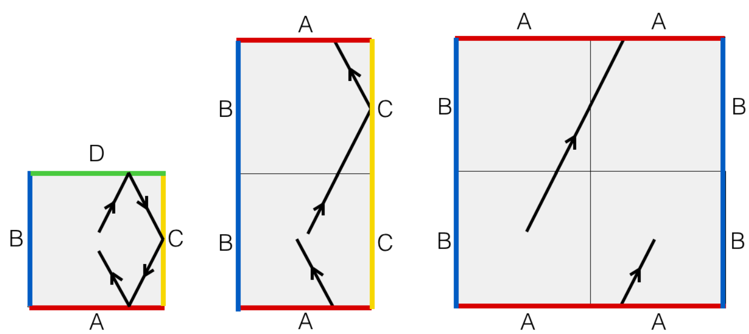
ST **1.** *A proof of the billiard reflection law, part II.* Now, let the mirror be a smooth curve ℓ; as before, our goal is to find the point X on ℓ that minimizes the length $AX + XB$. We will use two different methods to deduce the reflection law. If you haven't taken these courses, don't worry! Someone in the class surely has, and will explain.



(a) *If you have taken multivariable calculus:* Let X be a point in the plane, and define $f(X) = |AX| + |XB|$. The gradient vector of the function $f_A(x) = |AX|$ is the unit vector in the direction from A to X, and likewise for the function $f_B(x) = |BX|$. By the Lagrange multipliers principle, applied to the function __________ under the constraint __________ (fill these in), X is a critical point if and only if $\nabla f(X)$ is perpendicular to ℓ. Use vectors to deduce the billiard reflection law.

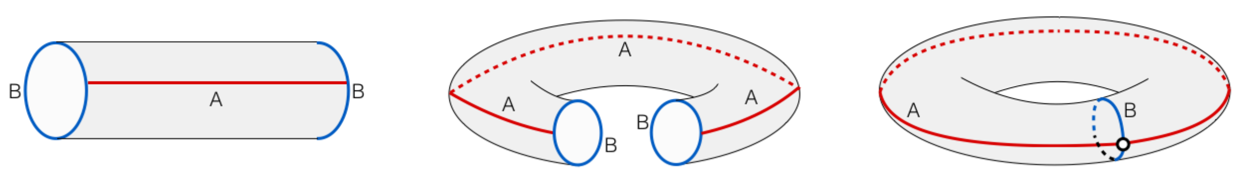
(b) *If you have taken physics:* Let ℓ be a wire, X a small ring that can move along the wire without friction, and AXB an elastic string fixed at A and B and passing through the ring. Use an equilibrium tension argument to deduce the billiard reflection law.

2. Here's another way that we can unfold the square billiard table. First, unfold across the top edge of the table, creating another copy in which the ball keeps going straight. The new top edge is just a copy of the bottom edge, so we now label them both A to remember that they are the same. Similarly, we can unfold across the right edge of the table, creating another copy of the unfolded table. The new right edge is a copy of the left edge, so we now label them both B. When the trajectory hits the top edge A, it reappears in the same place on the bottom edge A and keeps going. Similarly, when the trajectory hits the right edge B, it reappears on the left edge B.



(a) Label the top and bottom edges of a sheet of paper A, and the left and right edges B, and tape the identified edges together to create a surface. What does this surface look like?

(b) Explain why, if the paper were very stretchy, the instructions in **(a)** would create the steps in the figure below. The result is called a *torus*, the surface of a donut.



(c) The partial billiard trajectory shown on the left part of the top figure repeats after 6 bounces. Sketch in the rest of the trajectory in each of the three square pictures above. What is its corresponding *cutting sequence* for the surface on the right part of the figure?

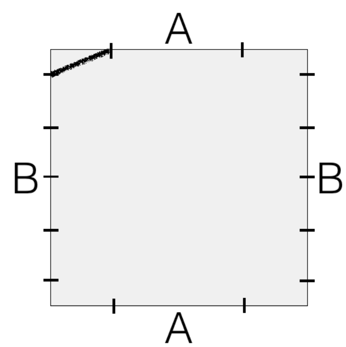
Billiards, Surfaces and Geometry

Contextual note. There is an entire field of mathematics devoted to the study of objects like the square torus that we just constructed: the study of *flat surfaces*. Your instructor is part of this community, as are a couple hundred other mathematicians, spread across the globe and particularly concentrated in France. Professor Amie Wilkinson (left) gave a phenomenal animation of how, as we did with the square, we can make an octagon into a flat surface. It is at 26:00 of her Fields Symposium public lecture from 2018, available here:
`https://www.youtube.com/watch?v=zjccKzHIniw&t=1560s`

DD **3.** Show that the cutting sequence corresponding to a line of slope $1/2$ on the square grid is periodic. Which other slopes yield periodic cutting sequences? What can you say about the period, from the slope?

DD **4.** Prove that every billiard trajectory on the square with irrational slope is non-periodic.

5. In problem 2, we ended up with a trajectory of slope 2 on the *square torus* surface. The picture to the right shows some scratchwork for drawing a trajectory of slope $2/5$ on the square torus. Starting at the top-left corner, connect the top mark on the left edge to the left-most mark on the top edge with a line segment, as shown. Then connect the other six pairs with parallel segments, down to the bottom-right corner.

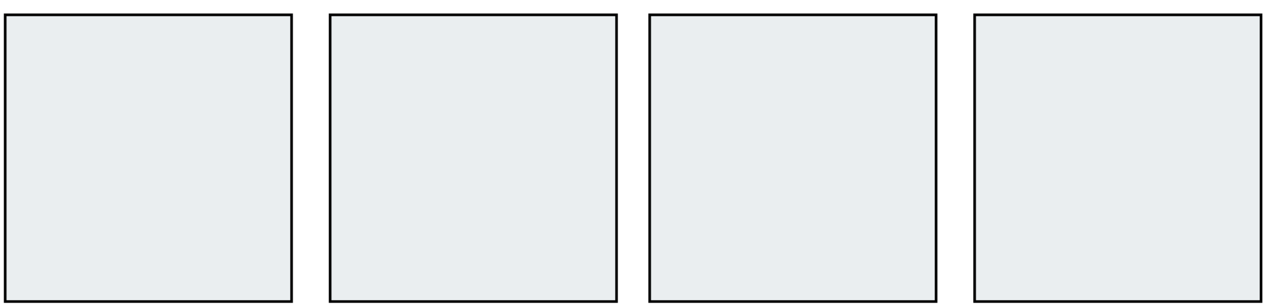


(a) Explain why, on the torus surface, these line segments connect up to form a continuous trajectory. Follow the trajectory along, and write down the corresponding cutting sequence of As and Bs.

(b) Exactly where should you place the tick marks so that all of the segments have the same slope? Create an accurate picture for a trajectory of slope $1/2$ and then $3/2$.

(c) Could you draw a trajectory of any other slope, using the same tick marks?

(d) Draw a picture of a *billiard* trajectory with slope $\pm 2/5$.

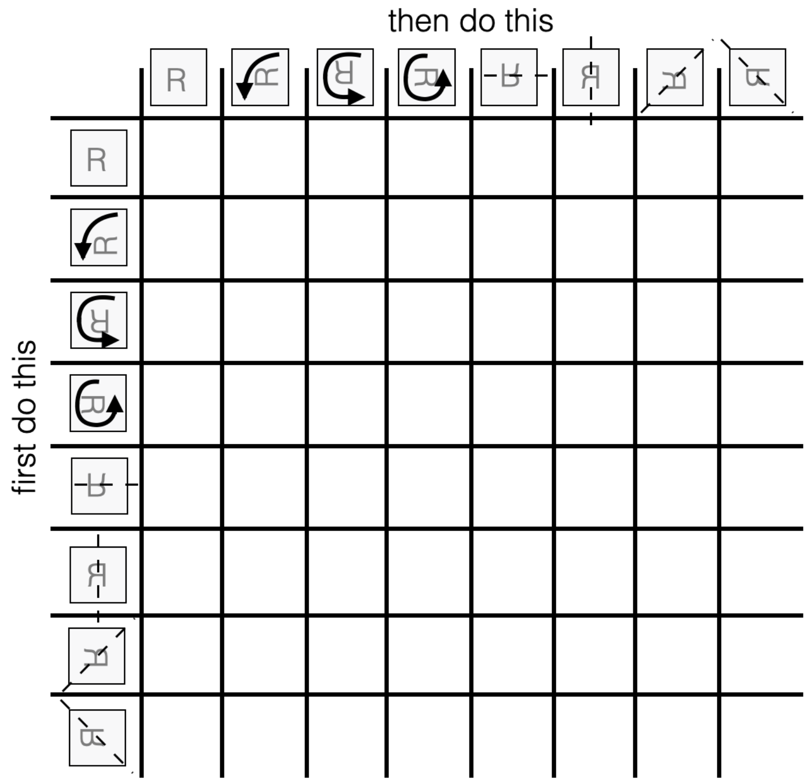


DD **1.** In Page 2 # 4, you found the eight symmetries of the square. It turns out that these eight elements form a *group*, called the *dihedral group* of the square. For a set of elements to be a group, it must have the following properties:

1. It contains an *identity element*, an element that doesn't change anything;
2. Each element has an *inverse*, an element that "undoes" its action;
3. The group is *closed*: composing two elements yields one that is already in the group;
4. Composing elements is *associative*, i.e. $a(bc) = (ab)c$ for elements a, b, c.

(a) Explain why parts (1) and (2) hold for the symmetries of the square.

(b) Fill in the following table to show that (3) holds. Note down any observations.



Part (4) seems tedious – is there a short way to prove it? Let's look at this instead:

(c) Does this group of symmetries commute, i.e. is $ab = ba$ always true for symmetries a, b?

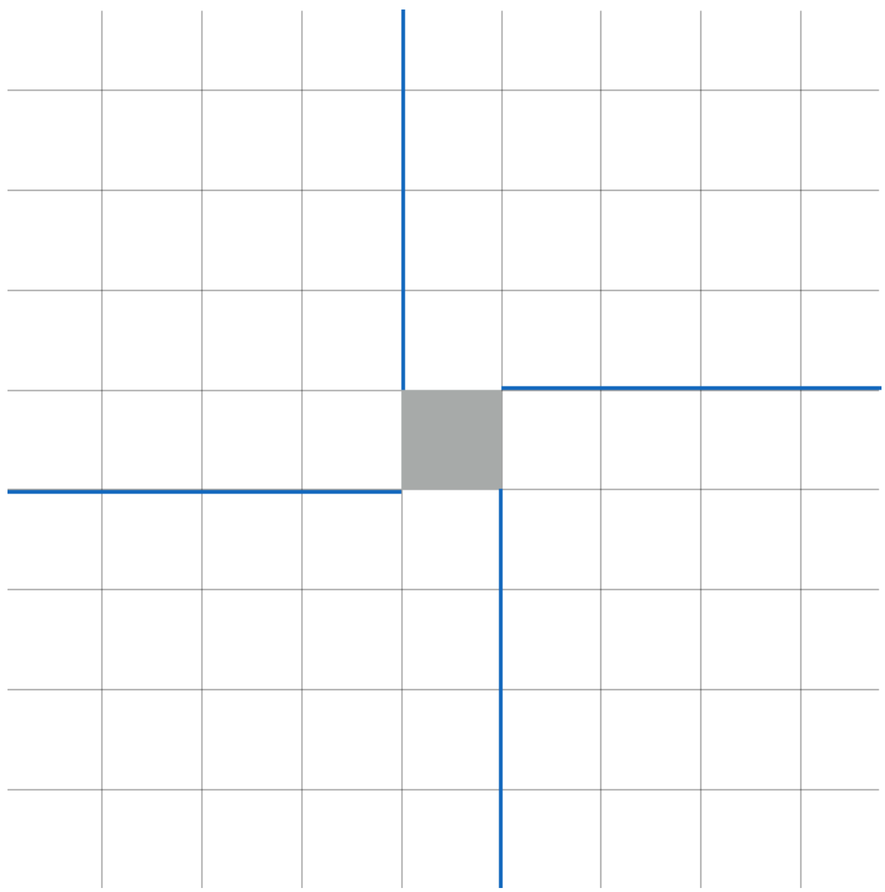
Contextual note. Some groups, such the group of integers under multiplication, are commutative. Others, like the one above, are not. For some sets, such as the set of integers, you can actually define *two* operations (e.g. addition and multiplication) on them, and this makes the set into a *ring*. Professor Haydee Lindo of Harvey Mudd College (left) studies commutative rings.

DD **2.** Consider again a billiard table in the shape of an infinite sector, with vertex angle α. Use unfolding to show that any billiard on such a table makes **(a)** finitely many bounces, and in fact **(b)** at most $\lceil \pi/\alpha \rceil$ bounces. *Hint*: Unfold the sector as many times as you can.

Here the notation $\lceil \cdot \rceil$ is the "ceiling" and means "round up," e.g. $\lceil \pi \rceil = 4$.

DD **3.** Draw an accurate picture of a trajectory on the square torus with slope $3/4$, and do the same for two other slopes of your choice. For each one, find the corresponding cutting sequence. Note down any observations.

DD **4.** Consider again outer billiards on the square table, in the counter-clockwise direction.



(a) Points p on the blue lines are not allowed, because their images p' are ambiguously defined. Explain.

(b) Points p whose image p' is on a blue line are also not allowed. Explain. These are the *inverse images* of the blue points. Color these points red.

(c) The inverse images of the red lines are also not allowed. Explain. Color these green.

(d) Color the inverse images of the green points black. Keep going, with different colors at each step. Describe the full set of disallowed points.

Note: The resemblance of the (incomplete) diagram to a swastika is unintentional. By the definition of the outer billiard map, it is unfortunately impossible to avoid. This symbol was first used 12,000 years ago; the fact that it arises in outer billiards shows that it is a natural construction that has sadly become synonymous with an odious regime.

Billiards, Surfaces and Geometry

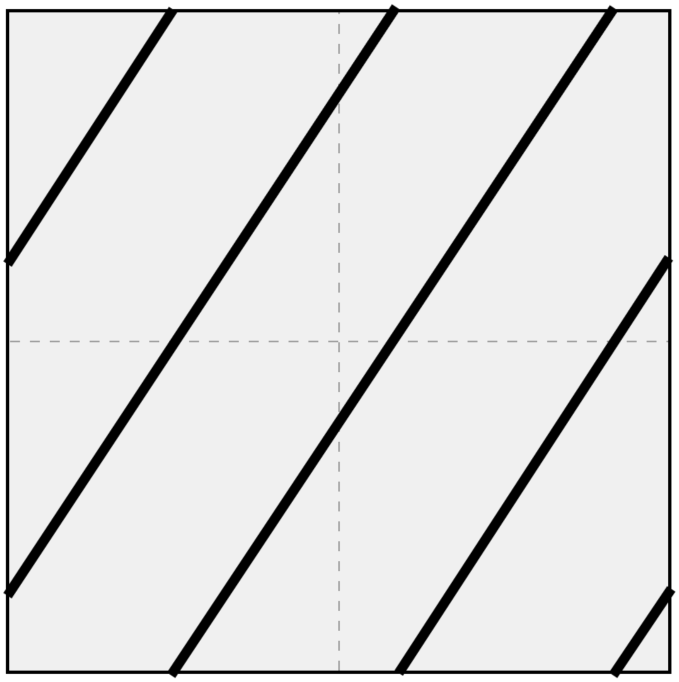
DD **1.** We saw that a billiard trajectory on the square table can be *unfolded* to a line on the square torus. Going the other way, a trajectory on the square torus can be *folded* to a billiard trajectory on the square table.

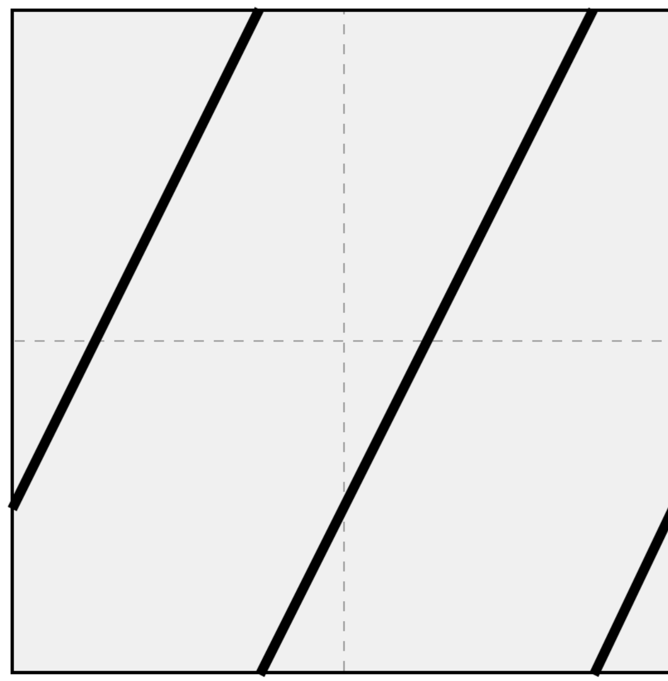
(a) Confirm that each of the trajectories below is a closed path on the square torus.

(b) Carefully trace the first figure onto a piece of *patty paper* (provided for you). Fold it in half as indicated by dashed lines, to transform it into a billiard trajectory!

Repeat for the second figure.

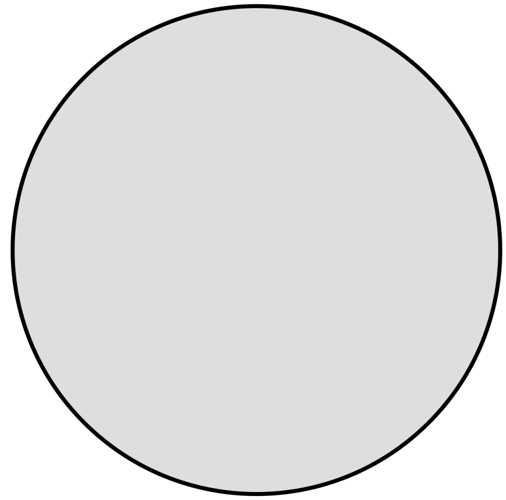
(c) Find the corresponding cutting sequences on the square torus, and on the square table. Note any observations.





DD **2.** In Page 4 # 5, we put 2 marks on edge A and 5 marks on edge B and connected up the marks to create a trajectory with slope $2/5$. Do the same procedure with 4 marks on edge A and 10 marks on edge B. Explain what you get.

ST **3.** Consider a billiard trajectory in the unit circle, where at each impact the trajectory makes angle α with the circle.



(a) Find the central angle θ from the circle's center, between each impact point and the next one, as a function of α.

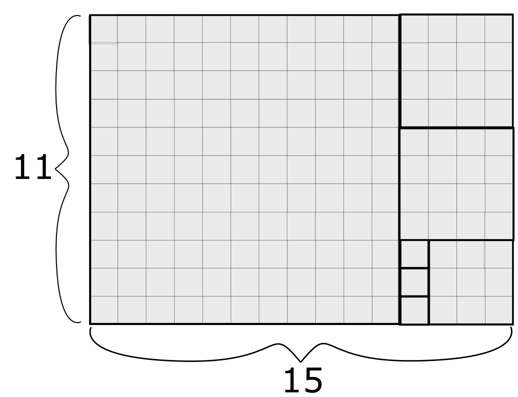
(b) Prove that if $\theta = 2\pi p/q$ for some $p, q \in \mathbf{N}$, then every billiard orbit is q-periodic and makes p turns around the circle before repeating.

(c) What happens if θ is *not* a rational multiple of π?

DD **4.** In Page 5 # 4, you showed that for outer billiards on the square, all of the points on the square grid lines are not allowed. Choose a point p that is *not* on one of the grid lines. Under the outer billiard map, this point reflects through a sequence of vertices $v_1, v_2, \ldots$ where each v_i is one of the four vertices of the square table. Explain why *every* point that is in the same (open) square as p reflects through that *same* sequence of vertices.

DD **5.** Geometrically, the continued fraction algorithm for a number x is:

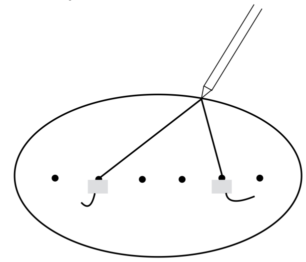
1. Begin with a $1 \times x$ rectangle (or $p \times q$ if $x = p/q$).
2. Cut off the largest possible square, as many times as possible. Count how many squares you cut off; this is a_1.
3. With the remaining rectangle, cut off the largest possible squares; the number of these is a_2.
4. Continue until there is no remaining rectangle. The continued fraction expansion of x is then $[a_1, a_2, \ldots]$ or possibly $[a_1; a_2, \ldots]$.



(a) Draw the rectangle picture for 5/7 to geometrically compute its continued fraction expansion, and **(b)** compute the continued fraction expansion for 5/7 in the way explained in Page 3 # 3, and check that your results agree. Explain why this geometric method is equivalent to the fraction method previously explained, for determining the continued fraction expansion.

Billiards, Surfaces and Geometry

DD **1.** Prove that a trajectory on the square torus is periodic if and only if its slope is rational.

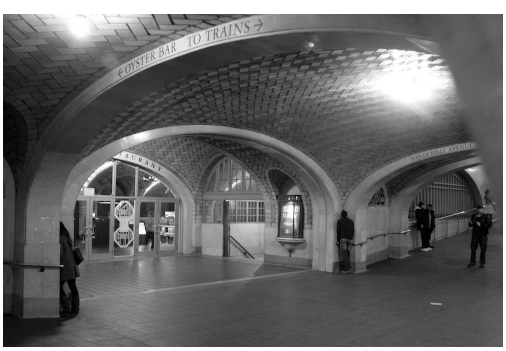


ST **2.** Geometrically, you can construct an ellipse as follows: Take a length of string and tape down the two endpoints, so that the string is somewhat loose. With your pencil, pull out the string until it is taut and trace out all the points the pencil can reach, as shown.

(a) Choose two dots as foci, and loosely tape down a piece of string. Use a pencil to make an ellipse as described above.

(b) Each of the two endpoints of the string is called a *focus* of the ellipse. Show that a billiard trajectory through one focus reflects through the other focus. In other words, the string is a billiard path in the ellipse.

DD **3.** An active area of research is to describe all possible cutting sequences on a given surface. On the square torus, that question is: "Which infinite sequences of As and Bs are cutting sequences corresponding to a trajectory?" Let's answer an easier question: How can you tell that a given infinite sequence of As and Bs is *not* a cutting sequence? You have computed many examples of cutting sequences that *do* correspond to a line on the square grid or square torus. Write down four of them. Then make up an example of an infinite sequence of As and Bs that *cannot* be a cutting sequence on the square grid or square torus, and justify your answer.



Contextual note. The reflection property of ellipses is well known, and appears in architecture as the *whispering gallery.* Several U.S. state house rotundas, and the Statuary Hall at the U.S. Capitol building, have ellipsoidal ceilings, so if you stand at one focus, you can hear someone whisper at the other. An accessible and impressive example of this is in Grand Central Station in New York City (left), where although the background noise is very loud, if you speak into one column, someone on the opposite column can hear you.

→ *more problems on the other side!* ←

DD **4.** An *automorphism* of a surface is a bijective action that takes the surface to itself. Two types of automorphisms of the square *torus* come from symmetries of the *square* itself: reflections and rotations, as we found in Page 2 # 4 and Page 5 # 1.

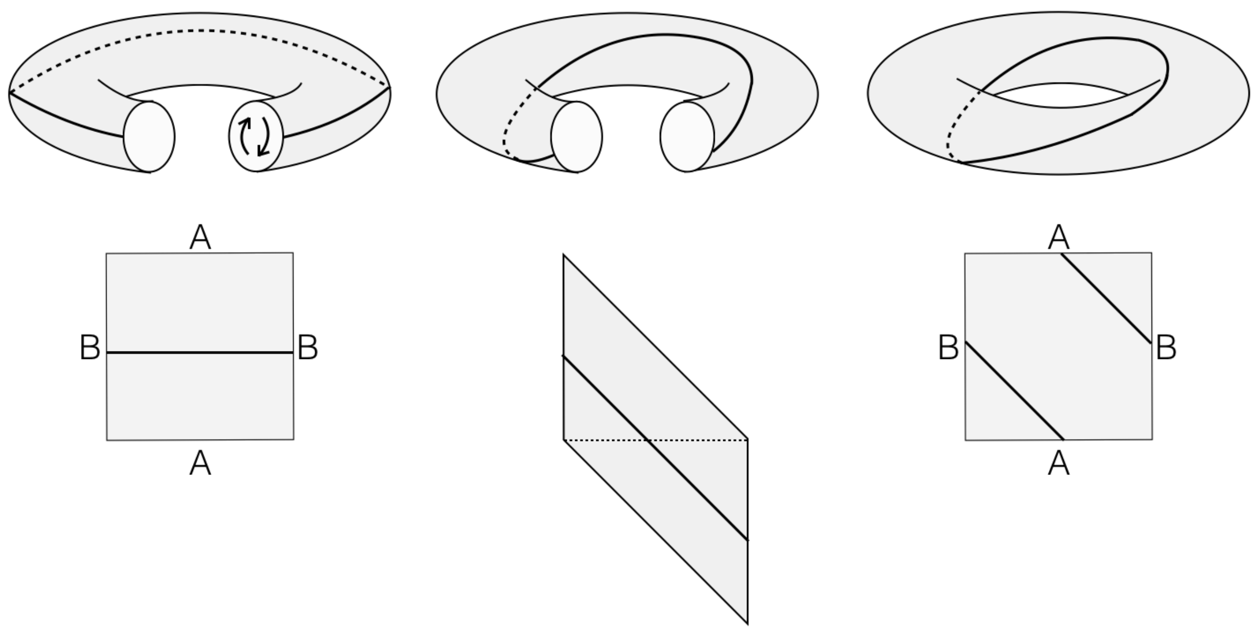
(a) Explain what a reflection of the square torus looks like on the torus surface. You might think about what it does to the surface, or to a closed path drawn on the surface.

(b) Do the same for a 90° rotation.

DD **5.** It turns out that there is one more automorphism of the square torus, that is *not* a symmetry of the square: a *shear*. The shear is shown below on the square on and the 3D surface, where its effect is to twist the torus.

(a) Explain the effect of this shear on the surface, and on a trajectory drawn on that surface.

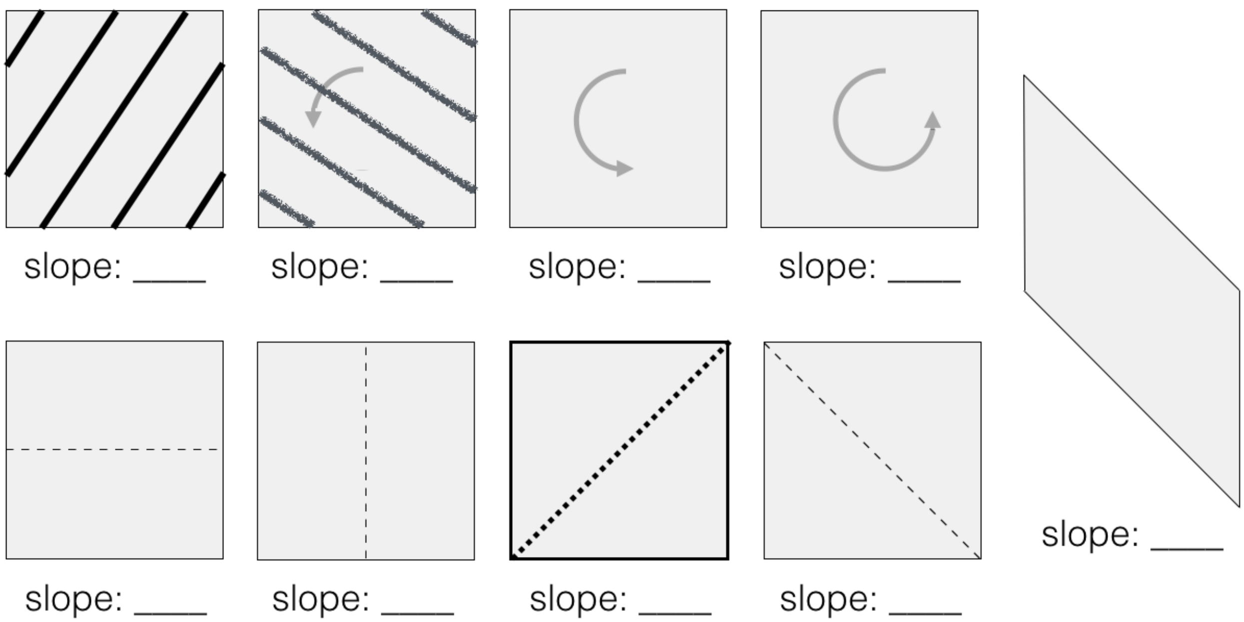
(b) What 2×2 matrix, applied to the "unit square" $[0,1] \times [0,1]$ shown in the left picture, gives the parallelogram shown in the middle picture?



Billiards, Surfaces and Geometry

Synthesis due – problems in class

DD **1.** Given a trajectory on the square torus, we want to know what happens to that trajectory if we apply a symmetry of the surface. To do this, we can sketch the trajectory before and after applying the symmetry. Do so below for each of the eight symmetries of the square, as indicated by the curved arrow or the reflection line, and for the shear. I've done one for you.



The flip across the positive diagonal is in bold because we will use it later.

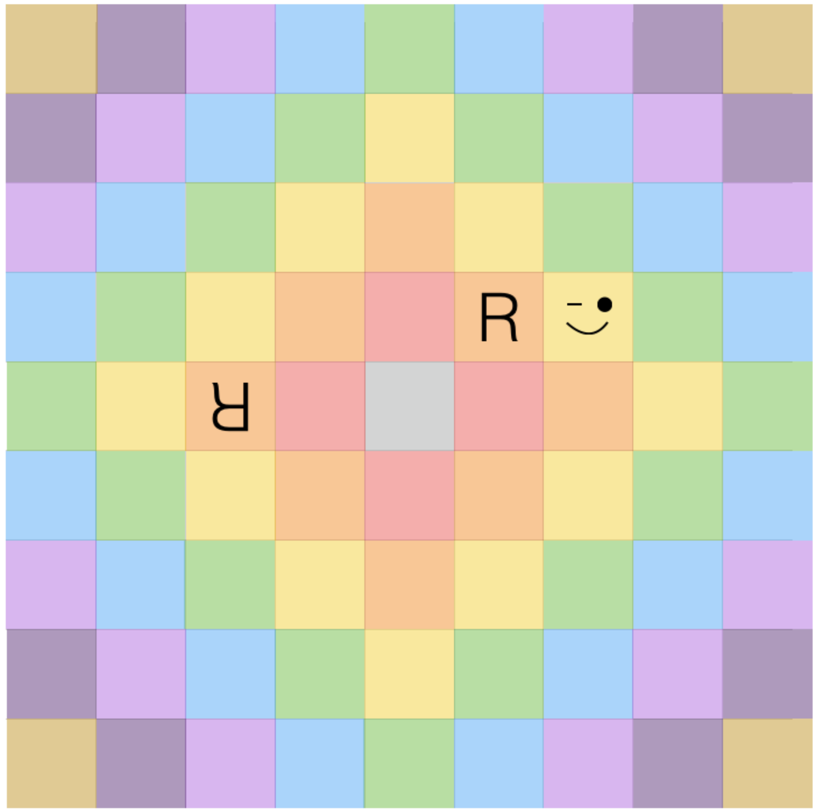
DD **2.** (Continuation) For each symmetry above, make a guess about what it does to a starting slope of the form p/q. Can you prove your answer correct?

→ *another problem on the other side!* ←

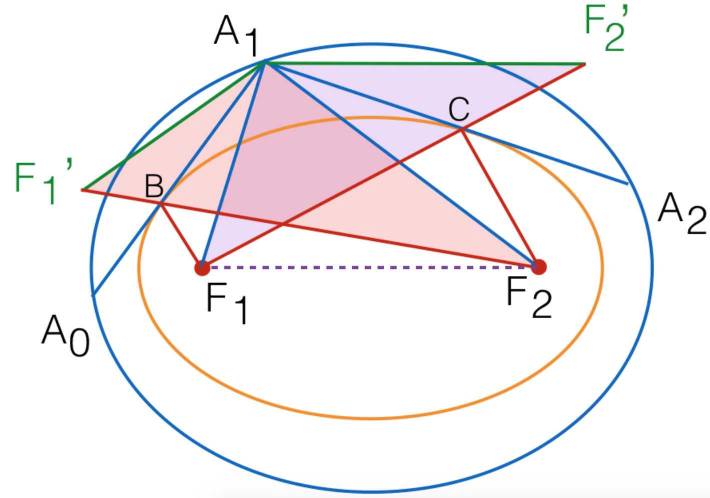
DD **3.** In Page 6 # 4, we showed that for the outer billiard map on the square, points in a given square in the grid move together. Now we will explore *how* they move.

(a) Plot the complete orbit (meaning, until you get back to where you started) of the R and of the winky face under the outer billiard map. One step is shown for the R.

(b) Prove that the square of the outer billiard map (this means that you apply it twice) is a translation.



ST **1. Theorem (billiards in an ellipse).** *Consider an ellipse E with foci F_1, F_2. If some segment of a billiard trajectory does not intersect the focal segment F_1F_2 of E, then no segment of this trajectory intersects F_1F_2, and all segments are tangent to the same ellipse E' with foci F_1 and F_2.*



(a) Consider the billiard trajectory $A_0A_1A_2$ in the larger ellipse E shown in the figure. Explain why $\angle A_0A_1F_1 = \angle A_2A_1F_2$.

(b) Reflect F_1 across $\overline{A_0A_1}$ to create F_1', and reflect F_2 across $\overline{A_1A_2}$ to create F_2'. Explain why $\angle A_0A_1F_1' = \angle A_0A_1F_1$ and $\angle A_2A_1F_2' = \angle A_2A_1F_2$.

(c) Show that $\Delta F_1'A_1F_2$ and $\Delta F_1A_1F_2'$ are congruent.

(d) Mark the intersection of $\overline{F_1'F_2}$ with $\overline{A_0A_1}$ as B, and the intersection of $\overline{F_1F_2'}$ with $\overline{A_1A_2}$ as C. Show that the *string length* $|\overline{F_1B}| + |\overline{BF_2}|$ is the same as the string length $|\overline{F_1C}| + |\overline{CF_2}|$.

(e) Prove the theorem as stated above.

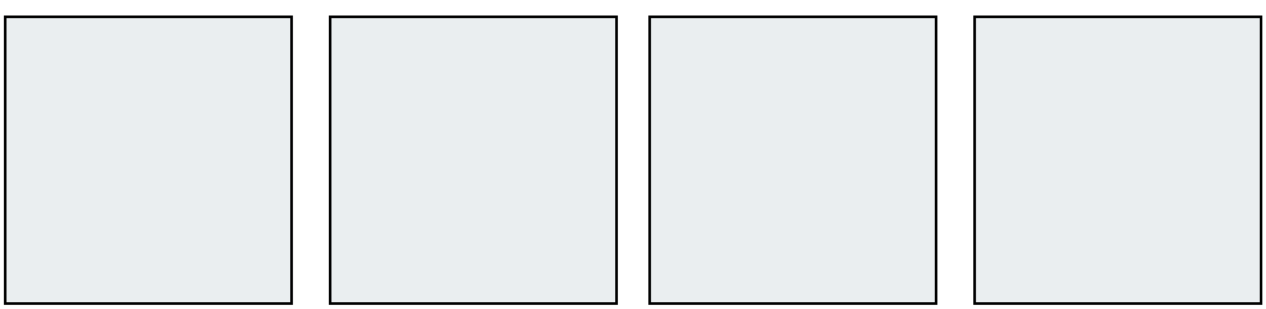
DD **2.** Given a trajectory on the square torus, we want to know what happens to that trajectory under an automorphism (symmetry) of the surface. One way to answer this question is to sketch the trajectory before and after applying the automorphism (Page 8 # 1). Another way is by comparing their cutting sequences: the cutting sequence $c(\tau)$ corresponding to the original trajectory τ, and the cutting sequence $c(\tau')$ corresponding to the transformed trajectory τ'.

(a) Let τ_2 be the trajectory of slope 2. Sketch τ_2, and find $c(\tau_2)$.

(b) For each automorphism (1)-(5) below, apply it to τ_2 to get a transformed trajectory τ_2', sketch τ_2', and compute $c(\tau_2')$.

(c) Explain how to obtain $c(\tau')$ from $c(\tau)$ for a general trajectory τ, for each automorphism.

(1) reflection across a horizontal line;
(2) reflection across a vertical line;
(3) reflection across the positive diagonal;
(4) reflection across the negative diagonal;
(5) rotation by 90°.



DD **3.** (Continuation) For each of the five automorphisms in the previous question:

(a) Find the 2×2 matrix that performs this automorphism. For the purpose of this question, assume that the square torus is centered at the origin.

(b) Find the determinant of each matrix and explain why they are all ± 1.

DD **4.** Explain why a cutting sequence on the square torus can have blocks of multiple As separated by single Bs, or blocks of multiple Bs separated by single As, but not both.

DD **5.** Find the continued fraction expansions of $3/2, 5/3, 8/5$, and $13/8$. Describe any patterns you notice, and explain why they occur.

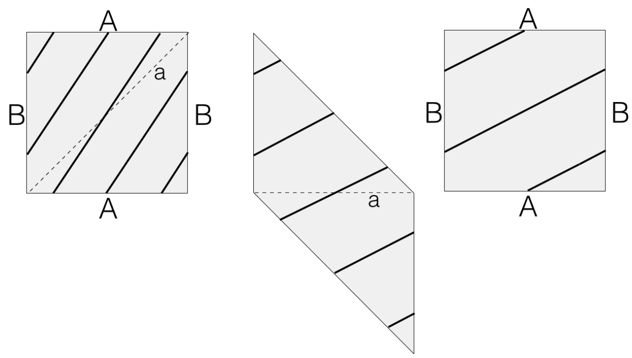
Billiards, Surfaces and Geometry

In the following problems, we will determine the effect of the shearing automorphism from Page 7 # 5 on a trajectory τ and its cutting sequence $c(\tau)$.

DD **1.** First, we will apply symmetry to reduce our work to just one set of trajectories. Show that, given a linear trajectory in any direction on the square torus, we can apply rotations and reflections so that it is going left to right with slope ≥ 1.

DD **2.** Since we have reduced to the case of slopes that are ≥ 1, we will analyze the effect of the vertical shear $\left[\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right]$, because these slopes work nicely with this shear. Later we will show that everything else can be reduced to this case.

As an example, we'll use the trajectory τ with slope $3/2$, with corresponding cutting sequence $c(\tau) = \overline{BAABA}$ (left picture). We shear it via $\left[\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right]$, which transforms the square into a parallelogram (middle picture), and then we reassemble the two triangles back into a square torus, while respecting the edge identifications (right picture). The new cutting sequence is $c(\tau') = \overline{BAB}$.



Do this geometric process for three different trajectories τ of your choice: Sketch a trajectory τ, sketch its image as a parallelogram after shearing by $\left[\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right]$, and then sketch the reassembled square with the new trajectory τ'. For each, record $c(\tau)$ and $c(\tau')$. Try to find the pattern: a rule to get $c(\tau')$ from $c(\tau)$. *Hint*: You can use the "edge marks" technique from Page 4 # 5 on the parallelogram edges to make an accurate picture.

DD **3.** Find the continued fraction expansion of $\sqrt{2} - 1$. Then solve the equation $x = \dfrac{1}{2 + x}$ and explain how these are related.

DD **4.** How many billiard paths of period 10 are there on the square billiard table? Of period 14? Construct (make a mathematically accurate sketch of) each of these.

DD **5.** We have identified the top and bottom edges, and the left and right edges, of a square to obtain a surface: the square torus. If we identify opposite parallel edges of a parallelogram, what surface do we get?

Billiards, Surfaces and Geometry

DD **1.** (Continuation of Page 10 # 2) Show that if we apply the shear $\left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right]$ to the square torus:

(a) The effect on the slope of a trajectory is to decrease it by 1.

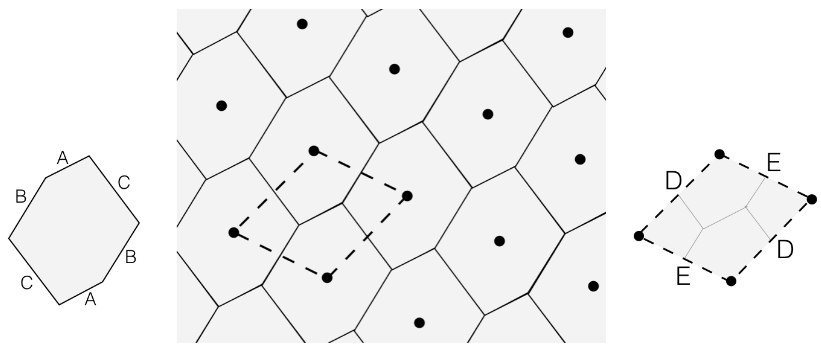
(b) The effect on the cutting sequence corresponding to a trajectory whose slope is greater than 1 is to remove one A between each pair of Bs.

DD **2.** Explain why a trajectory with slope p/q (in lowest terms) on the square billiard table has period $2(p+q)$.

DD **3.** Explain why the continued fraction expansion of a number terminates (stops) if and only if it is rational.

DD **4.** If we identify opposite parallel edges of a hexagon, what surface do we get? The figure to the right shows one way to figure it out, via a *cut-and-paste* approach. Explain.

An alternative approach is to sketch what it looks like to glue identified edges together, assuming that the hexagon is made out of stretchy material. Try this, too.



DD **5.** In the picture above, we tiled the plane with a hexagon that has three pairs of opposite parallel edges. Our "random" hexagon happened to be convex. Does a non-convex hexagon with three pairs of opposite parallel edges still tile the plane?

DD **1.** We need one more piece in order to relate trajectories on the square torus, continued fractions, and cutting sequences. Show that if we apply the flip $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$ to the square torus:

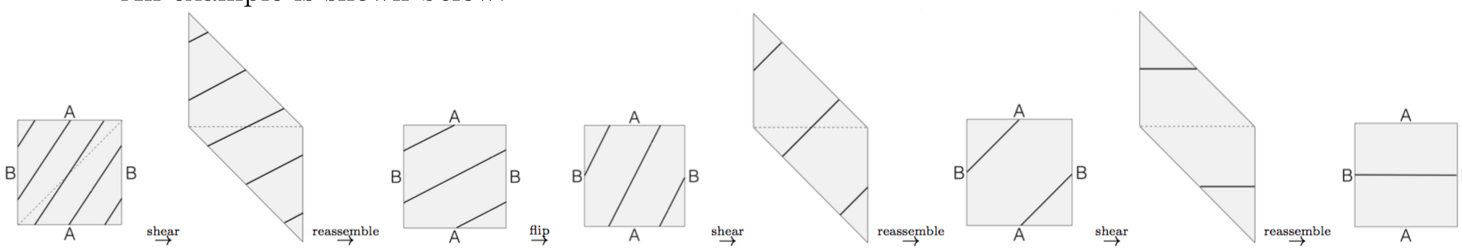
(a) The effect on the slope of a trajectory is to take its reciprocal.

(b) The effect on the cutting sequence corresponding to a trajectory is to switch As and Bs.

DD **2.** Starting with a trajectory on the square torus with positive slope, apply the following algorithm:

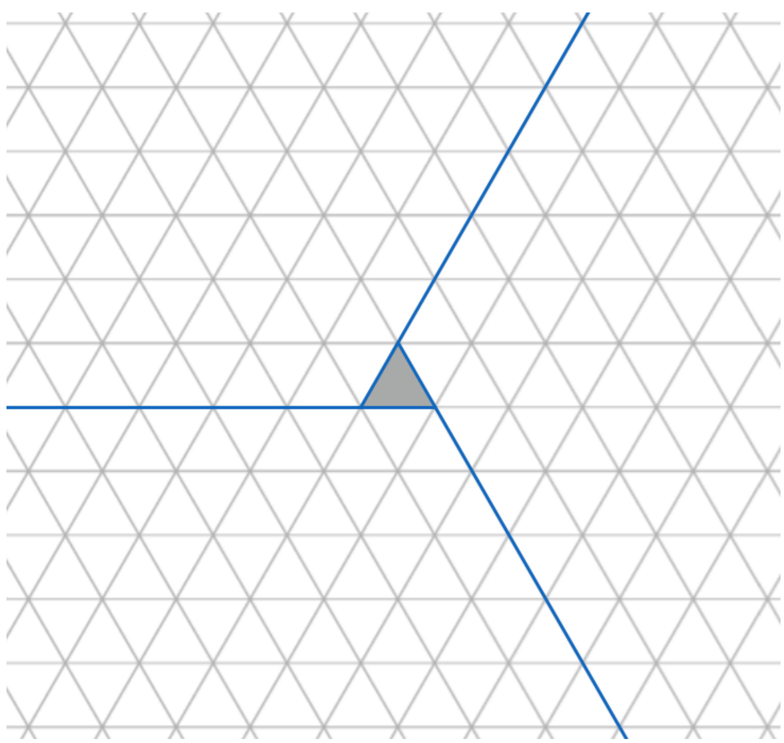
1. If the slope is ≥ 1, apply the shear $\left[\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right]$.
2. If the slope is between 0 and 1, apply the flip $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$.
3. If the slope is 0, stop.

An example is shown below.



We can note down the steps we took: shear, flip, shear, shear. We ended with a slope of 0. Work backwards, using this information and your work in Page 11 # 1 and Page 12 # 1, to determine the slope of the initial trajectory. Keep track of each step.

DD **3.** Consider the counter-clockwise outer billiard map on the *triangular* billiard table, as shown.

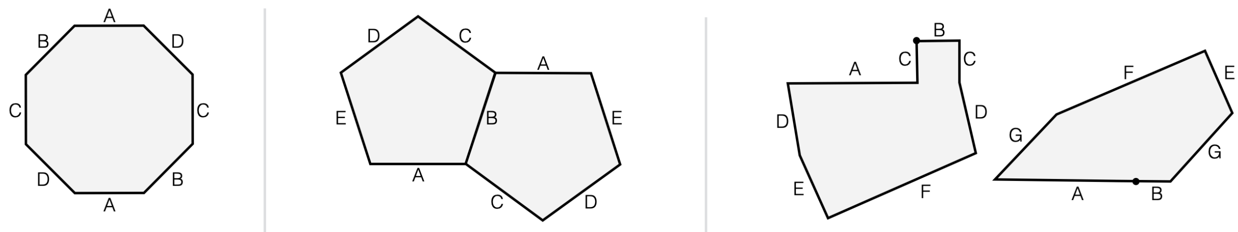


(a) Explain why points on the thick blue lines are not allowed. Then color the inverse images (red) of the blue lines, the inverse images (green) of the red lines, the inverse images (black) of the green lines, the inverse images (purple) of the black lines, and so on.

(b) Identify some *necklaces* of iterated images of triangles, and color each necklace a different color, like the picture in Page 8 # 5.

DD **4.** We can create a surface by identifying opposite parallel edges of a single polygon, as we have done with the square and hexagon. We'll call such a surface a *polygon surface. Parallel edges* must be parallel and also the same length. *Opposite edges* means that the polygon is on the left side of one of the edges, and on the right side of the other.

In a similar way, we can create a surface from two polygons, or from any number of polygons. Some examples are below. Edges with the same letter are identified, as with A and B on the square torus. The two polygons on the right side together form a single surface.



(a) Watch Amie Wilkinson's talk (the 3 minutes of it from 26 to 29 minutes) that shows how to wrap the flat octagon surface (far left) into a curved surface in 3-space. What is its *genus* – how many holes does it have?

YouTube: "Dr. Amie Wilkinson - Public Opening of the Fields Symposium 2018," available at `https://www.youtube.com/watch?v=zjccKzHIniw&t=1560s`

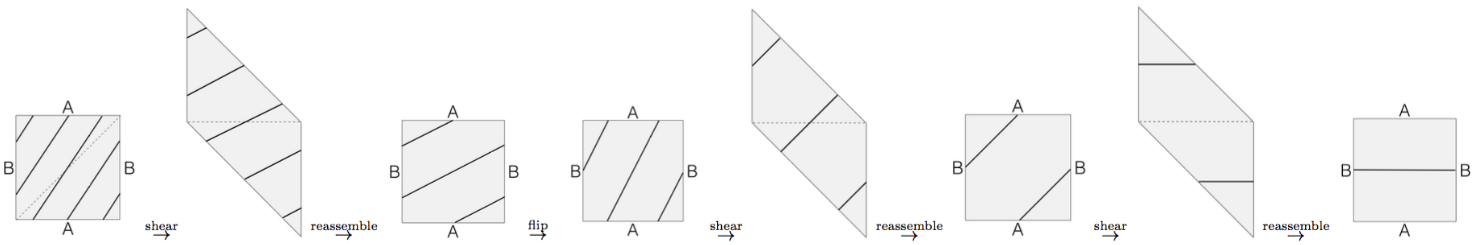
(b) Do your best to repeat her stretching methods for the double pentagon surface (center) to make it into a curved surface in 3-space.

(c) The flat octagon surface has 4 edges. How many edges do the other two surfaces have?

DD **5.** Suppose that a given periodic cutting sequence on the square torus has period n. Are there any values of n for which you can determine the cutting sequence (perhaps up to some symmetry) from this information?

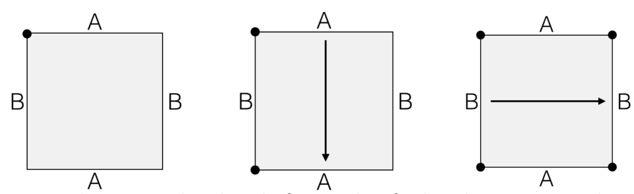
Billiards, Surfaces and Geometry

DD **1.** (Continuation of Page 12 # 2) Translate the given algorithm, which uses the *slope* of a trajectory, into an algorithm that uses the *cutting sequence* corresponding to a trajectory. You should translate each of the four sentences ("Starting with...," 1, 2 and 3.) Then apply your algorithm to the cutting sequence $\overline{ABAAB}$ and check that your results are consistent with the pictures in the figure.



DD **2.** Make up your own example of a polygon surface (recall Page 12 # 4) that will be different from everyone else's, made from *three* polygons. We will call your new surface S. How many edges does S have?

DD **3.** *Vertex chasing.* To explain how to count the vertices of a surface, we will use the square torus. First, mark any vertex (say, the top left). We want to see which other vertices are the same as this one. The marked vertex is at the left end of edge A, so we also mark the left end of the bottom edge A. We can see that the top and bottom ends of edge B on the left are now both marked, so we mark the top and bottom ends of edge B on the right, as well. Now all of the vertices are marked, so the square torus has just one vertex. (We already knew that – how?)



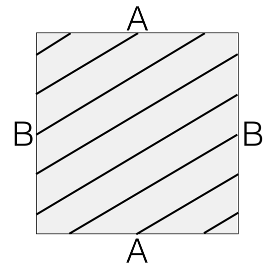
(a) Determine the number of vertices for a hexagon with opposite parallel edges identified.

(b) Do the same for each surface in Page 12 # 4, and

(c) for your surface S created in the previous problem.

ST **4.** Let P be a convex quadrilateral that has a 4-periodic inner billiard trajectory that reflects consecutively in all four sides. Prove that P is cyclic.

(Recall that a *cyclic* quadrilateral has a circle containing all four of its vertices.)



DD **1.** Apply the geometric algorithm from Page 12 # 2 and Page 13 # 1 to the trajectory shown to the right. Note down the steps you take (shears and flips) and use this information to work backwards from an ending slope of 0 to determine the slope of the initial trajectory. Show all of your steps.

DD **2.** (Continuation) Explain how shears and flips on the square torus are related to continued fraction expansions.

DD **3.** (Continuation) Find the cutting sequence corresponding to the trajectory above. Apply your algorithm from Page 13 # 1 to it, and check that your results at each step are consistent with each step of your work in problem 1.

DD **4.** Given that the continued fraction expansion of a particular number is $[0; 1, 2, 2]$, find the cutting sequence corresponding to a trajectory on the square torus with this slope.

DD **5.** Once we've made a surface, the *Euler characteristic* gives us a way of easily determining what kind of surface we obtain, without needing to come up with a clever trick like cutting up and reassembling hexagons into parallelograms.

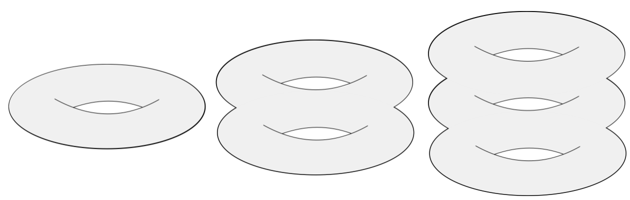
Given a surface S made by identifying edges of polygons, with V vertices, E edges, and F faces, its Euler characteristic is

$$\chi(S) = V - E + F.$$

Find the Euler characteristic of **(a)** the square torus, **(b)** the cube, **(c)** the tetrahedron, **(d)** the hexagon surface from Page 11 # 4 and **(e)** one of the surfaces from Page 12 # 4.

(f) Comment on any patterns you notice.

DD **6.** (Continuation) One of the main goals of the field of *topology* is to classify surfaces by their *genus*, which, informally speaking, is the number of "holes" it has. The surfaces to the right have genus $1, 2$ and 3.



We can use the Euler characteristic to determine the genus of a surface: A surface S with genus g has Euler characteristic $\chi(S) = 2 - 2g$. Use this to compute the genus of each of your surfaces from the previous problem, and check that your answer agrees with reality.

Billiards, Surfaces and Geometry

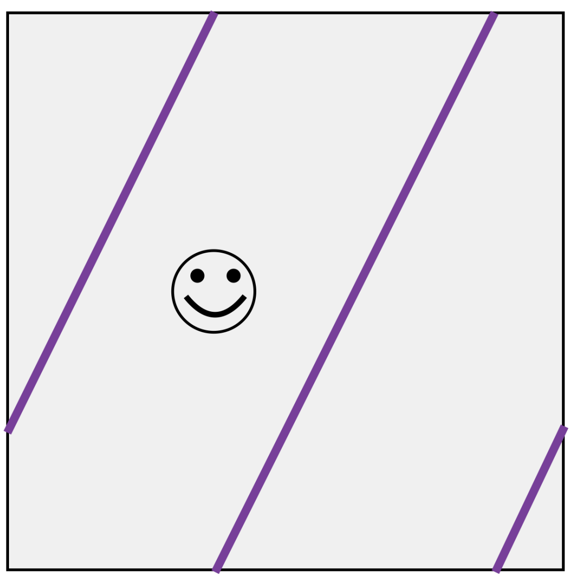
Synthesis due – in-class activity

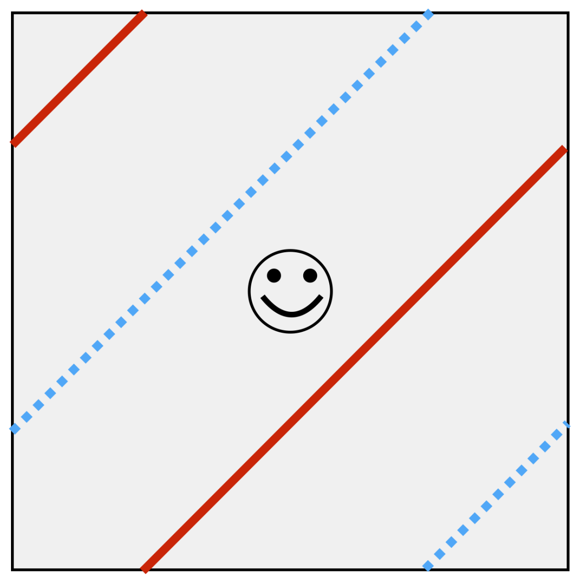
DD **1.** The two pictures below show linear trajectories on the square torus, as usual.

(a) Explain why the purple trajectory (left) is a single trajectory, while the red and blue trajectories (solid and dashed, right) are two different trajectories.

(b) The red and blue trajectories partition the square torus into two pieces. In other words, if the trajectories were walls, the smiley person could only explore half of the torus. Justify this statement.

(c) Also explain why the purple trajectory does *not* partition the torus into two pieces – the smiley person can explore the whole thing.





DD **2.** *Cutting a bagel into two linked rings.*
1. Obtain a bagel. Draw the blue and red trajectories on it.
2. Obtain a serrated knife. Your goal is that the pointy end of the knife follows the red trajectory, while the handle end follows the blue trajectory. Cut the bagel to make this happen. You will probably want to flip the roles of red and blue halfway through, to keep the handle on the outer part of the bagel.
3. Separate your bagel into linked rings!

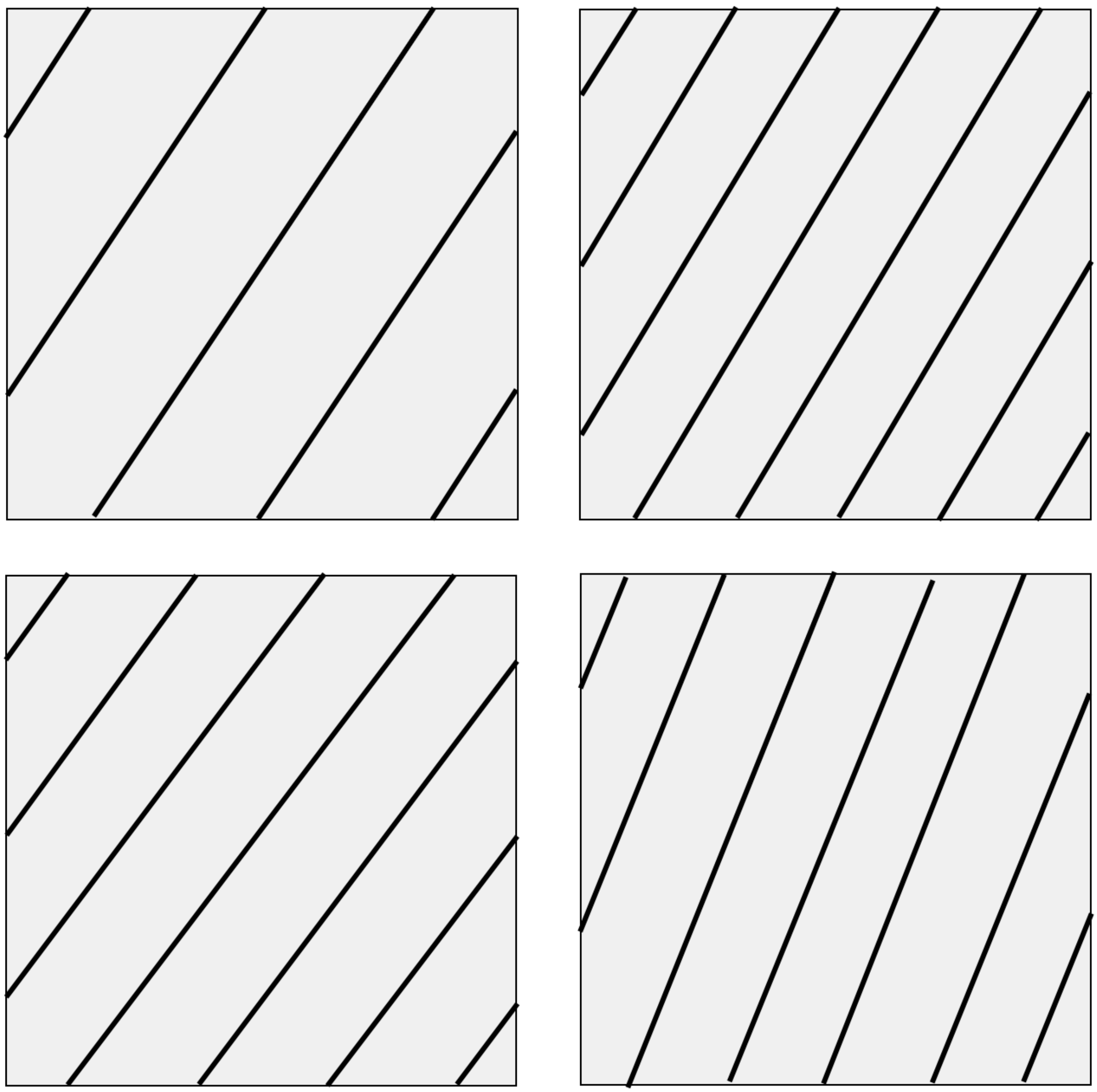
(a) Explain why the procedure above leads to linked rings.

(b) Explain what would have happened if you had cut along the purple trajectory instead.

If you finish that and want more, see the other side

DD **3.** In our classroom, we have bagels with trajectories corresponding to slopes 1, 1/2, 2 and 3/2. Draw a trajectory on a bagel corresponding to a slope of your choice. Some trajectories are below in case you need inspiration, but do feel free to use any slope you like.

Advice: Draw lots of guiding marks on the bagel before you start drawing in the trajectory!

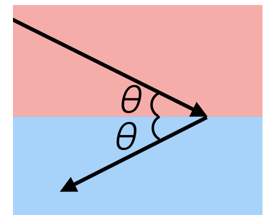


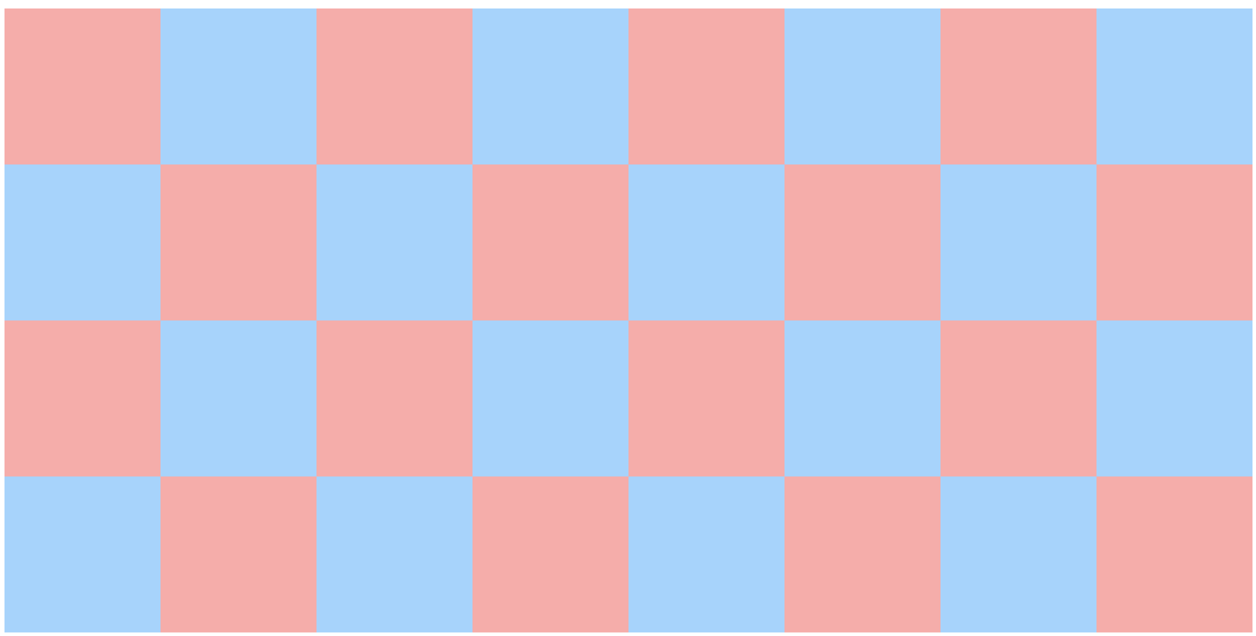
Connection to knot theory: Imagine that the bagel disappears, and all that is left is the trajectory, now made out of a piece of string. It turns out that the string is knotted up – you can't untangle it into a circle. The trajectory with slope p/q corresponds to the (p, q) *torus knot*, meaning that it goes through the center p times and around the outside q times.

Billiards, Surfaces and Geometry

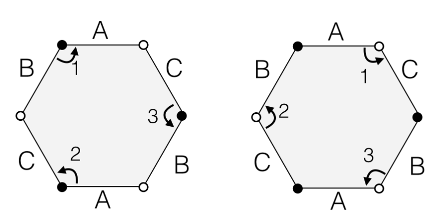
Now that we have explored the simplest case (the square) of classical billiards (inner billiards) in detail and understood it deeply, we will expand our view to other types of billiards.

DD **1.** *Tiling billiards.* Another type of billiards (besides inner and outer) that we will study is *tiling billiards*, where a trajectory refracts through a tiling of the plane. The *refraction rule* is that when the trajectory hits an edge of the tiling, it passes through in such a way that the angle of incidence is equal to the angle of reflection, and the trajectory has been reflected across the edge, as shown to the right. Sketch some trajectories on the square grid tiling. What kinds of behaviors can you find? Can you prove that these are the only ones?





DD **2.** *Walking around a vertex.* We can determine the angle around a vertex by "walking around" it, as shown in the figure for a hexagon with opposite parallel edges identified. The left picture shows that the angle around the black vertex is $3 \cdot \dfrac{2\pi}{3}$, and the right picture shows the same for the white vertex. Explain what is going on.

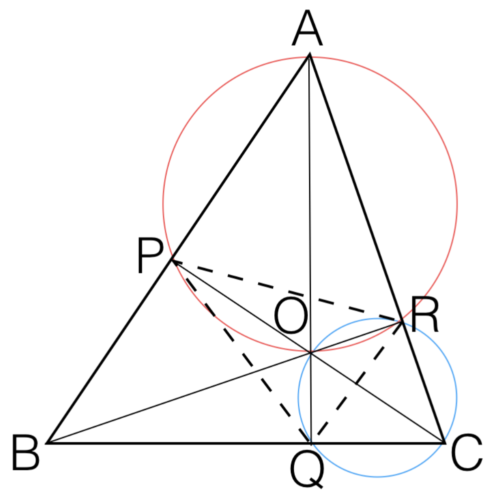


Since the black and white vertices each have 2π of angle around them, all the corners of the hexagon surface come together in a flat plane, as we have already seen.

Find the angle around each vertex of the surfaces in Page 12 # 4.

DD **3.** (Continuation) A surface is called *flat* if it looks like the flat plane everywhere, except possibly at finitely many *cone points* (vertices), where the angle around each vertex is a multiple of 2π. Prove that if a surface is created by identifying opposite parallel edges of a collection of polygons, then it is flat.

ST **4.** *The Fagnano trajectory.* You have constructed several periodic billiard paths in the square billiard table; other polygons also have periodic paths. A classical theorem says that *Fagnano trajectory* connecting the base points of the three altitudes of a triangle is a 3-periodic billiard trajectory. We will prove this by showing that angles ARP and CRQ are equal; the argument is the same for the other bounces.



(a) Opposite angles of a quadrilateral add up to π if and only if the quadrilateral is *cyclic*. Use this result to show that quadrilaterals $APOR$ and $CROQ$ are cyclic, as suggested by the diagram.

(b) Another classic theorem of geometry says that two angles supporting the same circular arc are equal. Use this to show that $\measuredangle PAO = \measuredangle PRO$, and $\measuredangle ORQ = \measuredangle OCQ$.

(c) Use triangles BAQ and BCP to show that $\measuredangle PAO = \measuredangle OCQ$.

(d) Show that $\measuredangle ARP = \measuredangle CRQ$, as desired.

DD **5.** An active area of research is to describe, or "characterize," all possible cutting sequences on a given surface. Now we can do this for the square torus.

Theorem. Valid cutting sequences on the square torus are those that do not fail under the following algorithm:

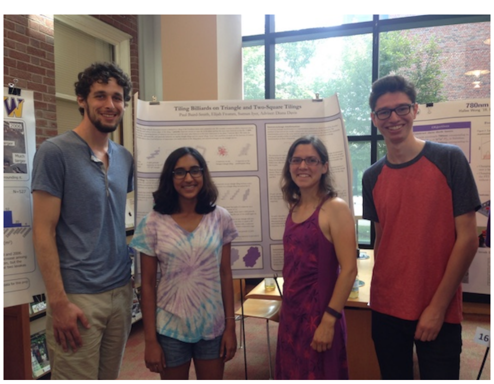
Starting with an infinite sequence of As and Bs, repeatedly apply the following algorithm:

(1) If there are multiple Bs separated by single As, switch As and Bs.

(2) If there are multiple As separated by single Bs, remove an A between each pair of Bs.

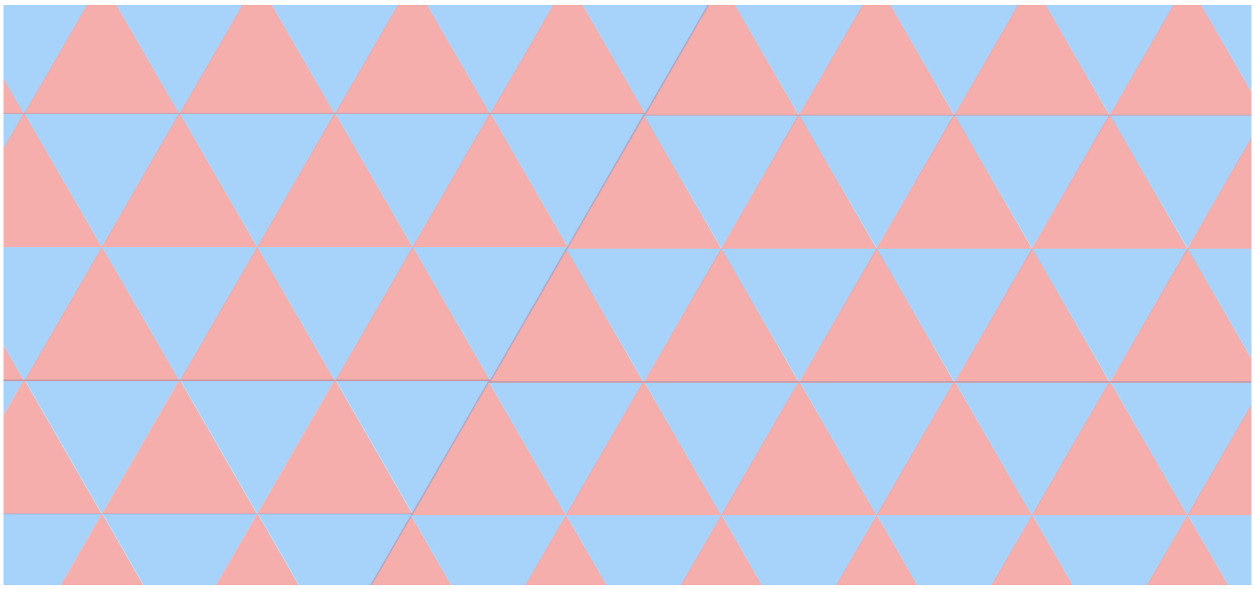
(3) If the sequence has AA somewhere and BB somewhere else, stop; it fails to be a valid cutting sequence.

Earlier in the course, students conjectured that a cutting sequence could only have two consecutive numbers of As, such as 2 and 3, between each pair of Bs, e.g. $\overline{BABAAA}$ is not allowed. Use the theorem to prove this conjecture true.



Contextual note. Tiling billiards is motivated by the existence of *metamaterials*, solids that have a negative index of refraction. Typical materials such as water and glass have a positive index of refraction; you have likely worked with these in physics, with *Snell's Law.* The idea is to create a two-colorable tiling out of materials with opposite indices of refraction. I (DD) named this type of billiards *tiling billiards* and coauthored the first two papers in this area, with two different groups of students; one group is shown to the left.

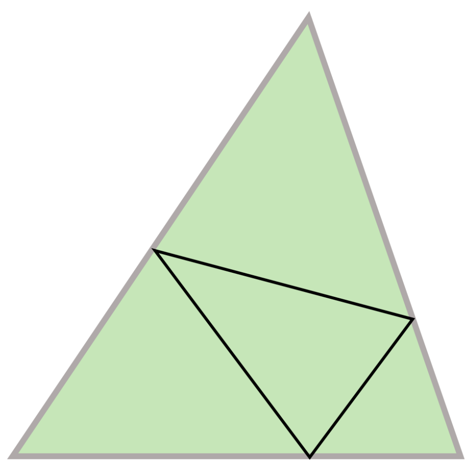
DD **1.** We saw that for tiling billiards on the square grid, there are only two types of trajectories: those that go to the opposite edge and zig-zag, and those that go to the adjacent edge and make a 4-periodic path. How many types of trajectories are there for tiling billiards on the triangular grid?



In billiards on the square, we *unfolded* a billiard trajectory into a line on the square grid, and onto a linear trajectory on the square torus. In an analogous way, *folding* is a powerful technique for understanding tiling billiards trajectories:

DD **2.** Consider a tiling billiards trajectory that crosses an edge e of the tiling. Show that, if you fold the tiling along edge e, the two pieces of trajectory that intersect edge e lie on top of each other.

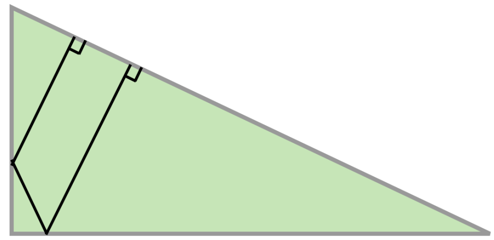
ST **3.** Consider again the 3-periodic Fagnano trajectory from Page 16 # 4. Sketch a trajectory that is parallel to the one in the construction and nearby. Show that this trajectory is also periodic, and find its period.



DD **4.** Consider again the theorem in Page 16 # 5.

(a) The vexing part of this characterization is that it doesn't have a step saying, "Stop! Congratulations; you have a valid cutting sequence." It only says, "Keep going; your cutting sequence hasn't proven to be invalid yet." But it turns out that it's the best we can do. Explain why this algorithm *does* stop for a *periodic* cutting sequence.

(b) I left out one technical point of the theorem: It actually characterizes the *closure* of the space of all cutting sequences. Valid cutting sequences are in the interior of the space, and cutting sequences such as $\ldots AAABAAA\ldots$ are on the boundary of the space. Explain why this cutting sequence does not fail in the algorithm, and why it is not a valid cutting sequence. Find another cutting sequence on the boundary, other than $\ldots BBBABBB\ldots$.



DD **1.** *Periodic trajectories in more triangles.*

(a) Explain why the Fagnano trajectory only works for *acute* triangles.

(b) The construction shown to the right was found by Rich Schwartz (DD's Ph.D. advisor). He calls it "shooting into the corner." Fill in the details, and show that it works for every right triangle.

(c) Find an example of a periodic path in an obtuse triangle.

Contextual note. The biggest open problem in billiards is: *does every triangular billiard table have a periodic trajectory?* The Fagnano trajectory shows that every *acute* triangle has a periodic billiard trajectory, and the construction above shows that every *right* triangle has one. Howie Masur showed that every polygon (including triangles) whose angles are *rational* numbers of degrees has a periodic path. Rich Schwartz used a computer-aided proof to show that every triangle whose largest angle is less than 100° has a periodic billiard trajectory, and in 2018 a team of four researchers extended that result to 112.3°. The problem is open in general for irrational-angled obtuse triangles with an angle larger than 112.3°.

DD **2.** Here is our dream: To understand the effect of *every* automorphism of the square torus, on the cutting sequence corresponding to a trajectory. Here is our progress so far:

(1) There are three types of automorphisms: rotations, reflections and shears. We understood the effects of rotations and reflections in Page ____ #____. (fill these in)
(2) Using rotations and reflections, we reduced our work, now only for shears, to the case of trajectories whose slope is greater than 1, in Page ____ #____.
(3) We understood the effect of the matrix $\left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right]$ on a trajectory on the square torus in Page ____ #____.

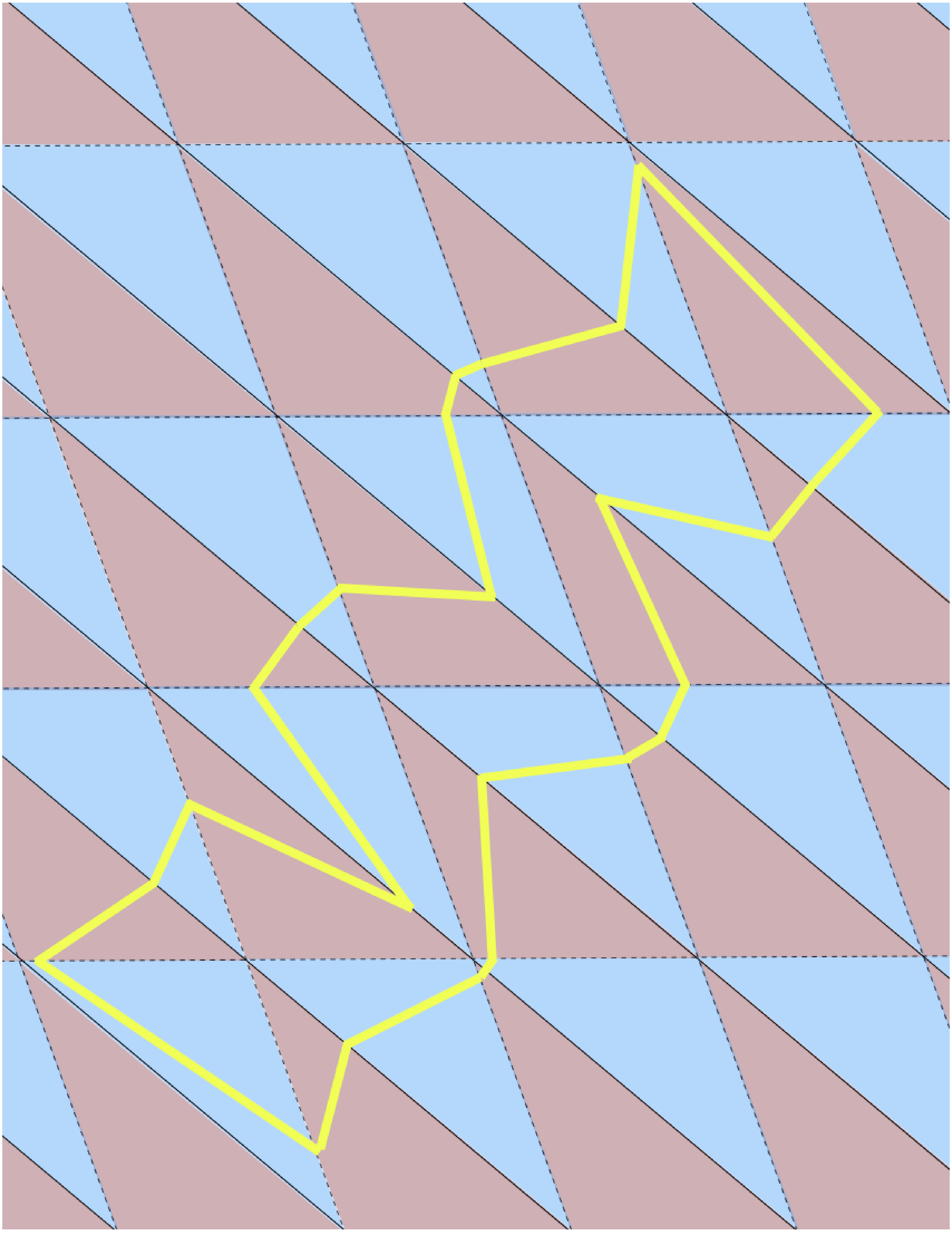
By the way, we used $\left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right]$ because it works nicely with trajectories whose slope is greater than 1: it makes them simpler, like taking a derivative in calculus, while $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ makes them more complicated, like taking an integral.

Find the analogous effects on slopes of trajectories, of the matrices $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]$.

DD **3.** (Continuation: challenge problem, optional) There is just one more step, to show that every shear can be reduced to the ones we understand. Prove the following:

(4) Every 2×2 matrix with nonnegative integer entries and determinant 1 is a product of powers of the shears $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$. For example, $\left[\begin{smallmatrix} 3 & 7 \\ 2 & 5 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]^2 \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]^2$.

DD **4.** The figure on the next page shows a periodic tiling billiards trajectory on a triangle tiling. Cut off the white part and then fold along all the edges of the tiling, in such a way that every part of the trajectory lies on a single line. The solid lines should be "mountain folds" and the dashed lines should be "valley folds." *Notes:* You can do it! Bring your folded paper to class. Save your folded paper, as we will use it in many subsequent problems.

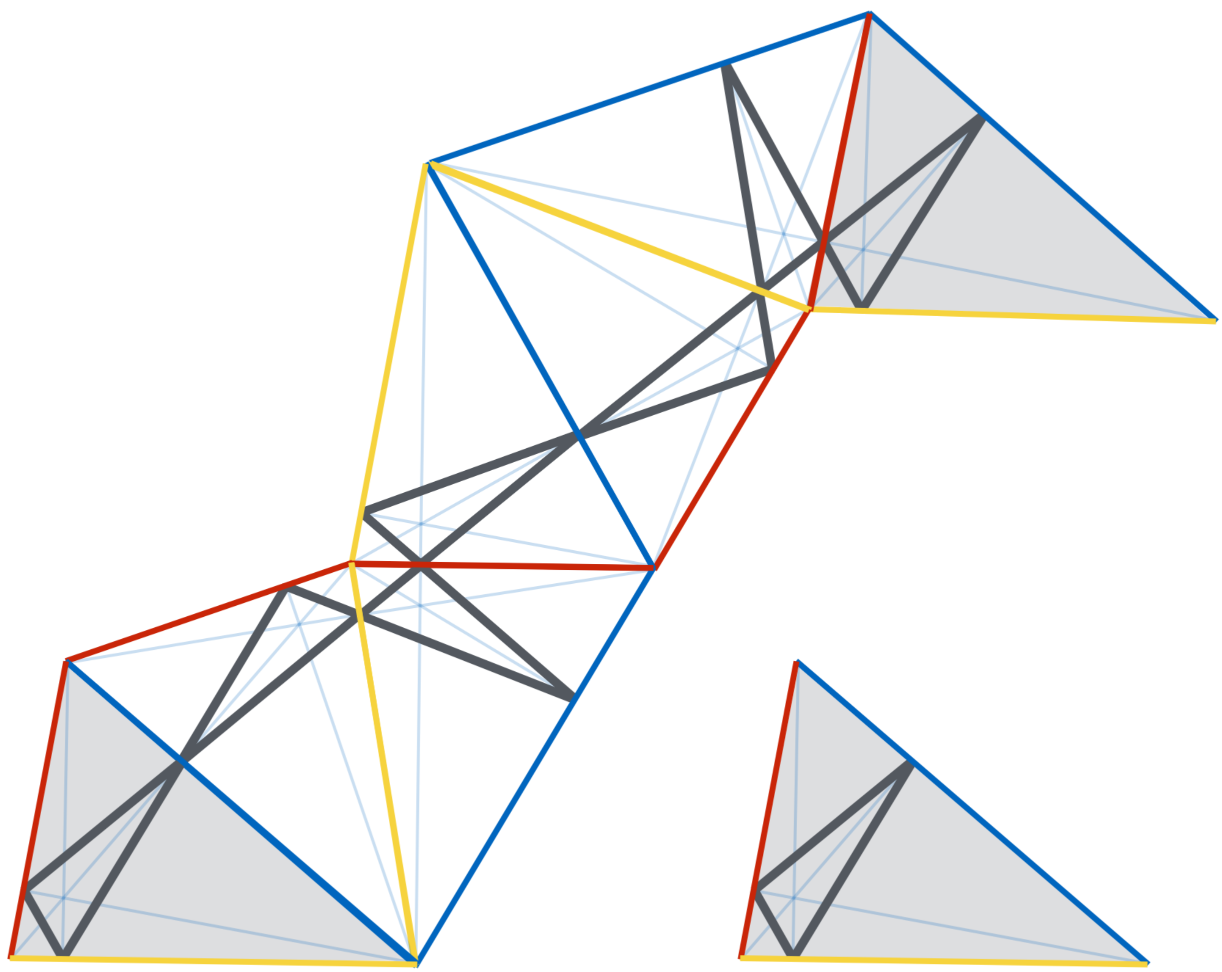


Billiards, Surfaces and Geometry

DD **1.** The figure below shows the Fagnano trajectory in the 40-60-80 triangle. In Page 17 # 3 we showed that there are nearby parallel billiard trajectories of period 6. It turns out that those trajectories form a "strip," or "family." The figure shows an unfolding of this trajectory, with a new copy of the triangle at each edge crossing, until the newest triangle is a translation of the original triangle.

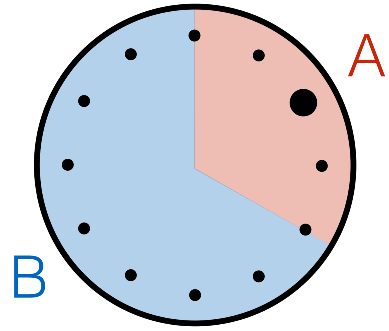
(a) Sketch the "strip" of parallel billiard trajectories on the unfolding. How wide can you make the strip – what constrains its width?

(b) Choose one trajectory in this strip, other than the original billiard path, and "fold" it back up, i.e. sketch it on the triangle in the lower right. Comment on any patterns you notice.



DD **2.** Start with a parallelogram, and add a diagonal to break it into two triangles T_1 and T_2. Fold the parallelogram along this diagonal. Prove that, in this folded state, the circumcenters of T_1 and T_2 coincide.

DD **3.** Consider a circle broken into a red arc and a blue arc, taking up 1/3 and 2/3 of the circle respectively, as shown. The game is to start with any point on the circle, repeatedly rotate it by a 1/3 turn, and each time note down which part of the circle it lands in – say, an A if it lands in the red arc and a B if it lands in the blue arc. Try this for several different starting points of your choice, and rotate each of them until you see a pattern.



DD **4.** One reason why people study cutting sequences on the square torus is that they have very low *complexity*: The *complexity function* $f(n)$ on a sequence is the number of different "words" of length n in the sequence. One way to think about complexity is that there is a "window" n letters wide that you slide along the sequence, and you count how many different things appear in the window.

(a) What is the highest possible complexity for a sequence of As and Bs? For this question, consider all possible sequences of As and Bs, not just cutting sequences.

(b) Confirm that the cutting sequence $\overline{ABABB}$ has complexity $f(n) = n+1$ for $n = 1, 2, 3, 4$ and complexity $f(n) = n$ for $n \geq 5$.

(c) Prove that a periodic cutting sequence on the square torus with period p has complexity $f(n) = n + 1$ for $n < p$ and complexity $f(n) = n$ for $n \geq p$.

(d) Aperiodic sequences on the square torus are called *Sturmian sequences*. Show that Sturmian sequences have complexity $f(n) = n + 1$.

DD **5.** The *Gauss-Bonnet Theorem* says that the total (Gaussian) curvature K of a closed surface S is

$$\int_{\partial S} k \; dA = 2\pi \; \chi(S).$$

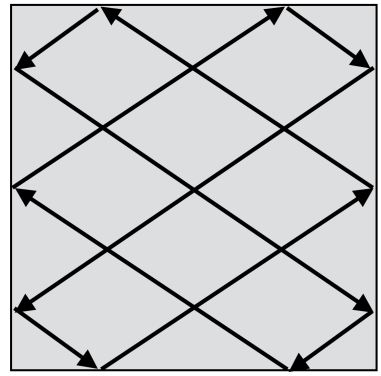
Here k is the curvature at each point of the surface and $\chi(S)$ is the Euler characteristic.

(a) Compute each side of this equation for the unit sphere S.

The *defect* of a vertex is 2π minus the sum of all the angles at the vertex. The *total defect* of a polyhedron is the sum of the defects of all of its vertices.

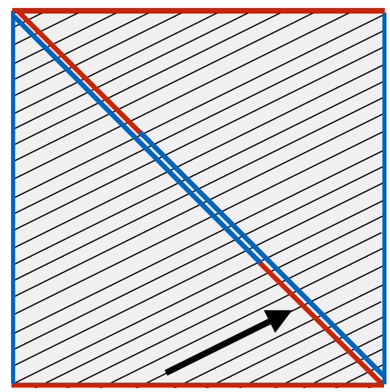
(b) Descartes' special case of the Gauss-Bonnet Theorem says that the total defect of a polyhedron is $2\pi \; \chi(S)$. Check this formula for the cube, the square torus, and the octagon surface.

Billiards, Surfaces and Geometry



ST **1.** Show that a periodic trajectory on a polygonal billiard table is never isolated: an even-periodic trajectory belongs to a 1-parameter family of parallel periodic trajectories of the same period and length, and an odd-periodic trajectory is contained in a strip consisting of trajectories whose period and length is twice as great. *Hint*: Think about a wide ribbon whose center line is the trajectory, wrapping around the triangle.

DD **2.** Given a tiling billiards trajectory on an obtuse triangle tiling (refer to your folded triangles from Page 21 # 3), show that, if the tiling is folded along each edge that the trajectory crosses, all of the triangles that the trajectory crosses are inscribed in the same circumcircle in their folded state. Is the same true on non-obtuse triangle tiling?



DD **3.** The picture to the right shows many trajectories of slope 1/2 on the square torus. As usual, we care about when a given trajectory crosses a horizontal or vertical edge, and we record such crossings with an A or B, respectively. In this picture, I've added a diagonal of the square, and colored it on both sides: on the bottom side to indicate whether an incoming trajectory comes from a red or blue side, and on the top side to indicate whether an outgoing trajectory will hit a red or blue side. Show how to use just the diagonal (copied larger below) to record the edge crossings of the indicated trajectory.

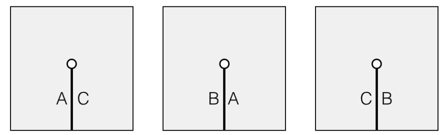
DD **4.** *Counting periodic trajectories, part I*

For a natural number p, how many periodic billiard paths (up to symmetry) of period $2p$ are there on the square billiard table? Check your answer with your previous results.

Billiards, Surfaces and Geometry

Synthesis due – METIC and problems in class

DD **1.** We saw that the octagon and double pentagon surfaces each have just one vertex, with 6π of angle around it. What does this even mean? What does it look like?

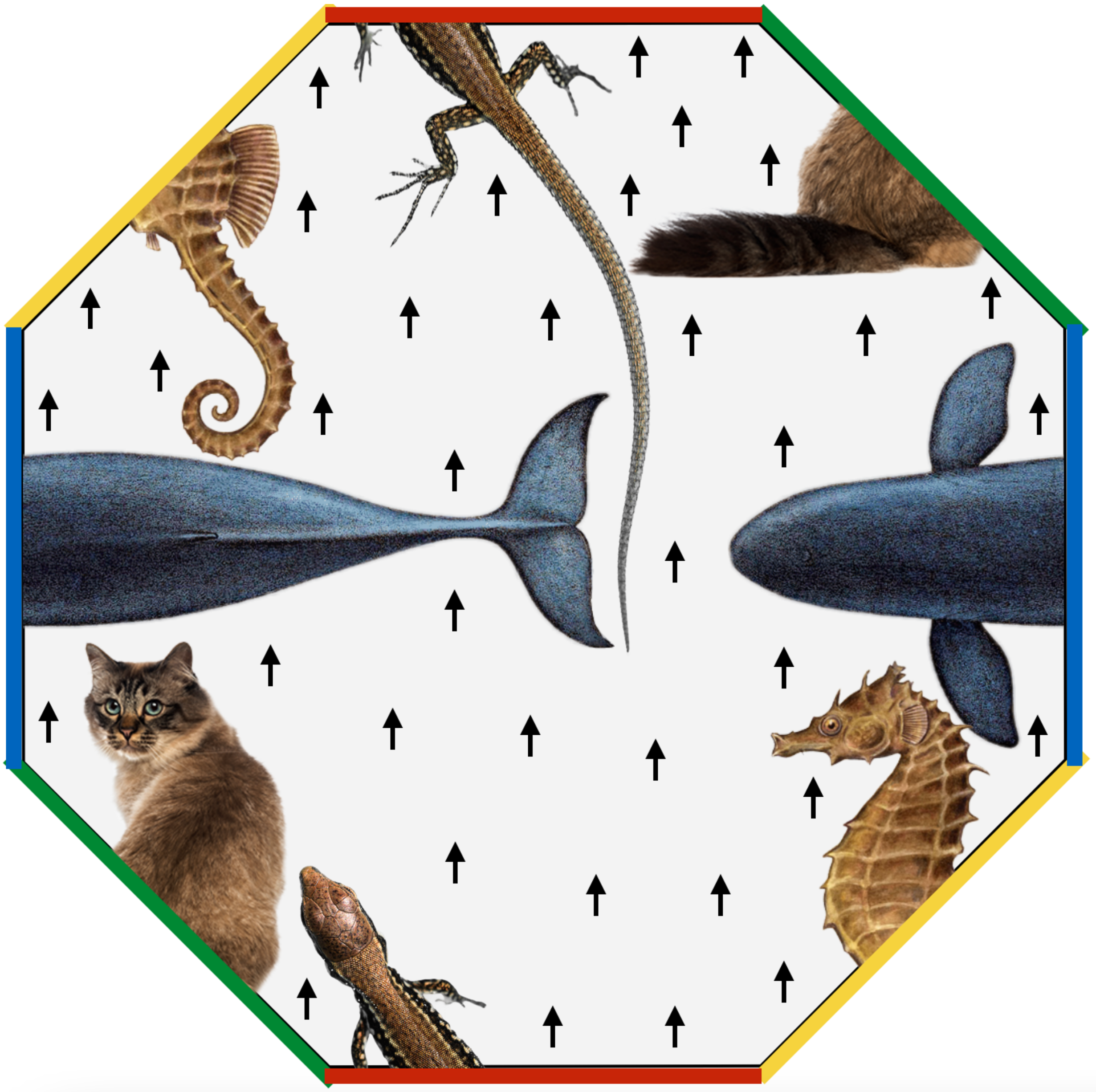


Choose one of the following activities and do it:

(a) Cut slits in three sheets of paper, and tape the edges together as shown above. The vertex angle at the white point is now that of three planes, which is 6π. Discuss.

(b) Cut out the double pentagon below (which includes Dr. Libby Stein, Brown '15, dancing on this surface). Tear off each corner, keeping each piece as large as you can. Tape them together according to which edges are identified. The angle at the vertex is now 6π. Discuss.

(c) Cut out the octagon on the next page. Do as described in **(b)**.



DD **2.** Given a tiling billiards trajectory on an obtuse triangle tiling (refer to your folded triangles from Page 21 # 3), show that, if the tiling is folded along each edge that the trajectory crosses, all of the triangles that the trajectory crosses are inscribed in the same circumcircle in their folded state. Is the same true on non-obtuse triangle tiling?

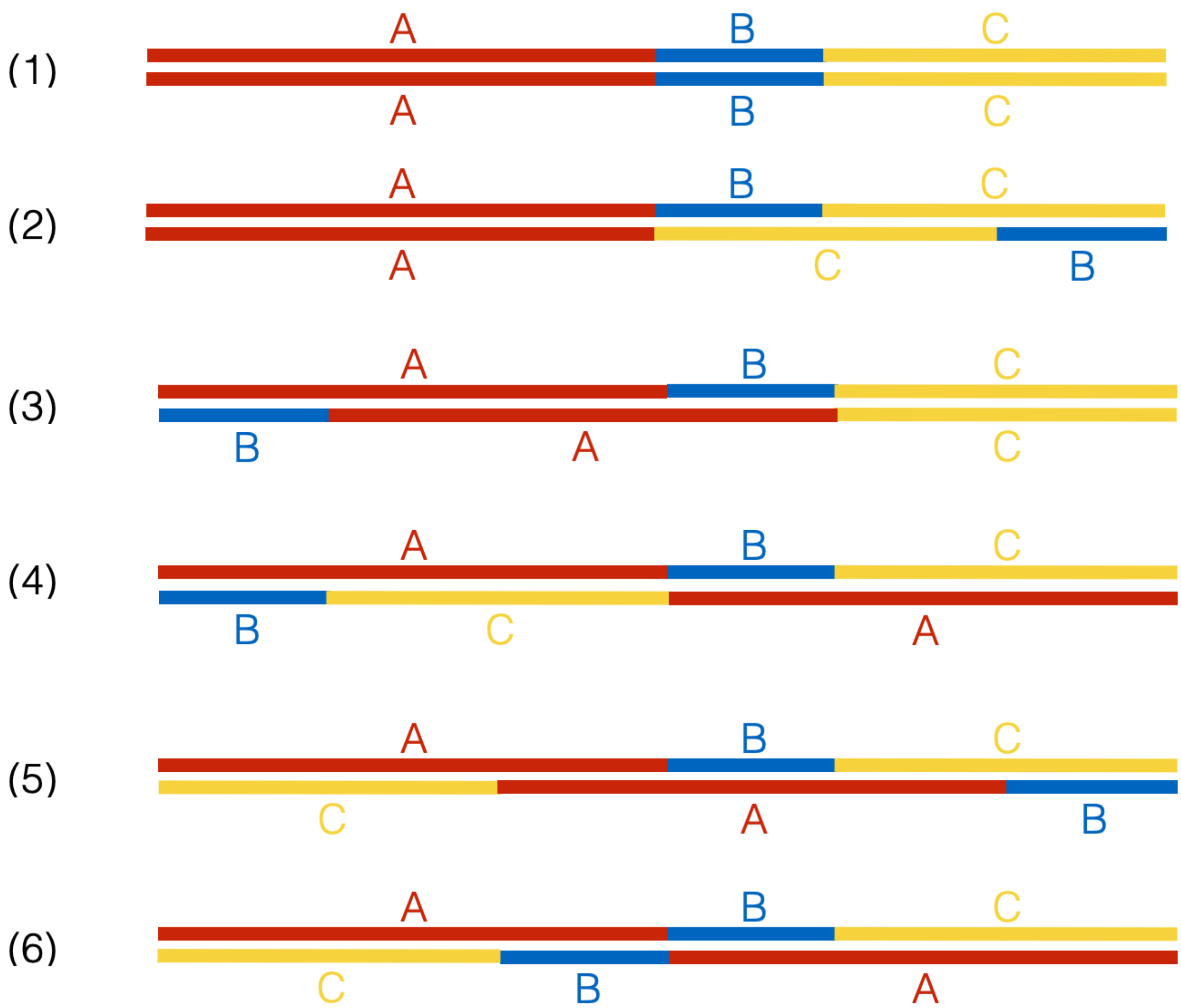
Billiards, Surfaces and Geometry

DD **1.** The construction in Page 20 # 3 showed how to represent a trajectory on a surface via the motion of a point on an *interval exchange transformation*.

(a) Explain why the interval exchange transformation (IET) in Page 20 # 3 is identical to the rotation in Page 19 # 3.

(b) Explain why every 2-interval IET is equivalent to a rotation.

(c) The figure below shows the six possible ways of rearranging three intervals. (1) is the identity, and (2) and (3) are just 2-IETs (rotations) on a smaller interval. Of the remaining three, two of these are also rotations, leaving just one true 3-IET. Which one?



DD **2.** For the 3-IET that you identified in the previous problem:

(a) Choose a point, mark all the places it goes (find its *orbit*), and find its period.

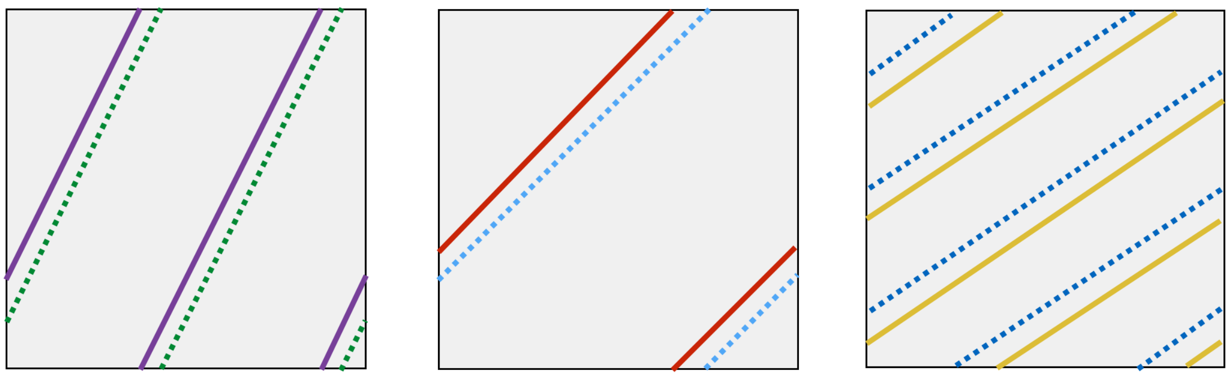
(b) The interval lengths for the IETs above are $1/2, 1/6, 1/3$. What if they were irrational?

DD **3.** Each picture below shows a trajectory on the square torus surface (solid line), and a parallel trajectory that is slightly shifted, or "perturbed," from the original (dotted line). Let's consider them to be in the same "family."

(a) For each picture, draw another trajectory that is slightly perturbed from the given ones, and is also in the same family.

(b) If you perturb a trajectory enough, it will eventually hit a vertex. A "singular trajectory" that hits a vertex is not allowed, and forms the boundary of the family of trajectories. Draw in these boundaries for each of the pictures.

(c) The union of a family of trajectories is called a *cylinder*. Can you guess why this name was chosen?



ST **4.** *Counting periodic trajectories, part II*

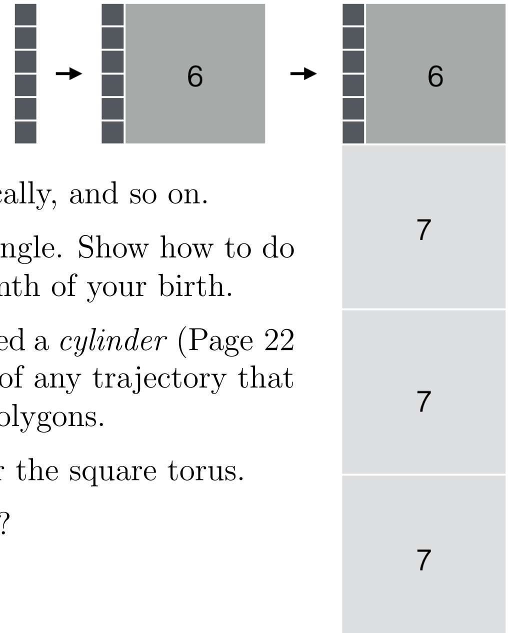
Another way to count billiard trajectories in the square is to ask how many periodic trajectories of length less than L it has. This question should be understood properly: Periodic trajectories appear in parallel families (as explored above); we will count the number of families.

(a) How long is the trajectory of slope 2? The trajectory of slope 3/4?

(b) Explain why the number of lattice points inside a disc of radius L is approximately πL^2, especially when L is large.

(c) Use the above to show that the number of periodic families of length less than L is approximately $\pi L^2/8$.

Billiards, Surfaces and Geometry



DD **1.** Here is a new game: make some number of 1×1 squares going vertically (here, six). Then make a big square that goes across all of them, and make some number of those going horizontally (here, one). Then make a big square that goes across all of *them*, and make some number of those going vertically, and so on.

The picture shows how to do this to end up with a 7×27 rectangle. Show how to do this to make a rectangle whose dimensions are the day and month of your birth.

DD **2.** Recall that the union of a family of parallel trajectories is called a *cylinder* (Page 22 # 3). For a polygon surface, a *cylinder direction* is a direction of any trajectory that goes from a vertex to another vertex, possibly crossing many polygons.

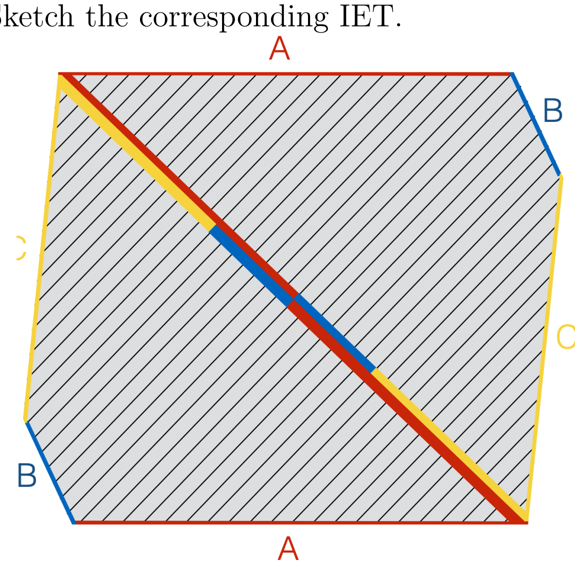
(a) Explain why slopes 1, 2, and 2/3 are cylinder directions for the square torus.

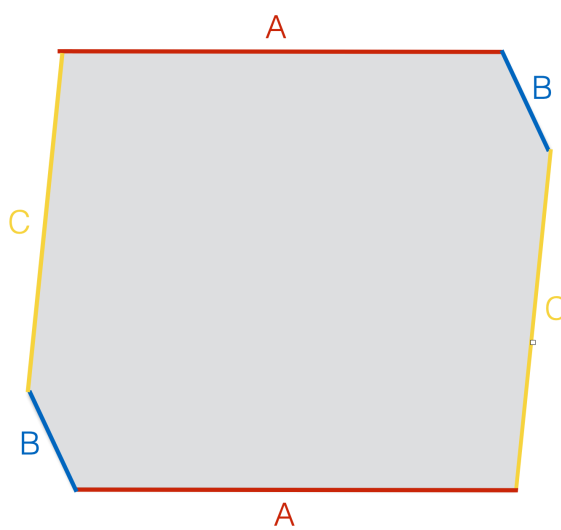
(b) What are all of the cylinder directions for the square torus?

DD **3.** The pictures below show a surface made from a non-regular hexagon.

(a) The first picture shows many trajectories in a given direction. Explain how any trajectory in this direction can be represented by the orbit of a point on a particular IET.

(b) In the second picture, draw a family of trajectories in a different direction of your choice. Sketch the corresponding IET.





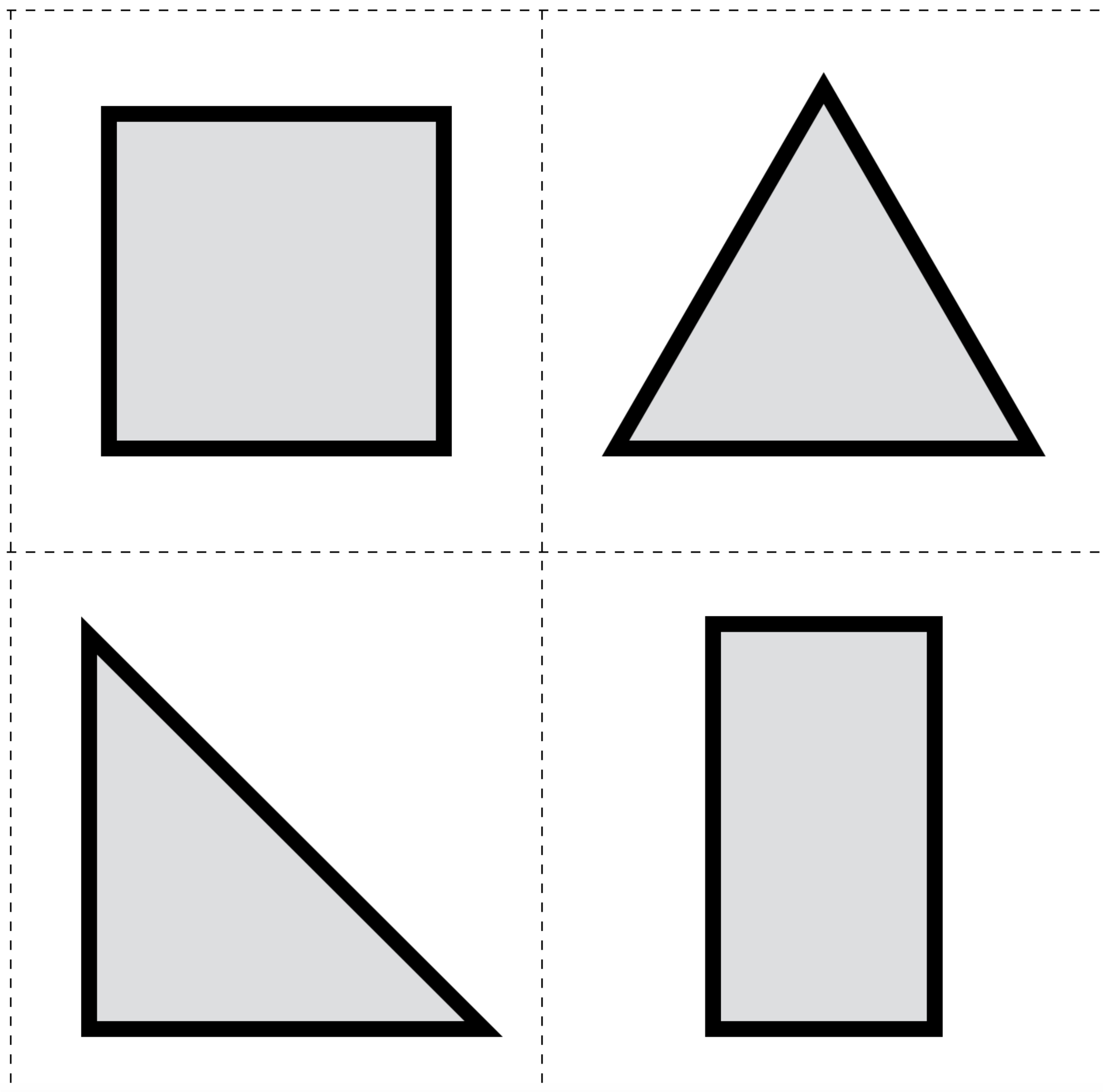
Contextual note. In mathematics, we often care about the *dimension* in which we are working. A torus is a 2D object, and if we look at it as the surface of a bagel, it is a 2D surface "embedded" in 3D space. The family of parallel trajectories in a given direction on the square torus looks like a 2D thing, but the problem above shows that the behavior of each one can be reduced to the orbit of a point on an IET, which is a 1D thing.

You all don't seem that enamored with tiling billiards on triangle tilings, so we're going to pivot it over to another topic (the Fold and Cut Theorem) and then maybe drop it.

DD **4.** We have shown that, when a triangle tiling is folded along every edge of the tiling, all of the pieces of trajectory line up (Page 17 # 2). You can see this in your folded triangles from Page 18 # 4.

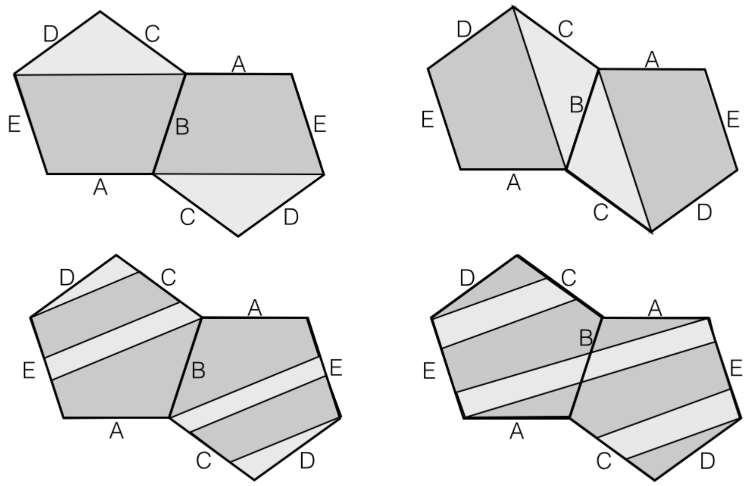
(a) If you have a tiling billiards trajectory on a tiling (such as your folded triangles), and you fold along every edge of the tiling, it is possible to make one single straight cut with a pair of scissors and exactly cut along all of the trajectory edges. Explain.

(b) Cut along the dashed lines, to give you four shapes, each on its own small piece of paper. For each shape, find a way to fold up the paper so that you could make one straight cut with a pair of scissors and exactly cut out the shape.



Billiards, Surfaces and Geometry

DD **1.** For the square torus, in every cylinder direction there is only one cylinder. For surfaces made from other polygons, there can be multiple cylinders. The double pentagon surface has *two* cylinders in each cylinder direction. Here are cylinders on the double pentagon surface in four directions.



(a) For each set of cylinders shown, consider a trajectory on the surface, in the cylinder direction. Write down the cutting sequence for the trajectory in the light cylinder and for the trajectory in the dark cylinder. Think about similarities and differences with our work on the square torus.

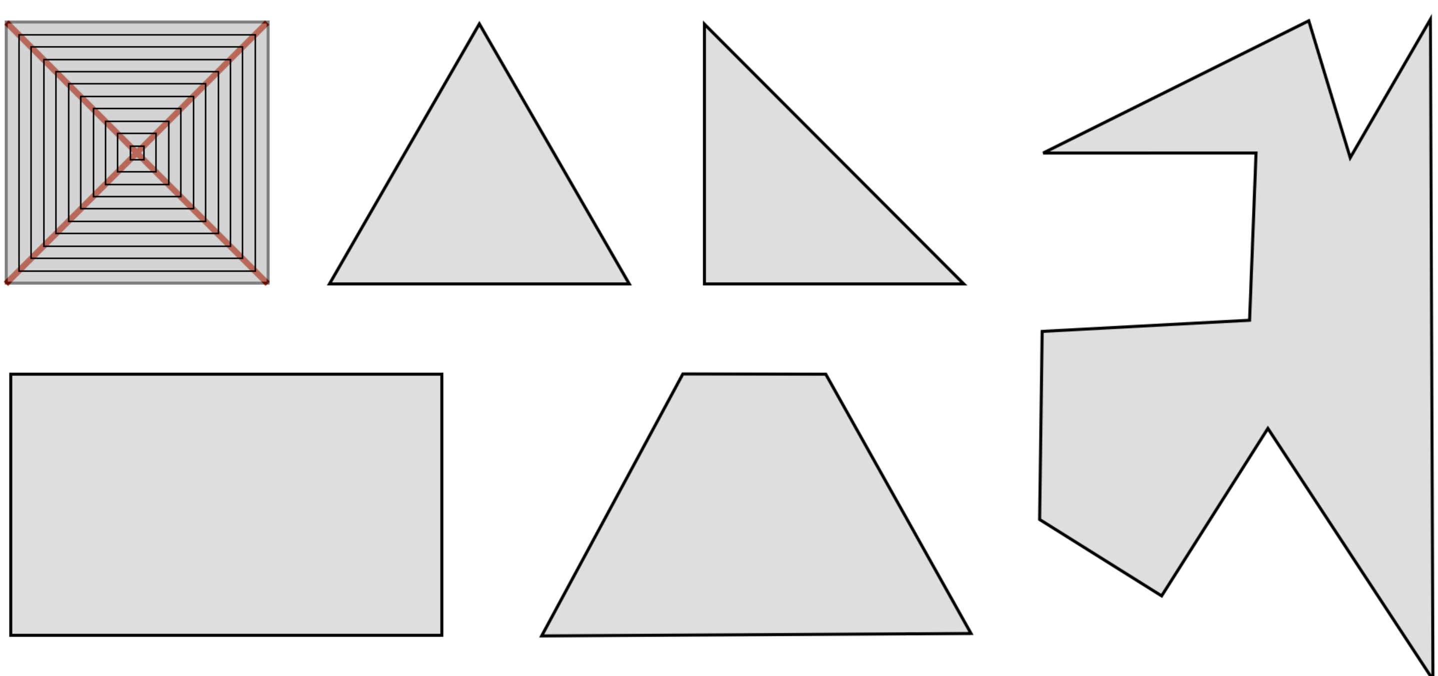
(b) The two cylinder decompositions in the top line of the picture are essentially the same, just in a different direction. Construct a vertical cylinder decomposition of the surface. Is it the same as any of those shown?

DD **2.** Given a polygon that we want to cut out with a single cut (Page 23 # 4), how do we know where to fold? An excellent first ingredient towards figuring this out is called the *skeleton* of the polygon. Here's how to construct it:

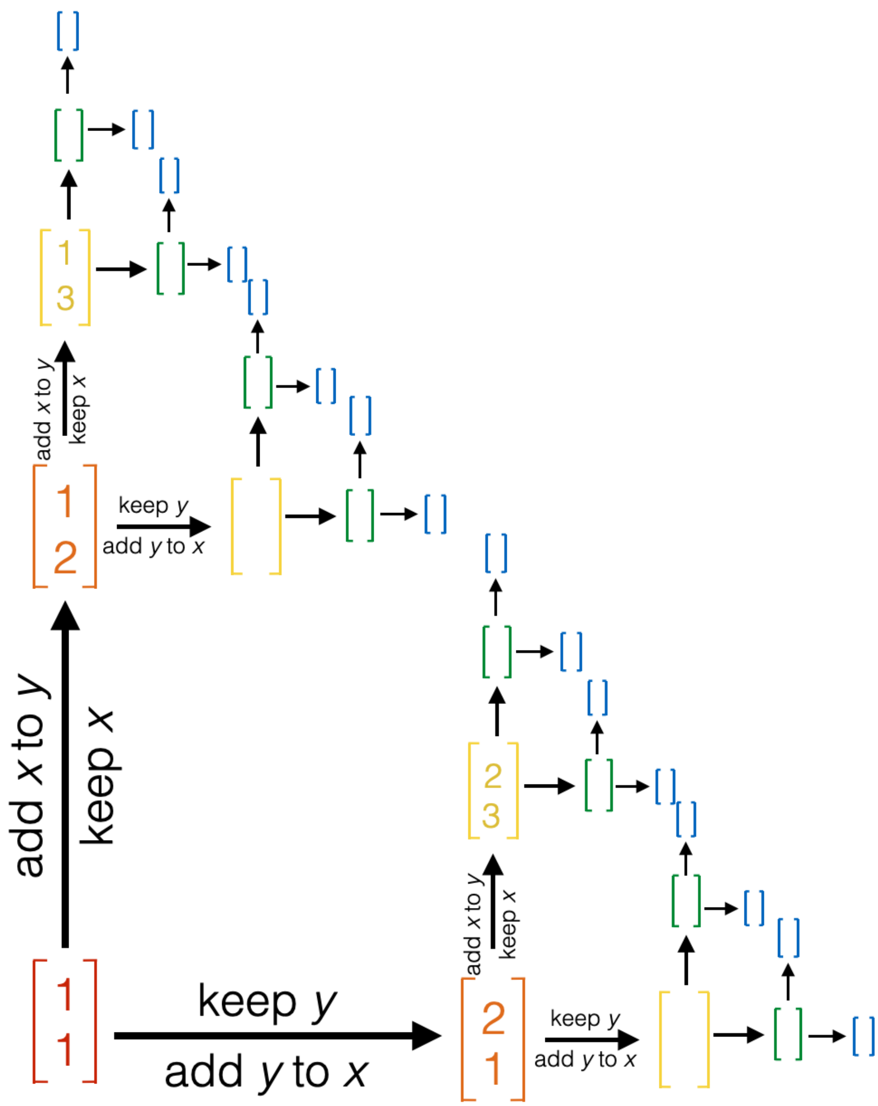
(1) Shrink the polygon in such a way that each edge of the shrunken version is *equidistant* from the original edge. The picture below shows this process for the square.

(2) Create line segments out of the vertices of the shrinking polygon. The skeleton consists of these line segments, plus any additional line segments that are left when a polygon shrinks to area 0. The skeleton for the square is shown in red.

Shrink away from the edges in several of the below polygons, and sketch in the skeleton.



DD **3.** The picture below shows a way of starting with simple vectors and generating more complicated vectors. Here is how we construct this tree: start with the vector $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ in the lower left. At each step, choose to either add the entries together to get a new x-value (moving right), or choose to add the entries together to get a new y-value (moving up). Fill in as many entries as you can.



The picture shows the first five levels of an infinite binary tree. A *binary tree* means that at this *node* of the tree, you have two choices of where to go – in this case, right or up. I made each level smaller than the previous one so that five levels would fit on the page.

DD **4.** (Continuation) Let's explore this tree a bit.

(a) Find $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}3\\5\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}8\\5\end{smallmatrix}\right]$ in the tree. Comment on any patterns.

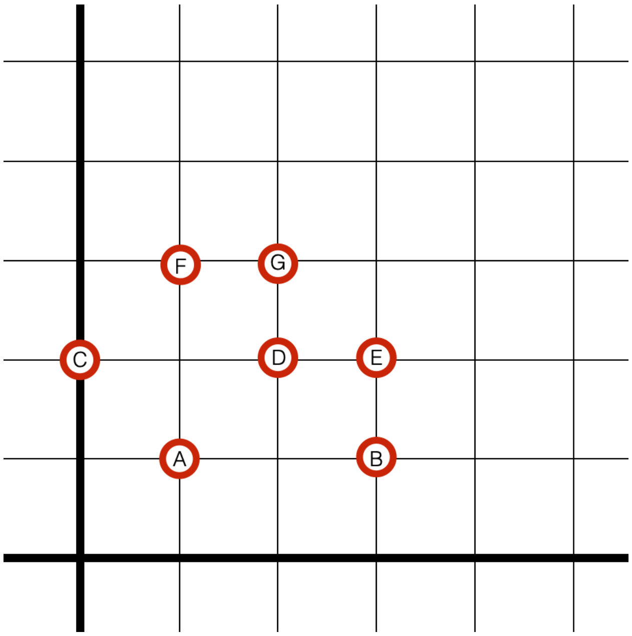
(b) What vectors appear in this tree? Does your birthday vector $\left[\begin{smallmatrix}\text{month}\\\text{day}\end{smallmatrix}\right]$ appear in the tree? At what level?

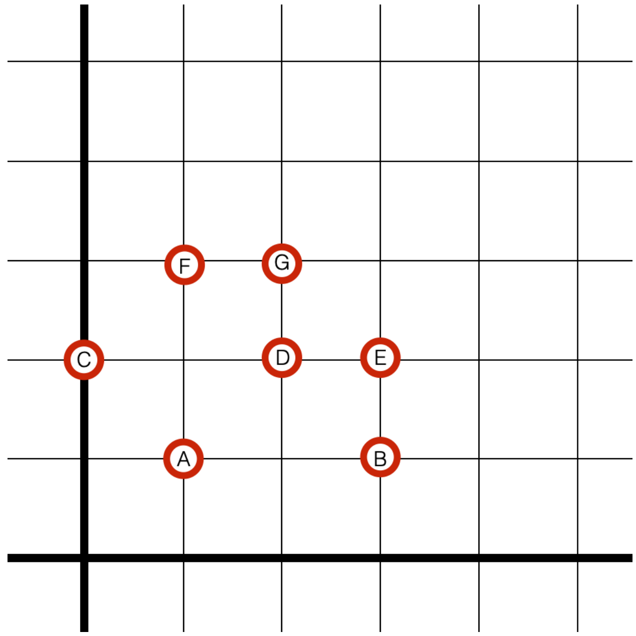
Billiards, Surfaces and Geometry

DD **1.** Let's make sure your shearing skills are sharp. (Local sheep, beware!)

(a) In the left picture, draw the image of each of the identified lattice points under the horizontal shear $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$.

(a) In the right picture, do the same for the vertical shear $\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]$.



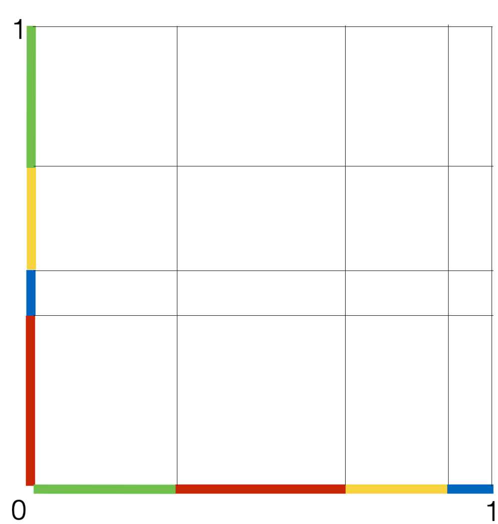


DD **2.** Ooh, let's graph more things!

(a) Here is a 4-IET. Find the orbit of a point of your choice for at least five or six iterations.

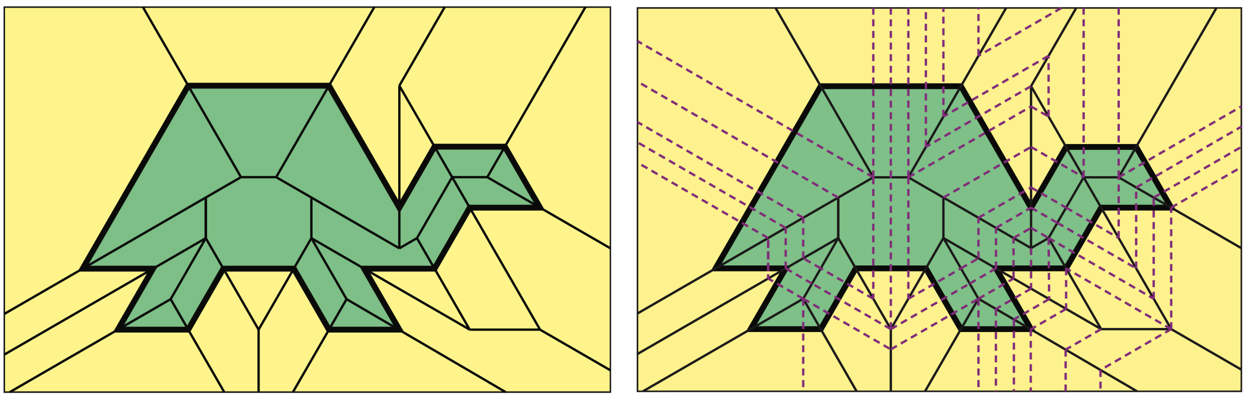
(b) An IET essentially cuts up an interval of points and reassembles them. So we can think of an IET as a function that maps points between 0 and 1 to points between 0 and 1. Graph the function corresponding to the above 4-IET on the axes to the right.

(c) Use the graph to find the orbit of the same point that you followed in part **(a)**.



DD **3.** The picture below shows a turtle, which we wish to cut out with a single cut. The left picture shows the skeleton of the turtle (green part), *and* the skeleton of the exterior of the turtle (yellow part), which we get by enlarging the polygon as we did with shrinking. The skeleton is a good start for how to fold, but it is not the whole story, because (for example) every vertex of a foldable folding pattern has an even number of edges meeting at it.

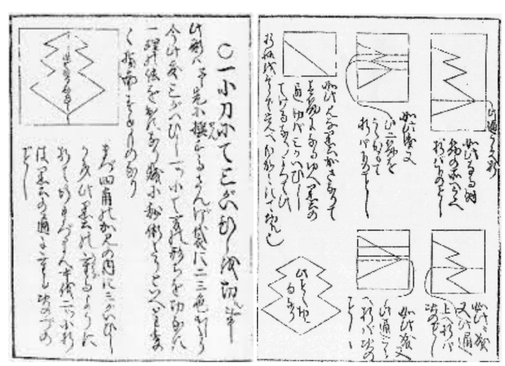
So the next part of the story is *perpendiculars* (right picture). From each vertex of the skeleton, the idea is that we draw more lines, perpendicular to edges of the turtle. When we hit an edge of the skeleton, we reflect as in tiling billiards. Look at the right picture and explain in more detail what is going on and why we want to do this.

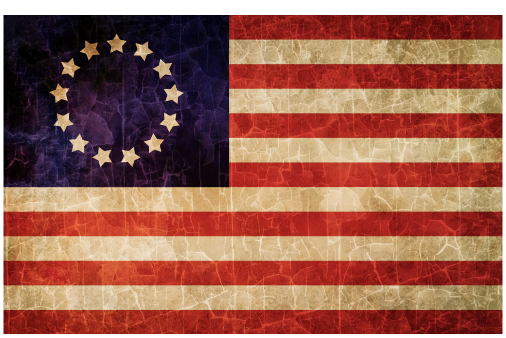


This turtle is from Demaine and O'Rourke, *Geometric Folding Algorithms* (2007) pp. 256–259.

DD **4.** The great magician Harry Houdini performed the following trick: take a piece of paper, fold it a few times times, make a single straight cut with scissors, and a five-pointed star falls out! Figure out how to do this magic trick and then practice it on your friends.

Contextual note 1. Fold and cut puzzles date to at least 1721, when Kan Chu Sen posed such a problem in a Japanese puzzle book. The first page gives the problem; the second gives the solution.

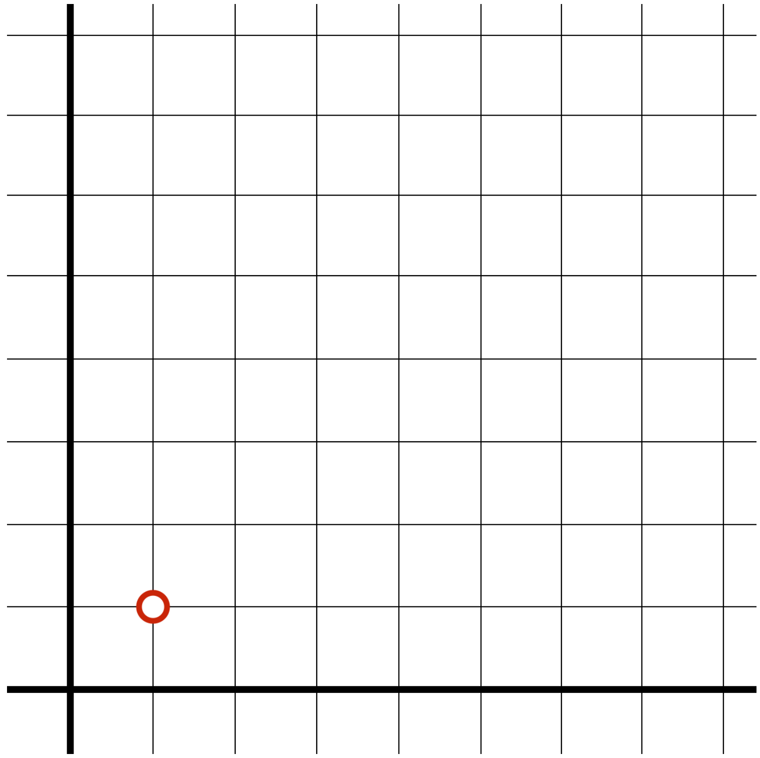




Contextual note 2. Legend has it that when the original U.S. flag was designed, someone commented to Betsy Ross that it was going to take forever to cut out all those stars. "No, it's easy," she said, picked up a piece of scrap cloth, folded it a few times, made a single cut, and a five-pointed star fell out.

Billiards, Surfaces and Geometry

DD **1.** You've built up rectangles from squares. You've filled in the binary tree of relatively prime vectors. Now let's look at a third way to generate all of the relatively prime vectors: *shears!*



(a) Start with $(1,1)$ as shown. If you apply the horizontal shear $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ to the red point, you'll get one new point, $(2,1)$ – draw this in orange. If you also apply the vertical shear $\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]$ to the red point, you'll get one new point $(1,2)$ – draw this in orange also.

(b) Now apply the horizontal shear $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ to *all* the existing points (red and orange) and draw these new points in yellow. Do the same for the vertical shear $\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]$, applying it to all of the pre-existing red and orange points.

(c) Now apply both the horizontal and vertical shears to the existing red, orange and yellow points. Draw these new points in green.

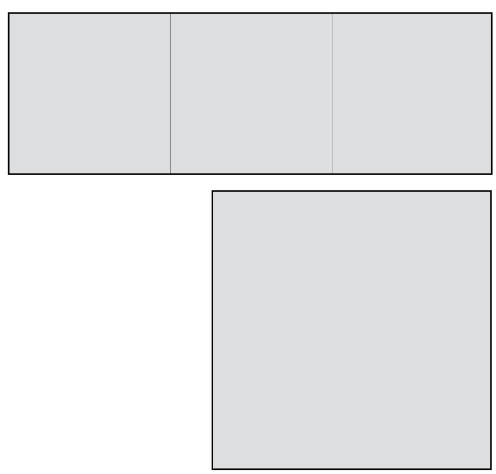
(d) Repeat the above for all the existing points. Draw the new points in blue. Continue as long as you like.

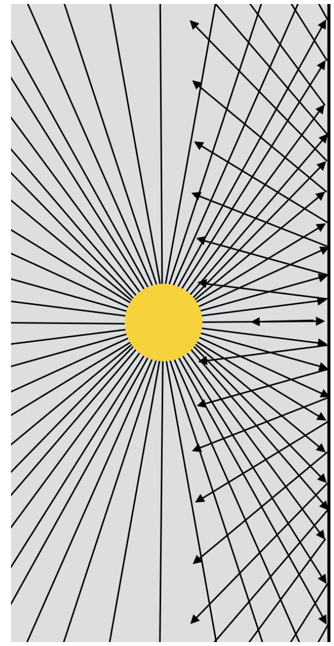
DD **2.** (Continuation)

(a) Explain the connections between these three ways of generating new points: adding squares, adding vectors, shearing the plane.

(b) A student asked if there is a name for the set of vectors we get in this way. Suppose that you are standing at the origin of an infinite orchard, and there is a tree at every lattice point. One name for the points generated using the method above is *visible points*. Explain.

DD **3.** Let us embark on a new magic trick: cutting and reassembling polygons. Show how to cut up a 1×3 rectangle into pieces that can be reassembled into a square. Use as few pieces as possible.



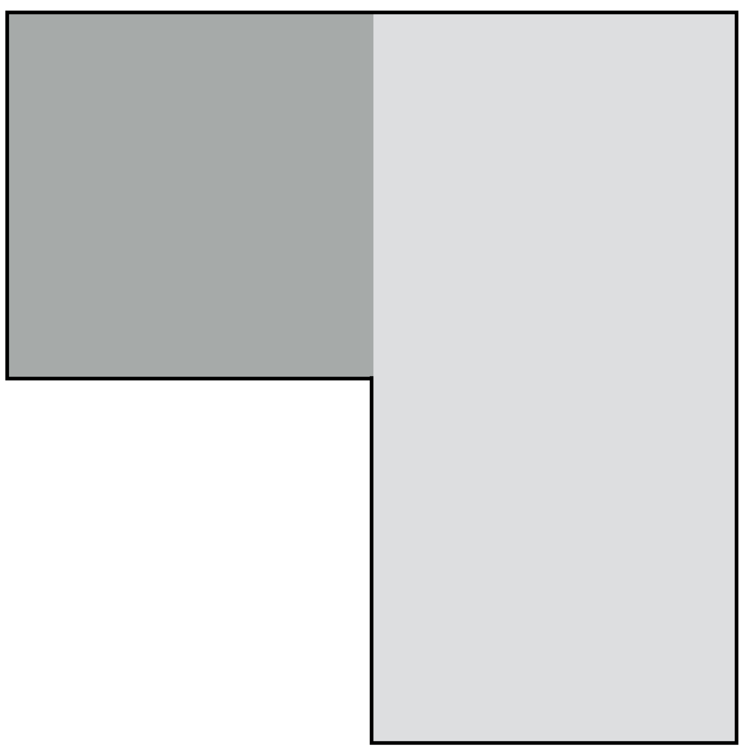


DD **4.** Let's get illuminated! The picture to the right shows what happens when you put a candle in a room: the light radiates out in every direction. Look closely at the right side of the picture: this room has a *mirror* on the wall, so the rays that hit the wall bounce off, following the billiard reflection law.

You live in a room whose walls are *all* mirrored. You wish to illuminate your entire room with a single candle.

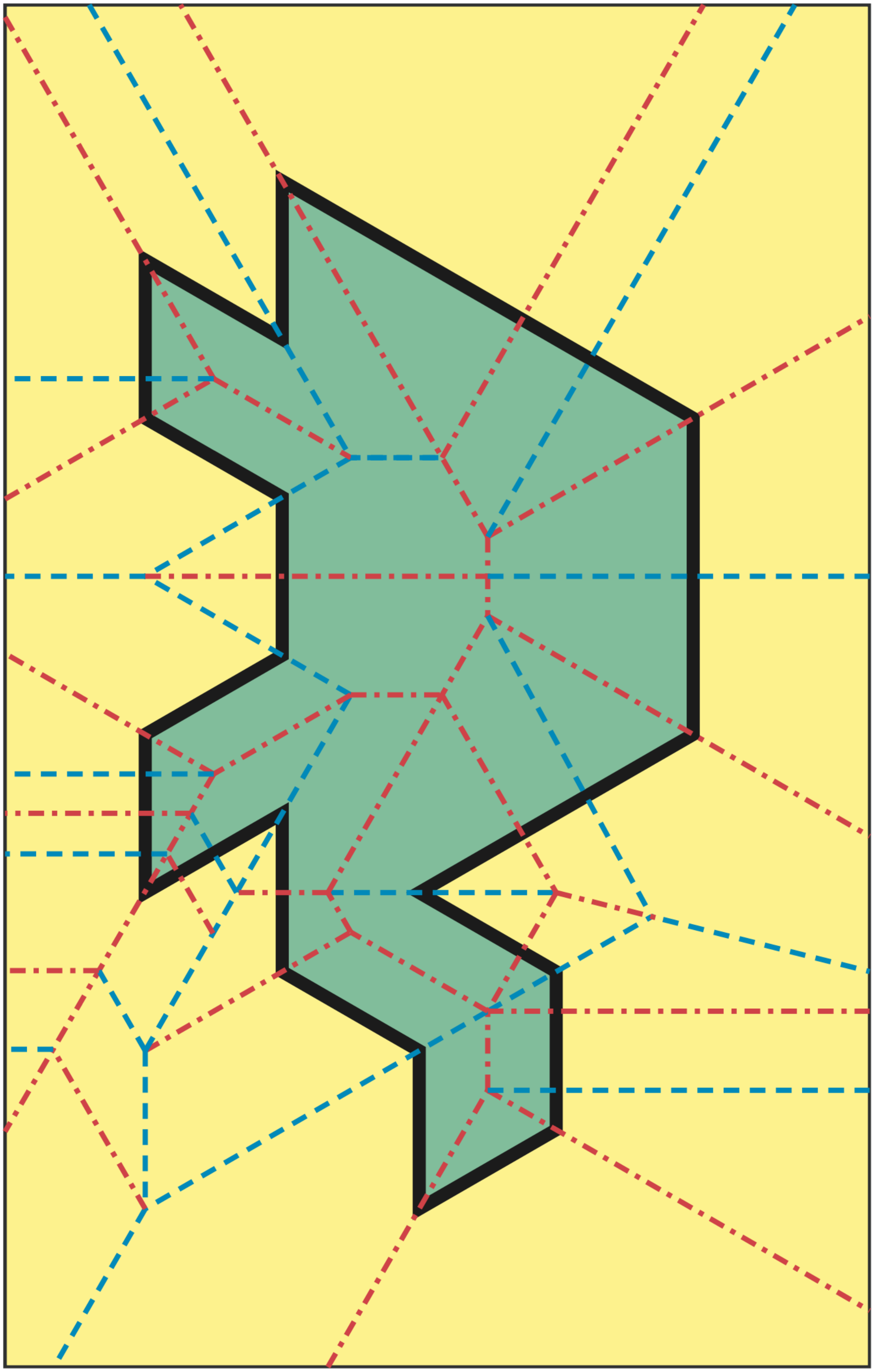
(a) Explain why this problem is easy when the room is convex.

(b) Suppose your room is an L-shape made of three squares, as shown below, and suppose you place the candle somewhere in the dark square. Does the candle illuminate the whole room? Explain why or why not.



DD **5.** (Optional folding challenge) Fold up the turtle so that you could cut it out with a single cut. The dashed lines show a folding pattern, which is a subset of the "skeleton + perpendiculars" construction described on the previous page, that suffices for this purpose.

This turtle is from Demaine and O'Rourke, *Geometric Folding Algorithms* (2007) pp. 256–259.



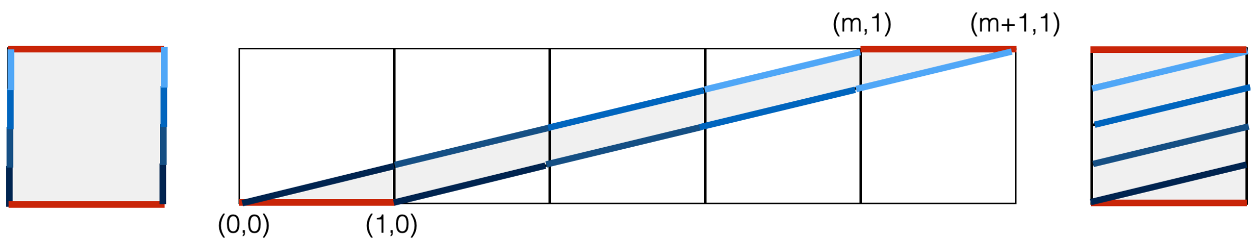
Billiards, Surfaces and Geometry

DD **1.** *More reassembly.*

(a) Show that, given *any* triangle, you can cut it into pieces and reassemble it into a parallelogram. What is the fewest number of pieces you can use?

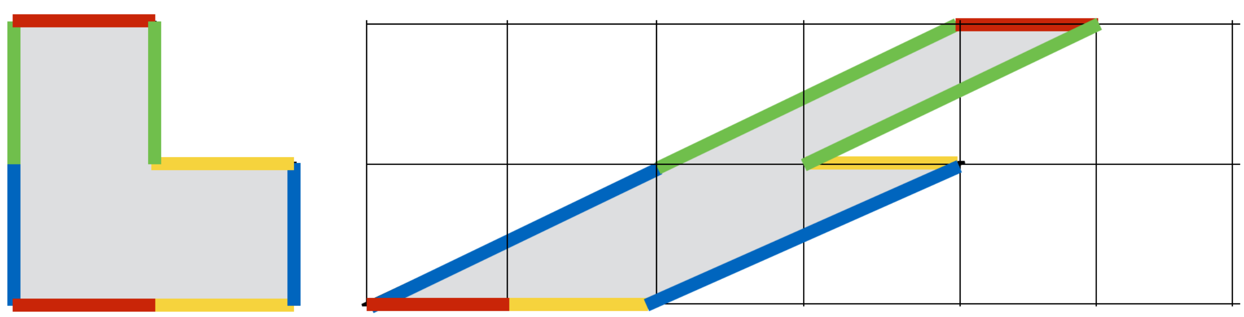
(a) Show that, given *any* parallelogram, you can cut it into pieces and reassemble it into a rectangle. What is the fewest number of pieces you can use?

DD **2.** In our earlier work, we sheared the square torus by the matrix $\left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right]$, which transformed it into a parallelogram, and then we reassembled the pieces back into a square, which was a twist of the torus surface. Below is another way of shearing the square torus (left), this time via the matrix $\left[\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right]$, and reassembling the pieces (right) in such a way that the reassembly respects the edge identifications. This is indicated with shades of blue.

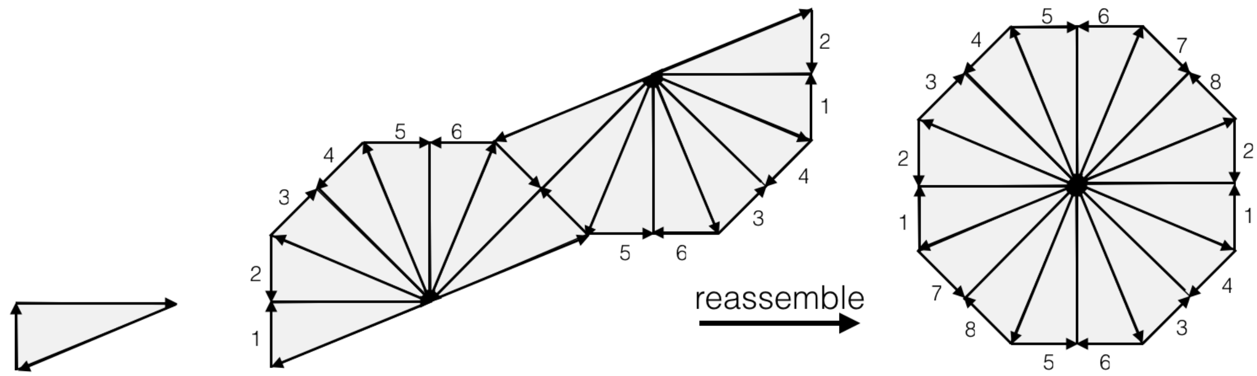


Consider the L-shaped surface made of three squares, with edge identifications as shown in the left picture below. We shear it by the matrix $\left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$, as shown.

Show how to reassemble the sheared surface back into the L surface. Make sure your reassembly respects the edge identifications.



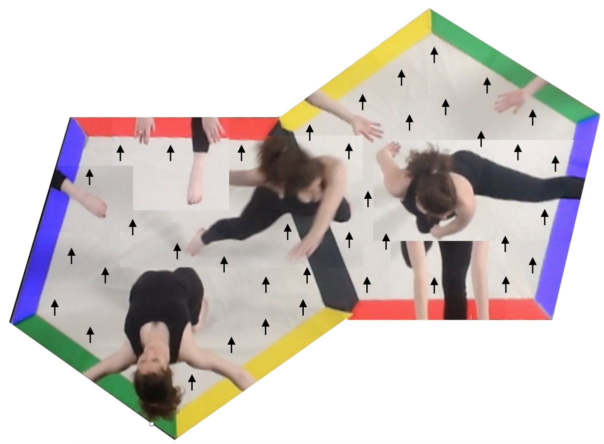
DD **3.** Our original motivation for studying the square torus was that it was the unfolding of the square billiard table. In fact, we can view *all* regular polygon surfaces as unfoldings of *triangular* billiard tables. We unfold the $(\pi/2, \pi/8, 3\pi/8)$ triangular billiard table until every edge is paired with a parallel, oppositely-oriented partner edge:



This gives us the regular octagon surface! So the regular octagon surface is the unfolding of the $(\pi/2, \pi/8, 3\pi/8)$ triangle.

(a) What triangle unfolds to the double regular pentagon surface? Make a guess, then draw the unfolding as above.

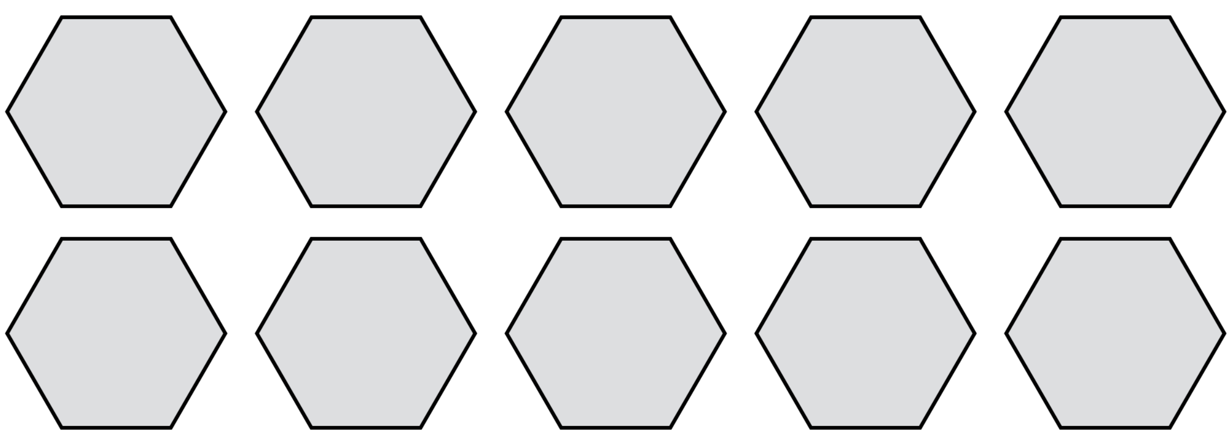
(b) Draw the "shooting into the corner" period-6 trajectory in the triangular billiard table on the left above. Then unfold it to a periodic trajectory on the regular octagon surface.



DD **4.** To *triangulate* a polygon P means to divide it into triangles, whose vertices are the vertices of P, that don't overlap and whose union is P.

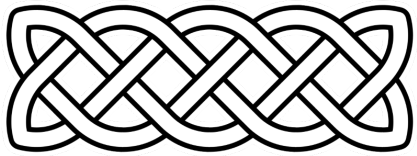
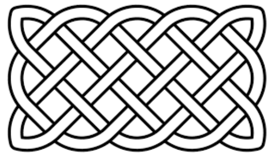
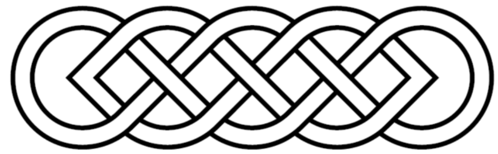
(a) How many different triangulations are there of a hexagon?

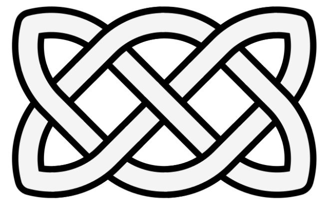
(b) Prove that every polygon, including non-convex polygons, can be triangulated.



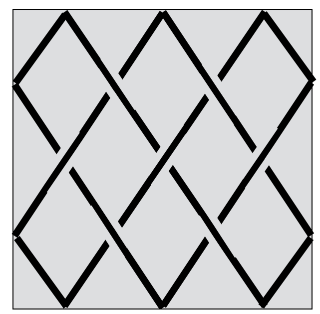
Billiards, Surfaces and Geometry

Synthesis due – problems in class



DD **1.** *Celtic knots* are a traditional form of decorative art associated with Ireland. They come in many different shapes, some of which are related to… periodic billiards on the square! Explain the relationship between billiards and Celtic knots.

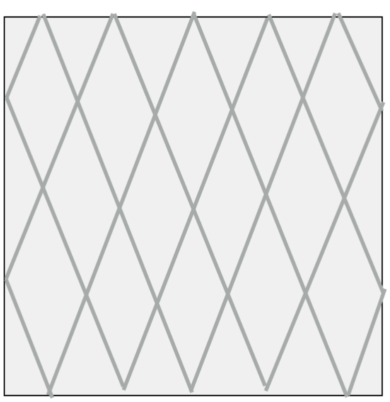
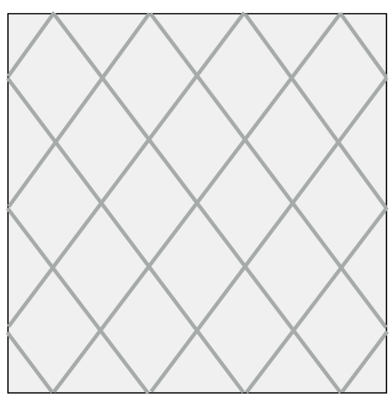
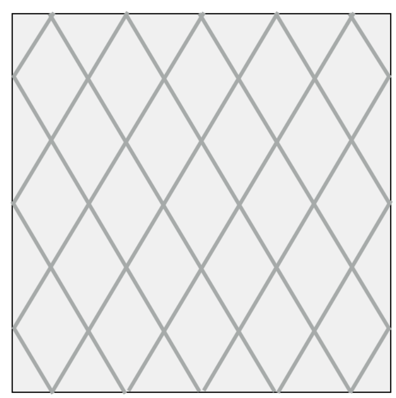


DD **2.** You are all pretty good at drawing diagrams of periodic billiard trajectories on the square. But how to turn it into a *knot*? All Celtic knots are *alternating*, meaning that if you follow a cord along its journey, it alternates over, under, over, under… as it crosses other parts of the cord.

(a) Look at the knot diagrams above and convince yourself that they are alternating.

(b) Some Celtic knots are *knots*, made from a single cord, and some are *links*, made from multiple cords. In the knot diagrams above, which are which?

(c) For the billiard trajectories below, draw in the "crossings" to make the knot alternating. Then fill in the rest of the parts of the line segments to get the entire knot.

DD **3.** With the provided cords, create Celtic knots based on periodic billiard trajectories. Can you make links ("knots" made from multiple cords) out of periodic billiard trajectories?

DD **4.** Can you make these out of billiards?



Billiards, Surfaces and Geometry

DD **0.** For the end of this class, each student will learn about a topic, and then tell the rest of us about it for 5-10 minutes in the last scheduled class plus the test week block. Please ponder the following topics and come to class on Friday with an idea of which one(s) interest(s) you.

• Any topic of your choice, related to topics from this class.

• Vertex-to-vertex paths on the tetrahedron *uses unfolding and tiling; read part of a paper*

• Vertex-to-vertex paths on the cube *uses the tree of visible points; read part of a paper*

• The space of all possible tori made from parallelograms is the upper half-plane *known as Teichmüller space; solve a set of problems by DD*

• The space of positions for two point masses colliding elastically on an interval is equivalent to billiards on a square *solve a set of problems from "Geometry and Billiards"*

• Rainbows are billiards in a raindrop *solve a set of problems*

• Tiling billiards on triangle tilings are equivalent to a family of IETs *read part of a paper*

• How satellite dishes / parabolic reflectors work *look it up*

• Billiards in a Harkness table shape ("stadium") is chaotic *look it up*

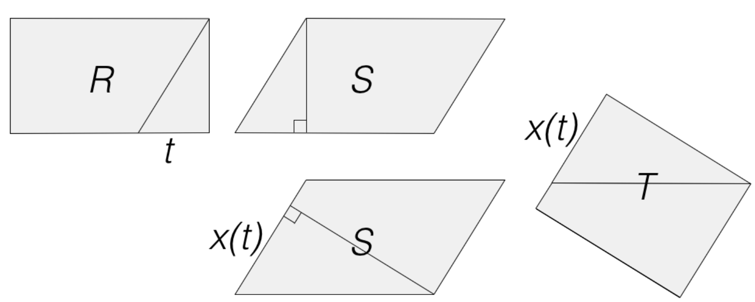
• Outer billiards on the half-disk has escaping trajectories *read part of a paper*

• Outer billiards on the Penrose kite has escaping trajectories *read part of a book*

• We used the skeleton & perpendiculars method for the Fold and Cut Theorem, which works *almost* all the time, but can fail spectacularly; there is a method with circle packings that works *all* the time *read part of a book*

• Any topic of your choice, related to topics from this class.

RS **1.** The picture shows a way to cut and reassemble one rectangle into another, based on a parameter t. The first step shows that R can be reassembled to S, and the second step shows that S can be reassembled to T. The middle two pictures are the same, but emphasize a different decomposition in each copy.



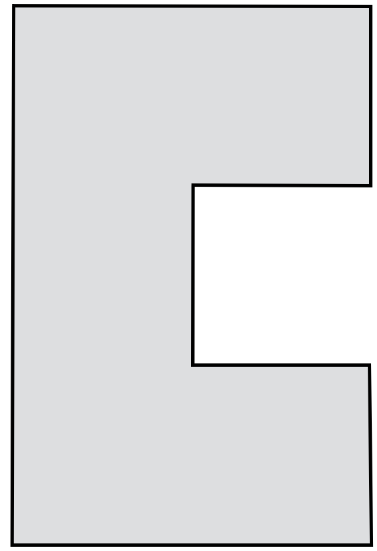
(a) Explain what is going on in the picture.

(b) Start with a 1×2 rectangle. For which values of t can you perform this construction? How about for a $1 \times r$ rectangle?

(c) What does the rectangle T look like, at the minimum and maximum values of t?

(d) Explain why the shape of the rectangle T varies continuously with t. In other words, if you change t a little bit, the shape of T changes a little bit.

(e) Prove that every rectangle can be cut and reassembled to a *square*.



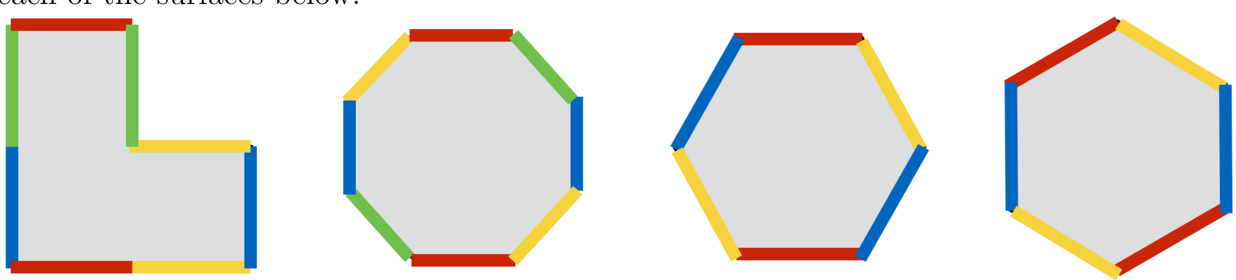
ST **2.** For a planar domain with reflective boundary (a.k.a a room with mirrored walls), the *illumination problem* asks: is it possible to illuminate the domain with a point source of light (a.k.a. a candle) that emits rays in all directions?

(b) Show that the C-shaped domain made from five squares can be illuminated. Is there a particular place where you have to put your candle, to illuminate the whole thing, or does any point work? Can you illuminate an Exeter "E"?

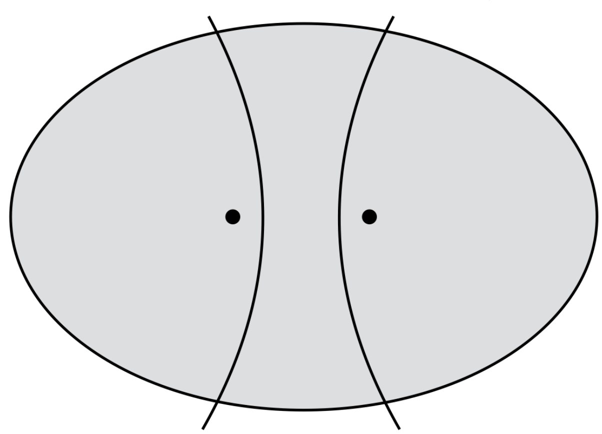
(c) *Big challenge.* Try to make up an example of a domain that cannot be illuminated. This means that, no matter where you put your candle, some part of the room will still be dark. *Note:* We will see one later; the point here is not to struggle all night or search the internet, but to construct some examples to see that the problem is hard.

DD **3.** We have seen that we can can partition a polygon surface into *cylinders*. The boundary of a cylinder is a cylinder direction (Page 23 # 2), and there are no vertices inside a given cylinder. To construct the cylinders, draw a line in the cylinder direction through each vertex of the surface, which might pass through many polygons before it reaches its ending vertex. These lines cut the surface up into strips, and then you can follow the edge identifications to see which strips are glued together. Several examples are shown on the next page for the double pentagon.

Sketch the *horizontal cylinder decomposition*, by shading each cylinder a different color, of each of the surfaces below.



ST **4.** An *ellipse* with foci F_1, F_2 and string length ℓ consists of all points X satisfying $|F_1X| + |XF_2| = \ell$. Similarly, a *hyperbola* with foci F_1, F_2 and "imaginary string length" ℓ consists of all points X satisfying $|F_1X| - |XF_2| = \pm\ell$.



In Page 7 # 1, we showed that a trajectory *through* the foci always passes through the foci. In Page 9 # 1, we showed that a trajectory *outside* the focal segment F_1F_2 stays outside and is tangent to an ellipse with the same foci. Show that every segment of a that passes *between* the foci F_1F_2 is tangent to a hyperbola with the same foci.

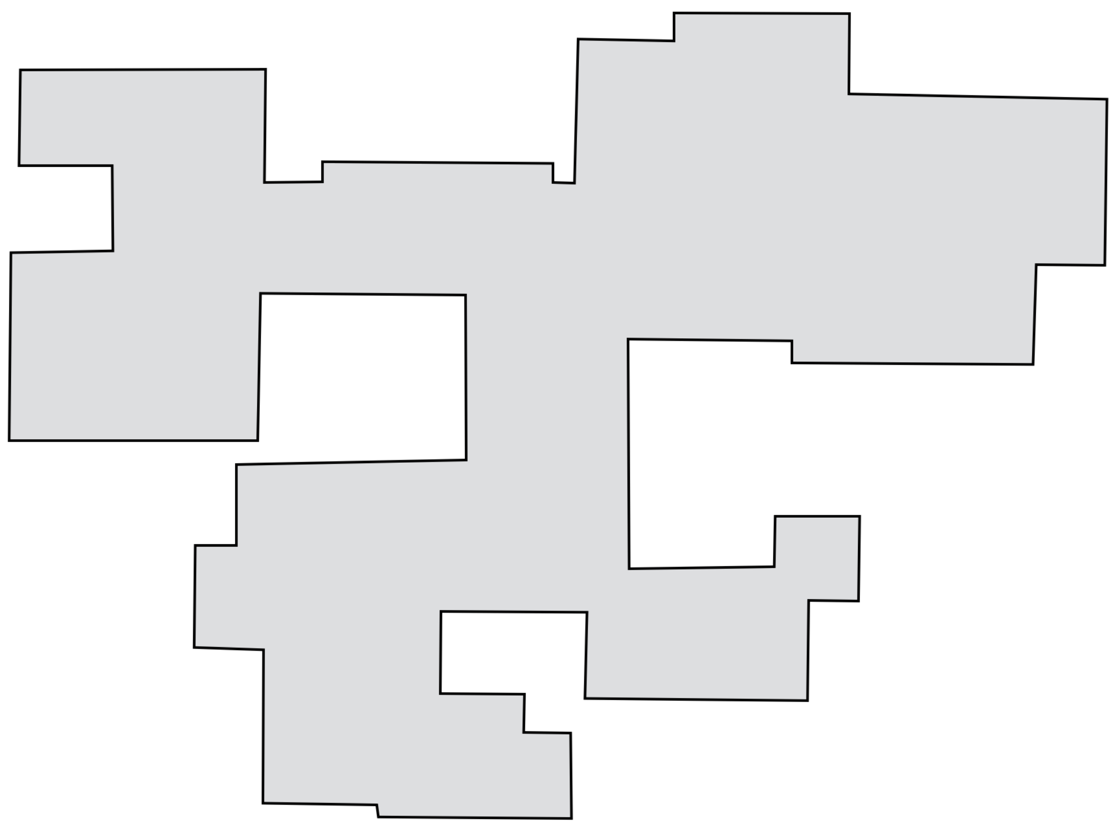
In the year 1900 at the International Congress of Mathematicians, David Hilbert described 23 problems that were unsolved at the time and, in his view, of major importance. Hilbert is thought to be the last person who understood all of the mathematics that was known during the time when that person was alive. His list of problems shaped much of 20th-century mathematics.

The first problem to be resolved (in 1901 by Max Dehn using "Dehn surgery") was the 3rd problem, which asks: *Given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces that can be reassembled into the second?*

DD **1.** The two-dimensional version of this problem asks: Given any two *polygons* of equal volume, is it always possible to cut the first into finitely many *polygonal* pieces that can be reassembled into the second? Answer this question: either prove that it is always possible, or give a counterexample with justification.
Hint: we have done a lot of work in this direction; now put it together.

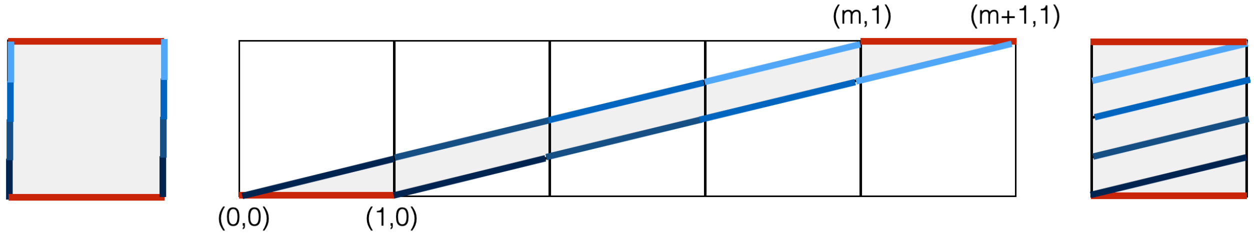
ST **2.** *The art gallery problem.* Suppose you have an art gallery with priceless masterpieces on all of the walls, so you must ensure that each wall is in the view of a security guard. You wish to employ the smallest possible number of security guards to accomplish this goal.

(a) For the Main Street Hall shape below, show that 8 guards are sufficient. Can you do better?



(b) Explain how this *art gallery problem* is different from the *illumination problem.*

DD **3.** The shears $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]$ *twist* the square torus once, as we have seen. The picture below shows the effect of the matrix $\left[\begin{smallmatrix}1&4\\0&1\end{smallmatrix}\right]$ on the square torus, including how to cut up and reassemble the pieces back into the square torus while respecting edge identifications. Explain why the shears $\left[\begin{smallmatrix}1&m\\0&1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1&0\\m&1\end{smallmatrix}\right]$ twist the square torus m times.

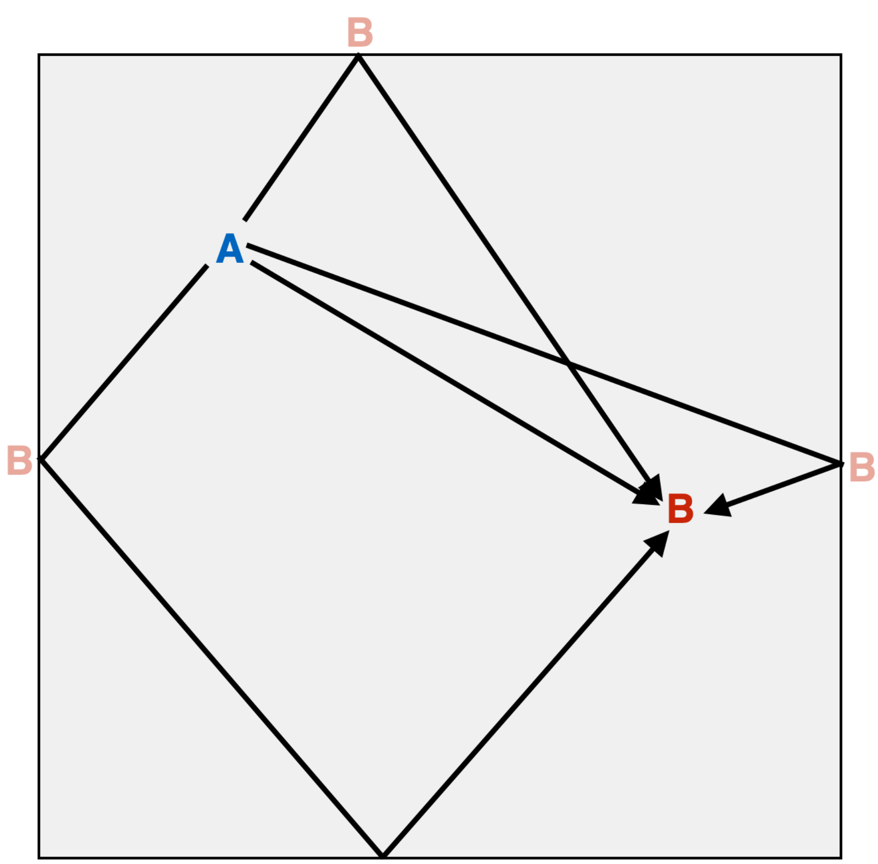


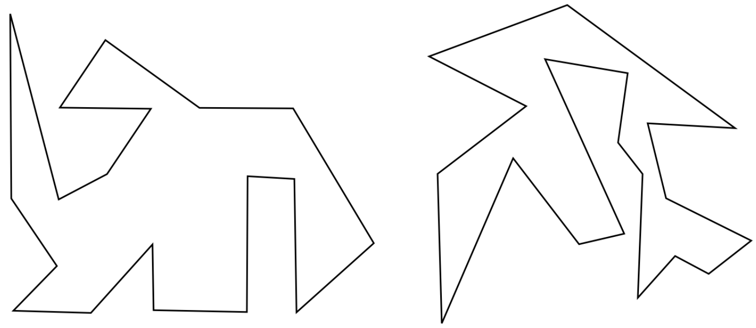
DD **4.** (Continuation) Consider the effect of the shear $\left[\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right]$ on the L-shaped table made of three squares shown on the left side of Page 29 # 3. How many times does this shear twist each of its two horizontal cylinders? Why?

DD **5.** (Challenge) Alice and Bob are in a square room with mirrored walls. They hate each other, and they don't want to see each other at all, through the room or in any reflection in the walls, looking in any direction. Show that it is possible for Alice and Bob to position a finite number of their friends in the room so that they cannot see each other (their friends block their view of the other person). *From the 1989 Leningrad Olympiad.*

A polygon where finitely many such friends suffice is called a *secure polygon*.

Hint: Three of the many places where Alice can see Bob in reflection are shown.





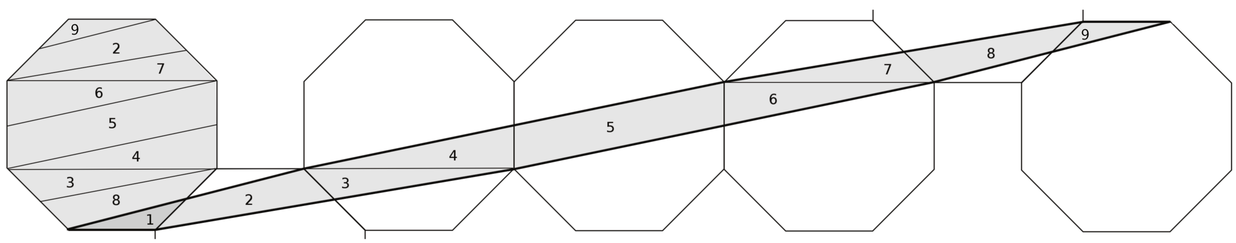
ST **1.** In this problem, we will give an *upper bound* on the number of guards required to guard the priceless artwork on the walls of an art gallery shaped like an n-sided polygon.

(a) Explain why a guard at the corner of a triangle can see the entire triangle.

(b) By triangulating your polygon and coloring each of the three vertices of each triangle a different color, prove that an upper bound for the number of guards required is $\lfloor n/3 \rfloor$.[1]

(c) Find the minimum number of guards for each polygon in the figure above.

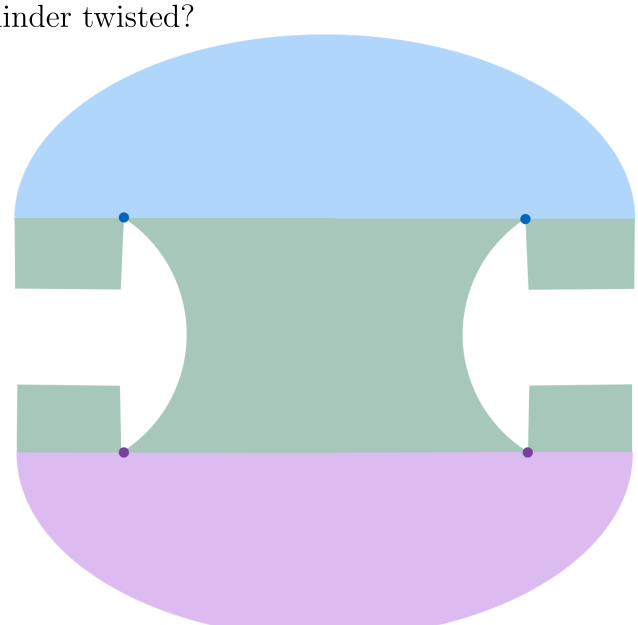
DD **2.** Amazingly, many surfaces made from regular polygons can be sheared, cut up and reassembled back into the original surface in the same way that we have done with the square and the L. One example is the regular octagon surface, shown below. The way to reassemble the sheared octagon pieces is indicated with tiny numbers in the pieces.



(a) By coloring each piece of each edge as in Page 30 # 3, show that this reassembly respects the octagon surface's edge identifications (shown in Page 29 # 3).

(b) How many times was each horizontal cylinder twisted?

ST **3.** One way to pose the illumination problem is: "Is every mirrored room illuminable from *some* point in the room?" Here is a counterexample, a room that cannot be illuminated from *any* point inside, shown to the right. The top and bottom are half-ellipses, whose foci are at the indicated points. Explain why this example works, by explaining which part of the room is illuminated when the candle is placed:



(a) in the interior of a half-ellipse,

(b) in the middle part, and

(c) in one of the rectangular parts.

Contextual note. Roger Penrose, who contributed to the creation of this "unilluminable room," is best known for creating the *Penrose tiling*, a set of two tiles that *only* tile the plane non-periodically. He was knighted in 1994 and won the Nobel Prize in fall 2020.

[1] Remember that this notation means "$n/3$, rounded down."

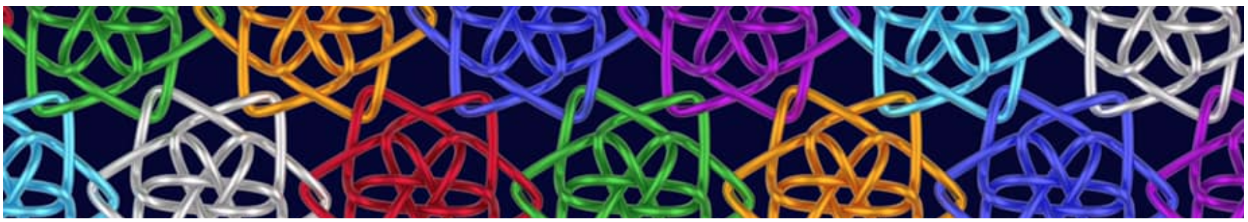
DD **4.** Do each of the following, by actually cutting up paper, and bring your pieces in to class:

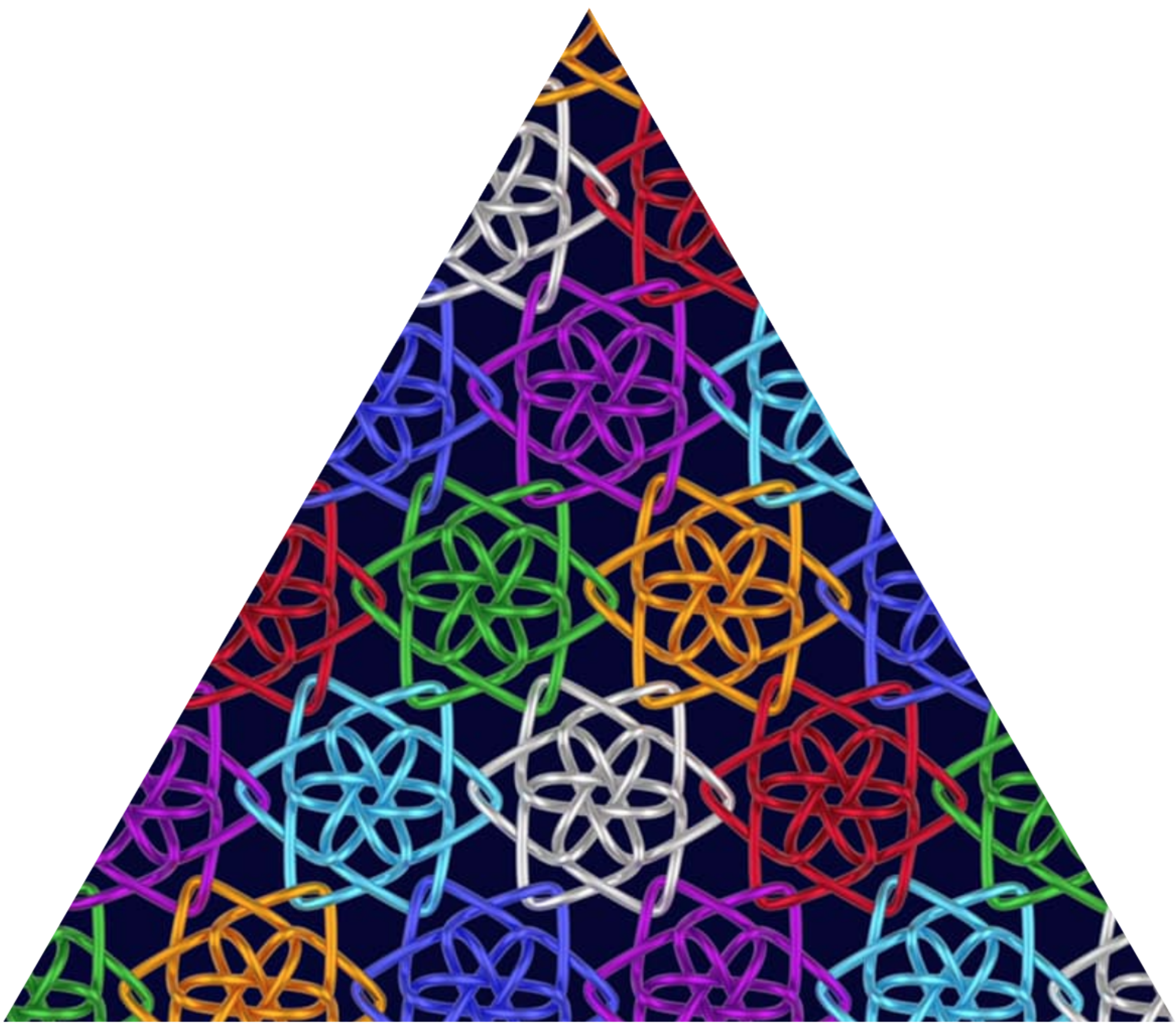
(a) Cut up and reassemble a 6×1 rectangle into a square.

(b) Cut up and reassemble a $30°$-$60°$-$90°$ triangle into a 3-4-5 triangle.

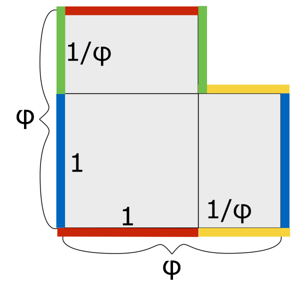
What is the smallest number of pieces you can use in each case?

Note. The knotted chain mail that bedazzles these shapes is by Frank Farris.





Billiards, Surfaces and Geometry

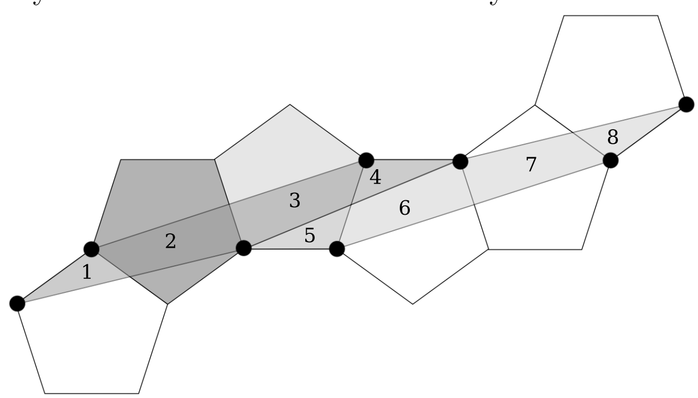


DD **1.** A particularly nice flat surface is the "Golden L," whose opposite parallel edges are identified as shown, and whose edge lengths are as shown in the picture. The indicated number φ satisfies the property that when you cut off the largest possible square from a $1 \times \varphi$ rectangle, the leftover rectangle has the same proportions as the original.

(a) Show that this number satisfies the relation $\varphi = 1 + 1/\varphi$.

(b) Find the continued fraction expansion of φ.

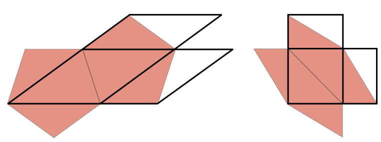
(c) Two numbers are *rationally related* if they are rational multiples of each other. Are the aspect ratios (known as "moduli") of the cylinders of the Golden L rationally related?



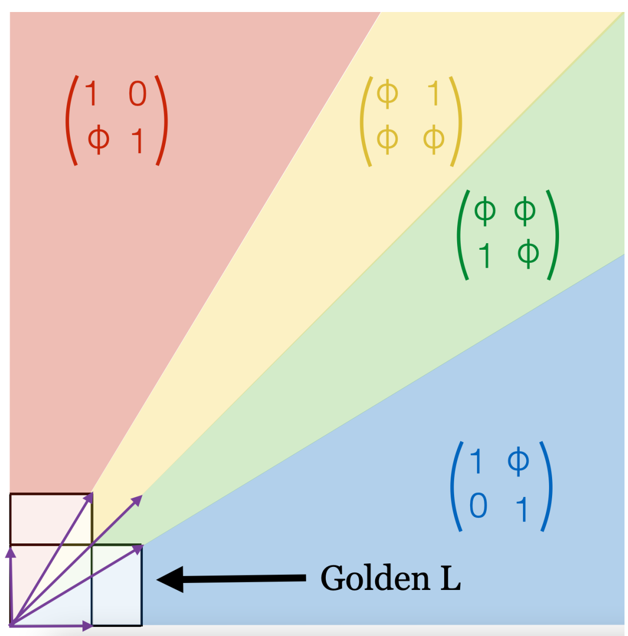
DD **2.** The picture shows the image of the double pentagon surface under the shear $\left[\begin{smallmatrix}1 & 2\cot\pi/5\\ 0 & 1\end{smallmatrix}\right]$.

(a) Show how to to reassemble (by translation, and respecting the edge identifications) the pieces back into the double pentagon.

(b) Explain why the even-numbered pieces end up in one pentagon and the odd-numbered pieces in the other.



(c) By comparing its two horizontal cylinders, show that the double pentagon surface is a cut, reassembled, and horizontally stretched version of the Golden L.



DD **3.** In your explanation of visible points, some of you asked, "what about surfaces other than the square?" Arguably, the next-simplest surface after the square torus is the golden L. Recall that the visible points in the square lattice exactly give us the periodic directions on the square torus.

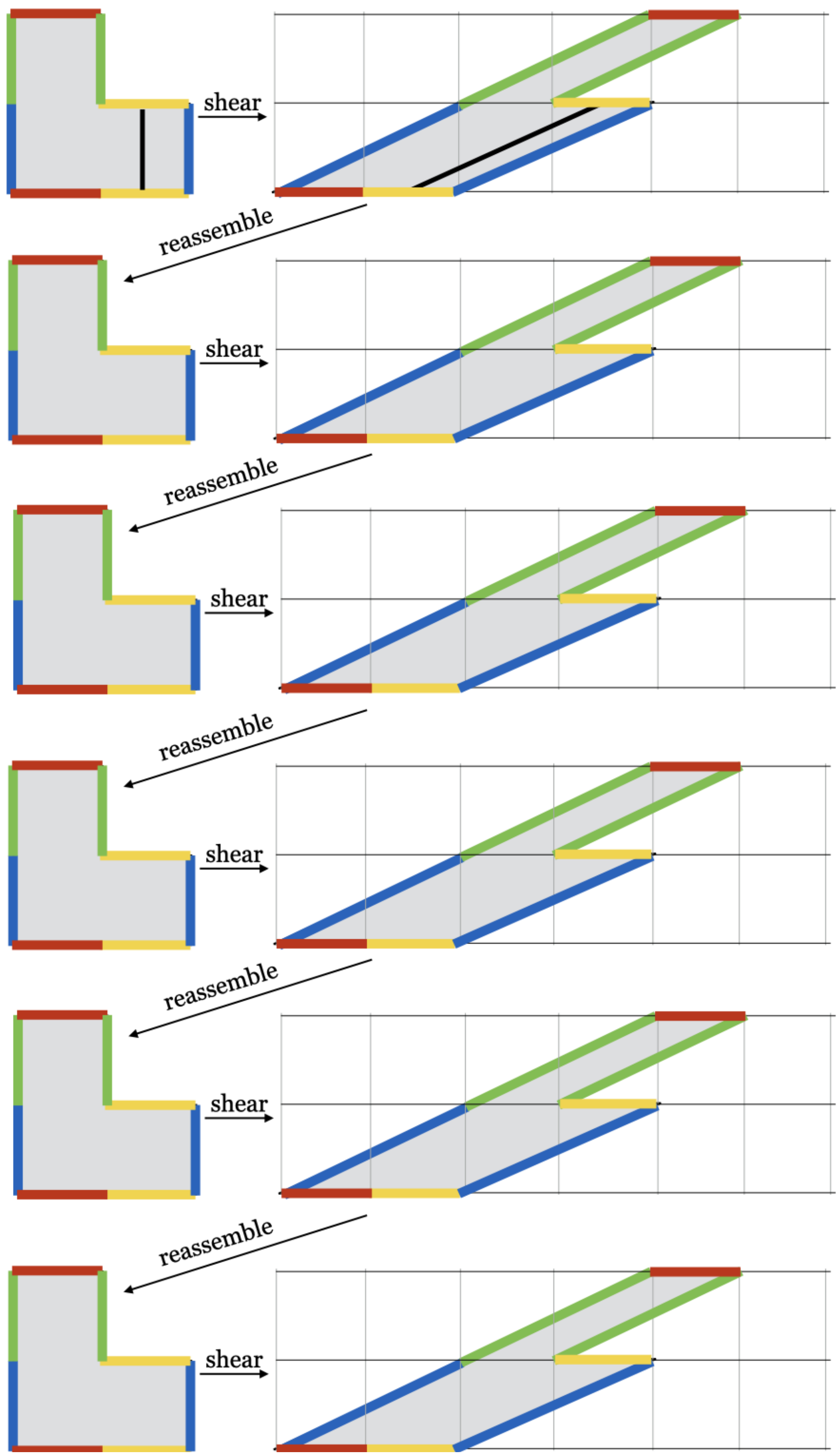
(a) The dimensions of the golden L are given above. Write down the components of the five purple vectors shown (hint: all their entries are 0, 1 and ϕ).

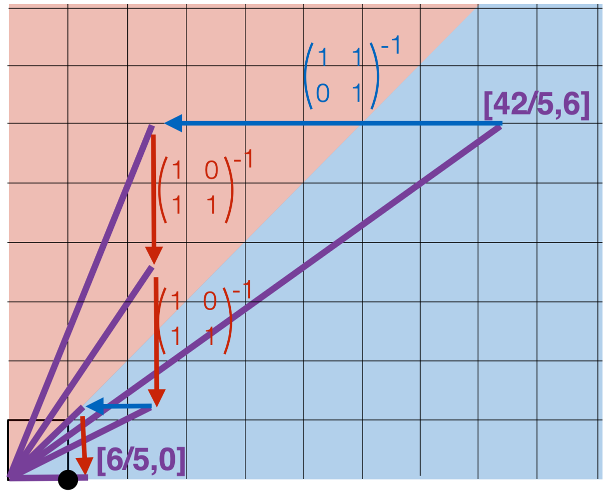
(b) Explain why the blue matrix takes the entire first quadrant to the blue sector. Repeat for the three other colors.

(c) To generate the set of periodic directions on the golden L, we start with the vector $\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]$ and repeatedly apply the blue, green, yellow and red shears. Write down the first two levels of the tree of visible points / periodic directions for the golden L (use symmetry whenever possible to reduce your work).

A big question in the study of trajectories on surfaces is: "what happens to a trajectory on a surface when you apply a symmetry of the surface?" In Page 8 # 1 we explored the effects of rotations and reflections on a trajectory on the square torus, and in Page 11 # 1 we showed that the effect of $\left[\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right]$ is to reduce a trajectory's slope by 1.

DD **4.** Let's see what happens when we apply the horizontal shear $\left[\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right]$ to the L-shaped table, with a short vertical trajectory on it. The diagram to the right provides a template for drawing the image of the trajectory under this shear. Fill it in and sketch the image of this trajectory under five applications of this shear. Then say what the trajectory would look like if you sheared the surface many times.





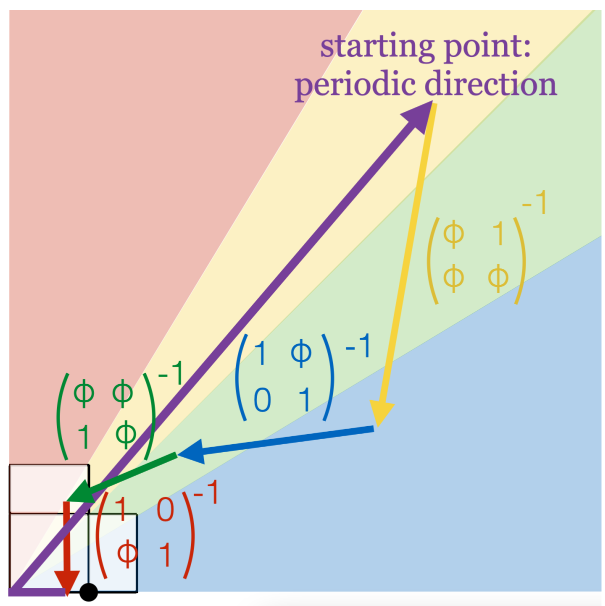
Here is another way to think about the continued fraction algorithm, one that generalizes to other surfaces: Suppose you are about to make a pool shot on a square billiard table. You have decided to shoot the ball in the direction of some vector, for example $[42/5, 6]$. You want to know (under idealized mathematical conditions) how many times the ball will bounce before it repeats. How do you do it?

- Divide the first quadrant into two sectors: blue below the line $y = x$ and red above. Draw your vector on the plane (shown here in purple).

- If your vector is in the blue sector, apply $\left[\begin{smallmatrix}1 & -1\\ 0 & 1\end{smallmatrix}\right]$ to it. If your vector is in the red sector, apply $\left[\begin{smallmatrix}1 & 0\\ -1 & 1\end{smallmatrix}\right]$ to it. Repeat this until your vector is horizontal.

DD **1.** Explain how to use this process to get the continued fraction expansion of your trajectory's slope. Under what conditions will the process terminate (give you a horizontal vector)?

DD **2.** How can you use this information to determine the period of your billiard trajectory in this direction?



We can do the same thing for periodic directions on the golden L: start with a periodic direction vector. If it is in the yellow sector, apply the inverse of the yellow matrix; if it is in the blue sector... and so on.

Note that while the picture for the square shown above is accurate, this picture for the golden L is only an illustration, as each of these shears greatly expands the size of vectors, and it's difficult to see what's going on if you draw them to scale.

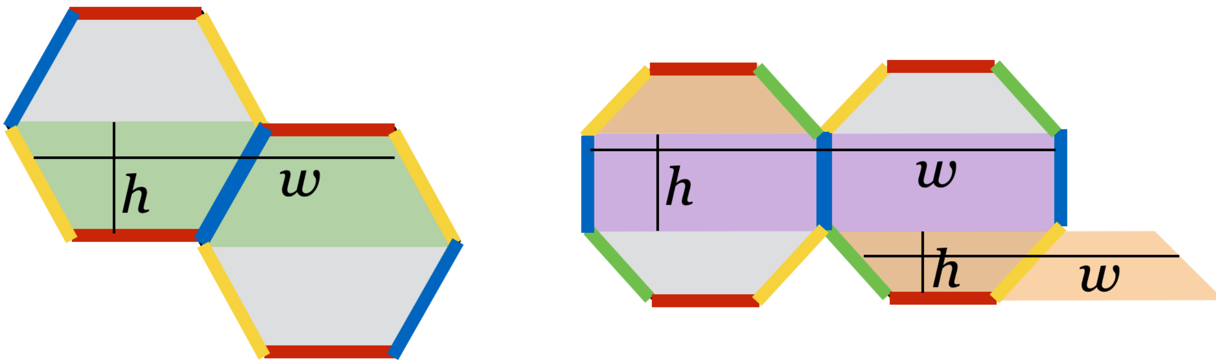
DD **3.** People say "this is essentially the continued fraction algorithm." Explain why. If you had to write down some kind of continued fraction representation of a periodic direction on the golden L, how would you do it?

Why do we care about the golden L so much? Because it's essentially the same surface as the double pentagon (as we showed in Page 32 # 2c). We want to study the double pentagon, but it has all those inconvenient angles, while the golden L is made of rectangles. So we translate our pentagon problem into a golden L problem, solve the problem on the golden L, and then translate the answer into a pentagon answer.

DD **4. Theorem** *(Modulus Miracle)*. Every horizontal cylinder of a double regular n-gon surface the same modulus ("aspect ratio"), which is $2\cot \pi/n$.

(a) Confirm this for the two surfaces shown, by calculating the ratio w/h for each cylinder.

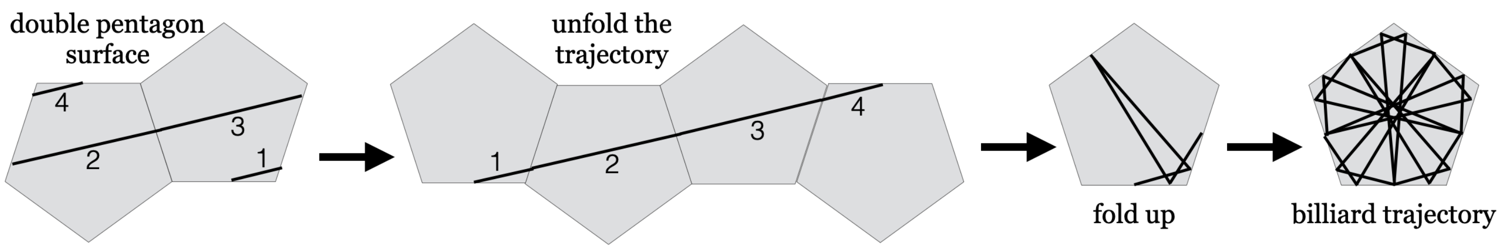
(b) Explain why this tells us that the horizontal shear $\left[\begin{smallmatrix} 1 & 2\cot\pi/n \\ 0 & 1 \end{smallmatrix}\right]$ is always a symmetry of the double regular n-gon surface.



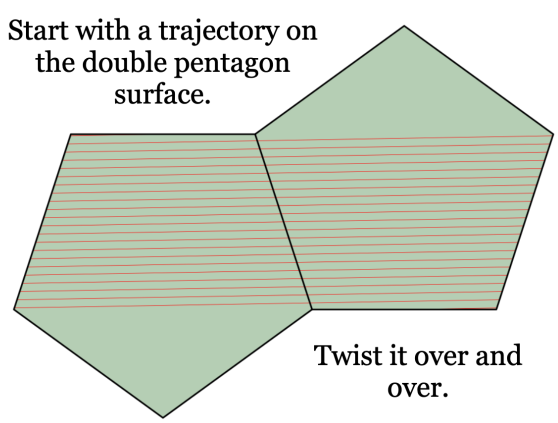
(c) Why do you think a double regular n-gon was used instead of a single one, like the familiar regular octagon surface? What happens with the moduli if you do use just one?

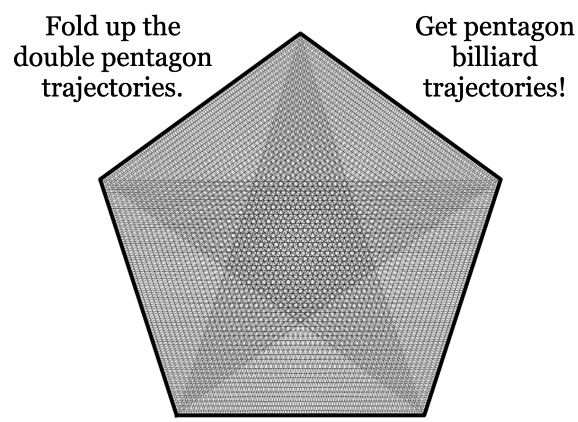
Billiards, Surfaces and Geometry

DD **1.** In Page 6 # 1, we used patty paper to fold up a trajectory of period 5 on the square torus into a billiard trajectory of period 10 on the square billiard table. Below is a diagram showing how to similarly fold up a trajectory of period 4 on the double pentagon surface, into a billiard trajectory of period 20 on the pentagon billiard table. Fill in the details.



DD **2.** In Page 32 # 4, we twisted the L-shaped table horizontally many times, filling up the large cylinder and leaving the small cylinder empty. Below left is a picture of what happens when you do the same thing to a trajectory on the double pentagon: the large cylinder is filled up and the small cylinder is empty. Below right is a picture of what happens when you fold up the trajectory on the left into a trajectory on the pentagon billiard table (as described in the previous problem). Explain why a star appears.

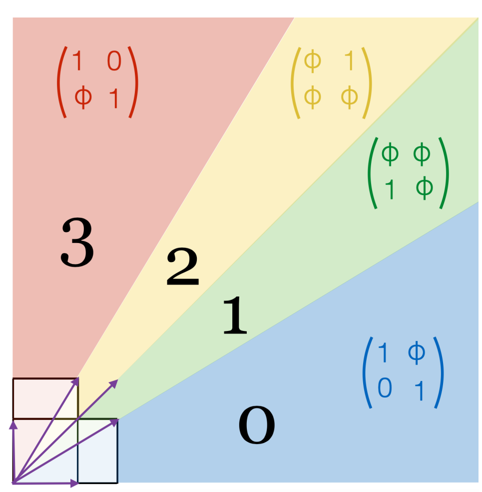




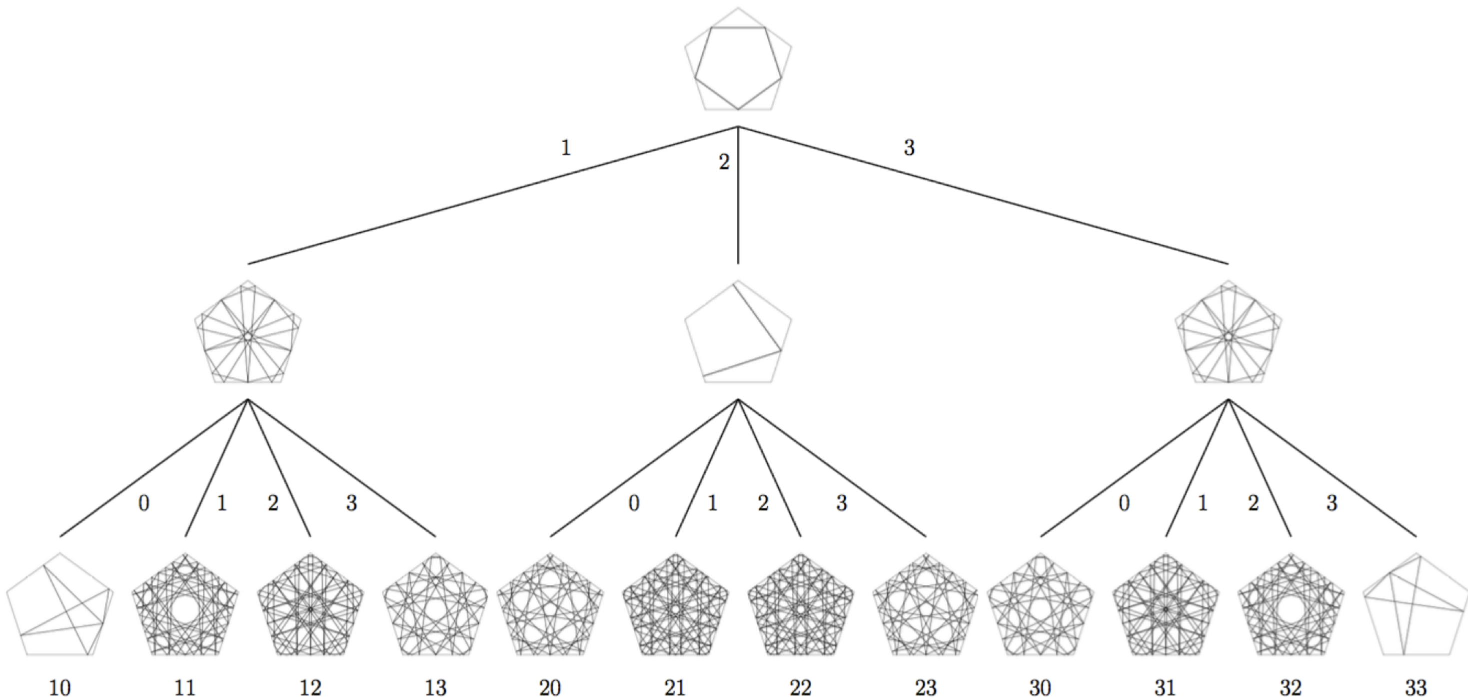
DD **3.** We can label each of the four sectors 0, 1, 2, 3 as shown, and use this labeling to specify a periodic direction on the Golden L.

(a) For example, the purple direction shown in Page 33 # 2 is recorded as "3102." Explain why.

(b) Let's go back to continued fractions, and shears with the square and its two sectors (now labeled 0 and 1, respectively) from Page 33 # 1. Express your birthday vector month/day using only the digits 0 and 1, according to the shears you use to get there.



DD **4.** In Page 32 # 3, we computed the first two levels of the tree of periodic directions on the Golden L, as direction vectors. Below is the same tree, drawn as the corresponding periodic billiard trajectories on the regular pentagon billiard table. The branch and node labels use the notation introduced in the previous problem. Comment on any patterns you notice.



DD **5.** So far, we have only drawn pictures of twisting a surface in the horizontal direction. Below left is a picture of a trajectory on the double pentagon that has been twisted many times in a direction with a slightly positive slope. The direction of this trajectory is $1000\cdots0002$. Again, it has filled in the large cylinder and left the small cylinder empty (because that is what your instructor finds most interesting). Below right is the same trajectory, folded up into a pentagon billiard trajectory. Trajectories like this seem to be people's favorite.

Can you figure out how to relate the surface trajectory with the billiard trajectory? The billiard trajectory seems to have different amounts of "shading" in different areas; can you say how much darker some are than others?