Real Analysis

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Real Analysis

The problems in this text

The method of instruction used with these problems is based on the curriculum at Phillips Exeter Academy, a private high school in Exeter, NH. Most of the beginning of the course (and some of the later) is based on *Real Analysis* by **Frank Morgan** (FM). Most of the end of the course (and some of the earlier) is based on **Aimee Johnson**'s lecture notes and worksheets, which in turn are based on *Introduction to Analysis* by **Maxwell Rosenlicht** (MR). The rest of the problems were written by **Diana Davis** (DD) specifically for this course. If you create your own text using these problems, please give credit as I am doing here, and note who wrote each problem, as I have done in the left margin.

About the course

This course meets two mornings a week for 75 minutes, for which the homework is the numbered pages, for a total of 26 "discussion sessions." We also meet once a week in the afternoon for a 75-minute "problem session," during which you will work in groups on the problem pages whose numbers end with P. You will type up (in LaTeX) and hand in solutions to two of these problems the following week. The course is designed to teach you how to write a proof, in addition to the content of analysis.

To the Student

Contents: As you work through this book, you will discover that the various topics of real analysis have been integrated into a mathematical whole. There is no Chapter 5, nor is there a section on sequences of functions. The curriculum is problem-centered, rather than topic-centered. Techniques and theorems will become apparent as you work through the problems, and you will need to keep appropriate notes for your records — there are no boxes containing important ideas. Key words are defined in the problems, where they appear italicized.

Your homework: The first day of class, we will work on the problems on page 1, and your homework is page 2 (2a and 2b); on the second day of class, we will discuss the problems on page 2, and your homework will be page 3 (3a and 3b), and so on for each day of the semester. You should plan to spend two to three hours solving problems for each class meeting.

Comments on problem-solving: You should approach each problem as an exploration. Draw a picture whenever appropriate. It is important that you work on each problem when assigned, since the questions you may have about a problem will likely motivate class discussion the next day. Problem-solving requires persistence as much as it requires ingenuity. When you get stuck, or solve a problem incorrectly, back up and start over. Keep in mind that you're probably not the only one who is stuck, and that may even include your teacher. If you have taken the time to think about a problem, you should bring to class a written record of your efforts, not just a blank space in your notebook. The methods that you use to solve a problem, the corrections that you make in your approach, the means by which you test the validity of your solutions, and your ability to communicate ideas are just as important as getting the correct answer.

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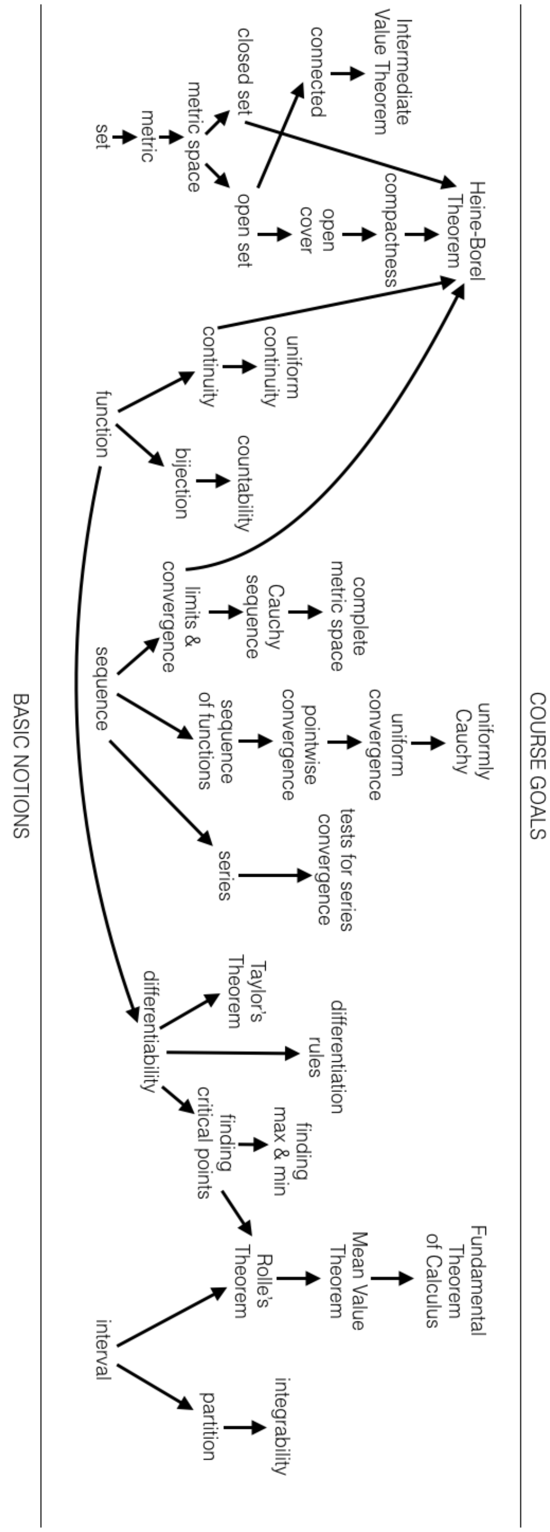
Below is a map of the ideas in this course, and how they connect, from the basic ideas at the bottom to the course goals at the top. An arrow goes from A to B if we need the ideas from A in order to understand B. I made this chart when I was constructing our curriculum.

- Circle topics that you feel you understand well.
- Periodically come back to this chart and circle topics as you master them.



Discussion Skills

1. Contribute to the class every day
2. Speak to classmates, not to the instructor
3. Put up a difficult problem, even if not correct
4. Use other students' names
5. Ask questions
6. Answer other students' questions
7. Suggest an alternate solution method
8. Draw a picture
9. Connect to a similar problem
10. Summarize the discussion of a problem

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sets / FM **Notation.**

- A *set* is a notion that we won't define, because any definition would end up using a word like "collection," which we'd then need to define. We'll just assume that we understand what is meant by a *set*, and let this notion of a set be fundamental.
- We use a capital letter to denote a set, e.g. "Let S be the set of even numbers."
- The symbol $\in$ means "is/be an element of," and $\notin$ means "is not an element of."
- We use a lower-case letter to denote an element of a set, e.g. "Let $s \in S$."
- To describe the elements of a set, use curly braces $\{\}$. For example, $S = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ or $S = \{x : x \text{ is an even number}\}$. The colon ":" means "such that," so that the latter set is read aloud as "S is the set of x such that x is an even number."

sets / DD **1**. Let $A = \{1, 2, 3, 4\}$. Which of the following are true statements?

(a) $3 \in A$ **(b)** $\{3\} \in A$ **(c)** $5 \in A$ **(d)** $2 \in a$ **(e)** $2 \notin A$

sets / FM **Talking about sets.**

- $X \subset Y$ is read "X is a subset of Y," and means that every x in X is also in Y:

$$x \in X \implies x \in Y.$$

- An equivalent notation to $X \subset Y$ is $X \subseteq Y$. If one wants to specify that $X \neq Y$, one can write $X \subsetneq Y$. Otherwise, $X \subset Y$ allows for the possibility that $X = Y$.

sets / DD **2**. Let S and A be as above, and let $B = \{1, 2, 3, 4, 5, 6\}$. Which are true? Explain.

(a) $A \subset B$ **(b)** $B \subset A$ **(c)** $A \in B$ **(d)** $A \subset S$ **(e)** $S \subset A$

sets / FM **Useful sets.**

- The *empty set* $\emptyset$, the set consisting of no elements.
- The *natural numbers* $\mathbf{N} = \{1, 2, 3, \ldots\}$. In Europe, $\mathbf{N}$ starts with 0.
- The *integers* $\mathbf{Z} = \{-3, -2, -1, -0, 1, 2, 3, \ldots\}$, from the German *zahl* for number.
- The *rationals* $\mathbf{Q} = \{p/q \text{ in lowest terms} : p \in \mathbf{Z}, q \in \mathbf{N}\}$, from *quotient* $= \{\text{repeating or terminating decimals}\}$.
- The *reals* $\mathbf{R} = \{\text{all decimals}\}$, with the understanding that $0.999\ldots = 1$, etc.

Note that these symbols are typeset as $\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}$ and written by hand as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

sets / DD **3**. Let A and S be as above. Which of the following are true? Explain.

(a) $B \subset \mathbf{N}$ **(b)** $S \subset \mathbf{Z}$ **(c)** $\mathbf{Q} \subset \mathbf{R}$ **(d)** $\mathbf{Z} \subset \mathbf{N}$ **(e)** $\emptyset \subset \mathbf{N}$ **(f)** $\emptyset \in A$

Real Analysis

The mathematical "or." In mathematics, "or" means one, or the other, *or both.*

- Shall we meet to do Real Analysis on Tuesday *or* Wednesday? Both!
- To take Math 29, a student must have taken a math course numbered 15 *or above.*

logic / FM **4**. **Implication.** There are many ways to say that one statement A implies another statement B. The following all mean exactly the same thing:

- If A, then B.
- A implies B, written $A \implies B$.
- A only if B.
- B if A, written $B \impliedby A$.
- not B implies not A. (This is the *contrapositive*)

logic / DD **5**. Let statement A be "Dale has a valid driver's license in Pennsylvania," and let statement B be "Dale is over age 16."

(a) Write out the five implications above, using these statements.

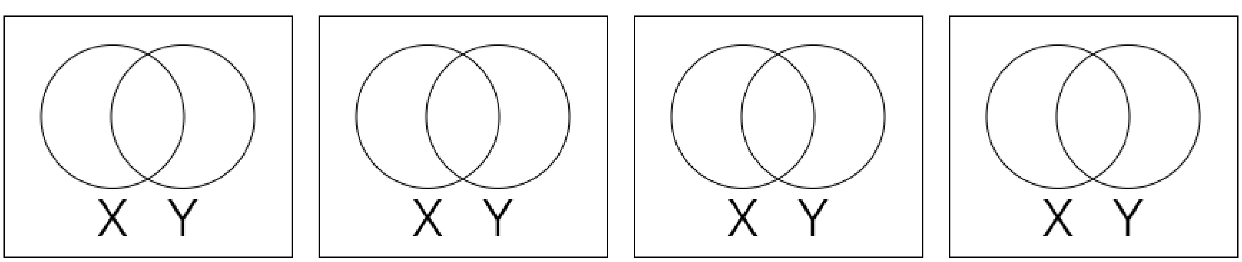
(b) Considering this example, do you agree that they are all logically equivalent?

(c) The *converse* is $B \implies A$. Is the converse true in this case?

sets / FM **Working with sets.**

- The *intersection* $X \cap Y$ of two sets X and Y is the set of all elements that are in X *and* in Y: $X \cap Y = \{x : x \in X \text{ and } x \in Y\}$.
- The *union* $X \cup Y$ of two sets X and Y is the set of all elements that are in X *or* in Y: $X \cup Y = \{x : x \in X \text{ or } x \in Y\}$.
- The *complement* X^C of a set X is the set of points *not* in X: $X^C = \{x : x \notin X\}$. For this to make sense, the "universal set" that X lives in must be understood.
- The set $X - Y$, or $X \setminus Y$, is the set of all points *in* X that are *not* in Y.

sets / DD **6**. Shade the regions corresponding to $X \cap Y$, $X \cup Y$, X^C, and $X - Y$, respectively.



sets / FM **7**. Let X be a subset of a universal set U, and let X and Y be subsets of U. Simplify:

(a) $(X \cup Y) \cap (U - X) \cap X$ **(b)** $X \cup (Y \cap X^C)$ **(c)** $(X \cap Y) \cup (X \cap Y^C)$

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Warm-up problems

logic / FM **Converse and logical equivalence.** The *converse* of the statement "A implies B" is the statement "B implies A." If a statement and its converse are both true, we say A and B are *logically equivalent*, or in other words $A \iff B$, or in other words "A if and only if B," sometimes abbreviated as "A iff B."

logic / DD **1**. Let statement A be "Quinn is eligible to vote in the United States" and let statement B be "Quinn is a United States citizen." Write out the implication $A \implies B$, its contrapositive, and its converse. Which of these implications are true?

logic / DD **2**. The lyrics to "This is why I'm hot" by Mims are shown to the right. Does the lyric "I'm hot 'cause I'm fly, you ain't 'cause you not" imply that the notions of "hot" and "fly" are logically equivalent?

[CHORUS]
This is why I'm hot
This is why I'm hot
This is why, this is why,this is why I'm hot
This is why I'm hot
This is why I'm hot
This is why, this is why, this is why I'm hot
I'm hot 'cause I'm fly, you ain't cause you not
This is why, this is why, this is why I'm hot
I'm hot 'cause I'm fly, you ain't cause you not
This is why, this is why, this is why I'm hot

Some things to prove

metric / DD **Metrics.** A *metric* on a set E is a rule that assigns, to each pair $p, q \in E$, a real number $d(p,q)$, called the *distance function*, which is a function $d : E \times E \to \mathbf{R}$, such that:

1. $d(p,q) \geq 0$ for all $p, q \in E$,
2. $d(p,q) = 0$ if and only if $p = q$, NOTE: There are two statements here.
3. $d(p,q) = d(q,p)$,
4. $d(p,r) + d(r,q) \geq d(p,q)$ for any $p, q, r \in E$.

metric / DD **3**. Show that each of the following is a metric on $\mathbf{R}^2$:

(a) The standard Euclidean metric (i.e. using the Pythagorean theorem),

(b) The "Taxicab metric": $d((a,b),(c,d)) = |c-a| + |d-b|$ (also explain the name).

sets / FM **4**. Find infinitely many nonempty sets $S_1, S_2, \ldots$ of natural numbers such that

$$\mathbf{N} \supset S_1 \supset S_2 \supset S_3 \cdots$$

and $\bigcap_{n=1}^{\infty} S_n = \emptyset$. Here the symbol $\bigcap_{n=1}^{\infty} S_n$ means $S_1 \cap S_2 \cap \cdots$, and is used to take the intersection of infinitely many sets.

limits / FM **5**. Consider the following sequences: $1, 1/2, 1/3, 1/4, 1/5, \ldots$ $\quad 3, 1, 4, 1, 5, 9, \ldots$ $\quad 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots$ $\quad 2.1, 2.01, 2.001, 2.0001, \ldots$

(a) Which of the sequences converge, and to what limit?

(b) Come up with a definition: a sequence *converges* to a *limit* p if...
(Don't look it up now; you'll work with the precise definition in your homework.)

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Metric spaces. A *metric space* is a set E, together with a *metric* $d(p,q)$ that gives the distance between any two points $p, q \in E$.

metric / FM **1**. Show that the following are metric spaces:

(a) The set $\mathbf{R}^n$, with the metric

$$d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \max\{|y_1 - x_1|, \ldots, |y_n - x_n|\}.$$

(b) Any set E, with the "don't touch me" metric $d(x,y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$.

(Technically, the "don't touch me" metric is called the *discrete* metric.)

Limits and convergence. Let $p_1, p_2, \ldots$ be a sequence of points in a metric space E.

- A point $p \in E$ is a *limit* of $\{p_i\}_{i=1}^{\infty}$ if, for any $\epsilon > 0$, there exists $N > 0$ such that, whenever $i > N$, $d(p, p_i) < \epsilon$.
- In such a situation, we say that the sequence p_i *converges to* p, and write $\lim_{i\to\infty} p_i = p$.

limits / DD **2**. I like to think of sequence convergence as a competition between me and an interlocutor:

ME: Consider -1, +1/2, -1/3, +1/4, -1/5, ... I claim that this sequence converges to 0.

INTERLOCUTOR: Nonsense! Show me that the terms get within 0.1 of 0.

ME: Okay, take $k > 10$. After that, the terms are all closer than 0.1.

INTERLOCUTOR: Hmm! Now show me that the terms get within 0.0001 of 0.

ME: Okay, then take $k > 10000$. After that, the terms are all closer than 0.0001.

INTERLOCUTOR: Hmm. Show me that, for any $\epsilon > 0$ that I might ever suggest, the terms get within ϵ of 0.

ME: Okay, take $k >$ ____________, and after that the terms are all closer than ϵ.

limits / DD **3**. Prove, from the definition (i.e. by finding an N that depends on the given ϵ, sometimes called $N(\epsilon)$ to emphasize this dependence), that the sequence $p_n = 1000/n^3$ converges to 0.

Open balls. Given a metric space E, a point $p_0 \in E$, and a real number $r > 0$, the *open ball* in E with center p_0 and radius r is $B_r(p_0) = \{p \in E : d(p_0, p) < r\}$.

open / DD **4**. Sketch the following open balls.

(a) In $\mathbf{R}$, the set $B_1(0)$. (Shade the included interval, with open circles for endpoints.)

(b) In $\mathbf{R}^2$, the set $B_{1/2}(1,1)$. (Shade the included region, with a dashed boundary curve.)

open / DD **5**. Express the open interval $(1,2) \in \mathbf{R}$ as an open ball as above. Then do the same for the general open interval (a,b).

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func / FM **Functions.**

- A *function* from X to Y is a rule that assigns, to each $x \in X$, exactly one $y \in Y$.
- We write $f : X \to Y$, and if $f(x) = y$, we write $x \mapsto y$ which is read "x maps to y."
- If f maps distinct points to distinct values, then f is called *one-to-one* or *injective*. Equivalently, f is injective if $f(x) = f(y)$ implies that $x = y$.
- X is called the *domain* of f, and Y is the *codomain* of f.
- The set of all outputs $f(X) = \{f(x) : x \in X\}$ is called the *image* of f. If the image is the entire codomain, f is called *onto* or *surjective*. Equivalently, f is surjective if, for each $y \in Y$, there exists an $x \in X$ such that $f(x) = y$.
- A function that is both injective and surjective is called *bijective*.

func / FM **6**. For each of the following, say whether it is injective, surjective, or both (bijective):

(a) $f(x) = -x$ **(b)** $f(x) = x^2$ **(c)** $f(x) = \sin x$ **(d)** $f(x) = e^x$ **(e)** $f(x) = x^3 + x^2$.

logic / FM **True and false.** An implication $A \implies B$ is true if B is true, or if A is false (in which case we say that the implication is "vacuously true.") For example, the statement "If 5 is even, then 15 is prime" is vacuously true. An implication is false *only* if A is true and B is false.

logic / FM **7**. Is the statement:

$$\text{If } x \in \mathbf{Q}, \text{ then } x^2 \in \mathbf{N}$$

true or false for the following values of x? Justify your answers.

(a) $x = 1/2$ **(b)** $x = 2$ **(c)** $x = \sqrt{2}$ **(d)** $x = \sqrt[4]{2}$

logic / DD **8**. Vacuously true statements can be used to make hilarious jokes, with other people who also understand vacuously true statements. Ruin each of the following hilarious jokes by writing each one in the form *if A, then B*:

- Every car I own is a Maserati.
- I've gotten As in all of my Sanskrit courses.
- Swarthmore Football, undefeated since 2000.

ONE MORE PROBLEM ON PAGE 3c! YOU WON'T WANT TO MISS THIS!

logic / DD The lyrics of Billy Joel's "Only the good die young" are shown to the right. Here I will describe how to *diagram the statement* "only the good die young." This task has four parts:

"Only The Good Die Young"

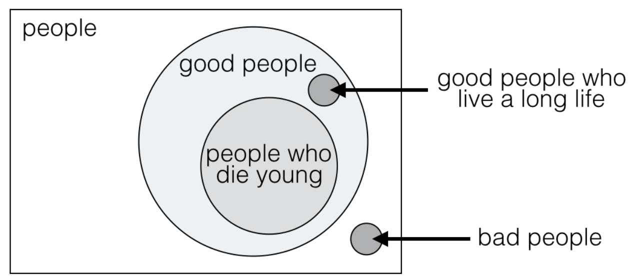
Originally by Billy Joel

Come out Virginia, don't let me wait
You Catholic girls start much too late
But sooner or later it comes down to fate
I might as well be the one

Well, they showed you a statue, and told you to pray
They built you a temple and locked you away
They never have told you the price you would pay
For things that you might have done

Only the good die young
Only the good die young
Only the good die young
Only the good die young

(a) Make a Venn diagram, where the outer box defines our universe, and there are circles representing the qualities in the statement. Here, I am assuming that the statement refers to good *people* (not good *rosebushes* or good *platypuses*) so my universe is people. The statement regards the qualities of *being good* and *dying young*, so I have made two circles, each containing the people who satisfy those qualities. The statement says that *only* the good die young, so the "people who die young" circle is contained within the "good people" circle.



(b) Make a circle anywhere other than inside the innermost circle, and give an example of something that is there. Here, within "good people" and outside of "people who die young" I have shown that there are "good people who live a long life." I have also shown that outside of "good people," there are "bad people." The way I drew it leaves open the possibility that, in addition to good people and bad people, there are people who are neither good nor bad.

(c) Write the statement as a formal logical statement. Here, that is "if a person dies young, then that person was good."

(d) Write the contrapositive of the statement, which will be a logically equivalent statement. For a statement "$A \implies B$," the contrapositive is "not $B \implies$ not A." Here, the contrapositive of the formal statement is "if a person is not good, then the person will not die young," or as a potential song lyric, perhaps "bad people just keep on living." (Contrary to my Venn diagram picture, this lyric assumes that people who are not good are bad.)

logic / DD **9**. (Lars Ojukwu) The lyrics to "If I were a rich man" from the musical *Fiddler on the roof* are shown to the right. For the lyric "All day long I'd biddy biddy bum / If I were a wealthy man," diagram the statement: do all four parts **(a)** – **(d)** as explained above.

[TEVYE]
"Dear God, you made many, many poor people.
I realize, of course, that it's no shame to be poor.
But it's no great honor either!
So, what would have been so terrible if I had a small fortune?"

If I were a rich man,
Yubby dibby dibby dibby dibby dibby dibby dum.
All day long I'd biddy biddy bum.
If I were a wealthy man.
I wouldn't have to work hard.
Ya ha deedle deedle, bubba bubba deedle deedle dum.
If I were a biddy biddy rich,
Idle-diddle-daidle-daidle man.

I'd build a big tall house with rooms by the dozen,
Right in the middle of the town.
A fine tin roof with real wooden floors below.
There would be one long staircase just going up,
And one even longer coming down,
And one more leading nowhere, just for show.

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Infinite sets. A set is *countable* if its elements can be listed.
More precisely, a set is *countable* if it is finite, or if its elements can be put in one-to-one correspondence with the natural numbers. Otherwise, the set is called *uncountable*.

count / DD **1**. Show that the set of even natural numbers is countable, by:

(a) Showing how to systematically list them;

(b) Explicitly constructing a bijective function from **N** to the even numbers.

count / DD **2**. Write a proof that the even numbers are countable, using your function from 1(b). The purpose of this problem is to practice constructing a clear, rigorous proof. Do this by filling in the following. In your notebook, write down the entire proof, not just the blanks.

Proof. We will show that ____.
We will do this by constructing ____________________________________,
and showing that it ____.
Let **N** be the set of natural numbers, and let S be the set of even numbers.
Define $f : \mathbf{N} \to S$ by $f(x) =$ __________ for each $x \in \mathbf{N}$.
First, we will show that f is injective. Suppose that $f(x) = f(y)$. Then __________
____, so $x = y$, as desired.
Now, we will show that f is surjective. Let $x \in S$. Then __________
______________________________, so $x = f(n)$ for some $n \in \mathbf{N}$, as desired.
Thus f is injective and surjective, so f is bijective, so there is a bijective function from **N** to the even numbers, so ______________________________________, as desired.

count / DD **3**. This result seems to be a contradiction: the set of even numbers seems to be a smaller set than **N** (half as big!), and yet the two sets have the same size. Explain.

Open sets. A set is *open* if there is an open ball around every point.
More precisely, a subset S of a metric space E is *open* if, for each $p \in S$, there exists an $r > 0$ such that $B_r(p) \subset S$.

open / DD **4**. Prove that the empty set is open. *Hint*: vacuously true

open / DD **5**. Let $S \subset R$ be defined by $S = [0, \infty)$. Show that S is *not* an open set. *Hint*: find a point in S about which there is *no* open ball that is completely contained in S.

open / DD **6**. Prove that, for any metric space E, the entire space E is an open set.

open / DD **7**. Let $E = [0, \infty)$. Prove that, in E, E is open.

Negating a statement. The contrapositive of "$A \implies B$" is "not $A \implies$ not B." The statement "not A" is the *negation* of statement A. I think of this as someone saying "A!" and someone replying "No, you're wrong, (negation of A)!" For example:

Person 1: Everyone in this class is named Matt.
Person 2: You're wrong! Not everyone in this class is named Matt. (true, but not useful)
Person 1: How do you know?
Person 2: There exists a person in this class not named Matt. (checkable! useful!)

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logic / DD **8**. Negate the following statements in a checkable, useful manner.

(a) All U.S. citizens can vote. **(b)** Every point of S has a ball around it.

(c) The ball around p contains a point of S. **(d)** One of my classes meets on Tuesday.

(e) Notice that the negation of an "all" or "none" statement is an existence statement (parts (a) and (b)), while the negation of an existence statement is an "all" or "none" statement (parts (c) and (d)). Explain why this is the case.

Precise notions to bound sets. Let A be a nonempty set of real numbers.

- A real number u is an *upper bound* for A if $a \le u$ for all $a \in A$.
- A real number l is a *lower bound* for A if $l \le a$ for all $a \in A$.
- A set is *bounded* if it has both an upper and a lower bound.
- A real number s is the *supremum* ("soo-PREE-mum") or *least upper bound* of A if s is an upper bound for A, and $s \le u$ for any other upper bound u of A. The supremum is denoted $\sup(A)$, pronounced "soup A," or l.u.b.(A).
- A real number t is the *infimum* ("in-FEE-mum") or *greatest lower bound* of A if t is a lower bound for A, and $l \le t$ for any other upper bound l of A. The infimum is denoted $\inf(A)$ or g.l.b.(A).
- A real number m is the *maximum* of A if $m \in A$ and $a \le m$ for all $a \in A$.
- A real number n is the *minimum* of A if $n \in A$ and $n \le a$ for all $a \in A$.

Note that, if you are just trying to show that a set is bounded, a crazy big bound like 1000 works just as well as a bound like 1. There is no need to do extra work to find a tight bound.

bound / AJ **9**. Complete the following table by filling in each box with a number, the letters DNE for "does not exist," or the word "Yes" or "No." Be prepared to justify your answers.

Set	L.B.	U.B.	min	max	sup	inf	is sup in set?	set bounded?
$\{x \in \mathbf{R} : 0 \le x < 1\}$								
$\{x \in \mathbf{R} : 0 \le x \le 1\}$								
$\{x \in \mathbf{R} : 0 < x < 1\}$								
$\{1/n : n \in \mathbf{Z} \setminus \{0\}\}$								
$\{1/n : n \in \mathbf{N}\}$								
$\{x \in \mathbf{R} : x < \sqrt{2}\}$								
$\{1, 4, 6, 16, 25\}$								
$\{(-1)^n(2 - 1/n) : n \in \mathbf{N}\}$								
$\{\ln(x) : x \in \mathbf{R}, x > 0\}$								
$\{e^x : x \in \mathbf{R}\}$								

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Warm-up problems

bound / AJ

1. For each of the following statements, either say it is true and explain why, or say it is false and provide a counterexample. *Hint*: Consider the examples from Page 4 # 5.

(a) Every set has a maximum.

(b) Every set has a minimum.

(c) If a set is bounded, then it has a supremum.

(d) If a set is bounded, then it has an infimum.

(e) If a set has an infimum, then it is bounded below.

(f) If a set has a supremum, then it is bounded above.

(g) If a set is bounded, then it has both a maximum and a minimum.

(h) If a set has a maximum, then it is bounded above.

(i) If a set is bounded above, then it has a maximum.

count / DD

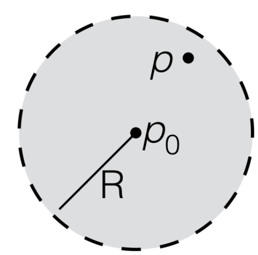
2. Show that the integers $\mathbf{Z}$ are countable.

seq / DD

3. For each of the following sequences, say whether it converges or diverges. For those that converge, prove that it converges by finding the limit p and finding an $N(\epsilon)$ for any $\epsilon > 0$.

(a) $a_n = \dfrac{\sin n}{n}$ **(b)** $b_n = 1 + (-1)^n$ **(c)** $1, 0, 1/2, 0, 1/4, 0, 1/8, 0, \ldots$

Some things to prove



open / DD

4. Prove that any open ball $B_R(p_0)$ in any metric space E is an open set. *Hint*: for any point p in the ball, find its distance to the boundary of the ball using p_0 and R, and choose a smaller number as the radius of the open ball around p.

open / FM

5. **Theorem.** The union of *any* collection of open sets is open.

Note: In order to even talk about this, we need a way of indexing an arbitrary collection of sets. We can't use $\bigcup_{n=1}^{k}$ or $\bigcup_{n=1}^{\infty}$, because the collection might be uncountable. The way we get around this is to use the symbol α, which represents an arbitrary indexing set: $\bigcup_{\alpha} A_\alpha$.

Proof. We will show that ____

We will do this by showing that, for any p in the union, there is an open ball around p contained in the union. Let $p \in \bigcup_{\alpha} A_\alpha$. Then for some α_0, $p \in A_{\alpha_0}$, because ____________________.

Since A_{α_0} is open, ... (complete the proof).

ONE MORE ON THE NEXT PAGE

Real Analysis

The following is an example of a *proof by contradiction*: We begin by supposing the *opposite* of what we want to prove, and we show that it leads to a contradiction (something that is clearly false). This shows that the thing we initially supposed was *false*, which shows that the thing we want to prove is *true*. A proof by contradiction takes the following form:

Proof. We will show A.
Suppose not A. (Steps of logical reasoning.)
Therefore, not A is false, so A is true.

seq / FM **6**. **Theorem.** A sequence $\{p_i\}_{i=1}^{\infty}$ of points in a metric space E has at most one limit.

Proof. We will show that a sequence of points in a metric space E has at most one limit. We will do this by contradiction, by supposing that it has two different limits, and showing that the two limits must be the same, by showing that the distance between them is 0.

Suppose that the sequence $\{p_i\}_{i=1}^{\infty}$ in metric space E has two different limits, p and p'. By definition of p and p' each being a limit point, we know that:

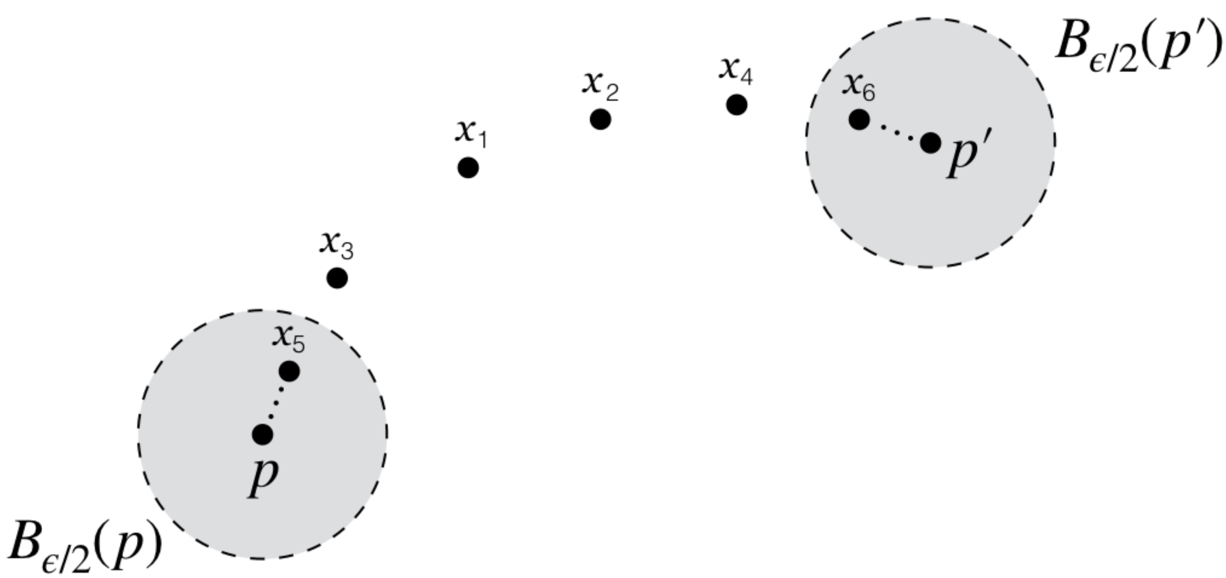
Given any $\epsilon/2 > 0$, there exists N such that $d(p, p_i) < \epsilon/2$ for all $i > N$, and

given any $\epsilon/2 > 0$, there exists M such that $d(p', p_i) < \epsilon/2$ for all $i > M$.

Given $\epsilon > 0$, choose a number $n > \max\{N, M\}$.

Finish the proof.

logic / DD **7**. We used $\epsilon/2$ to find N and M so that the bound comes out cleanly to ϵ at the end, but it would also have been fine to use ϵ and come out with a bound of 2ϵ at the end. Explain why a bound of 2ϵ would also prove that the two limit points coincide.



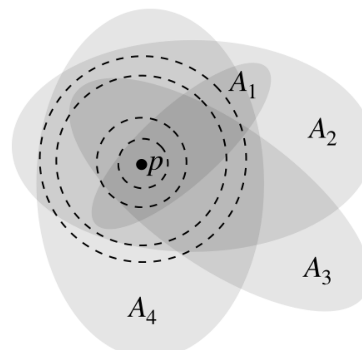
Real Analysis

Bounded sets. A set is *bounded* if it is contained in a finite ball.
More precisely, a subset S of a metric space E is *bounded* if there exists $p \in E$ and $r > 0$ such that $S \subset B_r(p)$.

seq / DD **1**. Show that the set $\{1000 - 500/n^2 : n \in \mathbf{N}\}$ is bounded by finding a suitable p and r.

open / FM **2**. **Theorem.** The intersection of a *finite* number of open sets is open.

Proof. We will show that **(a)** ______________. We will do this by showing that, for any p in the intersection, there is an open ball around p contained in the intersection. Let $p \in \displaystyle\bigcap_{k=1}^{n} A_k$. Then $p \in A_k$ for all $1 \leq k \leq n$, because **(b)** ______________. Thus, there exist $r_k \in \mathbf{R}$ such that $B_{r_k}(p) \subset A_k$ for each k. **(c)** (complete the proof.) *Hint*: See picture.



open / DD **3**. Consider the (false!) statement: *The intersection of infinitely many open sets is open.*

(a) Explain where the proof above breaks down for infinitely many sets.

(b) Give a counterexample to the statement.

The boundary, interior and closure. Let S be a subset of a metric space E. A point $p \in E$ is a *boundary point* of S if every open ball about p contains points of S and points of S^C. The *boundary* of S, denoted ∂S, is the collection of all of the boundary points of S. The *closure* of S, denoted $\overline{S}$, is $S \cup \partial S$. The *interior* of S, denoted $\overset{\circ}{S}$, is $S \setminus \partial S$.

open / DD **4**. Find $\partial S, \overline{S}$ and $\overset{\circ}{S}$ for each set S that is a subset of the given metric space, with the standard Euclidean metric:

(a) $(0, 1] \subset \mathbf{R}$ **(b)** $\mathbf{Z} \subset \mathbf{R}$ **(c)** $\mathbf{Q} \subset \mathbf{R}$ **(d)** $B_1(0, 0) \subset \mathbf{R}^2$

open / FM **5**. Prove that every point in a set is either a boundary point or an interior point.

seq / FM **6**. Prove that "the limit of a sum is the sum of the limits": If $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers, with $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$, then $\lim\limits_{n\to\infty} (a_n + b_n) = a + b$. *Hint*: At the end, you will want to show that $|(a_n + b_n) - (a + b)| < \epsilon$, so break it into two parts using rules of absolute values, and take n large enough that certain quantities are less than $\epsilon/2$.

seq / DD **7**. Explain why, in the previous problem, $\{a_n\}$ and $\{b_n\}$ had to be sequences of real numbers, rather than just sequences in an arbitrary metric space.

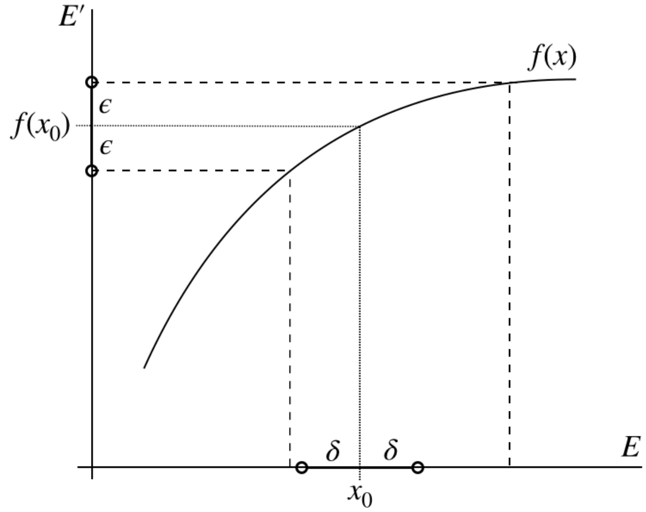
Continuity. "Nearby points are sent to nearby points." There are three (!) equivalent definitions of what it means for a function to be *continuous*. We will explore each of them. Then we will prove their equivalence.

The ϵ–δ definition of continuity.

Let E, E' be metric spaces with distance metrics d, d' respectively. Let $f : E \to E'$ be a function, and let $x_0 \in E$. We say that f is *continuous at* x_0 if, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for $x \in E$,

$$d(x, x_0) < \delta \implies d'(f(x), f(x_0)) < \epsilon.$$

We say that f is *continuous* if it is continuous everywhere in E.



cont / DD

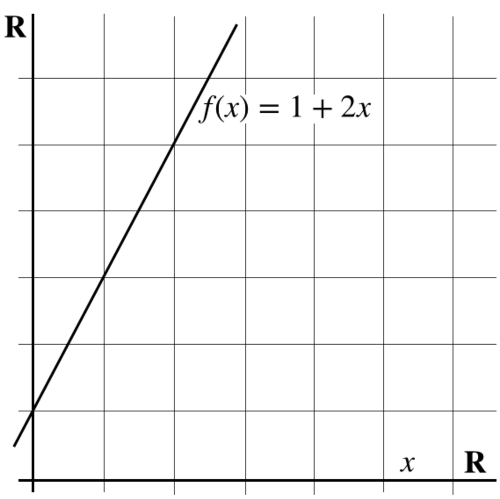
8. For the function $f(x) = 1 + 2x$, explore the definition of continuity by finding a δ for each given x_0 and ϵ so that, for any $x \in \mathbf{R}$, $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.

(a) $x_0 = 1, \epsilon = 1$

(b) $x_0 = 1, \epsilon = 0.1$

(c) $x_0 = 2, \epsilon = 0.001$

Hint: Draw in dashed lines as in the figure above.



logic / DD

9. Robert Frost's poem *Nothing Gold Can Stay* is shown to the right. Diagram the statement "nothing gold can stay": Do parts **(a)**–**(d)**; see Page 3c.

Nothing Gold Can Stay

Robert Frost - 1874-1963

Nature's first green is gold,
Her hardest hue to hold.
Her early leaf's a flower;
But only so an hour.
Then leaf subsides to leaf.
So Eden sank to grief,
So dawn goes down to day.
Nothing gold can stay.

Real Analysis

bound / AJ

1. In each of the following pairs, exactly one of the statements is true. For the one that is true, explain why; for the one that is false, provide a counterexample.

(A1) If a set of real numbers is bounded above, then it has a maximum.

(A2) If a set of real numbers is bounded above, then it has a supremum.

(B1) If a set of real numbers has a supremum, then it has a maximum.

(B2) If a set of real numbers has a maximum, then it has a supremum.

(C1) If a set of real numbers is has an infimum, then the infimum is in the set.

(C2) If a set of real numbers is has a minimum, then the minimum is in the set.

cont / AJ

2. Let's explore the *epsilon-delta definition of continuity.* For each function $f : \mathbf{R} \to \mathbf{R}$, compute $f(x_0)$ and draw a sketch of $f(x)$ in the vicinity of x_0. In the next columns, write out the "allowable output range" $(f(x_0) - \epsilon, f(x_0) + \epsilon)$ for the given x_0 and ϵ, and then write out the corresponding "permissible input range" $(x_0 - \delta, x_0 + \delta)$, if it exists, for each value of ϵ (in the same box). Finally, say if f is continuous at x_0.

function and x_0 value	$f(x_0)$	sketch	$\varepsilon = 1$	$\varepsilon = 0.1$	cont?
$f_1(x) = \lvert x \rvert$, $x_0 = 0$					
$f_2(x) = \begin{cases} x & x \leq 1 \\ 2x - 0.5 & x > 1 \end{cases}$, $x_0 = 1$					
$f_3(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q} \end{cases}$, $x_0 = 0$					
$f_4(x) = \begin{cases} 1/x & x \neq 0 \\ 2 & x = 0 \end{cases}$, $x_0 = 0$					
$f_5(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$, $x_0 = 0$					
$f_6(x) = \begin{cases} x \cdot \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$, $x_0 = 0$					

cont / AJ

3. When you *couldn't* find a δ for a particular ϵ, why not? When you *could* find a δ for a particular ϵ, was δ unique? If not, could you find a maximum value for δ? A minimum?

Closed sets. A subset S of a metric space E is *closed* if S^C is open.

A *closed ball* in a metric space E, with center p_0 and radius r, is the set $\{p \in E : d(p, p_0) \leq r\}$, or in other words $\overline{B_r(p_0)}$.

closed / DD **4**. Prove that the empty set is closed.

closed / DD **5**. Prove that, for any metric space E, the entire space E is closed.

closed / DD **6**. We have now proved that the empty set is both open and closed, and also that any entire space E is both open and closed. Are these contradictions? Explain.

Bounded sequences. A sequence of points $\{p_i\}_{i=1}^{\infty}$ in a metric space is *bounded* if it is bounded as a set, i.e. if it is contained in a ball.

seq / DD **7**. **Theorem.** Every convergent sequence is bounded.

Proof. We will show that every convergent sequence is bounded. We will do this by constructing a ball that contains all of the points of the sequence. Take $\epsilon = 1$. Then there exists N such that, for all $n > N$, $d(p_n, p) < 1$, because **(a)** __________________________________. Now take $r = \max\{1, d(p, p_1), d(p, p_2), \ldots, d(p, p_N)\}$. **(b)** (finish the proof)

seq / DD **8**. State the converse (note: converse, not contrapositive) of the theorem in Problem 7. Then either prove it or give a counterexample.

Continuity. "Nearby points are sent to nearby points." As stated before, there are three equivalent definitions of what it means for a function to be *continuous*, which we will later prove are equivalent. This is the second one.

The sequence definition of continuity.

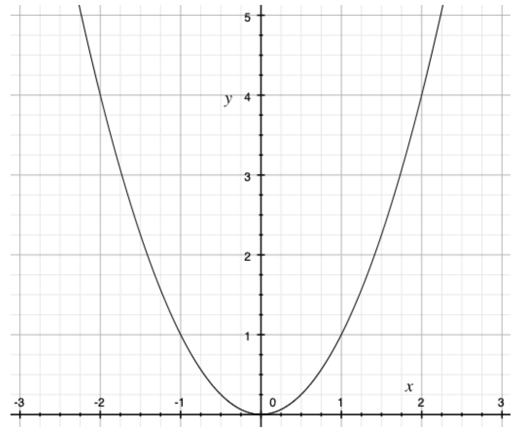
Let E, E' be metric spaces with distance metrics d, d' respectively. Let $f : E \to E'$ be a function, and let $x_0 \in E$.

We say that f is *continuous at* x_0 if, for every sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

As before, we say that f is *continuous* if it is continuous everywhere in E.

seq / DD **9**. Use the sequence definition of continuity to show that the function f_2 from the table earlier in this problem set is not continuous.

Inverses.



- If $f : X \to Y$ is one-to-one and onto (bijective), then we define the *inverse* of f to be the function $f^{-1} : Y \to X$, defined such that $f^{-1}(y) = x$ when $f(x) = y$.
- We define the *image* of a set $A \subset X$ as the collection of images of points in A, $f(A) = \{f(a) : a \in A\}$.
- No matter if f is bijective or not, we define the *inverse image* of a set $B \subset Y$ as the collection of points in X that map to points in B:

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

func / DD

10. Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x^2$, shown above. Find each of the following, or say why it is not possible:

(a) $f\big([-2,-1]\big)$ **(b)** $f^{-1}\big((0,2)\big)$ **(c)** $f^{-1}\big((-2,-1)\big)$ **(d)** $f^{-1}(x)$ for some $x \in \mathbf{R}$

logic / DD

11. (Annie Preis) Lyrics for the Spice Girls song "Wannabe" are shown to the right. Diagram the statement "If you wanna be my lover, you gotta get with my friends": Do parts **(a)**–**(d)**; see Page 3c.

Spice Girls Lyrics

Wannabe Lyrics

Yo I'll tell you what I want, what I really want
So tell me what you want, what you really really want
I'll tell you what I want, what I really really want
So tell me what you want, what you really really want
I wanna [huh], I wanna [huh], I wanna [huh], I wanna [huh]
I wanna really really really wanna zigazig ahh

If you want my future forget my past
If you wanna get with me better make it fast
Now don't go wasting my precious time
Get your act together, we could be just fine

I'll tell you what I want, what I really really want
So tell me what you want, what you really really want
I wanna [huh], I wanna [huh], I wanna [huh], I wanna [huh]
I wanna really really really wanna zigazig ahh

If you wanna be my lover you gotta get with my friends
[Gotta get with my friends]
Make it last forever, friendship never ends
If you wanna be my lover you have got to give
Taking is too easy but that's the way it is

What d'you think about that? Now you know how I feel
Say you can handle my love, are you for real? [are you for real?]
I won't be hasty, I'll give you a try
If you really bug me then I'll say goodbye

count / DD

12. *Dramatic foreshadowing.* Let

$$S_1 = \{1, 1/2, 1/3, 1/4, \ldots\} = \{1/n : n \in \mathbf{N}\},$$
$$S_2 = \{2, 2/2, 2/3, 2/4, \ldots\} = \{2/n : n \in \mathbf{N}\},$$
$$S_3 = \{3, 3/2, 3/3, 3/4, \ldots\} = \{3/n : n \in \mathbf{N}\}.$$

(a) Show that S_1, S_2, and S_3 are countable.

(b) Show that the set $S_1 \cup S_2 \cup S_3$ is countable, by showing how to systematically list its elements.

(c) Show that any union of finitely many countable sets is countable.

Real Analysis

Warm-up problems

bound / DD **1**. Give an example of each of the following:

(a) Sets $A \subset B$ with $\sup(A) < \sup(B)$;

(b) Sets $A \subsetneq B$ for which $\sup(A) = \sup(B)$.

Monotonicity. We use the following terms to describe sequences of real numbers:

- A sequence $\{a_i\}_{i=1}^{\infty}$ is *increasing* if $a_1 \leq a_2 \leq a_3 \leq \dots$
- A sequence $\{a_i\}_{i=1}^{\infty}$ is *decreasing* if $a_1 \geq a_2 \geq a_3 \geq \dots$
- A sequence is *monotone* if it is either increasing or decreasing.

seq / DD **2**. Prove or give a counterexample: Every convergent sequence of real numbers is monotone.

Some things to prove

count / DD **3**. Prove that the rational numbers are countable by showing how to systematically list them. *Hint*: see Page 7 # 12.

open / FM **4**. Prove that, for any set S, the interior of S is an open set.

open / FM **5**. Prove that, for any set S, the interior of S is the largest open set contained in S.

closed / DD **6**. Prove that a closed ball is a closed set: for any metric space E, any point $p_0 \in E$, and any radius $R > 0$, the closed ball $\overline{B_R(p_0)}$ is closed. *Hint*: use a strategy similar to Page 5 # 4.

seq / DD **7**. Let $\{a_i\}$, $\{b_i\}$ be convergent sequences of real numbers under the standard Euclidean metric, with limits a and b respectively. Prove that if $a_i \leq b_i$ for all i, then $a \leq b$.

seq / DD **8**. Prove or give a counterexample for the statement above, with "$\leq$" replaced by "$<$."

Real Analysis

Covers. Given a set S, a collection of sets $\{G_\alpha\}$ is a *cover* of S if $S \subset \bigcup_\alpha G_\alpha$.

$\{G_\alpha\}$ is an *open cover* if all of the G_α are open sets.

Note: recall (Page 5 # 4) that we use the symbol α to index an arbitrary indexing set, which may not be countable. A picture of an open cover of $\mathbf{R}^2$ by filled ellipses is in the background of the cover (ha!) of *Real Analysis* by Frank Morgan.

cpt / DD **1**. Construct an open cover that:

(a) covers $\mathbf{R}$, using unit intervals;

(b) covers $\mathbf{R}^2$, using unit balls;

(c) covers $\{1/n : n \in \mathbf{N}\}$.

cont / DD **2**. The open set definition of continuity (just below) uses *inverse* images. Let's think about *images*. Prove or give a counterexample: If f is a continuous function, and $\mathcal{U}$ is open, then $f(\mathcal{U})$ is open.

Continuity. Here is the third of the three equivalent definitions of what it means for a function to be *continuous*. We will prove their equivalence soon.

The open set definition of continuity.

Let E, E' be metric spaces with distance metrics d, d' respectively. Let $f : E \to E'$ be a function, and let $x_0 \in E$.

We say that f is *continuous* if, for every open set $\mathcal{U} \subset E'$, $f^{-1}(\mathcal{U})$ is open in E.

cont / DD **3**. Use the open set definition of continuity to show that the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$ is continuous.

More on open sets. So far, we have used the "every point is contained in an open ball" characterization of an open set. The following Theorem gives an alternative characterization.

open / DD **4**. **Theorem.** Let E be a metric space, and let S be a subset of E, considered as a metric space itself. For a subset $A \subset S$, the following are equivalent:

(1) A is open in S.

(2) There exists a set $\widetilde{A}$ that is open in E, such that $A = \widetilde{A} \cap S$.

Prove this. *Hint*: To prove that (1) $\implies$ (2), show that $\widetilde{A} \cap S \subset A$ and that $A \subset \widetilde{A} \cap S$.

Note: the symbol $\sim$ is "tilde," pronounced "TILL-duh," and $\widetilde{A}$ is read aloud as "A tilde."

closed / DD **5**. Prove that the union of a finite number of closed sets is closed. *Hint*: First, argue that $\left(\bigcup_{k=1}^{n} A_i\right)^C = \bigcap_{k=1}^{n} {A_i}^C$. Then argue that, if each A_i is closed, this intersection is open.

closed / DD **6**. Prove that the intersection of any collection of closed sets is closed.
Hint: apply a previous result.

Real Analysis

An *axiom* is a statement that we take as fact, without proof, generally because it is impossible to prove it from our other axioms. The following statement, conjectured on day 4 by members of our class who then (understandably!) failed to find a proof of the result, is an axiom:

Completeness axiom. A nonempty set of real numbers that is bounded from above has a least upper bound.

bound / DD **7**. **Theorem.** Let S be a nonempty, closed subset of $\mathbf{R}$ that is bounded from above. Then S has a maximum element.

Proof. We will show that any nonempty, closed subset $S \subset \mathbf{R}$ that is bounded from above has a maximum element. We will do this by showing that the least upper bound a of S is contained in S. The proof will be by contradiction: we will first suppose that $a \notin S$, and derive a contradiction. (Fill in the reasoning steps in the following proof.)

Let a be the least upper bound of S. We know that the least upper bound exists, by the completeness axiom. We want to show that $a \in S$. Suppose, for a contradiction, that $a \notin S$. S^C is open, because **(a)** ______________________________. Thus there exists some $r > 0$, such that $B_r(a) \subset S^C$, because **(b)** ______________________________. But then $a - r$ is also an upper bound for S, because **(c)** ______________________________. This contradicts a being the least upper bound for S, because **(d)** ______________________________. Thus $a \in S$, and thus S has a maximum element, as desired.

bound / DD **8**. (Continuation) The statement of the theorem contains the conditions that S is a nonempty, closed subset of R. Give a counterexample or explanation for why the conclusion "S has a maximum element" fails to be true if we remove the assumption that:

(a) S is nonempty; **(b)** S is closed; **(c)** S is a subset of $\mathbf{R}$.

Accumulation points. Let S be a subset of a metric space E. A point $p \in E$ is an *accumulation point* of S if, for every $\epsilon > 0$, $B_\epsilon(p)$ contains an infinite number of points from S. (An accumulation point is also called a *cluster point* or *limit point*.)

sets / DD **9**. For each of the following subsets of $\mathbf{R}$ with the Euclidean metric, describe its set of accumulation points.

(a) $\mathbf{Q}$ **(b)** the irrationals **(c)** $(a, b]$ **(d)** $\{1\}$ **(e)** $\{1/n : n \in \mathbf{N}\}$ **(f)** $\mathbf{Z}$

A new metric space.

Let $\mathcal{B} = \{\text{bounded, real-valued functions on } \mathbf{R}\}$

$$= \{f : \mathbf{R} \to \mathbf{R} \text{ such that there exists } M \in \mathbf{R} \text{ with } |f(x)| < M \text{ for all } x \in \mathbf{R}\}.$$

Define $d(f, g) = \sup\{|f(x) - g(x)| : x \in \mathbf{R}\}$.

metric / DD **10**. Let $f(x) = \cos(x)$ and $g(x) = 2$. Explain why $f, g \in \mathcal{B}$, and find $d(f, g)$.

metric / DD **11**. Show that d is a metric on $\mathcal{B}$.

Real Analysis

Isolated points. Let S be a subset of a metric space E. A point $p \in S$ is *isolated* if there exists $r > 0$ such that p is the only point of S in $B_r(p)$.

sets / FM

1. Prove that every boundary point is either an isolated point or an accumulation point.

closed / DD

2. For a point $p \in \mathbf{R}^n$, consider the set $S = \bigcap_{m=1}^{\infty} \{x \in \mathbf{R}^n : d(x,p) \leq 1/m\}$.

(a) Prove that S is closed. *Hint*: Quote a recent result you have proved.

(b) Give a simple description of the set S.

(c) Prove that any single point $\{p\}$, where $p \in \mathbf{R}^n$, is a closed set.

seq / FM

Subsequences. We like sequences to converge, but most don't. Fortunately, most sequences have subsequences that do converge. Given a sequence a_n, a *subsequence* a_{m_n} consists of some (infinitely many) of the terms, in the same order.

The lim sup and lim inf. Even for sequences that are not convergent, sometimes elements of the sequence do accumulate. The following *always* exist:

- The *lim inf* of a sequence is the smallest limit of any subsequence, or $\pm\infty$.
- The *lim sup* of a sequence is the largest limit of any subsequence, or $\pm\infty$.

Note: "lim sup" is pronounced "limm soup."

bound / AJ

3. In the following table, write out the first 8 terms of any sequence for which they are not already written out for you. Then find the lim inf and lim sup of each sequence.

sequence	first 8 terms	lim inf	lim sup
$a_n = 1/n$			
$a_n = \sin(n\pi/2)$			
$1, -1, 2, -2, 3, -3, 4, -4, \ldots$			
$a_n = -n^2$			
$a_n = 2 + \dfrac{(-1)^n}{n}$			
$a_n = \begin{cases} 3 - e^{-n} & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$			
$\frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, \frac{1}{4}, 1\frac{1}{4}, 2\frac{1}{4}, \frac{1}{8}, 1\frac{1}{8}, 2\frac{1}{8}, \ldots$			

bound / AJ

4. Make a conjecture as to what conditions ensure that the lim inf of a sequence equals its lim sup.

cont / FM **5**. Let $g : E \to E'$ and $f : E' \to E''$ be continuous functions. Prove that their composition $f \circ g : E \to E''$, i.e. $x \mapsto f(g(x))$, is continuous, using each definition of continuity:

(a) epsilon-delta definition **(b)** sequence definition **(c)** open set definition.

(d) Which way did you most prefer? Which did you least prefer?

cont / FM **6**. **Theorem.** The three definitions (1), (2), (3) of continuity are equivalent.

Proof. We will prove that the three are equivalent by proving (1) $\iff$ (2) and (1) $\iff$ (3).

(a) (1) $\iff$ (2). *Hint*: Prove (1) $\implies$ (2) directly, and (2) $\implies$ (1) using the contrapositive.

(b) (1) $\implies$ (3): Let $\mathcal{U}$ be an open set in E'. We wish to show that $f^{-1}(\mathcal{U})$ is open, so we need to show that, for any $p \in f^{-1}(\mathcal{U})$, there is an open ball about p in the set **(a)** ____________________. Let $p \in f^{-1}(\mathcal{U})$. Then $f(p) \in \mathcal{U}$, so there exists $\epsilon > 0$ such that $f(p)$ is contained in **(b)** ____________________ in the set **(c)** ____________. Since we assume (1), we can choose $\delta > 0$ such that

$$
\begin{aligned}
|x-p| < \delta &\implies |f(x) - f(p)| < \epsilon, \text{ and thus} \\
|x-p| \le \delta/2 &\implies |f(x) - f(p)| < \epsilon.
\end{aligned}
$$

Here we divided δ by 2 so that **(d)** ____________________.

Now we have shown that $B(p, \delta/2) \subset f^{-1}(B(f(p), \epsilon)) \subset f^{-1}(\mathcal{U})$, so

(e) (finish the proof).

(f) (3) $\implies$ (1): *Hint*: Since the inverse image of the open ball about $f(p)$ of radius ϵ is open and contains p, it contains some ball $B(p, \delta)$, so $|x - p| < \delta \implies |f(x) - f(p)| < \epsilon$. Fill in the details.

closed / DD **7**. Consider the (false!) statement: *The union of infinitely many closed sets is closed.*

(a) We proved in Page 9 # 5 that the union of *finitely* many closed sets is closed. Explain where the proof breaks down for infinitely many sets.

(b) Give a counterexample to the statement.

cpt / DD **8**. For each set $S_i \subset \mathbf{R}$, explain why $\mathcal{G}_i$ is an open cover of S_i.

(a) $S_1 = \{\pi\}$, $\mathcal{G}_1 = \{(1/(n+1), n) : n \in \mathbf{N}\}$

(b) $S_2 = [-3, 11]$, $\mathcal{G}_2 = \{(n, n+2) : n \in \mathbf{Z}\}$

(c) $S_3 = [2, \infty)$, $\mathcal{G}_3 = \{(n, n+2) : n \in \mathbf{N}\}$

(d) $S_4 = (0, 1)$, $\mathcal{G}_4 = \{(1/n, 1 - 1/n) : n \in \mathbf{N}\}$

An algorithm. Suppose that you are given a list of numbers. For example, maybe they are Dewey decimal numbers for books that are already in your library, and now you have a new book. So to make a number for the new book, you need to construct a number x that you can be sure is not already on the list.

Here is an algorithm for constructing a number x that is not on a given list:

- Make the 1$^{\text{st}}$ digit of x different from the 1$^{\text{st}}$ digit of the 1$^{\text{st}}$ number on the list.
- Make the 2$^{\text{nd}}$ digit of x different from the 2$^{\text{nd}}$ digit of the 2$^{\text{nd}}$ number on the list.
- Make the 3$^{\text{rd}}$ digit of x different from the 3$^{\text{rd}}$ digit of the 3$^{\text{rd}}$ number on the list.
- Continue in this way, for all of the numbers on the list.
- For concreteness, let the n^{th} digit of x be 1, unless the n^{th} digit of the n^{th} number on the list happens to be 1, and in that case let the n^{th} digit of x be 2.

count / DD **9**. Use the algorithm to construct a number that is not on the example list below.

1. 3.1415926
2. 5.0000000
3. 2.7182818
4. 1.6180339
5. 1.2121212
6. 1.4142135
7. 0.6931471

Then explain why this algorithm *always* produces a number that is not on the given list.

Real Analysis

Warm-up problems

metric / DD **1**. (Rick Muniu) Give an example of a distance function on a metric space that satisfies the triangle inequality (property 4 on Page 2P), but fails to satisfy one of the other three properties of a metric.

open / FM **2**. Is all of $\mathbf{R}$ the only open subset of $\mathbf{R}$ that contains all of $\mathbf{Q}$? Prove your answer correct.

closed / FM **3**. Prove that any finite set of points $\{p_1, \ldots, p_k\} \subset \mathbf{R}^n$ is closed.

metric / DD **4**. Consider a metric space E with the "don't touch me" (discrete) metric.

(a) What do the open balls look like in this space?

(b) What do the closed balls look like in this space?

(c) True or False: Any finite set of points, in a metric space E with this metric, is open.

Some things to prove

count / DD **5**. Prove that the real numbers are uncountable.
Hint: Show that the interval $[0, 1]$ is uncountable, using a proof by contradiction: If the set is countable, then its elements can be listed. Use the algorithm from Page 10 # 9 to derive a contradiction.

closed / DD **6**. Prove that, for any set S, $\overline{S} = \bigcap_{C \text{ closed}, S \subset C} C$. First write out the statement in words. Then prove it. Sometimes this is used as the *definition* of the closure of S.

seq / FM **7**. **Proposition.** (We use "Proposition" for a result that is smaller than a Theorem.) Every bounded sequence in $\mathbf{R}$ (with the usual Euclidean metric) has a convergent subsequence.

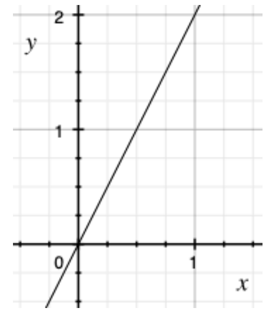
We will first show that every bounded sequence of *nonnegative* real numbers in $\mathbf{R}$ has a convergent subsequence. We will do this by explicitly constructing a convergent subsequence. Consider a nonnegative sequence $a_1, a_2, a_3, \ldots$. Each a_n starts off with a nonnegative integer before the decimal point, followed by infinitely many digits (possibly 0) after the decimal point. Since the sequence a_n is bounded, some integer part D before the decimal place occurs infinitely many times, because **(a)** ________________. Throw away the rest of the a_n. Among the infinitely many remaining a_n that start with D, some first decimal place d_1 occurs infinitely many times. Throw away the rest of the a_n.

(b) Complete the construction of a number $L = D.d_1d_2d_3\ldots$, and prove that there is a subsequence of a_n converging to L.

(c) Now show that *every* bounded sequence of real numbers has a convergent subsequence, to complete the proof of the Proposition as stated.

8. State and prove your conjecture from Page 10 # 4.

Real Analysis



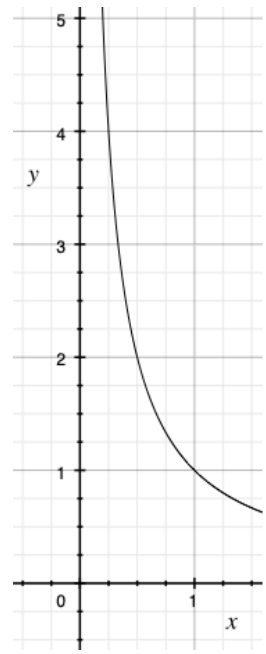
open / DD **1**. $S = [0, \infty)$ is a subset of the metric space $\mathbf{R}$ with the usual Euclidean metric, but S itself is also a metric space, with the inherited metric from $\mathbf{R}$. Which of the following are open sets in S?

(a) $(0, 1)$ **(b)** $[0, 1)$, **(c)** $(0, 1]$ **(d)** $[0, 1]$

cont / DD **2**. Show that the function $f(x) = 2x + 1$, $f : \mathbf{R} \to \mathbf{R}$ is continuous at $x = 1/2$ by explicitly finding a δ for each of the following values of ϵ:

(a) $\epsilon = 1$, **(b)** $\epsilon = 1/3$, **(c)** any given $\epsilon > 0$.

(d) Can you find an easier way to show that $f(x)$ is continuous?



Uniform continuity. $f : E \to E'$ is *uniformly continuous* if, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $p, q \in E$,

$$d(p, q) < \delta \implies d'(f(p), f(q)) < \epsilon.$$

uni-con / DD **3**. Let $E = (0, 1)$, and let $f_1(x) = 2x$ and $f_2(x) = 1/x$, where $f_1, f_2 : E \to \mathbf{R}$.

(a) Let $x_0 = 1/2$. Given any $\epsilon > 0$, find δ_1 so that

$$|x - x_0| < \delta_1 \implies |f_1(x) - f_1(x_0)| < \epsilon.$$

(b) Let $x_0 = 1/2$. Given any $\epsilon > 0$, find δ_2 so that

$$|x - x_0| < \delta_2 \implies |f_2(x) - f_2(x_0)| < \epsilon.$$

(c) Repeat part (a), for $x_0 = 1/10$. **(d)** Repeat part (b), for $x_0 = 1/10$.

(e) Both functions are continuous on $(0, 1)$, but only one is uniformly continuous on $(0, 1)$. Explain geometrically what causes the difference. See pictures above.

Cauchy sequences. A sequence $\{p_n\}_{n=1}^{\infty}$ in a metric space E is *Cauchy* ("CO-shee") if, for any $\epsilon > 0$, there exists N such that

$$m, n > N \implies d(p_m, p_n) < \epsilon.$$

You can think of a Cauchy sequence as one that is "trying" to converge, but to a limit that may be outside of its metric space.

Cauchy / DD **4**. Consider the sequence $\{a_n = 1/n : n \in \mathbf{N}\}$ in $R^+ = \{x \in \mathbf{R} : x > 0\}$ with the usual Euclidean metric.

(a) Show that $\{a_n\}$ is a Cauchy sequence.

(b) Show that $\{a_n\}$ does not converge in this metric space.

Cauchy / FM **5**. Prove that a convergent sequence in any metric space is Cauchy.

Cauchy / FM **6**. Prove that a Cauchy sequence in any metric space is bounded. *Hint*: The proof that shows that a convergent sequence is bounded works here, too.

Sequence definition of a closed set. So far, to show that a set is closed, we have to show that its complement is open. After proving the following Theorem, we will have another way. In fact, the following Theorem is sometimes used as the *definition* of a closed set.

closed / FM **7**. **Theorem.** Let S be a subset of a metric space E. Then S is closed if and only if every convergent sequence of points from S converges to a point in S.

Proof. This is an "if and only if" statement, so we need to show both directions of implication.

($\Longrightarrow$) Suppose S is closed. We will show that every convergent sequence converges to a point in S. We will prove this by contradiction: We will assume that some sequence of points in S converges to a point not in S, and show that some of the points in the sequence (in fact, the entire "tail of the sequence") are not in S, which will be the desired contradiction.

Let $\{p_n\} \subset S$ be a convergent sequence of points from S. Let $\lim_{n\to\infty} p_n = p$, and assume that $p \notin S$. Thus, $p \in S^C$. S^C is open, because **(a)** ______________, so there exists $\epsilon > 0$ such that $B_\epsilon(p) \subset S^C$, because **(b)** ______________. But by the definition of a limit, there exists N such that for all $n > N$, $d(p_n, p) < \epsilon$. **(c)** (finish the proof)

($\Longleftarrow$) Suppose that every convergent sequence of points in S converges to a point in S. We will show that S is closed. We will prove the contrapositive: We will assume that S is not closed, and find a convergent sequence of points in S that converges to a point not in S.

Suppose that S is not closed. Then S^C is not open. Thus, there exists a point p in S^C such that *every* open ball around p contains a point from S, because **(d)** ______________. So we can construct an infinite set of balls, $\{B_{1/n}(p)\}$, and an infinite sequence of points p_n, such that $p_n \in B_{1/n}(p)$ for each $n \in \mathbf{N}$, because **(e)** ______________. Then $\{p_n\} \subset S$, but $p_n \to p$ and $p \notin S$. **(f)** (finish the proof).

Given a cover $\mathcal{G}$ of a set S, a *subcover* is a collection consisting of some of the sets in $\mathcal{G}$, that also covers S. A *finite* subcover is a subcover consisting of finitely many of the sets in $\mathcal{G}$.

cpt / DD **8**. For each of the following covers $\mathcal{G}_i$ of the set S_i (copied from Page 10 # 4), find a subcover that still covers S_i. Can you find a *finite* subcover?

(a) $S_1 = \{\pi\}$, $\mathcal{G}_1 = \{(1/(n+1), n) : n \in \mathbf{N}\}$

(b) $S_2 = [-3, 11]$, $\mathcal{G}_2 = \{(n, n+2) : n \in \mathbf{Z}\}$

(c) $S_3 = [2, \infty)$, $\mathcal{G}_3 = \{(n, n+2) : n \in \mathbf{N}\}$

(d) $S_4 = (0, 1)$, $\mathcal{G}_4 = \{(1/n, 1 - 1/n) : n \in \mathbf{N}\}$

sets / DD **9**. Find the set of accumulation points in $\mathbf{R}^2$ for each of the following sets:

(a) $\{(p, q) : p, q \in \mathbf{Q}\}$ **(b)** $\{(m/n, 1/n) : m, n \in \mathbf{Z}, n \neq 0\}$

Real Analysis

The notion of a *compact* set is very important in analysis; it is why we have been thinking about open covers. In the Heine-Borel theorem (Page 15), we will show that, in $\mathbf{R}^n$, a compact set is just a set that is closed and bounded). The definition itself uses open covers:

Compactness. A subset X of a metric space E is *compact* if every open cover of X has a finite subcover.

However, this definition is rather difficult to check. Sure, we can find *an* open cover, but how do you check that *every possible* open cover has a finite subcover?

cpt / DD **1**. Show that $(0, 1]$ is not compact, by showing that the open cover $\bigcup_{n=1}^{\infty} U_n$, where $U_n = \{(1/n, \infty)\}$, has no finite subcover, i.e. that no finite subset of the U_n's covers the set.

cpt / DD **2**. Show that $\mathbf{R}$ is not compact, by finding an open cover that has no finite subcover.

We like *compact* sets because they tend to be the sets that have the properties we want. For example, eventually we will show that a continuous function on a compact set achieves a maximum and minimum, which is very useful in calculus.

uni-con / DD **3**. Let's recall the difference between *continuity* and *uniform continuity.*

(a) Explain why f uniformly continuous $\implies$ f continuous.

(b) Explain why f continuous $\not\Rightarrow$ f uniformly continuous by providing a counterexample.

In fact, f continuous $\implies$ f uniformly continuous in the special case when the domain of f is *compact*. Let's prove it. This will also give us some experience in applying the fact that every open cover has a finite subcover.

uni-con / FM **4**. **Theorem.** Let E, E' be metric spaces, and let $f : E \to E'$ be a continuous function. If E is compact, then f is uniformly continuous.

Proof. Given any $\epsilon > 0$, we will construct a δ such that, for all $p, q \in E$,

$$d(p, q) < \delta \implies \textbf{(a)}\underline{\hspace{6em}}.$$

Given $\epsilon > 0$, we know that for each $x_0 \in E$, there is a $\delta_{x_0} > 0$ such that

$$d(p, x_0) < \delta_{x_0} \implies d'(f(p), f(x_0)) < \epsilon/2,$$

because **(b)** ____________________.

Consider the open ball $\mathcal{U}_{x_0} = \{p : d(p, x_0) < \delta_{x_0}/2\}$. The collection $\{\mathcal{U}_x\}$ of all such open balls covers E, because **(c)** ____________________.

Since E is compact, it has a finite subcover $\{\mathcal{U}_{x_1}, \ldots, \mathcal{U}_{x_n}\}$. Let $\delta = \min\{\delta_{x_i}/2 : i = 1, \ldots, n\}$. We will show that this is the δ with the desired property.

Suppose that $d(x, x_0) < \delta$. Since $x_0 \in E$, $x_0 \in \mathcal{U}_{x_j}$ for some j in $\{1, \ldots, n\}$, so $d(x_0, x_j) < \delta_{x_j}/2$. Since $d(x, x_0) < \delta \leq \delta_{x_j}/2$, we have $d(x, x_j) < \delta_{x_j}$, by **(d)** ________________. Therefore,

$$d'(f(x_0), f(x_j)) < \epsilon/2 \text{ and } d'(f(x), f(x_j)) < \epsilon/2, \text{ so}$$

$$d'(f(x), f(x_0)) \leq d'(f(x_0), f(x_j)) + d'(f(x), f(x_j)) < \epsilon/2 + \epsilon/2 = \epsilon,$$

as desired.

Real Analysis

Complete metric spaces. A metric space E is *complete* if every Cauchy sequence of points in E converges to a point in E.

Cauchy / DD

5. Show that each of the following metric spaces (using the usual Euclidean distance metric) is not complete, by finding a Cauchy sequence of points in the space that does not converge to a point of the space:

(a) $\mathbf{R}^+ = \{x \in \mathbf{R} : x > 0\}$ **(b)** $\mathbf{Q}$

Cauchy / DD

6. Prove that a Cauchy sequence that has a convergent subsequence is itself convergent.

seq / FM

7. **Theorem.** If $a_n \in \mathbf{R}$ and $c \in \mathbf{R}$, and $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} c \cdot a_n = c \cdot a$.

Scratchwork: We want to show that, given $\epsilon > 0$, we can find an N large enough that, for all $n > N$, $|c \cdot a_n - c \cdot a| < \epsilon$. At the end of the proof, we'll need to show that $|c \cdot a_n - c \cdot a| = |c| \cdot |a_n - a| < \epsilon$, which will hold if $|a_n - a| < \epsilon/|c|$. Okay, ready for the proof.

Proof. We will show that, if the real sequence $a_n \to a$, then $c \cdot a_n \to c \cdot a$, for any $c \in \mathbf{R}$. We may assume that $c \neq 0$, because when $c = 0$, **(a)** ____________. Given any $\epsilon > 0$, choose N so that $|a_n - a| < \epsilon/|c|$ for all $n > N$. We can do this because **(b)** ____________. Then, for any $n > N$, we have $|c \cdot a_n - c \cdot a| =$ **(c)** ______________________ $< \epsilon$, as desired.

bound / DD

8. Prove that for all sequences $\{a_n\} \subset \mathbf{R}$,

$$\liminf \{a_n\} \leq \limsup \{a_n\}.$$

seq-fn / DD

9. Graph the following functions, for $n = 1, 2, 3, 4, 5$:

$$f_n(x) = \begin{cases} 1 - nx & \text{if } 0 \leq x < 1/n \\ 0 & \text{if } 1/n \leq x \end{cases}$$

Is there a "limit function" as $n \to \infty$? If so, say what it is.

logic / DD

10. (Charlie Fromm) Lyrics for "All Star" by Smash Mouth are shown to the right. Diagram the statement (do all four parts): "All that glitters is gold; only shooting stars break the mold."

All Star

Smash Mouth

Somebody once told me the world is gonna roll me
I ain't the sharpest tool in the shed
She was looking kind of dumb with her finger and her thumb
In the shape of an "L" on her forehead

Well the years start coming and they don't stop coming
Fed to the rules and I hit the ground running
Didn't make sense not to live for fun
Your brain gets smart but your head gets dumb
So much to do, so much to see
So what's wrong with taking the back streets?
You'll never know if you don't go
You'll never shine if you don't glow

Hey now, you're an all-star, get your game on, go play
Hey now, you're a rock star, get the show on, get paid
And all that glitters is gold
Only shooting stars break the mold

Real Analysis

Warm-up problems

cpt / DD **1**. Show that $(-1, 0)$ is not compact, by finding an open cover with no finite subcover.

cpt / AJ **2**. **Proposition.** $S = \{0\} \cup \{1/n : n \in \mathbf{N}\} \subset \mathbf{R}$ is a compact set.

Proof. We will show that S is compact, by explicitly constructing a finite subcover from any open cover. Consider an arbitrary open cover $\cup\, G_\alpha$ of S. 0 must be in some open set G_α, because **(a)** __________, so call this open set G_{α_0}. Then there exists an open ball $B_r(0) \subset G_{\alpha_0}$, because **(b)** __________. Then for all $n > 1/r$, we have $1/n \in B_r(0)$, because **(c)** __________. So $\{0\} \cup \{1/n : n > 1/r\} \subset G_{\alpha_0}$. **(d)** (finish the proof).

Some things to prove

Cauchy / FM **3**. **Theorem.** $\mathbf{R}$ is complete. (Prove this)
Hint: One method is to use the results of Page 11P # 7 and Page 12 # 6 and argue (using an $\epsilon - N$ proof) that every sequence converges to the same limit as its convergent subsequence.

Cauchy / AJ **4**. **Corollary.** $\mathbf{R}^n$ is complete. (We use "Corollary" for a result that follows directly from an existing Proposition or Theorem.) Prove this.
Hint: Repeat the same argument as above, for each coordinate.

sets / AJ **5**. **Proposition.** An infinite subset of a compact metric space has at least one accumulation point.

Proof. We will prove this by contradiction, by assuming that the subset has no accumulation point, and showing that the subset must be finite. Suppose that $A \subset E$ is an infinite set with no accumulation point. Then for each $p \in E$, there exists an r_p such that $B_{r_p}(p)$ contains only finitely many points of A, because **(a)** __________.

Then $\displaystyle\bigcup_{p \in E} B_{r_p}(p)$ is an open cover of E, because **(b)** __________.

Since E is compact, there is a finite subcover $B_{r_1}(p_1) \cup \cdots \cup B_{r_k}(p_k)$ that covers E.

(c) Finish the proof.

cpt / DD **6**. **Theorem.** Let K and S be subsets of a metric space E. Suppose $K \subset S \subset E$. Then K is compact in S if and only if K is compact in E.

Proof. ($\Longleftarrow$) Suppose that K is compact in E. We will show that K is also compact in S, by showing that every open cover of K in S has a finite subcover. Take any open cover $\{U_\alpha\}$ of K in S. By the previous Theorem, for each α, $U_\alpha = \widetilde{U}_\alpha$ for a set $\widetilde{U}_\alpha \cap E$ that is open in E. Then $\{\widetilde{U}_\alpha\}$ gives an open cover of S in E. Since K is compact in E, there exists a finite subcover of K in E, so …

(a) Finish the proof of this direction of implication.

(b) Prove the other direction of implication.

Real Analysis

Don't just read math; fight it! – Paul Halmos

Heine-Borel Theorem. The following are equivalent, for a set $S \subset \mathbf{R}^n$:

(1) Every sequence in S has a subsequence converging to a point of S.

(2) S is closed and bounded.

(3) S is compact: every open cover has a finite subcover.

cpt / FM **1**. *Proof.* We will prove that (3) $\implies$ (2) $\implies$ (1) $\implies$ (3). This will prove that all three criteria are equivalent, because ________________________.

cpt / FM **2**. (3) $\implies$ (2). We will prove the contrapositive: if S is not closed or not bounded, then there is some open cover that has no finite subcover. (Part 1) Suppose that S is not closed. Then some convergent sequence of points from S converges to a point a that is not in S, because **(a)** ____________. Then a is an accumulation point for S, because **(b)** ____________. Then the open cover $\{x : |x - a| > 1/n : n \in \mathbf{N}\}$ has no finite subcover, because **(c)** ____________. (Part 2) Suppose that S is not bounded. Then the open cover $\{x : |x| > n : n \in \mathbf{N}\}$ has no finite subcover, because **(d)** ____________.

cpt / FM **3**. (2) $\implies$ (1). We will show that, if S is closed and bounded, then every sequence in S has a subsequence converging to a point of S. Take any sequence in $S \subset \mathbf{R}^n$. First, just look at the first of the n components of each point. Since S is bounded, the sequence of the first components is bounded, because **(a)** ____________. So for some subsequence, the first components converge, because **(b)** ____________. Similarly, for a further subsequence, the second components converge. Eventually, for some further subsequence, each of the components converge, because **(c)** ____________. The limit point is in S, because **(d)** ____________.

cpt / FM **4**. (1) $\implies$ (3). We will show that, if every sequence in S has a subsequence converging to a point in S, then every open cover of S has a finite subcover. First, we will show that every open cover has a *countable* subcover, and then we will show, using a proof by contradiction, that it must actually have a finite subcover. Given an open cover $\{\mathcal{G}_\alpha\}$ of S, we will construct a countable subcover. Every point of S lies in a ball of rational radius about a rational point, because **(a)** ____________. Each of these countably many balls is contained in some $\mathcal{G}_{\alpha_0}$, because **(b)** ____________. So a countable open cover $\{V_i\}$ of S is given by **(c)** ____________.

cpt / FM **5**. Now suppose that $\{V_i\}$ has no finite subcover. Choose $x_1 \in S \backslash V_1$. Choose $x_2 \in S \setminus (V_1 \cup V_2)$. Continue, choosing x_n in $S \setminus \bigcup_{i=1}^{n} V_i$, which is always possible, because **(a)** ____________. For each value of i, there are only finitely many x_n, with $n < i$, contained in V_i, because **(b)** ____________. Because **(c)** ____________, the sequence x_n has a subsequence converging to some $x \in S$, with $x \in V_i$ for some i. Thus infinitely many x_n are contained in V_i, which is a contradiction because **(d)** ____________. Thus every open cover of S has a finite subcover, as desired.

cpt / DD **6**. Prove that a compact subset of a metric space is bounded. *Hint*: use the finite cover.

The following three results follow from the Proposition in Page 11P # 7.

cpt / DD **7**. **Corollary 1.** Every sequence of points in a compact metric space has a convergent subsequence.

Proof. We will prove the result by explicitly constructing the convergent subsequence. Let p_n be a sequence in a compact metric space E. There are two cases, depending on whether the number of different points in the sequence is finite or infinite. If $\{p_1, p_2, p_3, \ldots\}$ is a finite set, then there must be some point p that occurs infinitely many times.

(a) Finish the proof of the finite case.

On the other hand, if $\{p_1, p_2, p_3, \ldots\}$ is an infinite set, then it must have at least one accumulation point p, because **(b)** ________________. Choose n_1 so that $p_{n_1} \in B_1(p)$. Choose $n_2 > n_1$ so that $p_{n_2} \in B_{1/2}(p)$, and so on. Note that for each k, there are infinitely many points of the sequence in $B_{1/k}(p)$, because **(c)** ________________. **(d)** Finish the argument.

cpt / DD **8**. **Corollary 2.** A compact metric space is complete.

Proof. We will show that a compact metric space is complete, by showing that every Cauchy sequence converges to a point in the space.
Hint: combine Corollary 1 with Page 13 # 6.

cpt / DD **9**. **Corollary 3.** A compact subset of a metric space is closed.

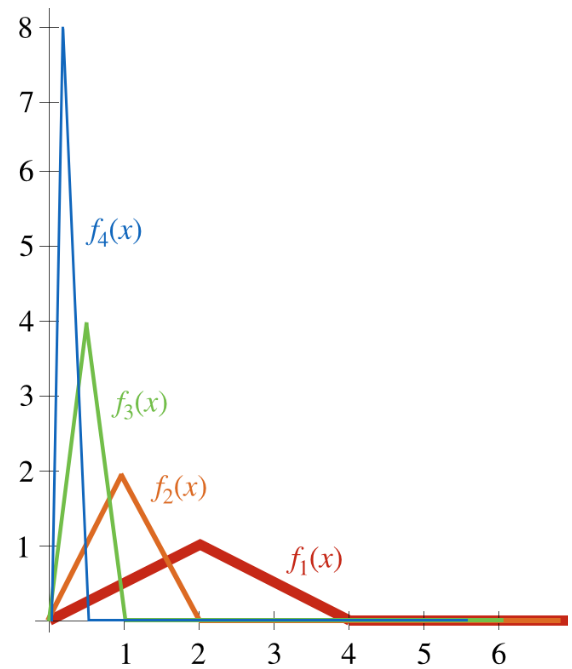
Proof. We will prove this by showing that every convergent sequence of points from a compact subset converges to a point in the compact subset (using the sequence definition of a closed set).
Hint: Combine Corollary 2 with Page 12 # 5 and Page 12 # 7.

seq-fn / DD **10**. The first four terms of a sequence f_n of functions $R_{\geq 0} \to R_{\geq 0}$ are shown to the right. The piecewise equations for the functions are given below, but the idea of this problem is to think about the picture.

(a) Is there some limit function as $n \to \infty$? If so, say what it is.

(b) What is the integral of f_n for each n?

$$f_n(x) = \begin{cases} 2^{2n-3}x & \text{if } \quad 0 \leq x \leq 2^{2-n} \\ 2^n - 2^{2n-3}x & \text{if } 2^{2-n} \leq x \leq 2^{3-n} \\ 0 & \text{if } 2^{3-n} \leq x \end{cases}$$



Real Analysis

Review for Midterm 1 – optional problems

vocab / FM **1.** Write the definition of each term, as a full sentence: **(a)** subset **(b)** countable **(c)** supremum **(d)** minimum **(e)** open **(f)** closed **(g)** metric space **(h)** bounded **(i)** limit **(j)** converge **(k)** subsequence **(l)** lim inf **(m)** monotone **(n)** accumulation point **(o)** isolated point **(p)** boundary **(q)** interior **(r)** closure **(s)** Cauchy **(t)** complete **(u)** cover **(v)** compact **(w)** continuous **(x)** uniformly continuous **(y)** inverse image

vocab / FM **2.** For each term in problem 1, write down a result (Theorem, etc.) that uses it.

sets / FM **3.** Consider the statement "Every **finite union** of **open** sets is **open**." For each entry in the following table, replace "finite union" and "open" with the other words as indicated, and decide whether the resulting statement is True or False. If it is false, give a counterexample.

set property	finite $\cup$	countable $\cup$	arbitrary $\cup$	finite $\cap$	countable $\cap$	arbitrary $\cap$
open						
closed						
countable						
uncountable						
bounded						
compact						

sets / FM **4.** Let A be a subset of a metric space E.

(a) Define the *boundary* of A.

(b) Define what it means for A to be *open*.

(c) Prove that A is open if and only if A contains none of its boundary, i.e. $A \cap \partial A = \emptyset$.

sets / DD **5.** Say whether the following are True or False; if false, give a counterexample.

(a) Every convergent sequence is Cauchy.

(b) Every bounded sequence in $\mathbf{R}$ has a convergent subsequence.

(c) Make up another statement of this kind, and then repeat the exercise.

seq / FM **6.** Prove that if $a_n \leq b_n \leq c_n$ for each $n \in \mathbf{N}$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

uni-con / DD **7.** (Peem Lerdputtipongporn) The Theorem in Page 13 # 4 says: A continuous function on a compact set is uniformly continuous.

(a) Write the converse. (*Hint*: refer to the precise language in the Theorem statement.)

(b) For the converse, prove it or find a counterexample.

/ DD **8.** In the topic dependence map on Page ii, circle the topics in which you are confident.

/ DD **9.** **Make a list of problems** (on any page or topic) that you would like to discuss in class.

Sequences of functions. Let $f_n : E \to E'$ and let $p \in E$. The sequence $f_1, f_2, \ldots$ *converges at* p if $f_1(p), f_2(p), \ldots$ converges as a sequence of points in E'. The sequence $f_1, f_2, \ldots$ *converges* (on E) if it converges at every point in E. If $f_1, f_2, \ldots$ converges, we define the *limit function* to be $f(p) = \lim_{n\to\infty} f_n(p)$ for each $p \in E$.

seq-fn / DD **1**. For each of the following sequences of functions, explain why the sequence converges, and give the limit function f:

(a) $f_n : \mathbf{R}^+ \to \mathbf{R}^+$ from Page 13 # 9

(b) $f_n : \mathbf{R}^+ \to \mathbf{R}^+$ from Page 15 # 10

Connected sets. A metric space E is *connected* if the only subsets of E that are both open and closed are E and $\emptyset$. A subset $S \subset E$ is connected if it is connected when considered as a metric space.

conn / DD **2**. Write a definition of the word "connected" in the non-mathematical sense (for, say, a subset of R^2). Then explain why the above definition is equivalent to that usual meaning.

conn / DD **3**. Give an example of a subset that is connected, and a subset that is not connected, in:

(a) $\mathbf{R}^2$ with the Euclidean metric, **(b)** $\mathbf{R}^2$ with the discrete metric.

Spaces of functions. Let $\mathcal{C}(E, E')$ be the set of all continuous functions from the metric space E to the metric space E'. Here $\mathcal{C}$ stands for *continuous*. It is a metric space under the distance metric

$$D(f, g) = \max\{d'(f(p), g(p)) : p \in E\},$$

where as usual d' is the distance metric in E'. Notice that for this metric to be well defined, the maximum must exist and be finite.

sp-fn / DD **4**. For each of the following, $f, g : \mathbf{R} \to \mathbf{R}$, find $D(f, g)$. *Hint*: draw a picture

(a) $f(x) = \sin(x), g(x) = 0$ **(b)** $f(x) = x + \sin(x), g(x) = x$

You will prove that D is a metric in your next homework.

The Heine-Borel Theorem says that, in $\mathbf{R}^n$ with the Euclidean metric, *compact* is equivalent to *closed and bounded*. The following two problems further explore the connection between compactness and closed-and-boundedness, for a general metric space.

cpt / DD **5**. Consider the metric space consisting of the integers $\mathbf{Z}$, with the discrete metric. Show that this metric space is closed and bounded, but not compact.

cpt / DD **6**. Prove that any compact subset of a metric space is closed and bounded. *Hint*: put together several previous results.

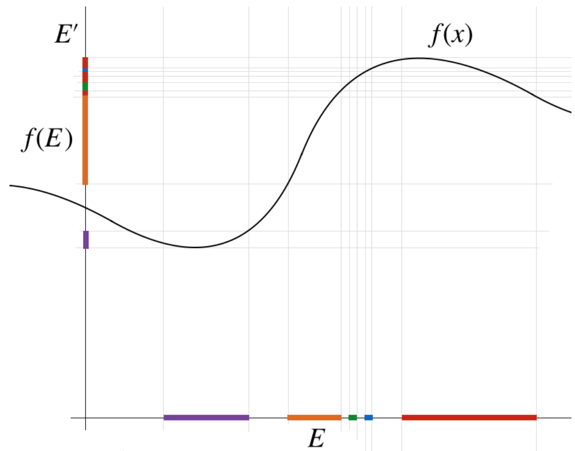
Other bases. The *decimal expansion* of a number between 0 and 1 tells, in each decimal place, the number of $1/10^1$s, $1/10^2$s, $1/10^3$s, etc. needed to sum to the number, using digits between 0 and 9. The *binary expansion* and *ternary expansion* do the same, with the number of powers of 1/2 and 1/3, respectively, using digits $\{0, 1\}$ and $\{0, 1, 2\}$, respectively.

Cantor / DD **7**. Write 3/8, 7/16 and 1/3 in binary. Write 5/9, 8/27 and 1/2 in ternary.

Real Analysis

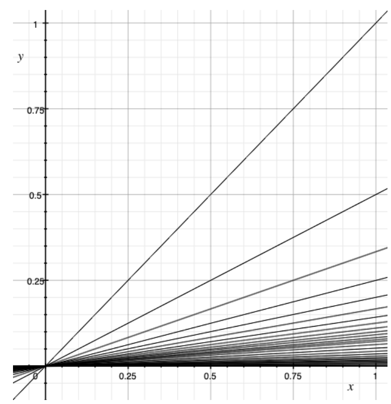
The following result is very important for finding maxima and minima in calculus.

cpt / DD **1**. **Theorem.** The continuous image of a compact set is compact.



Proof. Let $f : E \to E'$ be a continuous function, and let E be compact. We will show that its image $f(E)$ is compact by showing that, given any open cover of $f(E)$, we can construct a finite subcover. Let $\{\mathcal{U}_\alpha\}$ be an open cover of $f(E)$. Since each set $\mathcal{U}$ in $\{\mathcal{U}_\alpha\}$ is open in E', each $f^{-1}(\mathcal{U})$ is open in E, because **(a)** ______________. Consider the set $S = \{f^{-1}(\mathcal{U}) : \mathcal{U} \in \mathcal{U}_\alpha\}$. We claim that S is an open cover of E. To see this, take any $p \in E$. Then $f(p) \in \mathcal{U}$ for some $\mathcal{U} \in \mathcal{U}_\alpha$, so $p \in f^{-1}(\mathcal{U})$, so S covers E. Since E is compact, there is a finite subcover of S covering E, i.e. $E \subset f^{-1}(\mathcal{U}_1) \cup \cdots \cup f^{-1}(\mathcal{U}_k)$. Thus $f(E) \subset \mathcal{U}_1 \cup \cdots \cup U_k$. So for any open cover $\mathcal{U}_\alpha$ of $f(E)$, we have constructed a finite subcover **(b)** ______________, so $f(E)$ is compact, as desired.

seq-fn / DD **2**. Say whether the following sequences (also used in the following problems) converge, and if so, to what limit function:

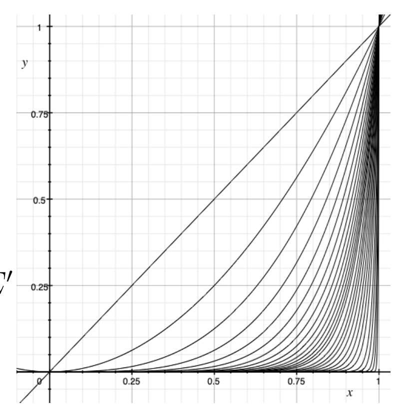


(a) $f_n : \mathbf{R} \to \mathbf{R}$ given by $f_n(x) = x/n$.

(b) $g_n : [0, 1] \to [0, 1]$ given by $g_n(x) = x^n$.

(c) $h_n : \mathbf{R} \to \mathbf{R}$ given by

$$h_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{n-1}\, x^{2n-1}}{(2n-1)!}.$$



It is a bit disturbing that, in some cases that we have seen, a limit of continuous functions is not continuous. The problem is that although each $f_n \to f$ at each point, for some functions, some points take longer to converge than others. We can express this difference precisely, as the difference between *pointwise convergence* (now) and *uniform convergence* (later):

Pointwise convergence. A sequence of functions $f_n : E \to E'$ *converges pointwise* to f on E if, for every $x \in E$, and given any $\epsilon > 0$, there exists an N such that

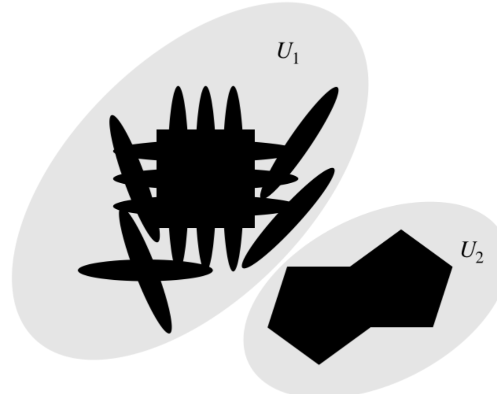
$$n > N \implies d'(f_n(x), f(x)) < \epsilon.$$

Notice that in this definition, N depends on both x and ϵ.

seq-fn / DD **3**. For the given x and ϵ, find N such that $n > N \implies |f_n(x) - f(x)| < \epsilon$.

(a) $x = 1/2, \epsilon = 0.01$ **(b)** $x = 0.9, \epsilon = 0.01$

seq-fn / DD **4**. Repeat problem 3 for $g_n(x)$ with $g(x)$.



Sets that are connected and not connected. A subset S of a metric space E is *not connected* if it can be separated by two disjoint open sets U_1 and U_2 into two nonempty pieces $S \cap U_1$ and $S \cap U_2$, such that $S = (S \cap U_1) \cup (S \cap U_2)$. Otherwise, it is *connected*.

conn / FM **5**. Prove, in two sentences, that any subset of $\mathbf{R}$ that contains two distinct points a and b, and does not contain all of the points between a and b, is not connected.

conn / DD **6**. Prove that this definition of *connected* is equivalent to the one on the previous page.

conn / FM **7**. Prove that an interval of real numbers is connected (perhaps by contradiction).

conn / FM **8**. Give a counterexample to the following statement: If $f : \mathbf{R} \to \mathbf{R}$ is continuous and S is connected, then $f^{-1}(S)$ is connected.

Cantor / DD **9**. Show that the set of all possible numbers such as $0.010100011101010\ldots$, with integer part 0 and decimal digits 0 and 1, is uncountable. *Hint*: binary.

The Cantor set. Start with the closed unit interval $[0, 1]$. Remove the open middle third $(1/3, 2/3)$, leaving two closed intervals of length $1/3$. Remove the open middle third of each of these, leaving four closed intervals of length $1/9$. Continue. At the n^{th} step, you have a set S_n consisting of 2^n closed intervals each of length $1/3^n$. Let $\mathcal{C} = \bigcap S_n$.

Cantor / DD **10**. Draw S_n for $n = 0, 1, 2, 3, 4$. S_0 should take up the entire width of your page.

Cantor / DD **11**. Find the total length of S_n as a function of n, and the total length of $\mathcal{C}$.

metric / DD **12**. Let $\mathcal{C}(E, E')$ be the set of all continuous functions from the metric space E to the metric space E'. Prove that the distance function $D(f, g) = \max\{d'(f(p), g(p)) : p \in E\}$ is a metric, assuming that the maximum exists and is finite.

Real Analysis

seq-fn / FM

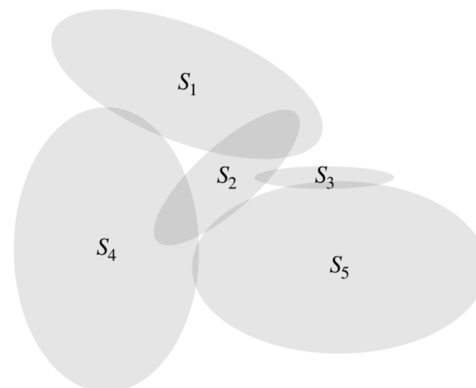
1. Consider continuous functions $f_n : [0,1] \to \mathbf{R}$ such that, for each $x \in [0,1]$,

$$0 \le f_1(x) \le f_2(x) \le f_3(x) \le \cdots ,$$

converging pointwise to $f(x)$. Must f be continuous? Prove it or give a counterexample.

conn / DD

2. **Proposition.** Let $\{S_i\}$ be a collection of connected subsets of a metric space E. Suppose that there exists k such that, for all i, $S_i \cap S_k \neq \emptyset$. Then $\bigcup_i S_i$ is connected.



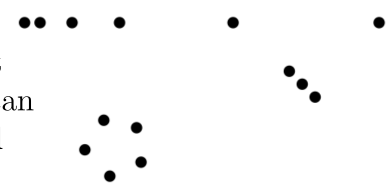
(a) The picture shows an example in the case $E = \mathbf{R}^2$. What is k in this case?

(b) *Proof.* We will show that $\cup S_i$ is connected using a proof by contradiction. Suppose that $\cup S_i$ is not connected. Then $\cup S_i$ is contained in the union of two disjoint open sets A, B. We will show that A or B must be empty. (Do this.)

Hint: write $S_k = (S_k \cap A) \cup (S_k \cap B)$ and use the fact that S_k is connected.

conn / FM

3. **Theorem.** The continuous image of a connected set is connected.
More formally: Let $f : E \to E'$ be continuous. If E is connected, then $f(E)$ is connected. (Prove this.)
Hint: Using the Page 18 definition, a one-sentence proof by contradiction is possible.



Totally disconnected. A set S is *totally disconnected* if it has at least two points, and if for every pair of distinct points $p_1, p_2 \in S$, the set S can be separated by disjoint open sets U_1 and U_2 into two pieces $S \cap U_1$ and $S \cap U_2$ containing p_1 and p_2, respectively.

disc / FM

4. Show that the following sets are totally disconnected: **(a) Z** **(b) Q**

seq / DD

5. **Theorem.** A bounded, monotone sequence of real numbers is convergent.

We will prove this for an increasing sequence; the proof for a decreasing sequence is similar.

Proof. Let $a = \sup\{a_k\}$. We will show that $\lim_{k\to\infty} a_k = a$. Given any $\epsilon > 0$, we need to show that there exists N such that, for any $n > N$, $|a - a_n| < \epsilon$. Now $a - \epsilon$ is too small to be a lower bound for $\{a_k\}$, because **(a)** _______________. So there exists N such that $a_N > a - \epsilon$, because **(b)** _______________. So we know that $a_i > a - \epsilon$ for all $i > N$, because **(c)** _______________. Thus $a - \epsilon < a_i < a + \epsilon$ for all $i > N$. **(d)** Finish the proof.

seq / DD

6. Consider $\mathbf{R}$ with the metric $d(p,q) = \begin{cases} 0 \text{ if } p = q \\ 1 \text{ if } p \neq q. \end{cases}$ Does the sequence $a_n = 1/n$ have a limit in this metric space? (In other words, does the previous Theorem apply?)

Real Analysis

seq-fn / DD **7**. Give an example of a continuous, *bounded* function on $(0,1)$ that is not uniformly continuous. *Hint*: One option is a "sawtooth" where the teeth get narrower and steeper.

Cantor / DD **8**. (Rick Parris) Consider again the Cantor set $\mathcal{C}$. Show that $1/4 \in \mathcal{C}$, in two ways:

(a) Using the decimal representation of points in $\mathcal{C}$, and

(b) using the self-similarity, or *fractal structure*, of $\mathcal{C}$, to show that $1/4$ is never removed.

Cantor / FM **9**. Explain why the Cantor set is totally disconnected.

diff / DD **10**. Consider a differentiable function $f : \mathbf{R} \to \mathbf{R}$, as in single-variable calculus. Write the equation of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$.

IVT / DD **11**. We (usually) do calculus in $\mathbf{R}^n$. You may wonder, why not on some other metric space?

(a) Write down the the Intermediate Value Theorem, and draw a picture to illustrate it.

(b) Explain why the Intermediate Value Theorem does not hold for a function $f : \mathbf{Q} \to \mathbf{R}$.

Real Analysis

disc / FM **1**. Give an example of a totally disconnected set $S \subset [0,1]$ whose closure is $[0,1]$.

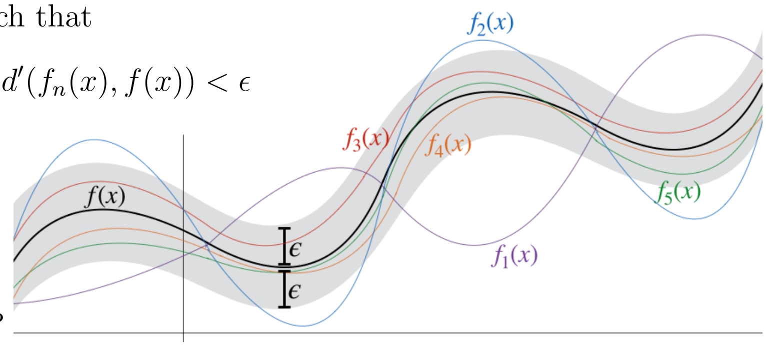
Intermediate Value Theorem: Let $f : E \to \mathbf{R}$ be continuous, let E be connected, and let $p_1, p_2 \in E$. Then f attains all values between $f(p_1)$ and $f(p_2)$.

IVT / DD **2**. Prove the Intermediate Value Theorem. *Hint*: put together previous results.

Uniform convergence. A sequence of functions $f_n : E \to E'$ converges *uniformly* to f on E if, given any $\epsilon > 0$, there exists an N such that

$$n > N \implies d'(f_n(x), f(x)) < \epsilon$$

for all $x \in E$.



seq-fn / DD **3**. Explain the difference between the definition of pointwise convergence and the definition of uniform convergence. In Page 18 # 2, which functions converge uniformly?

We saw in Page 18 # 2 that some limits of continuous functions are not continuous. For the limit to be continuous, *uniform* convergence is exactly what we need.

seq-fn / FM **4**. **Theorem.** *A uniform limit of continuous functions is continuous.*

Proof. We will show that, for a uniformly convergent sequence of functions $f_1, f_2, \ldots$, with $f_n : E \to E'$ for each n, $\lim_{n\to\infty} f_n(x) = f(x)$ is continuous. More precisely, we will show that, given any $\epsilon > 0$, we can find δ such that **(a)** ________. The idea is that because the sequence of functions converges uniformly, we can handle all points x near any particular point p by looking at one f_n (specifically f_{N+1}) that is uniformly near the limit function f.

Given any $\epsilon > 0$, choose N such that for all $x \in E$,

$$n > N \implies d(f_n(x), f(x)) < \epsilon/3. \tag{1}$$

We can find such an N because **(b)** ________.

We know that f_{N+1} is continuous because **(c)** ________. Since f_{N+1} is continuous, given any $p \in E$, we can choose $\delta > 0$ such that

$$d(x,p) < \delta \implies d'(f_{N+1}(x) - f_{N+1}(p)) < \epsilon/3. \tag{2}$$

We can find such a δ because **(d)** ________. So if $d(x,p) < \delta$, we have

$$\begin{aligned} d'(f(x), f(p)) &\le d'(f(x), f_{N+1}(x)) + d'(f_{N+1}(x), f_{N+1}(p)) + d'(f_{N+1}(p), f(p)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon, \end{aligned}$$

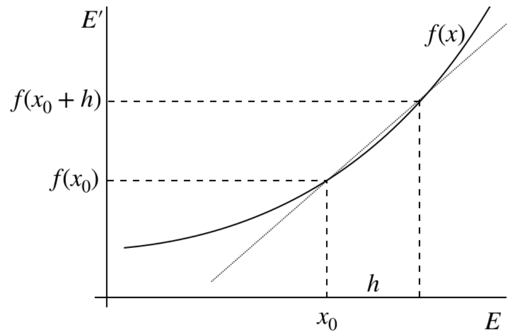
as desired. Here the inequality in the first line is true by **(e)** ________. On the right hand side, the first term is less than $\epsilon/3$ by equation **(f)** ________, the second term is less than $\epsilon/3$ by equation **(g)** ________, and the third term is less than $\epsilon/3$ by equation **(h)** ________.

cpt / DD **5**. **Corollary** to Page 18 # 1. If $f : E \to E'$ is continuous, and E is compact, then f is bounded. (Prove this.)

cpt / DD **6**. (Continuation) Show that it is necessary for E to be compact, by giving an example of an *unbounded* continuous function on a non-compact metric space.
If your example is a function from $\mathbf{R}$ to $\mathbf{R}$, for extra style points give one example where the space is not closed, and one where the space is not bounded.

diff / DD **7**. In single-variable calculus, for a function $f : \mathbf{R} \to \mathbf{R}$, to find $f'(x)$ at some point x_0, you took h to be a small number (positive or negative) and took the limit:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$



(a) Explain why this limit gives the instantaneous slope of f at x_0.

(b) Compute the above limit (meaning: write it out, expand, do arithmetic, take the limit of the expression as $h \to 0$) for $f(x) = x^2$ and $x_0 = 1$. Is the answer what you expected?

Cantor / DD **8**. Refer to the construction of the Cantor set $\mathcal{C}$ given on page 18.

(a) Explain why the set S_1 consists of all numbers between 0 and 1 that have either a 0 or a 2 (but not a 1) as their first digit after the "decimal" point, when expressed in ternary.

(b) Explain why $\mathcal{C}$ consists of all of the possible numbers expressed in ternary as infinite strings of 0s and 2s, e.g. $0.020200202202222\ldots$.

logic / DD **9**. Diagram the statement "What doesn't kill you makes you stronger."

Real Analysis

Warm-up problems

Inspired by bold claims made on Midterm 1: Give a counterexample to each of the following.

seq / DD **1**. If a sequence has an accumulation point, then it is convergent.

seq / DD **2**. A set (or sequence) has at most one accumulation point.

seq / DD **3**. A set (or sequence) has at most finitely many accumulation points.

bound / DD **4**. Let S be a set, with a metric d. If S is bounded, then given any $r > 0$, S can be covered by finitely many balls of radius r.

Some things to prove

IVT / FM **5**. Prove that your height in inches once equaled your weight in pounds.

open / DD **6**. Prove that any open subset of $\mathbf{R}$ is a union of disjoint open intervals.

cpt / FM **7**. **Proposition.** Any closed subset of a compact set is compact.

Proof. Let X be a compact subset of a metric space E, and let S be a closed subset of S. We will show that S is compact, by showing that **(a)** ______________. Let $\{G_\alpha\}$ be an open cover of S. We also know that S^C is open, because **(b)** ______________. Then the union $\{G_\alpha\} \cup S^C$ give an open cover of **(c)** ______________. Since X is compact... **(d)** Complete the proof.

Real Analysis

For a sequence of points, we wanted a way to discuss convergence without knowing what the limit is. This notion is "Cauchy" − for any $\epsilon > 0$, there exists N such that $n, m > N \implies d(a_n, a_m) < \epsilon$. We would like to have an analogous way to talk about a sequence of *functions* converging, without knowing what the limiting function is. This notion is *uniformly Cauchy,* combining the notion of the sequence of functions converging uniformly, with the notion of the function values being a Cauchy sequence at each point. The definition of *uniformly Cauchy* is in the beginning of the proof below.

Cauchy / DD **1. Theorem.** *A sequence of functions mapping to a complete metric space is uniformly convergent if and only if the sequence of functions is uniformly Cauchy.*

Proof. We will prove that, for a sequence of functions $f_n : E \to E'$, if E' is complete, then f_n is uniformly convergent if and only if, for any $\epsilon > 0$, there exists N such that

$$n, m > N \implies d'(f_n(p), f_m(p)) < \epsilon \text{ for all } p \in E,$$

or in other words, the sequence of f_n is uniformly Cauchy.

Proof. ($\Rightarrow$) We will assume that $f_n \to f$ uniformly, and show that f_n is uniformly Cauchy.

Suppose that f_n converges uniformly to f. Then, given $\epsilon > 0$, there exists $N > 0$ such that $d'(f(x), f_n(x)) < \epsilon/2$ for all $n > N$ and all $x \in E$, because **(a)** ____________________.
So for all $n, m > N$, we have

$$\begin{aligned} d'(f_n(x), f_m(x)) &\leq d'(f_n(x), f(x)) + d'(f(x), f_m(x)) \\ &< \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

as desired. Here the inequality in the first line is true by **(b)** ____________________.
Each of the terms on the right hand side of the first line are less than $\epsilon/2$ because **(c)** ____________.

($\Leftarrow$) We will assume that f_n is uniformly Cauchy, and show that $f_n \to f$ uniformly.

Suppose that the sequence f_n is uniformly Cauchy. Then for every $x \in E$, and for every n, $\{f_n(x)\}$ is a Cauchy sequence in E', because **(d)** ____________. Since E' is complete, $\{f_n(x)\}$ converges to a point in E' because **(e)** ____________________, and we will call this point $f(x)$. This shows that $f_n \to f$ pointwise. Now we need to show that $f_n \to f$ uniformly.

Given any $\epsilon > 0$, choose N such that

$$n, m > N \implies d'(f_n(x), f_m(x)) < \epsilon/2 \text{ for all } x \in E. \tag{1}$$

We can find such an N because **(f)** ____________________. We want to show that, for any $n > N$ and any $x \in E$, $d'(f_n(x), f(x)) < \epsilon$.

Fix a particular such choice of n and x. Now look at the ball $B_{\epsilon/2}(f_n(x))$. The sequence $f_1(x), f_2(x), \ldots$ eventually enters this ball and stays within it, by **(g)** ____________. So $f(x) \in \overline{B_{\epsilon/2}(f_n(x))}$, because **(h)** ____________________. Here we use the closure of the ball because **(i)** ____________________. Thus,

$$d'(f_n(x), f(x)) \leq \epsilon/2 < \epsilon,$$

as desired. Here the first inequality is true because **(j)** ____________________.

seq-fn / FM **2**. Let $f_n(x) = x/n$.

(a) Prove that $f_n(x) \to f(x) = 0$ uniformly on $[0, 1]$.

(b) Does $f_n(x) \to 0$ uniformly on $\mathbf{R}$?

Differentiability. In our continuing quest to put all of calculus on a rigorous mathematical basis, we will now study differentiability. Let $\mathcal{U} \subset \mathbf{R}$ be an open set, let $f : \mathcal{U} \to \mathbf{R}$, and let $x_0 \in \mathcal{U}$. We say that f is *differentiable at* x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{exists.}$$

We denote this limit by $f'(x_0)$ and call it *the derivative of* f *at* x_0.

diff / DD **3**. Answer these questions about the definition of differentiability.

(a) Explain why the above definition is equivalent to the "$x + h$" definition in Page 20 # 8 (when $\mathcal{U} = \mathbf{R}$).

(b) Why must the domain and range of $\mathbf{R}$ be real numbers?

(c) Why must the domain $\mathcal{U}$ be open?

bound / DD **4**. **Proposition.** A nonempty closed subset of $\mathbf{R}$ has a greatest element if it is bounded from above, and has a least element if it is bounded from below.

Proof. We will show that a nonempty closed subset S of $\mathbf{R}$ has a greatest element if it is bounded from above; the proof of the second part is similar.

Let $a = \sup S$. If $a \in S$, we are done, because **(a)** __________________________.

If $a \notin S$, then $a \in S^C$. Since S^C is open, there exists $r > 0$ such that $B_r(a) \subset S^C$. Thus, no element of S is greater than $a - r$. **(b)** Finish the proof.

Cantor / FM **5**. Prove that there is a bijection between elements of the Cantor set (*Hint*: ternary) and elements of the set $[0, 1]$ (*Hint*: binary). Are you surprised?!

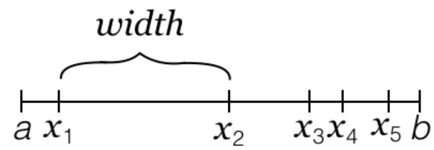
cont / FM **6**. A real-valued sequence $\{x_n\}$ can be thought of as a function $f : \mathbf{N} \to \mathbf{R}$. Prove or give a counterexample: Every sequence of real numbers is a continuous function.

cpt / FM **7**. (Candice Corvetti) Consider the statement: If $f : \mathbf{R} \to \mathbf{R}$ is continuous, then the image under f of every compact set is compact.

(a) Is the statement true or false?

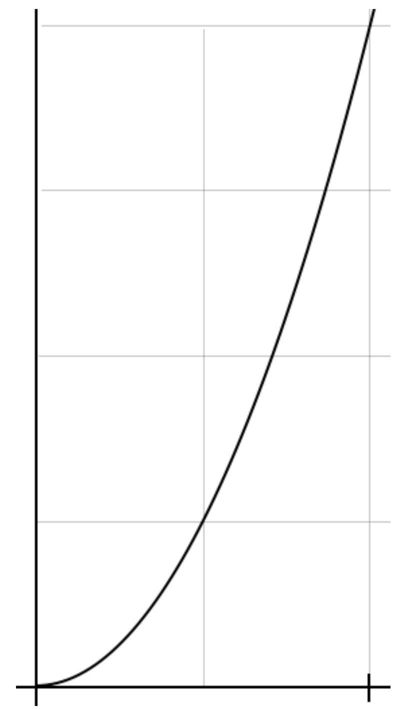
(b) Write the converse of the statement. Then prove it or give a counterexample.

Real Analysis



Partitions. Let $a, b \in \mathbf{R}$ with $a < b$. A *partition* of $[a, b]$ is given by a (finite) sequence $x_0, x_1, x_2, \ldots, x_N$ such that $a = x_0 < x_1 < \cdots < x_N = b$. The *width of the partition* is $\max\{x_i - x_{i-1} : i = 1, \ldots, N\}$.

part / DD **8**. If you partition $[a, b]$ using the partition $x_0, x_1, x_2, \ldots, x_N$, how many subintervals of the form $[x_{i-1}, x_i]$ do you get? If they are equally spaced, what is the length of each?



Riemann sums. A *Riemann sum* for f on $[a, b]$, corresponding to the partition $x_0, x_1, x_2, \ldots, x_N$ of $[a, b]$, is

$$\sum_{i=1}^{N} f(x_i') \cdot (x_i - x_{i-1}),$$

where $x_i' \in [x_{i-1}, x_i]$ is a representative point in each subinterval.

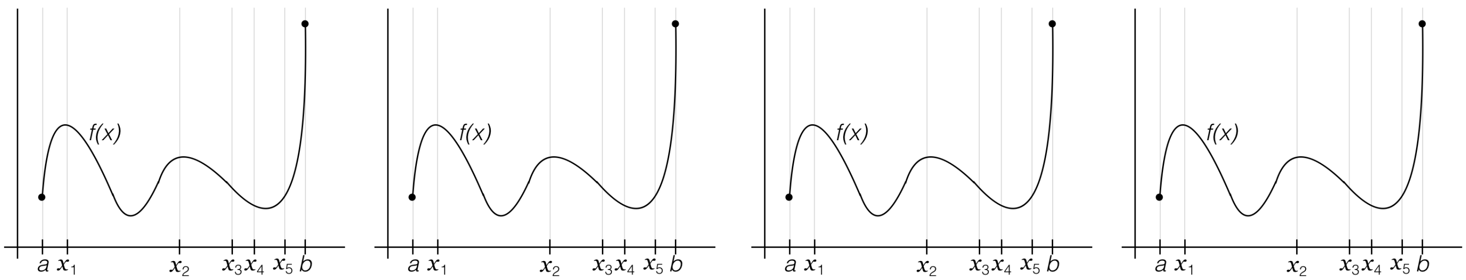
Riem / DD **9**. Compute the Riemann sum for $f(x) = x^2$ on the interval $[0, 2]$, with $N = 4$. Use a partition, and a representative point in each interval, *that no one else in the class will think of.* You are welcome to use a calculator for the computations. In the picture, draw in your partition, including the rectangles whose areas sum up to your Riemann sum value.

Real Analysis

Riem / DD **1**. On the graphs of $f(x)$ defined on $[a,b]$ below, draw in rectangles for the Riemann sum corresponding to each of the following choices for the representative point x_i' in $[x_{i-1}, x_i]$:

(a) $f(x_i') = \max\{f(x) : x \in [x_{i-1}, x_i]\}$ **(b)** $f(x_i') = \min\{f(x) : x \in [x_{i-1}, x_i]\}$

(c) $x_i' = x_{i-1}$ **(d)** $x_i' = x_i$ **(e)** What are the sums in parts (c) and (d) called?



Riem / DD **2**. Explain why, as the width of the associated partition decreases, the Riemann sum for f on $[a,b]$ approaches the area under $f(x)$ over the interval $[a,b]$.

Integrability. Let $a, b \in \mathbf{R}$ with $a < b$, and let $f : [a,b] \to \mathbf{R}$. Then f is *Riemann integrable* on $[a,b]$ if there exists a number A such that, for any $\epsilon > 0$, there exists δ such that, if we take *any* partition of width δ, and if we take S to be *any* Riemann sum value associated to such a partition, then $|A - S| < \epsilon$. In this case, we say that

$$A = \int_a^b f(x)\, dx$$

is the *Riemann integral* of f.

You can think of integrable functions from $\mathbf{R} \to \mathbf{R}$ as those that are bounded, and are not discontinuous everywhere on an interval.

Riem / DD **3**. Draw a picture of the function defined on $[0,3]$ by: $f(x) = \begin{cases} 0 \text{ if } x \in [0,1] \cup [2,3] \\ 1 \text{ if } x \in (1,2) \end{cases}$.

(a) Find a partition of $[0,3]$ of width $\leq 1/4$, and draw it on the interval $[0,3]$.

Find the value of each of the following Riemann sums, when the representative point x_i':

(b) is the left endpoint of each interval;

(c) yields the maximum value of f on its interval;

(d) yields the minimum value of f on its interval.

Refer to the definition of Riemann integrability, given above.

(e) For $\epsilon = 1/10$, can you find an A and a δ that satisfy the definition?

(f) What are the maximum and minimum values of a Riemann sum associated to a partition of width $1/4$? And what are the maximum and minimum values for width $1/10$?

(g) Is $f(x)$ Riemann integrable?

(h) Explain why, if the difference between the maximum and minimum Riemann sum values for a function f approaches 0 as the partition width approaches 0, then f is integrable.

MVT / FM **4. Proposition (finding critical points).** *If a function is differentiable at an interior minimum or maximum point, then its derivative is 0 there.*

This fact is the basis for finding maxima and minima in calculus. We will also need it to prove the Mean Value Theorem, which is likewise essential for calculus.

Proof. We will show that, for an open set $\mathcal{U} \subset \mathbf{R}$ and a function $f : \mathcal{U} \to \mathbf{R}$, if f attains a maximum or minimum at $x_0 \in \mathcal{U}$, and if f is differentiable at x_0, then $f'(x_0) = 0$. We will prove the result directly, showing that at a maximum or minimum, the limit is 0.

Since f is differentiable, we know that $\lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists.

Suppose that x_0 is a local minimum. For x near x_0, the numerator in the definition is non-negative, because **(a)** __________________________. If $t > x$, the denominator is positive, and thus $f'(x)$ is a limit of non-negative numbers, because **(b)** __________________________. For $t < x$, the denominator is negative, and thus similarly $f'(x)$ is a limit of non-positive numbers. Therefore, the only possibility is $f'(x) = 0$, because **(c)** __________________________. The proof when x_0 is a local maximum is similar.

diff / DD **5.** In this problem, we will justify the statement: *f is differentiable at x_0 if and only if it is well approximated by its tangent line there.*

(a) Write out the $\epsilon - \delta$ version of the statement $\lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$.
There should be a hypothesis involving ϵ, δ, x and x_0, and a conclusion that is an inequality with a difference on one side and an ϵ on the other; see the next step for a hint.

(b) Rearrange your conclusion inequality so that it becomes the following:

$$\left| f(x) - \Big(f(x_0) + f'(x_0)(x - x_0) \Big) \right| < \epsilon |x - x_0|.$$

(c) Explain how to interpret the above as saying that $f(x)$ is well approximated by its tangent line at x_0, when x is close to x_0. *Hint*: Refer to the picture for Page 20 # 7.

Derivatives and integrals with sequences of functions.

switch / DD **6.** Let $f_n(x) = x^n$ on $[0, 1]$ (recall Page 18 # 2b).

(a) Find $f(x) = \lim_{n \to \infty} f_n(x)$. Does $f_n \to f$ uniformly on $[0, 1]$?

(b) Find $\int_0^1 f_n(x)\, dx$, as a function of n, and $\int_0^1 f(x)\, dx$.

(c) Is $\displaystyle \lim_{n \to \infty} \int_0^1 f_n(x)\, dx = \int_0^1 \lim_{n \to \infty} f_n(x)\, dx$ true in this case?

Fun with pathological functions.

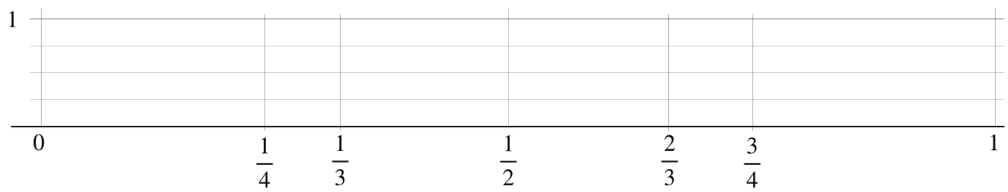
cont / DD

7. **The ruler function.** Define $f : \mathbf{R} \to \mathbf{R}$ $f(x) = \begin{cases} 1/q & \text{if } x = p/q \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$.

Here we assume as usual that $p \in \mathbf{Z}$, $q \in \mathbf{N}$, and p/q is in lowest terms.

(a) Sketch the function on $[0, 1]$, up to at least $q = 8$. Nice axes are provided below.

(b) Explain why it is called the ruler function.



cont / DD

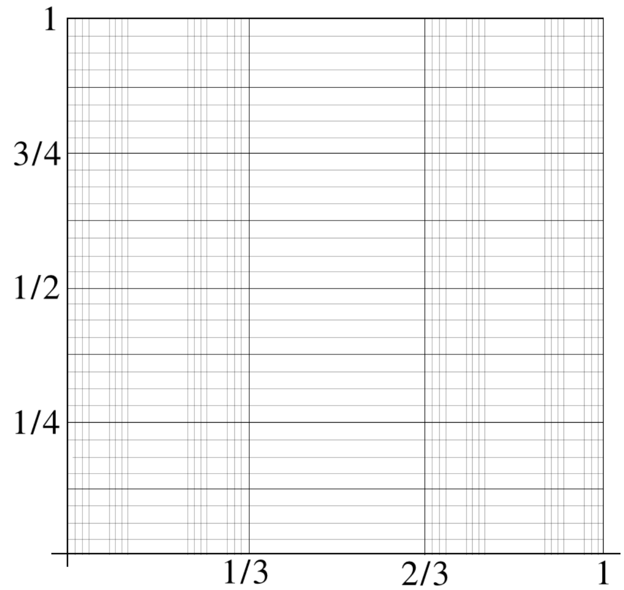
8. Prove that the ruler function is continuous at every irrational.

Cantor / DD

9. **The Cantor function.** Define $f : [0, 1] \to [0, 1]$ as follows: at 0 it is 0, and at 1 it is 1. On the middle third of the interval, define f to be 1/2. On the middle thirds of the two remaining intervals, define f to be 1/4 and 3/4. On the middle thirds of the remaining four intervals, define f to be $1/8, 3/8, 5/8, 7/8$. Continue. f extends to a continuous function on $[0, 1]$.

(a) Make a sketch of the graph of f on the axes to the right.

(b) Define a function *on* the Cantor set as follows: For any $p \in \mathcal{C}$, express it in ternary, giving an infinite decimal of 0s and 2s. Divide it by 2, yielding 0s and 1s. Interpret this number in binary; this is $f(p)$. How is this function related to the Cantor function? Is it onto? Is it continuous?



logic / DD

10. Lyrics to "Everybody Loves My Baby" are shown to the right. What can you conclude about my baby, from the lyric "Everybody loves my baby, but my baby don't love nobody but me"?

Note about double negatives in English:
"my baby don't love nobody but me" means
"my baby only loves me."

Everybody Loves My Baby

Words by Jack Palmer and Music by Spencer Williams

Verse: I'm as happy as a King, feelin' good n' everything,
I'm just like a bird in Spring, got to let it out.
It's my sweetie, can't you guess? Wild about her, I'll confess!
Does she love me? Oh my, yes! That's just why I shout:

Chorus 1: Everybody loves my baby,
but my baby don't love nobody but me.
Nobody but me.
Everybody wants my baby,
But my baby don't want nobody but me,
That's plain to see.

I am his sweet patootie and he is my lovin' man,
Knows how to do his duty, Loves me like no other can. That's why:
Everybody loves my baby,
But my baby don't love nobody but me.
Nobody but me!

Real Analysis

Warm-up problems

bound / DD **1**. Prove that a nonempty bounded subset of $\mathbf{R}$ has an infimum and a supremum.

count / DD **2**. Prove that if a set has an infinite subset, then it has uncountably many infinite subsets.

Riem / DD **3**. Prove, from the definition, that $f(x) = 3$ is Riemann integrable on $[0, 1]$.

Some things to prove

Riem / DD **4**. Prove, using the definition of the Riemann integral, that

$$\int_a^b \Big(f(x) + g(x)\Big)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx.$$

Hint: Such a proof will require ϵ (probably $\epsilon/2$), δ (probably $\delta = \min\{\delta_1, \delta_2\}$), and a chain of inequalities with summations. Use the sum $\int_a^b (f+g)\ dx$ as the number A in the definition.

Closed, complete and compact. We will explore relationships between these concepts.

cpt / DD **5**. Consider the set $A = [0, 1] \cap \mathbf{Q}$ in the metric space $E = \mathbf{Q}$.

(a) Show that A is closed in E. **(b)** Show that A is *not* complete. **(c)** Is A compact?

Cauchy / DD **6**. Let A be a subset of a metric space E. Show that if A is complete, then A is closed.

Cauchy / DD **7**. Let A be a subset of a *complete* metric space E. Show that A is complete if and only if A is closed.

uni-con / FM **8**. (For fun.) Consider the statement: If $f_n \to f$ uniformly, then ${f_n}^2 \to f^2$ uniformly.

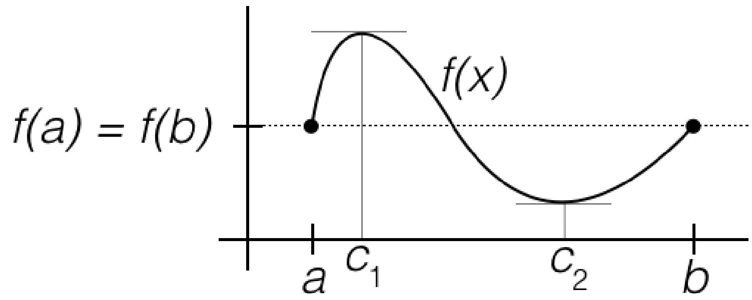
(a) Give a counterexample to the statement.

(b) Add a simple hypothesis, and prove the revised statement.

Real Analysis

MVT / DD **1. Rolle's Theorem.** Suppose that $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b), and that $f(a) = f(b)$. Then for some $c \in (a, b)$, $f'(c) = 0$.

Rolle's Theorem is the basis for the Mean Value Theorem, which we will see is very important in calculus.



Proof. We know that f attains a maximum and a minimum on $[a, b]$, by **(a)** ________________. If the maximum occurs at some point p on the interior, then by **(b)** ________________ $f'(p) = 0$, so let $c = p$ and we are done. The same argument holds for the minimum. If neither the maximum nor the minimum occurs on the interior, then they both occur at the endpoints, so... **(c)** Finish the proof.

Riem / FM **2**. Compute directly from the definition that $\displaystyle\int_0^1 x^2\, dx = 1/3$, as follows:

(a) Divide $[0, 1]$ into n subintervals of width $1/n$. Show that, if we evaluate the following Riemann sum at the right endpoint of each interval, the result is

$$\sum_{k=1}^{n} f(x)\frac{1}{n} = \frac{1}{n^3}\sum_{k=1}^{n} k^2.$$

(b) Use the formula $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(2n+1)(n+1)}{6}$ and take the limit as $n \to \infty$.

Riem / MR **3**. Show that, if $f : [a, b] \to \mathbf{R}$ is integrable, and $f(x) \geq 0$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\, dx \geq 0.$$

switch / DD **4**. Let $g_n(x) = \begin{cases} 0 & \text{if } \quad x = 0 \\ n & \text{if } \quad 0 \leq x \leq 1/n \\ 0 & \text{if } 1/n < x < 1 \end{cases}$ on $[0, 1]$. Repeat Page 23 # 6 for this function.

switch / DD **5**. Make a conjecture: For a sequence of functions f_n on $[a, b]$, when can you switch the limit and the integral? We will prove a Theorem about this after the exam.

diff / DD **6**. Consider again the ruler function $f : \mathbf{R} \to \mathbf{R}$ defined in Page 23 # 7. At which points is f differentiable?

MVT / DD **7**. **Corollary** to Page 18 # 1. *A continuous real-valued function on a nonempty compact metric space attains a maximum and minimum.* (This is essential for calculus!)

Proof. Let $f : E \to \mathbf{R}$ be a continuous function on a nonempty compact metric space E. Then $f(E)$ is closed and bounded because **(a)** ________ and nonempty because **(b)** ________. A nonempty bounded set has an infimum a and a supremum b, by Page __ # __. Furthermore, a and b are accumulation points of $f(E)$, so there are sequences in $f(E)$ converging to a and b. Since $f(E)$ is closed, the limits of these sequences are in $f(E)$, because **(d)** ________. Thus $a, b \in f(E)$, so $f(E)$ attains a maximum and minimum.

diff / DD **8**. **Proposition.** *If f is differentiable at x_0, then f is continuous at x_0.*

Proof. We will show that, if $\mathcal{U}$ is an open subset of $\mathbf{R}$, and $f : \mathcal{U} \to \mathbf{R}$ is differentiable at $x_0 \in \mathcal{U}$, then f is continuous at x_0.

Since f is differentiable at x_0, we know from **(a)** Page __ # __ that for any $\bar{\epsilon} > 0$, we can choose $\bar{\delta} > 0$ so that

$$|x - x_0| < \bar{\delta} \implies |f(x) - f(x_0) - f'(x_0)(x - x_0)| < \bar{\epsilon}|x - x_0|.$$

Thus, $|x - x_0| < \bar{\delta}$ implies

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f(x_0) - f'(x_0)(x - x_0)| + |f'(x_0)(x - x_0)| & (1) \\ &\leq (\bar{\epsilon} + |f'(x_0)|) \cdot |x - x_0|. & (2) \end{aligned}$$

Justify the inequalities in the **(b)** first and **(c)** second lines. Now choose $\delta = \min\left\{\bar{\delta}, \dfrac{\epsilon}{\bar{\epsilon} + |f'(x_0)|}\right\}$.

(d) Use the above to show that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$, as desired.

(e) Explain why we used the symbols $\bar{\delta}$ and $\bar{\epsilon}$ at the beginning instead of δ and ϵ.

diff / DD **9**. State the converse of the Proposition. Then either prove it or give a counterexample.

Series. Given a sequence of real numbers $\{x_n\}_{n=1}^{\infty} = \{x_1, x_2, \ldots\}$, we can construct a related sequence, called the *sequence of partial sums*, $\{S_n\}_{n=1}^{\infty}$, where $S_k = x_1 + x_2 + \cdots + x_k$.

series / DD **10**. For each of the following, give the first five partial sums of the series. Then give a formula for the n^{th} partial sum:

(a) $\{x_n\}_{n=1}^{\infty} = 1, 1, 1, \ldots$

(b) $\{x_n\}_{n=1}^{\infty} = 1/2, 1/4, 1/8, \ldots$

(c) $\{x_n\}_{n=1}^{\infty} = 0.1, 0.01, 0.001, \ldots$

Real Analysis

Review for Midterm 1 – optional problems

1. Write the definition of each term, as a full sentence: **(a)** The terms from Page 16 # 1
(b) connected **(c)** pointwise convergence **(d)** uniform convergence

(e) Cantor set **(f)** totally disconnected **(g)** uniformly Cauchy

(h) bounded function **(i)** differentiable **(j)** partition **(k)** Riemann sum

(l) integrable **(m)** Cantor function **(n)** ruler function

2. For each term in problem 1, write down a result (Theorem, etc.) that uses it.

2. State the following theorems:

(a) Heine-Borel Theorem

(b) Intermediate Value Theorem

(c) Rolle's Theorem

3. For each of the following, say whether it is True or False. For those that are false, give a counterexample.

(a) A bounded function on a bounded set is uniformly continuous.

(b) Every continuous function is uniformly continuous.

(c) If $f(x)$ is differentiable at p, then $f(x)$ is continuous at p.

(d) The continuous image of a compact set is compact.

(e) If a subset of $\mathbf{R}$ is uncountable, then it contains at least one interval.

5. Prove that a differentiable function on $[a, b]$ with bounded derivative is uniformly continuous.

6. Let $\mathcal{U}, \mathcal{V}, \mathcal{W} \subset \mathbf{R}$ be open sets, and let $f : \mathcal{V} \to \mathcal{W}$ and $g_n : \mathcal{U} \to \mathcal{V}$ be continuous functions. Prove or give a counterexample:

(a) If $g_n \to g$, then $f \circ g_n \to f \circ g$.

(b) If $g_n \to g$ uniformly, then $f \circ g$ is continuous.

(c) If $g_n \to g$ uniformly, then g is continuous.

7. Write down questions or problems that you wish to discuss in class.

8. In the topic dependence map on Page ii, circle the topics in which you are confident.

9. **Make a list of problems** (on any page or topic) that you would like to discuss in class.

Real Analysis

Day after exam – numerous entertaining problems

Riem / MR **1**. Show that, if $f, g : [a, b] \to \mathbf{R}$ are integrable, and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx.$$

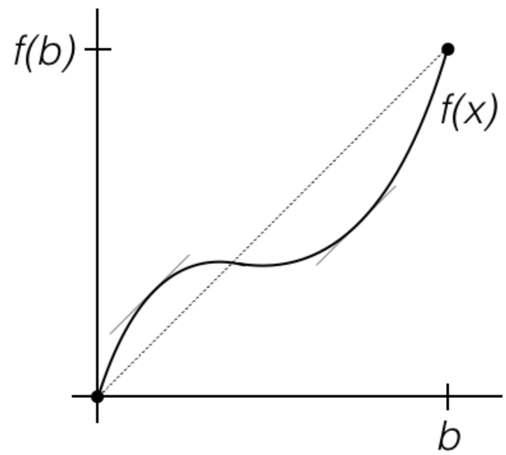
Hint: You can prove this from scratch, but it is easier to apply a previous result.

MVT / DD **2**. The hypothesis for the preceding Rolle's Theorem, and for the upcoming Mean Value Theorem, is that f is continuous on $[a, b]$ and differentiable on (a, b).

(a) Why did we need f to be continuous on $[a, b]$ instead of just (a, b)?

(b) Why don't we ask for f to be differentiable on $[a, b]$ instead of just on (a, b)?

MVT / DD **2**. The picture on the right is meant to illustrate the Mean Value Theorem. On the same axes, sketch the function $g(x) = f(x) - x$, under the assumption that $f(b) = b$.



MVT / DD **3**. **Mean Value Theorem.** Suppose that $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then for some $c \in (a, b)$,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The MVT is essentially the same as Rolle's Theorem, just "tilted," or perhaps we could call it "vertically sheared."

Proof. By horizontal scaling and translation, we may assume that $[a, b] = [0, 1]$, because **(a)** ____________________. If $f(0) = f(1)$, then we are done, because **(b)** ____________________. If not, then by vertical scaling and translation, we may assume that $f(0) = 0$ and $f(1) = 1$, because **(c)** ____________________. Let $g(x) = f(x) - x$. (d) (finish the proof)

MVT / DD **4**. Check the Mean Value Theorem for the function $f(x) = x^3$ on $[0, 1]$. (This means: determine a, b, $f(a)$, and $f(b)$ in this case, and find the c that satisfies the equation above.)

Fundamental Theorem of Calculus. Let f be a continuous function on $[a, b]$.

I. $\displaystyle \frac{d}{db} \int_a^b f(x)\ dx = f(b).$

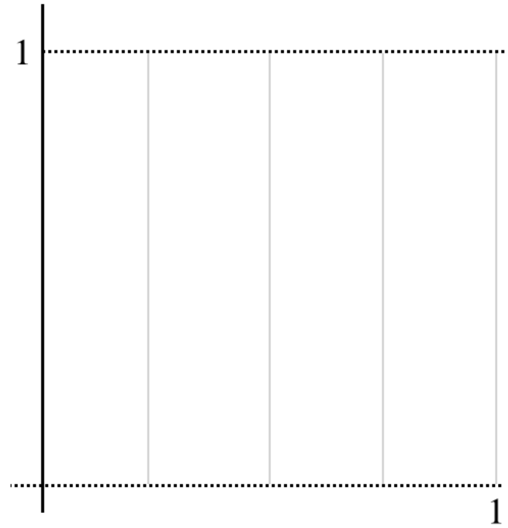
II. If there exists F such that $f(x) = F'(x)$, then $\displaystyle \int_a^b f(x)\ dx = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a).$

FTC / FM **5**. Use the FTC to compute $\displaystyle \int_0^1 x^2\ dx$. Was this easier or harder than Page 25 # 2?

FTC / FM **6**. Compute $\displaystyle \frac{d}{db} \int_a^b x^2\ dx$, using **(a)** FTC (I) **(b)** FTC (II).

Riem / DD **7. The characteristic function of the rationals.** Consider the function defined on $[0,1]$ by:

$$\chi_{\mathbf{Q}}(x) = \begin{cases} 1 \text{ if } x \in \mathbf{Q} \\ 0 \text{ if } x \notin \mathbf{Q} \end{cases} .$$



For a partition of $[0,1]$ of width $1/4$, find the value of each of the following Riemann sums, when the representative point x_i':

(a) is the left endpoint of each interval;

(b) yields the maximum value of $\chi_{\mathbf{Q}}$ on its interval;

(c) yields the minimum value of $\chi_{\mathbf{Q}}$ on its interval.

(d) Find a partition of $[0,1]$ of width $\leq 1/10$, and repeat parts (b)-(d).

(e) For $\epsilon = 1/2$, can you find an A and a δ that satisfy the definition of Riemann integrability? How about for $\epsilon = 1/10$?

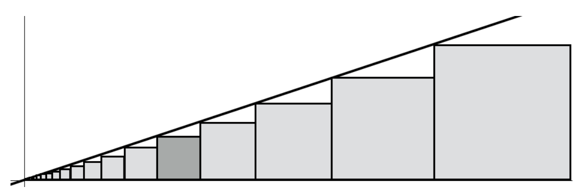
(f) What are the maximum and minimum values of a Riemann sum associated to a partition of width $1/4$? How about for width $1/10$?

(g) Is it possible to get a value of 0.123456789 for a Riemann sum of $\chi_{\mathbf{Q}}(x)$ on $[0,1]$? Describe all possible values that you can get as a Riemann sum.

(h) Is $\chi_{\mathbf{Q}}(x)$ Riemann integrable on $[0,1]$?

Geometric series. A series of the form $a + ar + ar^2 + ar^3 + \cdots$ is called a *geometric series.*

series / DD **8.** In the figure, the line has slope $1/3$, the squares are *inscribed* between the line and the x-axis, and the largest square has area a.



(a) Find the area of the second-largest square.

(b) Find the area of the dark square.

(c) Write an expression for the total area of all of the squares in the picture.

(d) Explain the terminology "geometric series."

Real Analysis

In Page 23 # 6, we saw that for $f_n(x) = x^n$ on $[0,1]$, $\lim \int f_n = \int \lim f_n$, but in Page 25 # 5, we saw an example of a sequence of functions where the two are *not* equal. You may wonder, based on the above examples, *when* you can switch the limit and the integral. The following Theorem tells you that, as usual, *uniform* convergence is the key!

switch / AJ **1**. **Theorem.** For a uniformly convergent sequence of continuous functions f_n on $[a,b]$, you can switch the limit and the integral:

$$\lim_{n\to\infty} \int_a^b f_n(x)\ dx = \int_a^b \lim_{n\to\infty} f_n(x)\ dx.$$

Proof. Given $\epsilon > 0$, choose N such that, for all $x \in [a,b]$, $n > N \implies |f_n(x) - f(x)| < \epsilon$. We can do this because **(a)** ______________________________. Then

$$\left|\int_a^b f_n(x)\ dx - \int_a^b f(x)\ dx\right| = \left|\int_a^b (f_n(x) - f(x))\ dx\right| \tag{1}$$

$$\leq \left|\int_a^b |f_n(x) - f(x)|\ dx\right| \tag{2}$$

$$< \int_a^b \epsilon\ dx = \epsilon(b-a). \tag{3}$$

Justify lines **(b)** (1), **(c)** (2), and **(d)** (3) above. **(e)** Complete the proof.

switch / AJ **2**. Give a counterexample to the above Theorem when each of the following hypotheses is omitted: **(a)** f_n are continuous, **(b)** f_n converge uniformly.

switch / AJ **3**. Write the converse of the above Theorem. Then prove it or give a counterexample.

Dense. Let S be a subset of a metric space E. We say that S is *dense* in E if every ball about every point of E contains a point of S.

dense / FM **4**. Which of the following sets are dense in $\mathbf{R}$?

(a) rationals **(b)** irrationals **(c)** rationals with powers of 2 in the denominator

series / DD **5**. Consider the *harmonic series* $1 + 1/2 + 1/3 + 1/4 + \cdots$.

(a) Write down the sum of the first 10 terms, the first 100, and the first 1000. Does it seem to be converging to some number? Please answer this question even if you have already read the next sentence below.

(b) Show that the series diverges to infinity. *Hint*:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \overbrace{\frac{1}{3} + \frac{1}{4}}^{\geq 1/4} + \overbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}^{\geq 1/8} + \cdots$$

FTC / DD **6**. In the problem session, you will show that if $f : \mathbf{R} \to \mathbf{R}$ satisfies $f(0) = 0$ and $|f'(x)| \leq M$, then $|f(x)| \leq M|x|$. Check that this result holds for $f(x) = \sin(x)$.

The following result is equivalent to the Mean Value Theorem (MVT). In words, it says that "how a function varies on an interval depends on the length of the interval, multiplied by some bound on the derivative in that interval."

FTC / MR **7**. **Corollary 1 to the MVT.** Suppose that $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then for some $c \in (a, b)$, $f(b) - f(a) = f'(c) \cdot (b - a)$. (Prove this.)

FTC / FM **8**. Show that, if $|f(x)| \leq M$ for all $x \in [a, b]$, then $\left| \int_a^b f(x)\, dx \right| \leq M(b - a)$.

Hint: You can do this from scratch, but it is easier to use a previous result.

Proof of the Fundamental Theorem of Calculus.

FTC / FM **9**. *Proof of I.* We have

$$\frac{d}{db} \int_a^b f(x)\, dx = \lim_{h \to 0} \frac{\int_a^{b+h} f(x)\, dx - \int_a^b f(x)\, dx}{h} \tag{1}$$

$$= \frac{\int_b^{b+h} f(x)\, dx}{h}. \tag{2}$$

(a) Justify equation (1) . For (2), we use the fact that if f is integrable on $[a, c]$ and $a < b < c$, then $\int_a^c f = \int_a^b f + \int_b^c f$, whose proof is straightforward; we omit it here. Now if $h > 0$, we have

$$\min_{|x-b| \leq |h|} f(x) \leq \frac{\int_b^{b+h} f(x)\, dx}{h} \leq \max_{|x-b| \leq |h|} f(x), \tag{3}$$

by **(b)** ________________. **(c)** Justify why equation (3) also holds when $h < 0$.

(d) Explain why, as $h \to 0$, the left and right sides of (3) both approach $f(b)$.

(e) Finish the proof, that

$$\frac{d}{db} \int_a^b f(x)\, dx = \frac{\int_b^{b+h} f(x)\, dx}{h} = f(b).$$

FTC / FM **10**. *Proof of II.* By (I), we have

$$\frac{d}{db} \left(F(b) - \int_a^b f(x)\, dx \right) = F'(b) - f(b) = f(b) - f(b) = 0.$$

(a) Justify the equalities above. By **(b)** ________________, there is a constant C such that

$$\frac{d}{db} \left(F(b) - \int_a^b f(x)\, dx \right) = 0 \implies F(b) - \int_a^b f(x)\, dx = C.$$

(c) Finish the proof by setting $b = a$ and deducing the desired statement.

FTC / FM **11**. Let $F(x) = \int_0^x e^{-t^2}\, dt$. Compute $F'(x)$ and $F'(0)$.

Real Analysis

Warm-up problems

series / DD

1. Prove the formula for the sum of a geometric series: If $a, r \in \mathbf{R}$ and $|r| < 1$,

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}.$$

Hint: telescoping

Riem / DD

2. Is $f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ integrable on $[0, 1]$? Prove your answer correct.

Some things to prove

FTC / FM

3. Suppose that $f : \mathbf{R} \to \mathbf{R}$ satisfies $f(0) = 0$ and $|f'(x)| \leq M$. Prove that $|f(x)| \leq M|x|$.

Prove the following Corollaries to the Fundamental Theorem of Calculus. Each one is stated in a colloquial way that is easy to remember, and then restated in a precise form that is easier to prove.

FTC / MR

4. **Corollary 1.** A continuous function has an antiderivative.

If f is a continuous, real-valued function on an open interval $\mathcal{U} \subset \mathbf{R}$, then there exists a real-valued function F on $\mathcal{U}$ such that $F'(x) = f(x)$.

FTC / MR

5. **Corollary 2.** Antiderivatives differ by a constant.

If F and G are both antiderivatives of f, then $F - G = C$ for some constant C.

Real Analysis

Prove the following (easy) Corollary to the Fundamental Theorem of Calculus:

FTC / MR

1. **Corollary 3 to the FTC.** If F is the antiderivative for f, then $\displaystyle\int_a^b f(x)\,dx = F(b)-F(a)$.

More formally: If $\mathcal{U} \subset \mathbf{R}$ is open and $F : \mathcal{U} \to \mathbf{R}$ has continuous derivative f, then for $a, b \in \mathcal{U}$, $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

The following result is the reason why we need the Mean Value Theorem to do calculus.

FTC / FM

2. **Corollary 2 to the MVT.** On an open interval where f' is always 0, f is constant. (Prove this.)

Note: Corollary 2 may seem obvious, but it is *almost* false! Recall the Cantor function, defined in Page 17 # 4. This is a non-constant, continuous function on $[0,1]$, with derivative 0 everywhere except on the Cantor set, which has measure (total length) 0.

Riem / AJ

3. **Theorem.** Every continuous, real-valued function is integrable on $[a,b]$.

Proof. We will show that any sequence of Riemann sums whose partition widths $\to 0$ is Cauchy. This will prove the result, because the sequence of Riemann sums is a Cauchy sequence of real numbers, which converges because **(a)** ____________________. The limit it converges to is then the Riemann integral $\displaystyle\int_a^b f(x)\,dx$.

We know that f is uniformly continuous, because **(b)** ____________________. So, given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x-y| < \delta \implies |f(x) - f(y)| < \epsilon. \tag{1}$$

Consider two Riemann sums, each with width less than $\delta/2$. Their subintervals intersect (if we break up $[a,b]$ at all the places where either of the partitions has a subinterval break) in smaller subintervals of width also less than $\delta/2$, because **(c)** ____________. On each subinterval, the values $f(x)$ from the two Riemann sums S_1 and S_2 come from points at distance at most $\delta/2$ from a point in the intersection, and hence at distance at most δ from each other, because **(d)** ____________. By (1), these values differ by at most ϵ. Let x_i^1 and x_i^2 be the representative values chosen for S_1 and S_2, respectively. Summing over the smaller subintervals, we see that the Riemann sums can differ by at most

$$\begin{aligned}
\left|\sum \big(f(x_i^1) - f(x_i^2)\big)(x_i - x_{i-1})\right| &\le \sum \left|f(x_i^1) - f(x_i^2)\right|(x_i - x_{i-1}) && (2)\\
&\le \sum \epsilon(x_i - x_{i-1}) && (3)\\
&= \epsilon \sum (x_i - x_{i-1}) && (4)\\
&= \epsilon(b-a). && (5)
\end{aligned}$$

Justify the lines **(e)** (2) **(f)** (3) **(g)** (4) **(h)** (5). Since $b-a$ is finite, we can make $\epsilon(b-a)$ as small as we like, so the sequence is Cauchy, proving the result.

Real Analysis

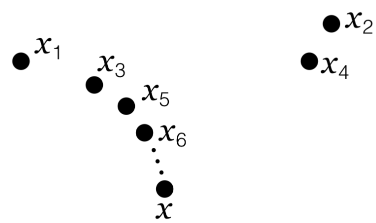
Infinite Series. An *infinite series* is an expression of the form $x_1 + x_2 + x_3 + \cdots = \sum_{i=1}^{\infty} x_i$. An infinite series with partial sums S_n *converges* to S (or "has sum S") if $S = \lim_{n\to\infty} S_n$.

In this case, we write $\sum_{i=1}^{\infty} x_i = S$.

series / DD **4**. For the following three infinite series, say whether the series converges, and if so, to what limit. Also express the series and its sum in Sigma notation ($\sum$).

(a) $1/4 + 1/16 + 1/64 + \cdots$ **(b)** $0.1 + .01 + .001 + \cdots$ **(c)** $1 - 2 + 4 - \cdots$

Convergence and divergence of sequences. We have already seen that a sequence x_n *converges* to x if $\lim_{n\to\infty} x_n = x$, and diverges otherwise.



series / DD **5**. Prove that the sequence $x_n = 1/2,\ 3/4,\ 7/8,\ 15/16,\ \ldots$ converges to 1, by finding an N that depends on the given ϵ.

series / DD **6**. The definition that "a sequence diverges if it does not converge" seems a bit difficult to check in practice. Here is a proposed alternative definition: A sequence x_n *diverges* if, for any $M > 0$, there exists $N > 0$ such that $n > N \implies |x_n| > M$. What do you think of this definition? If you don't like it, write a better one.

series / DD **7**. **Lemma.** (We use "Lemma" for a result that we are proving because it is one step needed in the proof of a later result. It is like breaking up a big computer program into smaller functions, and then calling them.) Let $x_1 + x_2 + \cdots$ be an infinite series, and let $S_1, S_2, \ldots$ be its associated sequence of partial sums. The sequence S_n converges if and only if, for any $\epsilon > 0$, there exists N such that $n, m > N \implies |S_n - S_m| < \epsilon$. (Prove this.)

series / DD **8**. **Proposition.** The infinite series $x_1 + x_2 + \cdots$ converges if and only if, for any $\epsilon > 0$, there exists N such that, for all $m > n > N$, $|x_{n+1} + \cdots + x_m| < \epsilon$. (Prove this.)

Riem / DD **9**. Consider again the *ruler function* $f(x) = \begin{cases} 1/q & \text{if } x = p/q \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$, on $[0, 1]$.

(a) For how many values of x is it true that $f(x) > 1/4$?

(b) Explain why, given any $N > 0$, there are only finitely many x for which $f(x) > 1/N$.

(c) Show that, in fact, given any $N > 0$, the number of values of x with $f(x) > 1/N$ is bounded by $N^2/2$.

(d) What should the value be for $\int_0^1 f(x)\, dx$? We will calculate it next time.

Real Analysis

As the definition of the Cantor function in Page 23 # 9 was slightly confusing, here is a clearer definition. Define the *Cantor function* $f_{\mathcal{C}} : [0,1] \to [0,1]$ as follows.

First, on the complement of the Cantor set $\mathcal{C}$, define $f_{\mathcal{C}}$ as follows: On the open middle third of the interval, f is $1/2$. On the open middle thirds of the two remaining intervals, f is $1/4$ and $3/4$, respectively. On the open middle thirds of the remaining intervals, f is $1/8$, $3/8$, $5/8$ and $7/8$, respectively. Continue in this manner to assign a value to every point in $\mathcal{C}^C$.

Then, on $\mathcal{C}$, define $f_{\mathcal{C}}$ as follows: For any point $p \in \mathcal{C}$, express p in ternary as a decimal point followed by an infinite string of 0s and 2s. Divide this number by 2 to yield a decimal point followed by an infinite string of 0s and 1s, and interpret it in binary; this is $f_{\mathcal{C}}(p)$.

Cantor / DD **1**. (Brian Jenike) Where is $f_{\mathcal{C}}$ continuous? Prove your answer correct.

Cantor / DD **2**. (Hari Srinivasulu) Where is $f_{\mathcal{C}}$ differentiable? Prove your answer correct.

Taylor / DD **3**. We showed in Page __ # __ that f is differentiable at x_0 if and only if it is well-approximated by its tangent line, $L(x) = f(x_0) + f'(x_0)(x - x_0)$. Show that the tangent line $L(x)$ matches the value and the first derivative of f at x_0, i.e. $L(x_0) = f(x_0)$ and $L'(x_0) = f'(x_0)$. Also draw a picture to illustrate this.

If f has enough derivatives, we can approximate f with a polynomial that matches not only the value and the first derivative of f, but also as many higher-order derivatives as we like, using a polynomial of degree n. This is the idea of Taylor's Theorem:

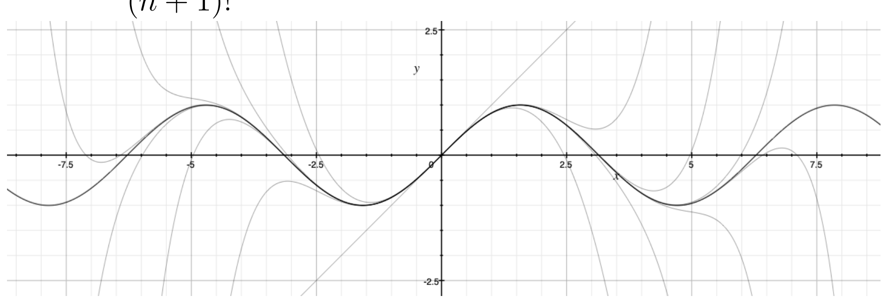
Taylor's Theorem. Let $\mathcal{U} \subset \mathbf{R}$ be an open set, and let $f : \mathcal{U} \to \mathbf{R}$ be $n+1$ times differentiable. Then for each pair $a, b \in \mathcal{U}$, there exists $c \in (a, b)$ such that

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Here, the last term $\dfrac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$ is the *error term*, i.e. the difference between $f(x)$ and its polynomial approximation of degree n.

Taylor / AJ **4**. Show that the Mean Value Theorem is a corollary of Taylor's Theorem, with $n = 0$.

Taylor / AJ **5**. Use Taylor's Theorem to show that the function $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots \pm \dfrac{x^n}{n!}$, for n odd, differs from $\sin x$ by at most $\dfrac{\pi^{n+1}}{(n+1)!}$ on the interval $[-\pi, \pi]$.



Real Analysis

logic / DD

6. Negation. Let $a, b \in \mathbf{R}$ with $a < b$, and let $f : [a, b] \to \mathbf{R}$.

(a) Write the definition: The function f is *Riemann integrable* if...

(b) Negate the definition: The function f is *not* Riemann integrable if...

int / DD

7. Let $f(x)$ be the ruler function. Prove that $\displaystyle\int_0^1 f(x)\, dx = 0$.

Hint: Fix $\epsilon > 0$, choose N such that $2/N < \epsilon$, and then choose $\delta < 1/N^3$. Take a partition of width less than δ, note that there are N^3 subintervals in the partition, and note that the number of subintervals contain a point x with $f(x) > 1/N$ is bounded by $N^2 = 2 \cdot N^2/2$. Then bound the possible Riemann sum values between 0 and ϵ to complete the proof.

series / FM

8. Rearrangements.

(a) Compute partial sums of the following series (maybe write them below):

$$\frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} + \cdots$$

(b) Compute partial sums of the following series:

$$\frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} - \frac{1}{3} - \frac{1}{3} - \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \cdots$$

(c) Does the sequence in part (a) converge? Does the sequence in part (b)? Explain.

sets / DD

9. (Alec Barreto) Suppose you have two colors, red and blue.

(a) Find a coloring scheme for points in $\mathbf{R}$ so that there is no interval, no matter how short, that is all one color.

(b) Find a coloring scheme for points in $\mathbf{R}^2$ so that there is no line segment or curve in the plane, no matter how short, that is all one color.

Taylor / DD **1**. *Proof of Taylor's Theorem.* We will show that, if $\mathcal{U} \subset \mathbf{R}$ is an open set, and $f : \mathcal{U} \to \mathbf{R}$ is $n+1$ times differentiable, then for any $a, b \in \mathcal{U}$, there exists $c \in (a, b)$ such that

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Our goal is to find a number $c \in (a, b)$ to make the above equation true. If $a = b$, the equation holds trivially because **(a)** ________________________.

Now we will assume $a \neq b$. Let P be the real number such that

$$f(b) - \left[f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n\right] = \frac{P}{(n+1)!}(b-a)^{n+1}. \tag{1}$$

We can find such a P because **(b)** ________________________. We want to show that P has the form $f^{(n+1)}(c)$ for some $c \in (a, b)$, because **(c)** ________________________. Let

$$\varphi(x) = f(b) - \left[f(x) + \frac{f'(x)}{1!}(b-x) + \cdots + \frac{f^{(n)}(x)}{n!}(b-x)^n\right] - \frac{P}{(n+1)!}(b-x)^{n+1}. \tag{2}$$

Here the difference between equations (1) and (2) is **(d)** ________________________. Now, $\varphi(x)$ is differentiable on $\mathcal{U}$ because it is a sum of differentiable functions: All of the terms of the form $(b-x)^k$ are differentiable because they are polynomials, and the $f', f'', \ldots, f^{(n)}$, are differentiable because **(e)** ________________________.

Now $\varphi(b) = 0$ because **(f)** ________________________, and $\varphi(a) = 0$ because **(g)** ________________. So by **(h)** ______________ Theorem, there exists $c \in (a, b)$ such that $\varphi'(c) = 0$. Now find $\varphi'(x)$:

$$\varphi'(x) = \frac{d}{dx}\left(f(b) - \left[f(x) + \frac{f'(x)}{1!}(b-a) + \cdots + \frac{f^{(n)}(x)}{n!}(b-x)^n\right] - \frac{P}{(n+1)!}(b-x)^{n+1}\right)$$

$$\begin{aligned}
= 0 &- \Big(f'(x) + f'(x)(-1) + (b-x)f''(x)\Big) \\
&- \left(f''(x)\frac{2(b-x)}{2}(-1) + \frac{(b-x)^2}{2!}f'''(x)\right) \\
&- \cdots \\
&- \left(f^{(n)}x)\frac{n(b-x)^{n-1}}{n!} + \frac{(b-x)^n}{n!}f^{(n+1)}(x)\right) \\
&+ \frac{P}{(n+1)!}(n+1)(b-x)^n(-1)
\end{aligned} \tag{3}$$

$$= -\frac{(b-x)^n}{n!}f^{(n+1)}(x) - \frac{P}{(n+1)!}(n+1)(b-x)^n(-1). \tag{4}$$

Here each line of equation (3) is an application of the **(i)** ________________, and we can simplify it to equation (4) by **(j)** ________________________. So we have $c \in (a, b)$ such that

$$0 = \varphi'(c) = -\frac{(b-c)^n}{n!}f^{(n+1)}(c) - \frac{P}{(n+1)!}(n+1)(b-c)^n(-1),$$

which we can solve for $P = f^{(n+1)}(c)$ **(k)** (show the steps to do this), as desired.

Real Analysis

Cantor / DD **2**. Let $\chi_{\mathcal{C}}$ be the characteristic function of the Cantor set: It is 1 on $\mathcal{C}$, and 0 otherwise. Compute $\displaystyle\int_0^1 \chi_{\mathcal{C}}\, dx$.

Hint: Given $\epsilon > 0$, choose n such that $(2/3)^n < \epsilon$ and $\delta < 1/3^n$, and use the partition of $[0,1]$ determined by the intervals of the Cantor set in the n^{th} step of construction.

int / AJ **3**. **Theorem.** Integrable functions are bounded.

Proof. We will show that, if $f : [a, b] \to \mathbf{R}$ is integrable, then f is bounded on $[a, b]$.
Hint: One way is to prove it directly, showing that integrability implies that the function value is bounded at every representative point. Another way is to prove the contrapositive, that an unbounded function is not integrable because the Riemann sums do not converge.

Absolute and conditional convergence.

The series $\displaystyle\sum_{n=0}^{\infty} x_n$ is *absolutely convergent* if $\displaystyle\sum_{n=0}^{\infty} |x_n|$ is convergent. A series that converges, but does not converge absolutely, is *conditionally convergent.*

series / DD **5**. For each of the following, first compute enough partial sums so that you have a good idea what is going on. Then guess whether it is absolutely convergent, conditionally convergent, or divergent. If it is convergent, try to guess what the limit is.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$ **(b)** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ **(c)** $2\ln 2 - 3/\ln 3 + 4/\ln 4 - 5/\ln 5 + \cdots$

series / FM **6**. **Proposition.** *If a series converges conditionally, then its terms may be rearranged to converge to* any *limit, or to diverge to* $\pm\infty$*, or to diverge by oscillation.*

In all of the rearrangements that follow, we'll keep the positive terms in the same order, and the negative terms in the same order. The idea is that we'll "front load" positive terms to get big positive limits, and we'll "front load" negative terms to get big negative limits.

(a) Explain (probably by contradiction) why the sum of all of the positive terms must be $+\infty$, and the sum of the negative terms must be $-\infty$.

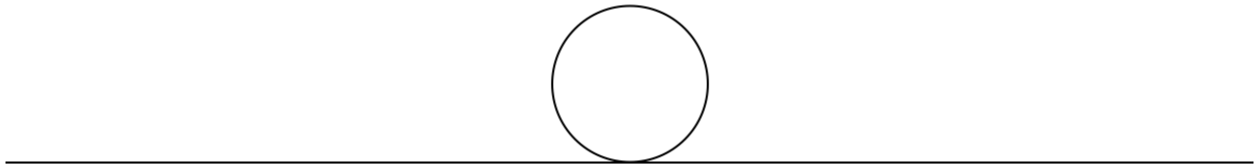
(b) Show that the terms of the series go to 0.

(c) Given some desired limit L, first take positive terms until you pass L heading right, and then take negative terms until you pass L heading left, and repeat this over and over. Explain why this rearrangement converges to L.

(d) Make a rearrangement as follows: Take positive terms until you pass 1, then take one negative term; then take positive terms until you pass 2, then take one negative term, etc. Explain why this rearrangement diverges to $+\infty$.

(e) Explain how to make a rearrangement that diverges to $-\infty$.

We'll prove divergence by oscillation in the problem session.



cpt / DD **7**. *Dramatic foreshadowing.*

(a) Show that the x-axis $\{(x, y) \in \mathbf{R}^2 : y = 0\}$ is not compact in $\mathbf{R}^2$.

(b) Show that the circle $\{(x, y) \in \mathbf{R}^2 : x^2 + (y - 1)^2 = 1\}$ is compact in $\mathbf{R}^2$.

Product spaces. Let S and T be sets. We define the *product space*

$$S \times T = \{(s, t) : s \in S, t \in T\}.$$

sets / DD **8**. Explain why $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$.

sets / DD **9**. Let S^1 denote a circle. What does $S^1 \times [0, 1]$ look like, geometrically? How about $S^1 \times S^1$?

Real Analysis

Warm-up problems

sets / DD **1**. (Andy Zhang, Joy Zhang) Prove that, between any two real numbers, there is:

(a) a rational number, and **(b)** an irrational number.

Cantor / DD **2**. Recall the Cantor function $f_{\mathcal{C}}(x)$ on $[0,1]$, which you sketched on Page 23. Find $\int_0^1 f_{\mathcal{C}}(x)\,dx$.
Hint: refer to your picture

Some things to prove

series / FM **3**. Prove that if a series converges conditionally, then its terms may be rearranged to diverge by oscillation.

sets / DD **4**. (Jake Springer) Prove that $[0,1]\times[0,1]$ is the same size as $[0,1]$.
Hint: to do this, either construct a bijection between the sets and prove that it is a bijection, or construct a surjection in each direction and prove that both are surjections.

uni-con / DD **5**. (Angelina Cao) Prove that a continuous function with bounded derivative is uniformly continuous.

Real Analysis

cpt / DD

1. Compactification.

(a) Show that the set $\left\{ \frac{(-1)^n}{n} : n \in \mathbf{N} \right\} \subset \mathbf{R}$ is not compact.

(b) Show that, by adding just one point to the set, you can make it compact.

(c) This is called *one-point compactification.* Can you think of any other sets that can be "compactified" in this manner?

int / DD

2. Let $f_n = \begin{cases} 1 \text{ if } x = p/q \in \mathbf{Q} \text{ and } q \leq n \\ 0 \text{ otherwise} \end{cases}$ on $[0, 1]$, where p/q is in lowest terms as usual.

(a) Plot f_n for $n = 1, 2, 3, 4, 5$ in five different colors on the same axes.

(b) For a given n, is f_n is integrable on $[0, 1]$? Why or why not?

(c) Find a function f so that $f = \lim f_n$.

(d) Is $\lim f_n$ integrable? Why or why not?

The purpose of the "fill-in" theorems was to teach you how to read proofs: by fighting with every single statement and justifying to yourself why it is true. Practice this skill by reading the following theorem, copied – with no blanks this time – out of *Introduction to Analysis* by Maxwell Rosenlicht (p. 140).

Theorem. Let $f_n : \mathcal{U} \to \mathbf{R}$, where $\mathcal{U} \subset \mathbf{R}$ is open. Suppose that f_n' is continuous for all n, f_n' converges uniformly on $\mathcal{U}$, and for some $a \in \mathcal{U}$, $\{f_n(a)\}$ converges. Then

- $\lim f_n$ exists,
- $\lim f_n$ is differentiable, and
- $(\lim f_n)' = \lim f_n'$.

Proof. By the Fundamental Theorem of Calculus, we have that, for all $x \in \mathcal{U}$ and all $n \in \mathbf{N}$,

$$\int_a^x f_n'(t)\ dt = f_n(x) - f_n(a).$$

Let $\lim\limits_{n\to\infty} f_n' = g$. By the Theorem in Page 28 # 1, $\lim\limits_{n\to\infty} (f_n(x) - f_n(a))$ exists for any $x \in U$, and equals $\int_a^x g(t)\ dt$. Since $\lim\limits_{n\to\infty} f_n(a)$ exists, so does $\lim\limits_{n\to\infty} f_n(x)$. Setting $\lim\limits_{n\to\infty} f_n(x) = f(x)$ we have

$$f(x) - f(a) = \int_a^x g(t)\ dt$$

for each $x \in U$. A second use of the Fundamental Theorem of Calculus gives $f' = g$, which is what was to be proved.

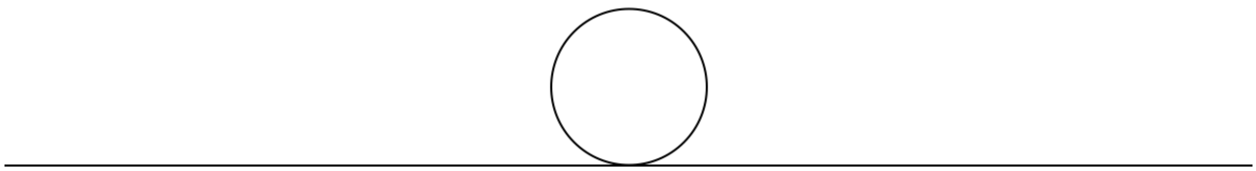
switch / DD

3. Work through the proof, statement by statement, figuring out why each part is true and how it all works. In class, you will each present this proof to each other.

uni-con / DD **4**. We have studied:

- continuous (Page 6) and *uniformly* continuous functions (Page 12),
- pointwise convergent sequences of functions (Page 19) and *uniformly* convergent sequences of functions (Page 20), and
- *uniformly* Cauchy sequences of functions (Page 22 # 1).

Look at all the definitions, and explain what "uniformly" means in general.



cpt / DD **5**. Consider the following function $f : \mathbf{R} \to S^1$ from points on the x-axis to points on the circle $S^1 = \{(x, y) \in \mathbf{R}^2 : x^2 + (y-1)^2 = 1\}$: For any point $(p, 0)$ on the x-axis, draw the line segment from $(p, 0)$ to the top point $(0, 2)$ of the circle. Let $f(p, 0)$ be the point where the line segment intersects the circle. Draw several such segments in the picture above.

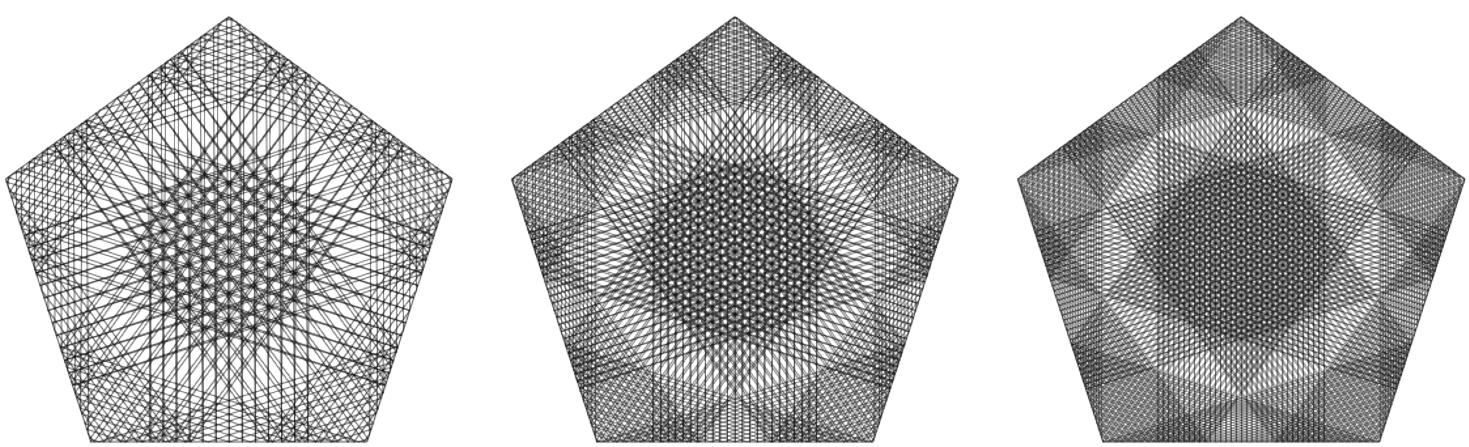
(a) Find $f(0, 0)$, $f(2, 0)$, $f(-2, 0)$, and f of one more point of your choice.

(b) Show that f is continuous. Is f a bijection?

Arbitrarily dense. Let S_n ($n \in \mathbf{N}$) be subsets of a metric space E. We say that the family $\{S_n\}$ is *arbitrarily dense* in E if, for every $r > 0$, there exists N such that for all $n > N$, for each $p \in E$ there is a point of S_n in the ball of radius r around p.

dense / DD **6**. Explain why the family $\{S_n\}$, where each $S_k = \{\text{rationals with denominator at most } k\}$, is arbitrarily dense in $\mathbf{R}$, by finding an N for each r.

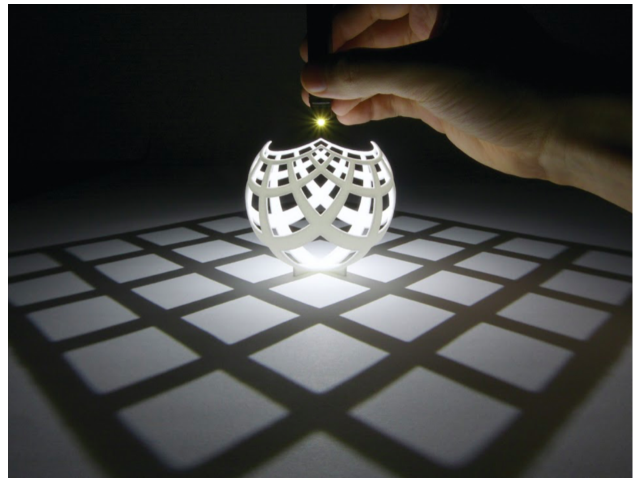
dense / DD **7**. In my (DD) research, I study billiard trajectories, which are the paths that a billiard ball takes in a billiard table. I found a sequence $\{P_n\}$ of increasingly long periodic billiard trajectories on the regular pentagon billiard table. The picture shows P_5, P_{10}, and P_{15}.



(a) Explain (geometrically) why $\{P_n\}$ are arbitrarily dense in the pentagon billiard table.

(b) Explain the 2019 math/stat T-shirt slogan "Periodically dense, always beautiful."

cpt / DD **1**. **One-point compactification.**



We showed that the x-axis in $\mathbf{R}^2$ is *not* compact, while a unit circle in $\mathbf{R}^2$ *is* compact, in Page __ # __ . We also know that the continuous image of a compact set is compact, by Page __ # __ . Show that the set $\mathbf{R} \cup \{\infty\}$ is compact. *Terminology:* Here $\{\infty\}$ is known as "the point at infinity."

The picture at right illustrates the same idea, for compactifying the xy-plane. This picture, and the design and creation of the object pictured, are by Henry Segerman. You can watch his 1-minute YouTube video called "Stereographic projection," which I promise is wicked awesome.

int / DD **2**. Prove or find a counterexample: the limit of integrable functions is integrable.

cont / DD **3**. Read "Law & Order: MVT" by Evelyn Lamb on pages 35c–35d. Then give an example of a function that occurs *in your real life* that is:

(a) continuous; **(b)** not continuous.

sets / DD **4**. How many subsets are there, of a set of 3 elements? of n elements?

sets / DD **5**. Prove that the set of subsets of $\mathbf{N}$ is uncountable. *Hint*: binary

sets / DD **6**. The size of $\mathbf{N}$ is denoted by $\aleph_0$ ("aleph-nought"). The notation $\{0,1\}^{\aleph_0}$, or even $2^{\aleph_0}$, is sometimes used to represent the set of subsets of $\mathbf{N}$. Explain why this notation makes sense.

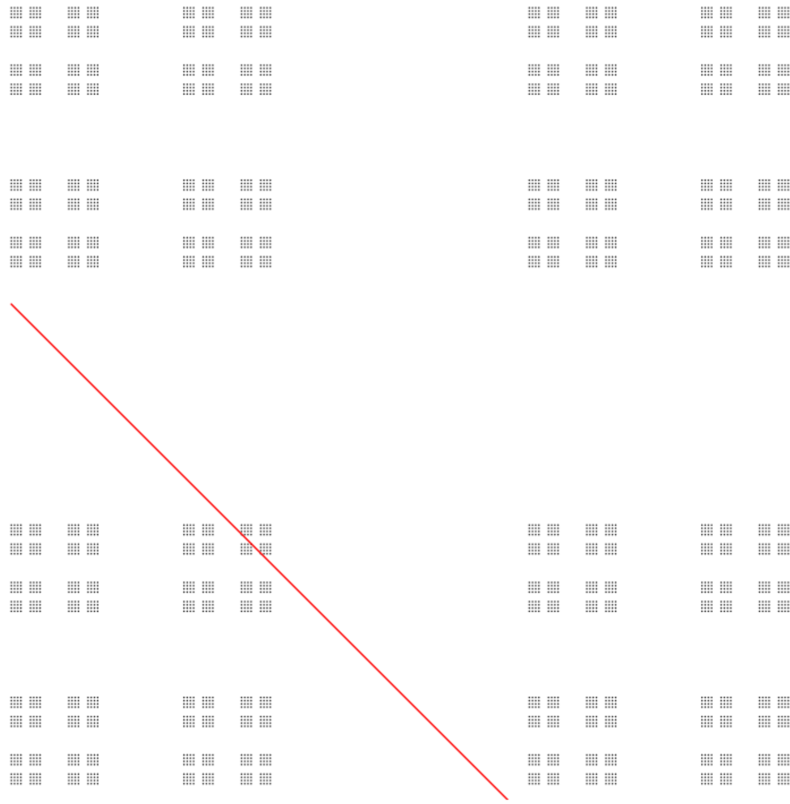
So far in this course, we have seen that there are three sizes (cardinalities) of sets: finite (which encompasses many sizes), countable and uncountable. The *continuum hypothesis* states that there are no sizes between the size of $\mathbf{N}$ and the size of $\mathbf{R}$; it turns out that it is not *possible* to prove this. After the size of $\mathbf{R}$, there are multiple sizes of "uncountable." One way to find a set that is larger than the one you have is to take the set of its subsets:

sets / DD **7**. Prove that, for any set X, the set of subsets of X (the "power set") is strictly larger.

Hint: Suppose that there exists a bijection, and show that this leads to a contradiction. Read the argument, which is explained in the illustrated book *Gallery of the Infinite*, by my (DD) Ph.D. advisor Richard Schwartz, on pages 35e–35f. Then write down a mathematical proof. You can find the whole book here: http://www.math.brown.edu/~res/infinity.pdf

Cantor / DD

8. (Amie Wilkinson)
Show that every number in $[0, 2]$ is the sum of two elements of the Cantor set.
Hint: One way is to show that for every $t \in [0, 2]$, the line $x + y = t$ touches $\mathcal{C} \times \mathcal{C}$.
Hint: self-similarity

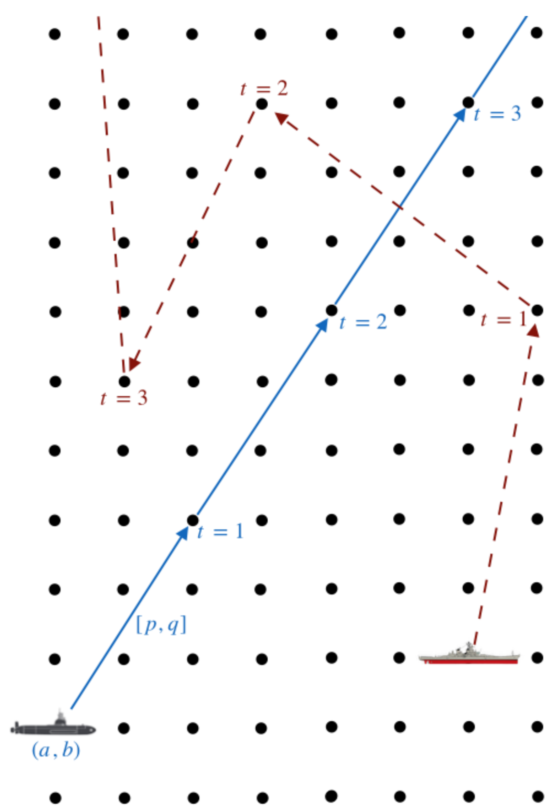


sets / DD

9. (Theo Johnson-Freyd) We will play a version of the game *Battleship* with the following rules. One player is a submarine, and the other is a battleship. The game board is the integer lattice in the plane, i.e. $\mathbf{Z}^2 = \{(a, b) \in \mathbf{R}^2 : a, b \in \mathbf{Z}\}$. The submarine chooses a starting point (a, b) and a direction vector $[p, q]$, both in $\mathbf{Z}^2$. The battleship has no knowledge about the initial position or direction vector of the submarine.

- At time $t = 0$, the submarine is at (a, b). Each second, the submarine travels along the vector $[p, q]$, and it repeats this process for all time.
- The battleship can teleport to any point on the lattice instantaneously. Each second, the battleship teleports to a lattice point and drops a depth charge, which explodes immediately.

Is there a strategy the battleship can employ so that it will be sure to destroy the submarine?



Law & Order: MVT, by Evelyn Lamb, in *Scientific American*, Oct 13, 2019:

In the criminal justice system, velocity-based offenses are considered especially unimportant. In New York, the dedicated detectives who investigate these minor misdemeanors are members of an elite squad known as the Moving Violation Team. These are their stories.

[Open with aerial shot of the New York State Thruway. It is a beautiful fall day. Traffic is on the heavy side but moving freely. Zero in on a car passing below a set of cameras on the road.]

DUN DUN

[One week later, in Syracuse, NY. MICHELLE ROLLINS is bringing the mail inside. Their wife CARLA GOFF is sitting on the couch.]

MICHELLE ROLLINS: What's this? [opens envelope] Really? A ticket? But I didn't see any cops when I was driving last week.

CARLA GOFF: They have those cameras mounted above the roads now.

MICHELLE ROLLINS: I saw those. They were right near the toll plazas. I was never speeding when I was near one of the cameras. This is garbage! They can't prove I was speeding.

[A few days later, in the Moving Violation Team office. Detective DOROTHY BERNSTEIN is going through papers, filing some and tossing others. Her colleague EDDIE WILLIAMS looks on.]

DOROTHY BERNSTEIN: We've got another driver contesting the ticket.

EDDIE WILLIAMS: They just don't stop, do they? They have no idea what they're in for.

DUN DUN

[Inside the courtroom. Judge CHARLOTTE SCOTT presiding. Another *DUN DUN* for good measure.]

BAILIFF: Please rise.

CHARLOTTE SCOTT: You may be seated. What do we have today? Ah, a contested speeding ticket. Plaintiff, opening statement, please.

MICHELLE ROLLINS: Your honor, I received a speeding ticket, but I was never pulled over.

DOROTHY BERNSTEIN: Are you familiar with the cameras we have to record license plates for tolls?

MICHELLE ROLLINS: Sure.

DOROTHY BERNSTEIN: They also record your location and time.

MICHELLE ROLLINS: Of course, but I don't see how that's relevant.

DOROTHY BERNSTEIN: There are multiple cameras. We recorded you driving here, at mile marker 192, [holds up blurry photo of a car passing under a camera on the road] at 12:47 pm on October 12. Then we took this photograph of you at mile marker 148. Can you read the timestamp on that photograph for me?

MICHELLE ROLLINS: Are these theatrics really necessary?

CHARLOTTE SCOTT: Just answer the question.

MICHELLE ROLLINS: It says [squints] 1:21 pm.

DOROTHY BERNSTEIN: In 34 minutes, you traveled 44 miles. Is that correct?

MICHELLE ROLLINS: Yes.

DOROTHY BERNSTEIN: The speed limit for this entire portion of the highway is 65 miles per hour. Would you agree that your average speed was above 65 miles per hour?

MICHELLE ROLLINS: [muttering] 68 minutes, 88 miles, 60 minutes, 65 miles, plus 8 is 73, the extra is less than a mile… [regular voice] Yes, it was.

Real Analysis

DOROTHY BERNSTEIN: As a matter of fact, it was 77.65 miles per hour.

MICHELLE ROLLINS: But that doesn't prove anything. The speed limit is not an average speed limit. You have to show I was traveling above 65 miles per hour at some point.

DOROTHY BERNSTEIN: Mx. Rollins, are you familiar with the Mean Value Theorem?

DUN DUN

[But no scene change]

MICHELLE ROLLINS: Yeah, I took calculus. That's the theorem that says that if your average rate of change between two endpoints is M, then your instantaneous rate of change at some point between two endpoints must have been M, if –

CHARLOTTE SCOTT: [bangs gavel] Case closed!

MICHELLE ROLLINS: Wait a minute, I didn't finish! That's *if* the function is a continuous function on the whole closed interval and differentiable on the open interval!

CHARLOTTE SCOTT: Are you saying the function describing your position was somewhere discontinuous or non-differentiable?

MICHELLE ROLLINS: I didn't say that, but, with all due respect, it's not my responsibility to prove they weren't but Detective Bernstein's to prove they were. Detective Bernstein, can you show that time and position are continuous, rather than discrete, quantities?

DOROTHY BERNSTEIN: Oh, please! Your honor, all widely-used modern and classical physical theories that are used to make predictions about real-world behavior use the assumption of continuous time. If time is not continuous, it is close enough on a practical level to assume such.

MICHELLE ROLLINS: By the same token, though, all numbers can be practically represented – to any degree of accuracy we desire – by rational numbers, can they not?

DOROTHY BERNSTEIN: Objection, your honor, irrelevant.

CHARLOTTE SCOTT: Mx. Rollins, where are you going with this?

MICHELLE ROLLINS: I promise it is highly relevant. At all points of my journey, we can assume the time and my position were rational numbers, using Detective Bernstein's "close enough on a practical level" criterion. Therefore my position was a function of time defined on the rational numbers. The mean value theorem does not hold for functions defined over the rationals! Take, for example, the function that is 0 for all rational numbers q such that q^2 is less than 2 and 1 for all rational numbers whose squares are larger than 2. The average value of this continuous function on the $[0, 2]$ interval is strictly between 0 and 1, but the function only takes the values 0 and 1.

[A gasp ripples through the courtroom, which somehow is full of an audience of people despite the fact that this is a very boring traffic case.]

CHARLOTTE SCOTT: [bangs gavel] Order! Order! Detective Bernstein?

DOROTHY BERNSTEIN: [Stammering] Wait – I – what – You can't be serious!

CHARLOTTE SCOTT: If the detective cannot counter Mx. Rollins' argument, I have no choice but to dismiss the ticket.

[DOROTHY BERNSTEIN sinks into her chair. EDDIE WILLIAMS brings her a cup of coffee. MICHELLE ROLLINS leaves the courtroom to a flock of reporters outside.]

DOROTHY BERNSTEIN: Thanks, Eddie. I can't believe they're getting away with it.

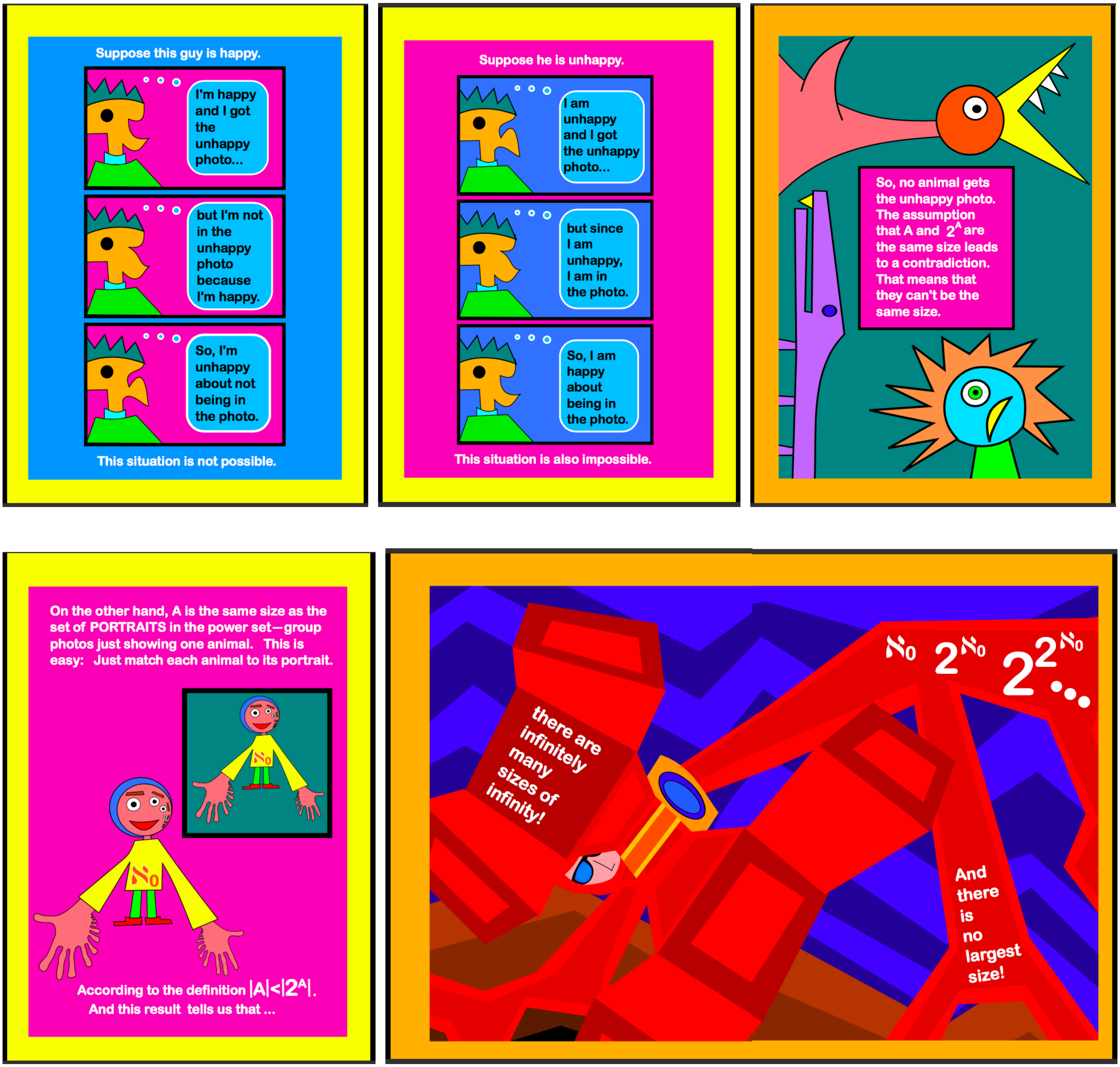
EDDIE WILLIAMS: We can only hope the next one hasn't thought so deeply about the mean value property.

DUN DUN

Law & Order: MVT, by Evelyn Lamb, in *Scientific American*, Oct 13, 2019.

Real Analysis

Excerpted from *The Gallery of the Infinite*, by Richard Evan Schwartz, AMS 2016.



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