Multivariable Calculus

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Spring 2020

Multivariable Calculus

The problems in this text

This set of problems is based on the curriculum at Phillips Exeter Academy, a private high school in Exeter, NH. Many of the problems and figures are taken directly from the *Mathematics 5* book, written by Rick Parris and other members of the PEA Mathematics Department. A few of the problems are adapted from *Calculus*, by Jon Rogawski and Colin Adams, and *Vector Calculus* by Susan Colley. The rest were written by me – Diana Davis. These problem sources are labeled in the margin as PEA, R-A, SC or DD, respectively. Anyone is welcome to use this text, and these problems, so long as you do not sell the result for profit. If you use these problems, please give appropriate attribution, as I am doing here.

About the course

This course meets twice a week for 50 minutes for a total of 41 classes, for which the homework is the numbered pages, usually with several pages (a and b, and possibly c) per class.

To the Student

Contents: As you work through this book, you will discover that the various topics of multivariable calculus have been integrated into a mathematical whole. There is no Chapter 5, nor is there a section on the gradient. The curriculum is problem-centered, rather than topic-centered. Techniques and theorems will become apparent as you work through the problems, and you will need to keep appropriate notes for your records — there are no boxes containing important theorems.

Your homework: Each page number of this book contains the homework assignment for one night. The first day of class, we will work on the problems on page 1, and your homework is page 2; on the second day of class, we will discuss the problems on page 2, and your homework is page 3, and so on for the 41 class days of the semester. You are not required to solve every problem before class, but *you are required to think hard about every problem and try to solve it*, including writing it down and drawing a picture in your notebook. Plan to devote at least two hours to solving problems for each class meeting.

Comments on problem-solving: You should approach each problem as an exploration. Reading each question carefully is essential, especially since definitions, highlighted in italics, are routinely inserted into the problem texts. It is important to make accurate diagrams whenever appropriate. Useful strategies to keep in mind are: draw a picture, create an easier problem, guess and check, work backwards, and recall a similar problem. It is important that you work on each problem when assigned, since the questions you may have about a problem will likely motivate class discussion the next day.

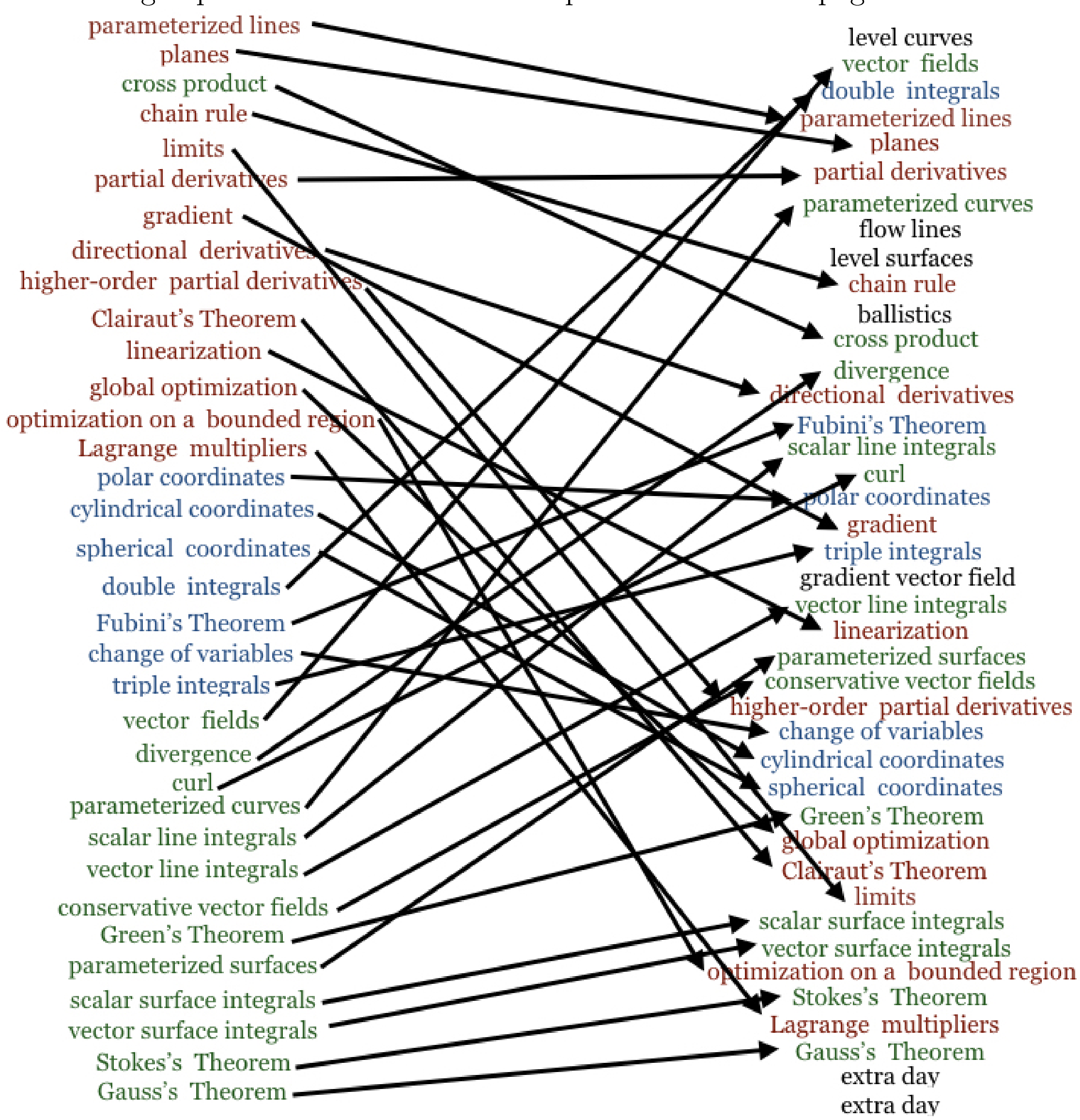
Problem-solving requires persistence as much as it requires ingenuity. When you get stuck, or solve a problem incorrectly, back up and start over. Keep in mind that you're probably not the only one who is stuck, and that may even include your professor. You should bring to class a written record of your efforts, not just a blank space in your notebook. The methods that you use to solve a problem, the corrections that you make in your approach, the means by which you test the validity of your solutions, and your ability to communicate ideas are just as important as getting the correct answer.

Multivariable Calculus

About this curriculum

We can roughly divide the topics of "multivariable calculus" into three categories: derivatives, integrals, and calculus on vector fields. Most multivariable calculus courses are taught in that order (left column below). One issue with this is that each topic is discussed, and then left behind. The other is that calculus on vector fields, and its associated big theorems – Green's Theorem, Stokes' Theorem and Gauss's Theorem – are the most challenging parts of the course, and they are left for the end (sometimes even for the very last day of the course!), when the student doesn't have much time to absorb them.

Instead, I have rearranged the topics (right column below), with the goal of studying the ideas of vector calculus as early as possible, so that we have plenty of time to explore them. A chart showing dependencies of the various topics is on the next page.

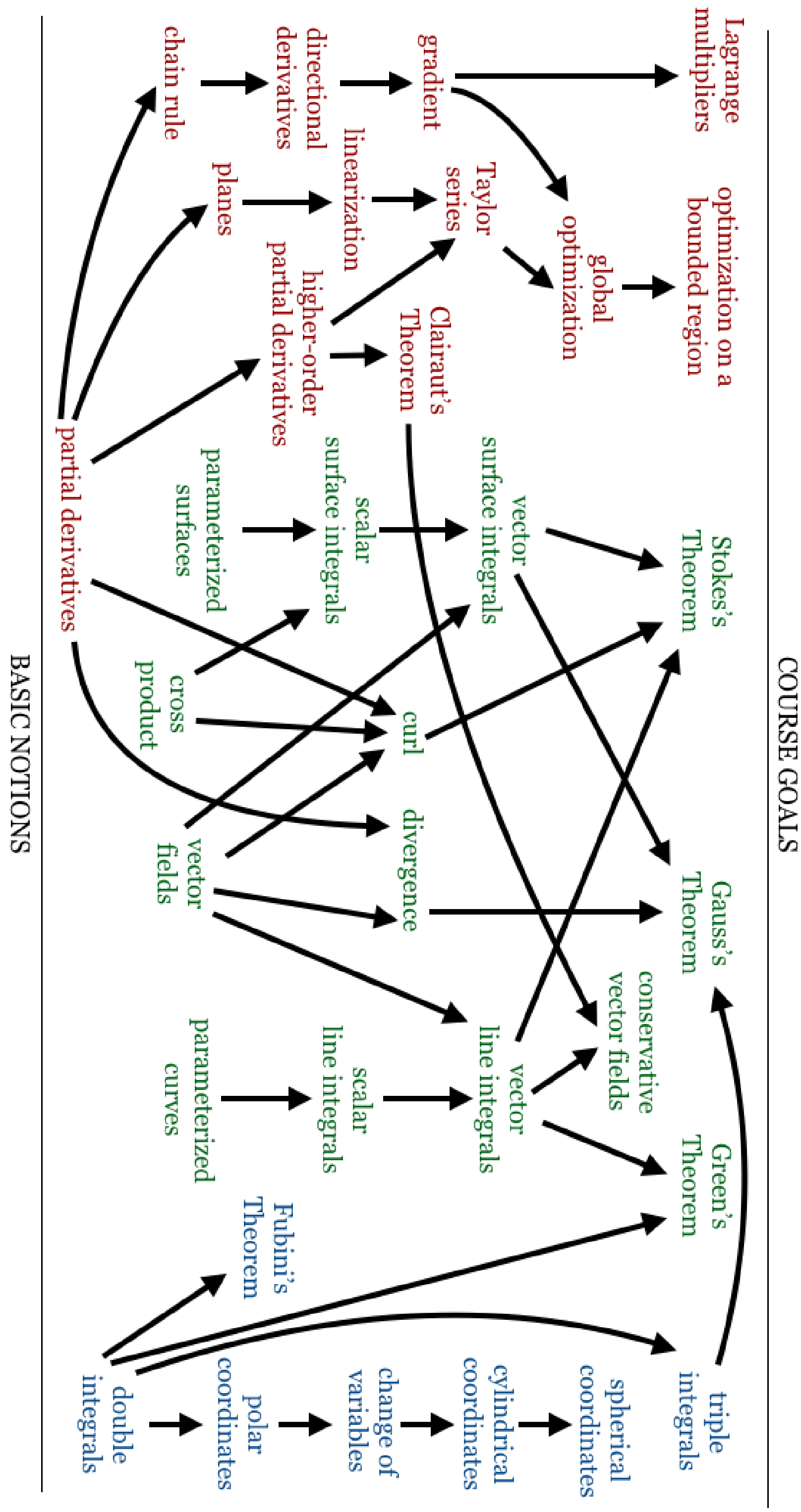


The big idea here is that we are developing derivatives, integrals, and calculus on vector fields *simultaneously*, so that we have time to absorb each of them. By popular request, each problem is labeled in the margin with its topic.

Multivariable Calculus

Below is a map of the ideas in this course, and how they connect, from the basic ideas at the bottom to the course goals at the top. An arrow goes from A to B if we need the ideas from A in order to understand B. I made this chart when I was constructing our curriculum.

- Circle topics that you feel you understand well.
- Periodically come back to this chart and circle topics as you master them.



Acknowledgements

Thank you to the instructors in the mathematics department at Phillips Exeter Academy for developing this curriculum and writing so many of the problems. Special thanks to department chair Gwyn Coogan for teaching me how to write my own curriculum, and for sharing the source code with me.

Thank you to David Merola at the Solebury School for doing every problem in the book in summer 2019, pointing out errors, and making helpful suggestions. Thank you to Brian Katz for using this book for a course at Smith College in fall 2019 and doing the same, sending errors and suggestions. The same thanks go to each of my students, in each of the semesters I have used previous versions of these materials:

- Spring 2016 at Northwestern University
- Spring 2017 at Williams College
- Fall 2018 at Swarthmore College
- Spring 2019 at Swarthmore College

Online materials

In Spring 2018, I taught a course very similar to this one in a lecture format, and videotaped my 40 lectures and posted them on YouTube. The topics go in the standard order, as listed on the left side of page ii. You can find the videos using a YouTube search, via a link from my web site, or using the following extremely long link: `https://www.youtube.com/watch?v=Fnqq39chaH4&list=PL3e7J-yDTmahubJZoHodT0DLw64gRxLGT`

Contact me

If you use these materials – as a student, instructor, or in any other capacity – I would love to know about it. Please send me errata, suggestions, laudations, etc. at `dianajdavis@gmail.com`.

I have written a guide on how to write your own problem-centered curriculum, which you can find at `https://www.swarthmore.edu/NatSci/ddavis3/davis-how-to-write-a-pbc.pdf`.

Discussion Skills

1. Draw a picture
2. Ask questions
3. Connect to a similar problem
4. Speak to classmates, not to the instructor
5. Use other students' names
6. Explain a difficult problem, even if your solution may not be correct
7. Answer other students' questions
8. Suggest an alternate solution method
9. Summarize the discussion of a problem
10. Contribute to the class every day

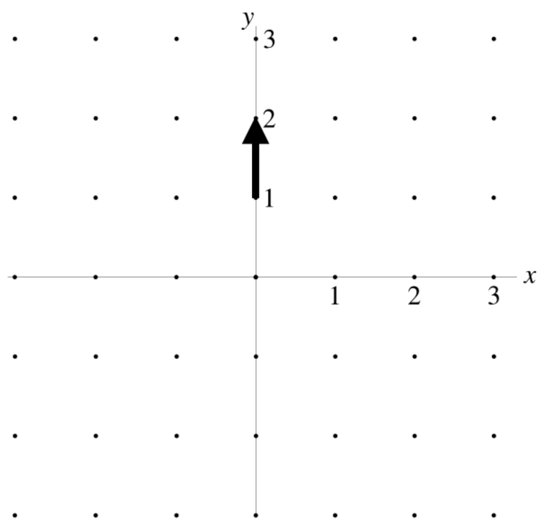
Multivariable Calculus

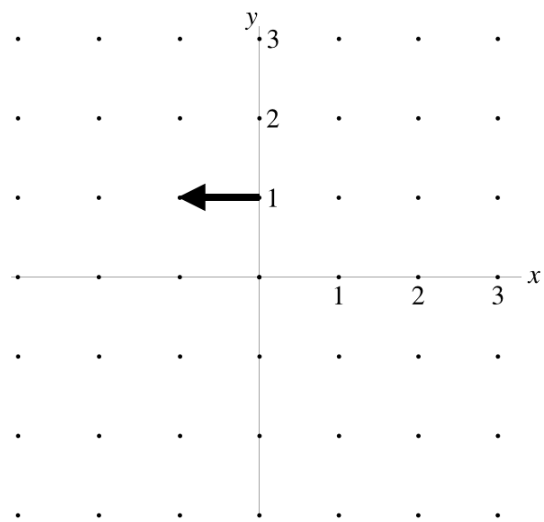
First day - in class

ParEq / PEA **1.** A bug moves linearly with constant speed across my graph paper. I first notice the bug when it is at $(3,4)$. It reaches $(9,8)$ after two seconds and $(15,12)$ after four seconds. Draw a clear, accurate diagram of this situation. Then predict the position of the bug after six seconds; after nine seconds; after t seconds.

VecFie / DD **2.** A *vector field* is a function $\mathbf{F} : \mathbf{R}^2 \to \mathbf{R}^2$. You can think of it as giving the wind speed and direction at each point on a windy field, or the water speed and direction at each point in a turbulent stream. You input a *point* (x,y), and the output $\mathbf{F}(x,y)$ is a *vector*.

(a) On the axes below, sketch the vector fields $\mathbf{F}(x,y) = [x,y]$ and $\mathbf{G}(x,y) = [-y,x]$. "Sketch the vector field $\mathbf{F}(x,y)$" means "For lots of points (x,y) of your choice, draw the vector $\mathbf{F}(x,y)$ with its tail at (x,y)." I have drawn the vectors $\mathbf{F}(0,1)$ and $\mathbf{G}(0,1)$ for you.





(b) Make sure that you sketched enough vectors to get a good idea of what is going on. Then describe, in words, what the vector field $\mathbf{F}$ does, and what the vector field $\mathbf{G}$ does.

Planes / DD **3.** For a *line* in the familiar coordinate plane $\mathbf{R}^2$, you are familiar with the notion of its *slope*. We would like to define an analogous measure for a *plane* in 3-space, $\mathbf{R}^3$. What geometric information would you want this number to encode?

Multivariable Calculus

ParEq / PEA **1.** The x- and y-coordinates of a point are given by the equations below. The position of the point depends on the value assigned to t. Use graph paper to plot points corresponding to the values $t = -4, -3, -2, -1, 0, 1, 2, 3,$ and 4. Do you recognize any patterns? Describe them.

$$\begin{cases} x = -2 + 2t \\ y = 10 - t \end{cases}$$

ParEq / PEA **2.** (Continuation) The path of the bug in Page 1 # 1 intersects the line given by the equations above. At what point? First answer this question by making a careful sketch on graph paper, and then find a way to solve it using a system of equations. You will have to think carefully about t.

ParEq / PEA **3.** (Continuation) Another way to write an equation for the line above is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} -2 \\ 10 \end{pmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} t.$$

(a) Explain. This is called a *parametric equation*, and t is called the *parameter*.

(b) What does the point $\left(\begin{smallmatrix} -2 \\ 10 \end{smallmatrix}\right)$ represent? What does the vector $\left[\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right]$ represent?

(c) Give two other parametric equations whose graph is this same line. Can you find one that uses a starting point other than $\left(\begin{smallmatrix} -2 \\ 10 \end{smallmatrix}\right)$? One that uses a vector other than $\left[\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right]$?

VecFie / DD **4.** You know how to compute the dot product of two vectors, e.g. $[a, b] \bullet [c, d] = ac + bd$ . . . but what does it *mean*? We'll give two answers, one now and one later.

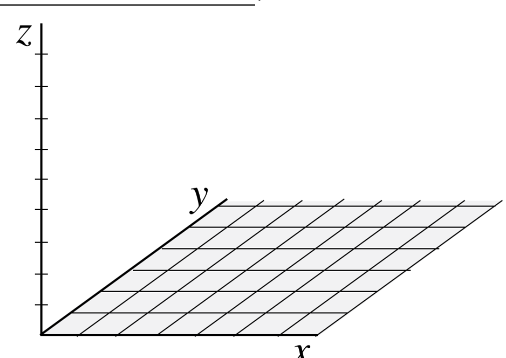
Answer 1. The dot product measures how much two vectors point in the "same direction."

Carefully sketch the vectors $\mathbf{v}_1 = [5, 1], \mathbf{v}_2 = [-1, 5], \mathbf{v}_3 = [-3, 2]$, and another vector $\mathbf{v}_4$ of your choice. Compute all of the pairwise dot products, and use your observations from this data to fill in each blank below with one of the following characterizations:

are perpendicular   point in similar directions   point generally in opposite directions

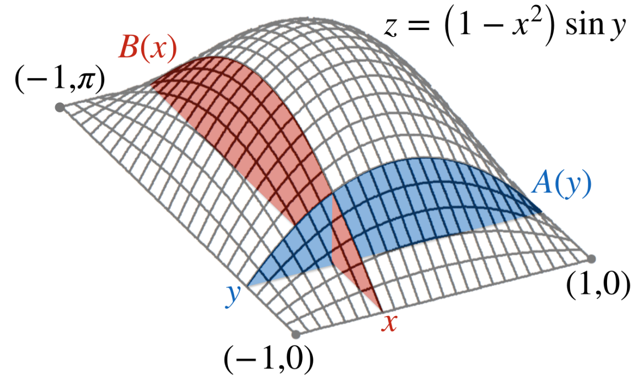
- $\mathbf{u} \bullet \mathbf{v} > 0$ when $\mathbf{u}$ and $\mathbf{v}$ ______________________________.
- $\mathbf{u} \bullet \mathbf{v} = 0$ when $\mathbf{u}$ and $\mathbf{v}$ ______________________________.
- $\mathbf{u} \bullet \mathbf{v} < 0$ when $\mathbf{u}$ and $\mathbf{v}$ ______________________________.

Gra / DD **5.** Find four different points (x, y, z) that satisfy the equation $x + 2y + 3z = 6$. Make a clear, accurate diagram in your notebook of the x-, y- and z-axes, like the one shown to the right, and plot the four points on your sketch. What kind of object do you think this equation represents?



PROBLEMS 6 AND 7 ON THE OTHER SIDE!

DbInt / PEA **6.** The diagram shows $z = (1-x^2)\sin y$ for the rectangular domain defined by $-1 \le x \le 1$ and $0 \le y \le \pi$. This surface and the plane $z = 0$ enclose a solid region $\mathcal{R}$. It is possible to find the volume of $\mathcal{R}$ by integration:



(a) Notice first that $\mathcal{R}$ can be sliced neatly into sections by cutting planes that are perpendicular to the y-axis — one for each value of y between 0 and π, inclusive. Explain why the area $A(y)$ of the slice determined by a specific value of y is given by $A(y) = \displaystyle\int_{x=-1}^{x=1} (1-x^2)\sin y \, dx$. Then evaluate this integral, treating y as a constant.

(b) Explain why the integral $\displaystyle\int_{y=0}^{y=\pi} A(y)\, dy$ gives the volume of $\mathcal{R}$. Then evaluate it.

(c) Notice also that $\mathcal{R}$ can be sliced into sections by cutting planes that are perpendicular to the x-axis — one for each value of x between -1 and 1. As in **(a)**, use ordinary integration to find the area $B(x)$ of the slice determined by a specific value of x.

(d) Integrate $B(x)$ to find the volume of $\mathcal{R}$.

DD **7. A note on notation.** Vector quantities, such as vector fields and vectors, are typeset in bold, like $\mathbf{F}$ and $\mathbf{v}$, and handwritten with an arrow above them, like $\vec{F}$ and $\vec{v}$. When you see $\mathbf{F}$, you should write it as $\vec{F}$. Copy down the following in proper handwritten notation:

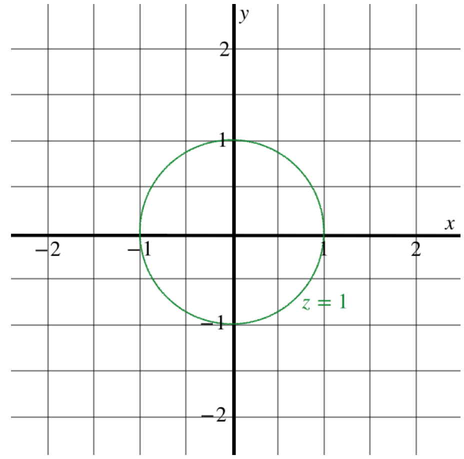
(a) $\mathbf{F}$ ______ **(b)** $\mathbf{u}$ ______ **(c)** $\mathbf{v}_1$ ______ **(d)** $\mathbf{i}, \mathbf{j}, \mathbf{k}$ ____________

(e) Go back and fix your notation for problem 4, if your vectors $\mathbf{v}_i$ do not have arrows!

Remember that when you come to class, you must have a written record in your notebook of your thoughts about each of the problems, including:

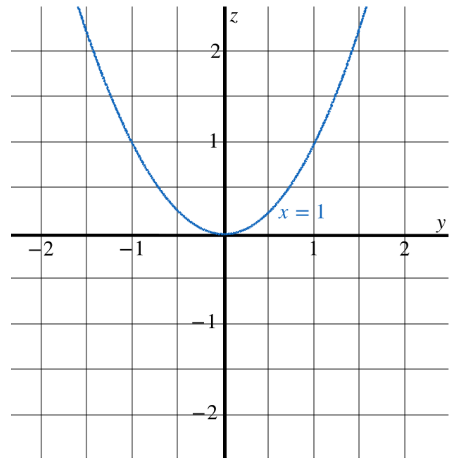
- *a picture,*
- *the relevant information given in the problem, and*
- *a full record of your solution, or all of your efforts towards one.*

Multivariable Calculus



Gra / DD **1.** Define the *function* $f : \mathbf{R}^2 \to \mathbf{R}$ by $f(x, y) = x^2 + y^2$. This function takes in a point (x, y) from the plane, and outputs a number $f(x, y)$. To visualize this function, we can sketch the associated *surface* $z = x^2 + y^2$. To figure out how to sketch the surface, we can first intersect the surface with planes, to find *level curves* and *vertical sections.*

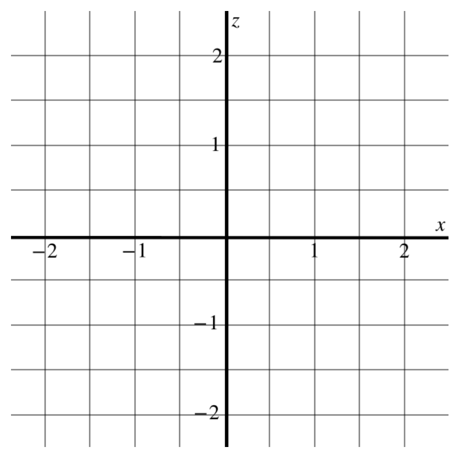
(a) *Level curves.* Choose a horizontal level, such as $z = f(x, y) = 1$, and plug it into the equation. This gives you a curve in the xy-plane, in this case the curve $1 = x^2 + y^2$, shown to the right in green. Sketch this curve in your notebook, and also the level curves corresponding to levels $2, 3, 4, 5, 6, 7, 8, 9, 10, 0$, and -1, all on the same picture of the xy-plane.



(b) *Vertical sections.* Now intersect the surface with vertical planes, choosing a constant c and setting $x = c$ for several choices of c, including 0. Graph the resulting curves in the yz-plane.

(c) Repeat **(b)** for the xz-plane, slicing with vertical planes of the form $y = c$.

(d) Use the information from the slices to sketch a 3D picture of the surface, which is called a *paraboloid.*



Planes / PEA **2.** Find coordinates for two points that belong to the plane $2x + 3y + 5z = 15$, trying to choose points that no one else in the class will think of. Show that the vector $[2, 3, 5]$ is perpendicular to the segment that joins your two points.

Planes / PEA **3.** (Continuation) Now you know that $[2, 3, 5]$ is perpendicular to the plane. Explain how you know this.

VF / DD **4.** To represent a vector field $\mathbf{F}$ on $\mathbf{R}^2$, one way to do it (as in Page 1 # 2) is to draw little arrows at many representative points, to show the direction and magnitude of the vector field at each point. Instead of doing that, this time we'll sketch the *flow lines*, which show trajectories of particles under the effect of the vector field, as though you drop a feather into the flowing wind and see where it goes. The idea of the flow lines is that the vector field arrows that we drew before are *tangent vectors* to the flow lines.

Draw the flow lines for the vector fields

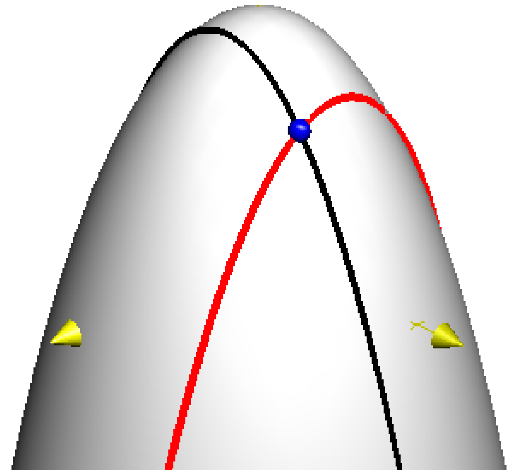
(a) $\mathbf{F} = [x, y]$ and **(b)** $\mathbf{G} = [-y, x]$ and **(c)** $\mathbf{H} = [1, 2]$.

(d) Are the flow lines for $\mathbf{G}$ circles or spirals? How do you know?

Remember that $\mathbf{F}$ *is handwritten as* $\vec{F}$*, with an arrow above it.*

ParDer / DD

5. The picture to the right shows the surface $z = 5 - x^2/3 - y^2/3$, along with the curves cut through this surface by the vertical planes $x = 1$ (black) and $y = 2$ (red). The positive x-axis (left) and y-axis (right) are pointing towards you out of the surface. Label them now with x and y.

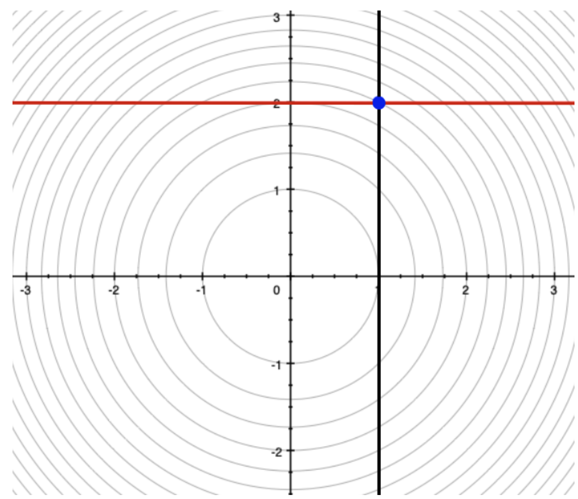


(a) What are the (x, y, z) coordinates of the blue point of intersection of the two curves?

(b) Imagine that you are a hiker standing at the blue point. If you walk due north, which is the direction of the positive y-axis, will you be ascending or descending?

(c) If you walk due east, which is the direction of the positive x-axis, will you be ascending or descending? Will this eastward walk be steeper or less steep than walking north?

(d) Explain how to use the *level curves* picture to the right, to help you answer parts **(b)** and **(c)**.



Gra / DD

6. *Multivariable calculus is about understanding three-dimensional objects.* Anytime you are investigating a function, you should graph it. Consider $z = 9 - x^2 - y^2$.

- Easiest way: Type the equation into Google. Try it now: `z=9-x^2-y^2`.
- Next-easiest: Type the same equation into WolframAlpha, a super powerful web site.
- If you are using a Mac, search for "Grapher" – it comes standard on the Mac, and allows you to plot multiple graphs on the same axes, zoom and rotate. If you have Grapher, use it to draw the surface; if not, see if your computer has another such program.
- There are many free 3D graphing apps for a mobile device – download one of them and draw the surface using it. Some even have augmented reality.

(a) Sketch a graph of the surface $z = 9 - x^2 - y^2$ in your notebook.

(b) Which of the four graphing tools worked best for you? Be prepared to report to your group which graphing tool is your favorite, and why.

Remember that when you come to class, you must have a written record in your notebook of your thoughts about each of the problems, including a picture, the relevant information given in the problem, and a full record of your solution, or all of your efforts towards one.

Multivariable Calculus

DirDer / DD **1.** We will figure out how to explicitly compute the two *directional derivatives* of the function $f(x,y) = 5 - x^2/3 - y^2/3$ at the point $(1,2)$ that we estimated in Page 3 # 5.

(a) Along the red curve, $y = 2$, the function is just a function of x: $f(x,2) = 5 - x^2/3 - 4/3$. Take the derivative of this function with respect to x, plug in $x = 1$, and thereby find the slope of the hiker's eastward walk from the blue point $(1, 2, 10/3)$.

(b) The notation for this is:

$$f_x(x,y)\Big|_{y=2} = \frac{\partial}{\partial x} f(x,y)\Big|_{y=2} = \frac{d}{dx} f(x,2) = \frac{d}{dx}(5 - x^2/3 - 4/3) = -2x/3.$$
$$f_x(1,2) = -2x/3\Big|_{x=1} = -2/3.$$

The vertical line indicates that we are evaluating the expression at a certain point or value. The symbol ∂ is for a *partial derivative*, which we use for a function of more than one variable, while the symbol d is for a *total derivative* of a function of only one variable. Justify each of the equalities above, in words, and write your explanation in your notebook.

(c) Find $f_y(1,2)$, which is the slope that the hiker would experience when walking north from the blue point, along the black curve where $x = 1$.

2. (Continuation) Refer to the picture for Page 3 # 5.

(a) Explain why the vector $[1, 0, f_x(1,2)] = [1, 0, -2/3]$ is in a direction tangent to the red curve at the blue point.

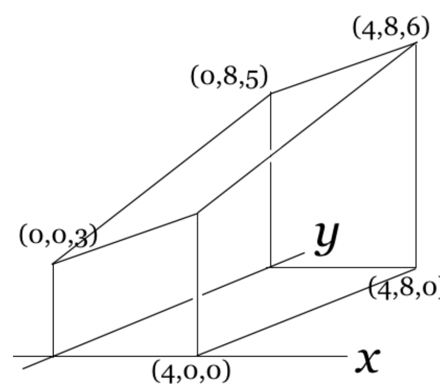
(b) Give a vector that is in the direction tangent to the black curve at the blue point.

(c) Find a vector that is perpendicular to both $[1, 0, -2/3]$ and the vector you found in (**b**).

DbInt / DD **3.** Suppose that the base of your storage shed is the rectangle $0 \le x \le 4$, $0 \le y \le 8$, and its slanted roof is formed by the plane $z = x/4 + y/4 + 3$, as shown. Explain why the following integral gives the *volume* of the shed (a useful number to know, if you wish to store things inside), and calculate the integral.

$$\int_{y=0}^{y=8} \int_{x=0}^{x=4} (x/4 + y/4 + 3)\; dx\; dy$$

Can you express it as a *triple* integral?



ParEq / DD **4.** A little fishie swims towards the water's surface according to the parametric equation

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} t \\ 1+2t \\ -14+4t \end{bmatrix}.$$

A fishing net hangs in the water in the shape of the surface $z = x^2 + y^2 - 20$.

(a) Draw a picture of this situation.

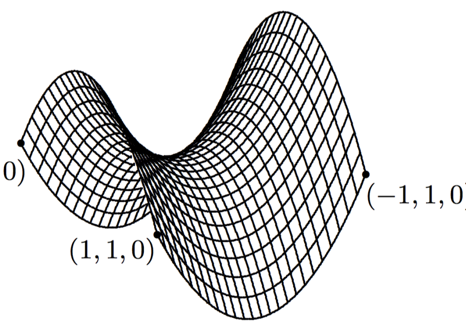
(b) Does the fishie pass through the net? Where? When?

ParEq / DD **5.** (Continuation) An equivalent expression of the fishie's path is $\mathbf{r}(t) = [t, 1 + 2t, -14 + 4t]$. Here t is measured in seconds, and distance is measured in centimeters (cm).

(a) How fast is the fishie moving (in cm/sec) in the x-direction? How fast is it moving in the y-direction? How fast is it moving in the z-direction? What is the speed of the fishie?

(b) The *velocity vector* of the fishie's path is $\mathbf{r}'(t) = [x'(t), y'(t), z'(t)]$. Find this vector, and explain what it means in the context of the fishie's swim.

Gra / DD **6.** The figure shows the surface $z = f(x, y)$, where $f(x, y) = x^2 - y^2$, and $-1 \le x \le 1$ and $-1 \le y \le 1$. Fifty curves have been traced on the surface, twenty-five in each of the coordinate directions. This saddle-like surface is called a *hyperbolic paraboloid*, to distinguish it from the elliptical (or circular) paraboloids you have already encountered. It has some unusual features.



(a) What do all fifty curves have in common?

(b) Confirm that the line through $(1, 1, 0)$ and $(-1, -1, 0)$ lies entirely on the surface.

Hint: write a parametric equation for this line, and plug it into the surface equation.

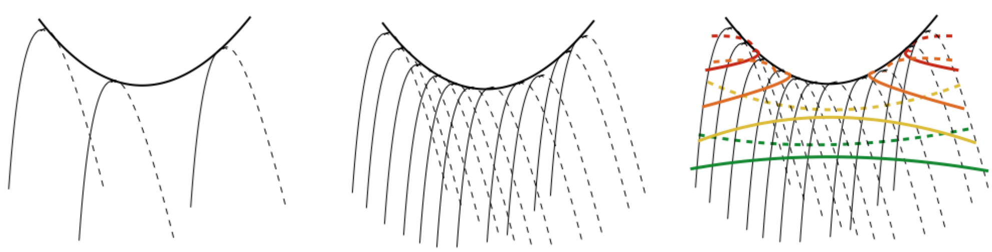
(c) In addition to the line given, there is another line through the origin that lies entirely on the surface. Identify it.

(d) Explain the name "hyperbolic paraboloid."

Gra / DD **7.** *How to draw a hyperbolic paraboloid in three easy steps.* See picture below.

1. Draw an upward-facing parabola, and hang some downward-facing parabolas from it.
2. Hang more downward-facing parabolas, with the back part of each dashed or light.
3. (Optional) Sketch in some horizontal cross sections in different colors.

Draw a hyperbolic paraboloid in your notebook.



LevCu / DD **8.** Make a sketch in the xy-plane of the level curves of the hyperbolic paraboloid. These correspond to the colored horizontal cross sections in the graph above. Use colors!

Experiment with different ways of drawing surfaces. Draw in enough features to show what is going on, but not so many that the picture is confusing. This is an art that requires practice!

ParDer / DD **1.** People often say: "when you take a partial derivative with respect to x, you just treat y as a constant," and similarly, "when you take a partial derivative with respect to y, you just treat x as a constant."

(a) With this in mind, find the partial derivatives $f_x(x,y)$ and $f_y(x,y)$ of $f(x,y) = x^4y + y^3$.

(b) Notice that each partial derivative is actually a *function* of x and y, which takes different values at different points. Find $f_x(3,2)$ and $f_y(-1,4)$ and explain what these numbers mean geometrically. *Remember to look at the surface $z = x^4y + y^3$ on a computer.*

Planes / PEA **2.** We have a guess, from Page 3 # 2–3, of how the equation of a plane relates to the vector perpendicular to it. With this in mind, write an equation for:

(a) a plane that is perpendicular to the plane $2x - y + 3z = 6$ and passes through the origin;

(b) the plane that is perpendicular to the vector $[4, 7, -4]$ and goes through the point $(2, 3, 5)$.

(c) Explain why part (a) says "a plane" and part (b) says "the plane."

ParEq / DD **3.** If you put a ball in a long sock and whirl it around above your head, the position of the ball at time t will be something like $\mathbf{r}(t) = [\cos t, \sin t, 2]$, where t is measured in seconds and distance is measured in meters.

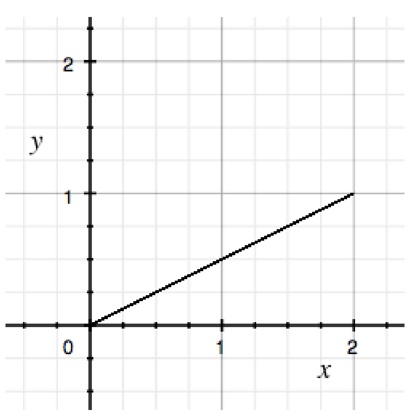
(a) Draw a picture of this situation.

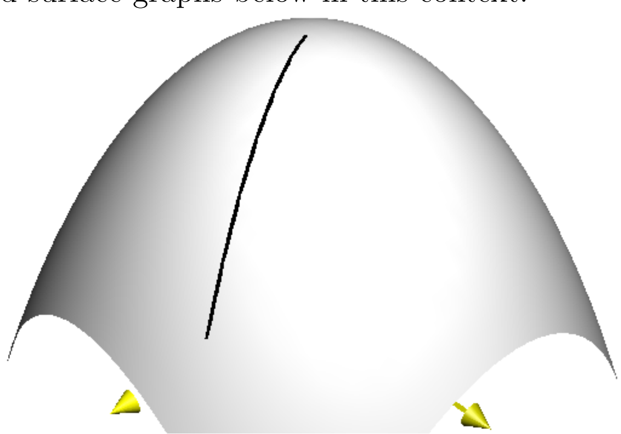
(a) Velocity is the derivative of position: it measures how position is changing. Compute the *velocity vector* $\mathbf{r}'(t)$, add it to your picture, and explain its meaning.

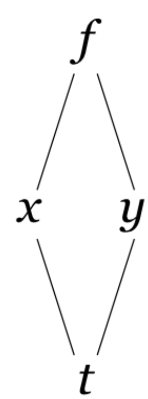
(b) Acceleration is the derivative of velocity: it measures how velocity is changing. Compute the *acceleration vector* $\mathbf{r}''(t)$, add it to your picture, and explain its meaning.

ChRule / DD **4.** Suppose that you have just hiked Mount Davis (the highest point in Pennsylvania), and on your map (which is in the xy-plane), the path you took can be parameterized by $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2-2t \\ 1-t \end{bmatrix}$, where t is measured in hours from $t = 0$ to $t = 1$, and distance is measured in miles east and north from the summit. Further suppose that Mount Davis's surface can be modeled by the elevation function $f(x,y) = 5 - x^2/3 - y^2/3$.

(a) Explain the line segment and surface graphs below in this context.







(b) Explain why the elevation f is a function of x and y, while x and y are each functions of t. Using the hiking story and the figures, explain why elevation f is a function of t.

DbInt / PEA **5.** Page 2 # 6 illustrates how a problem can be solved using *double integration.* Justify the terminology (it does not mean that the problem was actually solved twice). Notice that the example was made especially simple because the limits on the integrals were constant — the limits on the integral used to find $A(y)$ did not depend on y, nor did the limits on the integral used to find $B(x)$ depend on x. The method of using cross-sections to find volumes can also be adapted to other situations:

(a) Sketch the region of the xy-plane defined by $0 \le x$, $0 \le y$, and $x + y \le 6$.

(b) Sketch the 3D region $\mathcal{R}$ enclosed by the surface $z = xy(6 - x - y)$ and the plane $z = 0$ for $0 \le x$, $0 \le y$, and $x + y \le 6$.

Remember to look at the surface $z = xy(6 - x - y)$ *on a computer or other device.*

(c) Find the volume of $\mathcal{R}$.

FSV / DD

6. We can graph a function $f(x)$ of one variable as a curve in two dimensions, $y = f(x)$. We can graph a function $f(x, y)$ of two variables as a surface in three dimensions, $z = f(x, y)$. It's more difficult to graph a function $f(x, y, z)$ of three variables! One way to think about such a function is that it gives the *temperature* at each point (x, y, z) in space. A good way to visualize the function is to draw its *level surfaces*, the surfaces of the form $f(x, y, z) = c$.

The temperature of a candle flame is about $1500°F$. The temperature of a typical room is $70°F$. Sketch a couple of representative *level surfaces* around the candle flame, which are surfaces for which all points on the surface have the same temperature, and label the temperature of each.

VF / PEA **7.** We will prove the *Vector Law of Cosines*, using the SSS version of the Law of Cosines that you may remember from a geometry course:

For a triangle where the sides with lengths a and b come together to form angle C, and the side opposite angle C has length c, we have $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

Draw vectors $\mathbf{u}$ and $\mathbf{v}$ tail-to-tail so that they make a θ-degree angle. Draw the vector $\mathbf{u} - \mathbf{v}$, the third side of the triangle, and check to see that it points in the right direction.

(a) Solve for $\cos\theta$ using the SSS version of the Law of Cosines, expressing all lengths in terms of $\mathbf{u}, \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$.

(b) If you use vector algebra to simplify the numerator as much as possible, you will discover the relationship between $\mathbf{u} \bullet \mathbf{v}$ and $\cos\theta$.

Hint: Use proper notation! $|\mathbf{u}| \cdot |\mathbf{v}|$ and $\mathbf{u} \bullet \mathbf{v}$ are different, and "$\mathbf{uv}$" is meaningless.

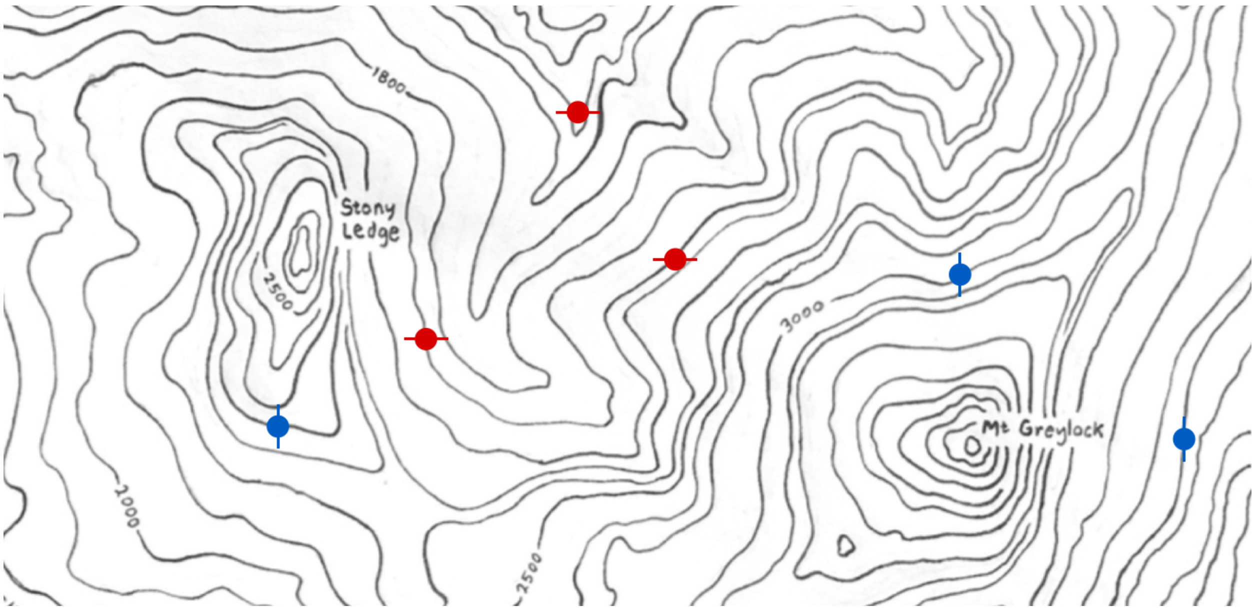
DirDer / DD **8.** To specify a direction, you can use a vector of any length. Give vectors of length 1, 5 and 14 in: **(a)** the direction of vector $[3, -4]$, **(b)** the direction of vector $[-2, 3, 6]$.

A vector of length 1 is called a *unit vector*; it is sometimes convenient to use these.

ParDer / DD **1.** The picture below is a topographic map of a small portion of western Massachusetts, near Williams College, containing Mount Greylock (the highest point in Massachusetts) and the nearby mountain Stony Ledge. We could say that this map shows the level curves of the elevation function $f(x,y)$, which gives the elevation of a point (x,y), measured in feet. Level curves are drawn in at multiples of 100 feet. You may wish to label more of the curves.

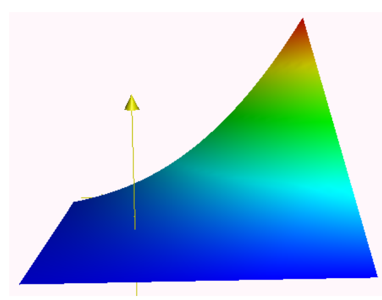
(a) For each blue point on the map, say whether f_y is positive, negative or 0 at that point. In other words, say whether you are ascending, descending, or neither as you walk in the positive y-direction (north).

(b) For each red point on the map, do the same for f_x.



DbInt / PEA **2.** Evaluate the double integrals: **(a)** $\displaystyle\int_{x=1}^{x=3}\int_{y=0}^{y=2} x^3 y\,dy\,dx$ **(b)** $\displaystyle\int_{x=-1}^{x=2}\int_{y=x^2}^{y=x+2} 1\,dy\,dx$

3. (Continuation) In part (a), you integrated to find the volume under the *surface* $z = x^3 y$ over a rectangular region of the plane, as shown to the right. In part (b), you integrated to find the volume under the surface $z = 1$ over a non-rectangular region of integration. Sketch this region. Then explain why, if you want to find the area of a region of the plane, you can integrate the function $f(x,y) = 1$ over that region.



DirDer / DD **4.** Find the rate of change of the function $f(x,y) = 5 - x^2/3 - y^2/3$, at the point $(1,2)$, in the direction of the vector $[-3,-4]$, as follows:

(a) Parameterize a line $[x(t), y(t)]$ through $(1,2)$ with direction $[-3,-4]$ that moves at unit speed. Then substitute in $x(t)$ and $y(t)$ for x and y in the function $f(x,y)$ to get the function $f(t)$ as a function of t. Take the derivative of $f(t)$ with respect to t, at $t=0$.

(b) Explain why the work you did in **(a)** gives the rate of change of f in direction $[-3,-4]$.

Later, we will find an easier way, using intuition from this problem.

CrPr / PEA **5.** *The cross product.* Given two vectors $\mathbf{u} = [p, q, r]$ and $\mathbf{v} = [d, e, f]$, there are infinitely many vectors $[a, b, c]$ that are perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. It is a routine exercise in algebra to find one, and it requires that you make a choice during the process. It so happens that there is a "natural" way to make this choice, and an interesting formula results.

(a) Confirm that $\mathbf{w} = [\,qf - re, rd - pf, pe - qd\,]$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

(b) It is customary to call $\mathbf{w}$ the *cross product* of $\mathbf{u}$ and $\mathbf{v}$, and to write $\mathbf{w} = \mathbf{u} \times \mathbf{v}$. There is an easier way to remember the formula: if we allow ourselves to use $\mathbf{i} = [1, 0, 0]$, $\mathbf{j} = [0, 1, 0]$, $\mathbf{k} = [0, 0, 1]$ as entries in a matrix (!!!), then the cross product is the matrix determinant

$$\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ p & q & r \\ d & e & f \end{bmatrix}.$$

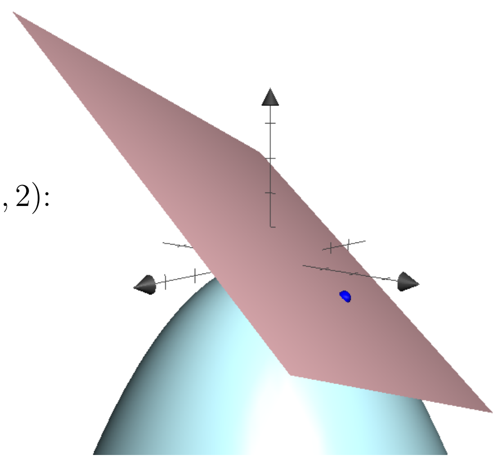
Use this to find a vector that is perpendicular to $[2, -3, 6]$ and $[-6, 2, 3]$.

(c) The *direction* of $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. It so happens that the *length* of $\mathbf{u} \times \mathbf{v}$ is the area of the parallelogram spanned by the vectors $\mathbf{u}$ and $\mathbf{v}$. Confirm this fact for vectors $\mathbf{u} = [a, b, 0]$ and $\mathbf{v} = [c, d, 0]$ of your choice, using vectors that no one else in the class will think of. *Hint*: Use integers a, b, c, d and draw your parallelogram on graph paper.

(d) Is it true that $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$?

(e) Give three explanations of why $\mathbf{u} \times \mathbf{u} = \mathbf{0}$. Also explain why the zero is a vector.

Lin / DD **6.** Let $f(x, y) = -x^2 - y^2$, and let $L(x, y) = 2x - 4y + 5$.



(a) Show that $f(-1, 2) = L(-1, 2)$: the *values* of the functions agree at $(-1, 2)$.

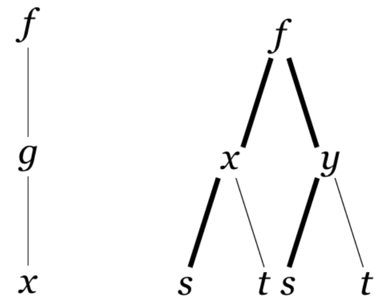
(b) Show that $f_x(-1, 2) = L_x(-1, 2)$ and $f_y(-1, 2) = L_y(-1, 2)$: the *partial derivatives* of the functions agree at $(-1, 2)$.

(c) The function $L(x, y)$ is called the *linearization* of f at $(-1, 2)$, or the *best linear approximation* of f at $(-1, 2)$. Explain the terminology. It may help to notice that the graph of $z = L(x, y)$, shown in pink, is the *tangent plane* to the surface $z = f(x, y)$, shown in blue, at the point $(-1, 2, -5)$, shown as a ball.

Multivariable Calculus

For problems 1 and 2, let $f(x,y) = x^2 y$, and let $x(s,t) = st$ and $y(s,t) = e^{st}$.

ChRule / DD **1.** Solve for f as a function of s and t. Then find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.



ChRule / DD **2.** The *Chain Rule* from single-variable calculus says:

$$\frac{d}{dx} f\Big(g(x)\Big) = f'\Big(g(x)\Big) \cdot g'(x).$$

We write this as $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$, and draw a "dependence tree" (left side of picture): f depends on g, which depends on x. In multivariable calculus, a function f can depend on several variables (say, x and y), which themselves each depend on several variables (say, s and t). The dependence tree for this example is on the right side of the picture. In this case, the multivariable chain rule says:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \text{ (shown in thick lines)} \qquad \text{and} \qquad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

The idea is, to find $\frac{\partial f}{\partial s}$, you go down every "branch" of the "tree" that connects f to s.

Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ by solving for $\frac{\partial f}{\partial x}, \frac{\partial x}{\partial s}, \ldots$ – each of the four parts of the right sides of each equation above – and substituting for x and y until your expression is entirely in terms of s and t. Compare your answer with that of #1, and discuss which approach you prefer.

Ball / DD **3.** The USA women's soccer team made penalty kicks in their quarterfinal match against Sweden in the 2016 Olympics. Let's assume that the line between the kicker and the goalie is the x-axis, and that the y-coordinate measures height above the ground in feet. The kicker kicks the ball at $(0,0)$ with an initial velocity vector of $[40, 32]$, measured in feet per second.

(a) Make a sketch of this situation.

(b) There is no wind, so the only force acting on the ball is gravity, which has a force of -32ft/sec^2, so the ball's acceleration vector is $\mathbf{r}''(t) = [0, -32]$. Integrate this to find $\mathbf{r}'(t)$.

(c) You should have a integration constant in your answer to (**a**). The initial velocity $\mathbf{r}'(0)$ was given in the problem; use this to find the constants and give an expression for $\mathbf{r}'(t)$.

(d) Integrate $\mathbf{r}'(t)$, and use the initial position $\mathbf{r}(0)$ given in the problem to determine what the integration constant should be, and thus give an expression for $\mathbf{r}(t)$.

(e) The goal is 75 feet from the kicker. Will the ball go into the goal?

Hint: In addition to calculations, you may also need to apply common sense.

ChOr / PEA **4.** Evaluate the double integral $\int_{x=0}^{x=1} \int_{y=x}^{y=1} \cos\left(y^2\right)\, dy\, dx$ without using a calculator. You will need to describe the domain of the integration in the xy-plane in a way that is different from the given description. This is called *reversing the order of integration.*

Gra / PEA **5.** Let $P_0 = (p, q, r)$ be a given point, $\mathbf{n} = [a, b, c]$ be a direction vector, and $X = (x, y, z)$ be a point. Write an equation that says that the vector $\overrightarrow{P_0X}$ is perpendicular to $\mathbf{n}$. Then simplify your equation as much as possible, so that the variables x, y, z are on one side, and the constants a, b, c are on the other side. Sketch an example P_0 and $\mathbf{n}$, and an example X. What does the configuration of *all* such points X look like?

The gradient. We think about $f_x(a, b)$ as being "the rate of change of the function $f(x, y)$ in the positive x-direction at (a, b)," and similarly for $f_y(a, b)$. It turns out that the *vector* whose entries are these two numbers has some meaning. We call this vector the *gradient*:

$$\text{gradient of } f = \nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix}.$$

$$\text{gradient of } f \text{ evaluated at the point } (a, b) = \nabla f(a, b) = \begin{bmatrix} f_x(a, b) \\ f_y(a, b) \end{bmatrix}.$$

Lin / DD **6.** Given a function $f(x, y)$ and a point (a, b), consider the related function

$$L(x, y) = f(a, b) + \nabla f(a, b) \bullet \begin{bmatrix} x - a \\ y - b \end{bmatrix}.$$

(a) Find $L(a, b)$. Then explain why the *values* of the two functions agree at (a, b).

Hint: to find $L(a, b)$, plug in $x = a$ and $y = b$ and simplify.

(b) Find $L_x(a, b)$ and $L_y(a, b)$. Then explain why the *first derivatives* of the two functions agree at (a, b).

(c) What is the relationship between the surface $z = f(x, y)$ and the surface $z = L(x, y)$?

Div / DD **7.** For a vector field $\mathbf{F} = [P, Q]$, its *divergence* div($\mathbf{F}$) is defined as $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$. Compute this for our favorite vector fields **(a)** $\mathbf{F} = [x, y]$ and **(b)** $\mathbf{G} = [-y, x]$ and **(c)** $\mathbf{H} = [1, 2]$. We can think of divergence as measuring the net amount of "stuff" emitted (if positive) or absorbed (if negative) at each point of a vector field. With this in mind:

(d) Referring to your flow line pictures from Page 3 # 4, explain why the divergence values for $\mathbf{G}$ and $\mathbf{H}$ are both 0.

(e) Sketch a vector field that has *negative* divergence. If you can, find an equation for your vector field and check your answer algebraically.

Multivariable Calculus

DirDer / DD **1.** The numbers $f_x(a,b)$ and $f_y(a,b)$ give us rates of change of f at (a,b) in the positive x- and y-directions. What if we want to know the rate of change of f in some *other* direction?

(a) Suppose that you are at the point (a,b), headed in the direction of the vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.
Explain why your position can be described by the equation $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} a + u_1 t \\ b + u_2 t \end{bmatrix}$.

(b) Explain why each of the following equalities is true:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(a,b)\ u_1 + f_y(a,b)\ u_2 = \nabla f(a,b) \bullet \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

(c) The symbol ∂ is for a *partial* derivative (when a function depends on more than one variable), while the symbol d is for a *total* derivative (for a function of one variable.) Explain why some of the derivatives in part (b) are ∂ and some are d.

(d) Explain why, if we want to use the equation in part (b) to answer the question "what is the rate of change of f in the direction of the vector $\mathbf{u}$?" we must use a *unit vector* for $\mathbf{u}$.

HOP / DD **2.** For the function $g(x) = x^2 + \sin x - e^x$, find $g'(x), g''(x)$, and $g'''(x)$.

For problems 3 and 4, use the function $\qquad f(x,y) = x^2 y + 2x + x \sin y.$

HOP / DD **3.** Find the partial derivatives of f with respect to x and y.

HOP / DD **4.** Just as you can take multiple *derivatives* $g'(x), g''(x), g'''(x)$ of a function $g(x)$ of one variable, you can take multiple *partial derivatives* of a function of several variables.

(a) For example, the *second partial derivative* f_{xx} means "the partial derivative of f_x with respect to x." For the function $f(x,y)$, compute f_{xx} and also f_{yy}.

(b) You can also compute a *mixed partial derivative* f_{xy}, which means "the partial derivative of f_x with respect to y." For the function $f(x,y)$, compute f_{xy} and also f_{yx}.

More problems on the next page.

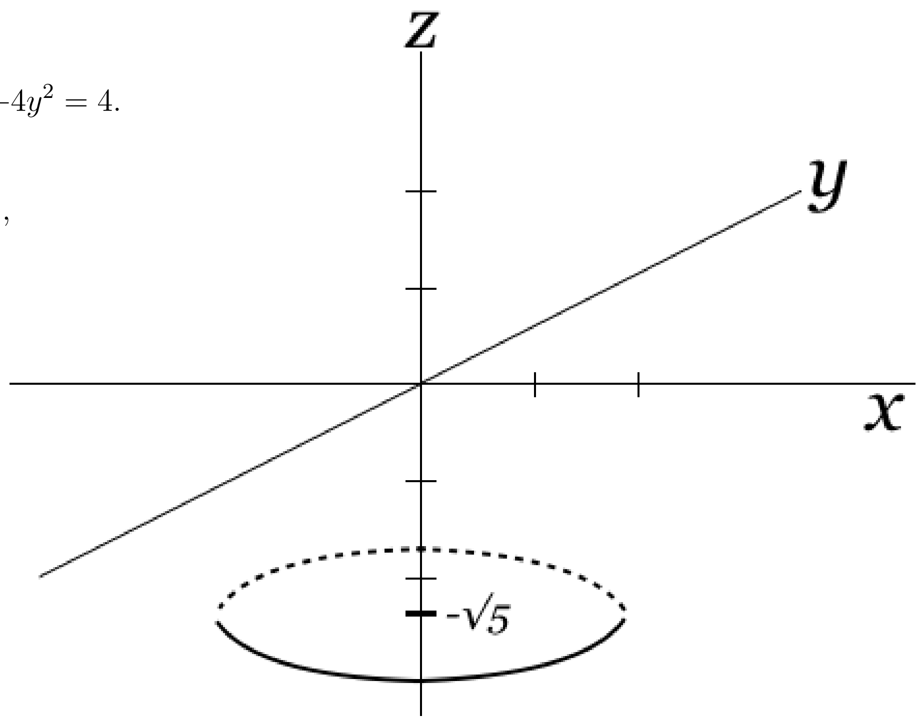
Gra / DD **5.** Verify that the point $P = (2, 1, 3)$ is on the *hyperboloid* $x^2 + 4y^2 + 1 = z^2$.

(a) Sketch cross sections of this surface for $z = -\sqrt{5}, -\sqrt{2}, -1, 0, 1, \sqrt{2}, \sqrt{5}$.

For example, we plug in $z = -\sqrt{5}$ and get

$x^2+4y^2+1 = 5 \implies x^2+4y^2 = 4$.

This is an ellipse, which contains the points $(2, 0)$, $(-2, 0)$, $(1, 0)$, $(-1, 0)$, so it has a width of 2 in the x-direction and a width of 1 in the y-direction. I have drawn this ellipse in the picture at a height of $z = -\sqrt{5} \approx -2.2$. Now you do the rest.



(b) After sketching in all of the cross-sections, describe what the surface looks like.

Gra / PEA **6.** Verify that the point $Q = (7, 2, 8)$ is on the hyperboloid $x^2 + 4y^2 - 1 = z^2$.

(a) Show that for this hyperboloid, *every* level curve $z = k$ is an ellipse.

(b) Conclude that this hyperboloid is a connected surface, in contrast to the preceding example, which had two separate parts. We call this one a *one-sheeted* hyperboloid, and the preceding example is a *two-sheeted* hyperboloid. Make a sketch of the surface.

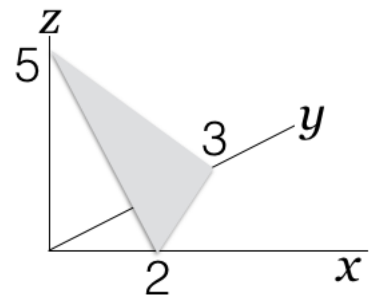
SLI / PEA **7.** If $[x(t), y(t)]$ is a parametric curve, then $\left[\frac{dx}{dt}, \frac{dy}{dt}\right]$ is its velocity and $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is its speed. Find a parameterized curve whose speed is $\sqrt{t^4 - 2t^2 + 1 + 4t^2}$.

☺ / PEA **8.** The integral $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ is a template for what type of problem?

Multivariable Calculus

Planes / DD

1. Find an equation for the plane that contains the triangle shown at right.

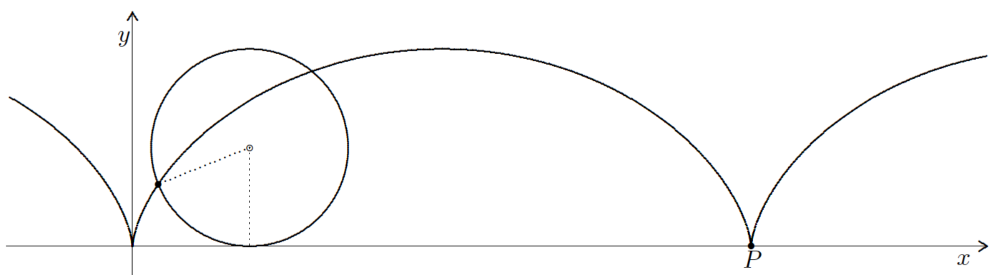


Planes / DD

2. Sketch the plane given by the equation $6x + 2y - 3z = 12$.

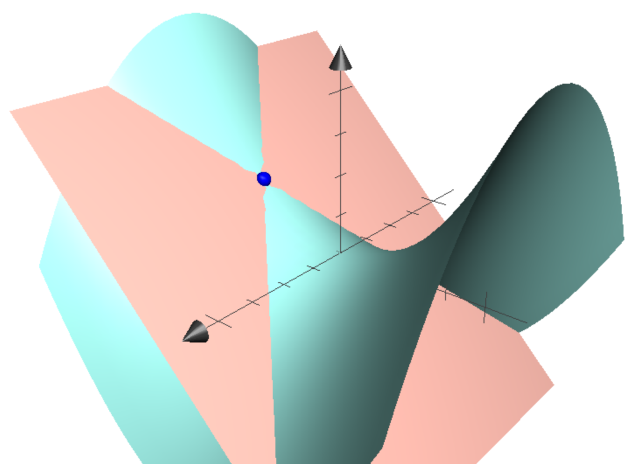
SLI / PEA

3. The *cycloid* $(x, y) = (t - \sin t, 1 - \cos t)$ is the path followed by a point on the edge of a wheel of unit radius that is rolling along the x-axis. The point begins its journey at the origin (when $t = 0$) and returns to the x-axis at $x = 2\pi$ (when $t = 2\pi$), after the wheel has made one complete turn. What is the length of the cycloidal path that joins these x-intercepts? This length is called the *arclength*.



Lin / DD

4. Just as a tangent *line* to a curve gives a good linear approximation to the curve near the point of tangency, a tangent *plane* to a surface gives a good linear approximation near the point of tangency. The picture for Page 6 # 6 shows one example of a tangent plane, and another is to the right.



(a) For a surface S given by $z = f(x, y)$, and a point $P = (a, b, f(a, b))$ on the surface, explain why the vectors $[1, 0, f_x(a, b)]$ and $[0, 1, f_y(a, b)]$ give tangent directions to S at P. *Hint*: Page 4 # 2

(b) Use these two tangent vectors to find a *normal vector* to S at P.

(c) Find an equation for the tangent plane at the point $P = (1, 2, 10/3)$ to the familiar surface $z = 5 - x^2/3 - y^2/3$ shown in Page 3 # 5.

(d) Find a way to check your answer, and do so.

VF / DD **5.** (Continuation of Page 2 # 4) What does the dot product *mean*?

Answer 2. By Page 5 # 7, $\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Informed by this knowledge, fill in the following statements with right/obtuse/acute:

- $\mathbf{u} \bullet \mathbf{v} > 0$ when the angle between $\mathbf{u}$ and $\mathbf{v}$ is ____________.
- $\mathbf{u} \bullet \mathbf{v} = 0$ when the angle between $\mathbf{u}$ and $\mathbf{v}$ is ____________.
- $\mathbf{u} \bullet \mathbf{v} < 0$ when the angle between $\mathbf{u}$ and $\mathbf{v}$ is ____________.

DbInt / PEA **6.** *Fubini's Theorem* states that

$$\int_a^b \int_c^d f(x,y)\, dy\, dx \;=\; \int_c^d \int_a^b f(x,y)\, dx\, dy$$

is true whenever f is a function that is continuous at all points in the rectangle $a \le x \le b$ and $c \le y \le d$. Despite the intuitive content of this statement, a proof is not easy, and this will be left for a later course in real analysis. It suffices to do examples that illustrate its non-trivial content.

(a) Sketch the region of integration for each integral above.

(b) Verify the conclusion of the theorem using $f(x,y) = x\sin(xy)$ and the rectangle $1 \le x \le 2$ and $0 \le y \le \pi$.

Hint: To compute the integral in the order $dx\ dy$, you will need to use integration by parts *multiple times.* Don't give up!

Multivariable Calculus

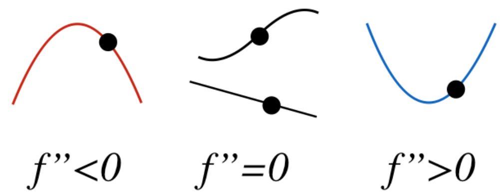
Lin / DD **1.** Find the tangent plane to the surface $z = x^2y + 2x + x \sin y$ at the point $(1, 0, 2)$.

ChOr / DD **2.** For the following double integral, first sketch the region of integration, and then change the order of integration to $dx\ dy$. *Hint*: You will have to use *two* integrals!

$$\int_{x=0}^{x=1} \int_{y=-x}^{y=x} f(x, y)\ dy\ dx$$

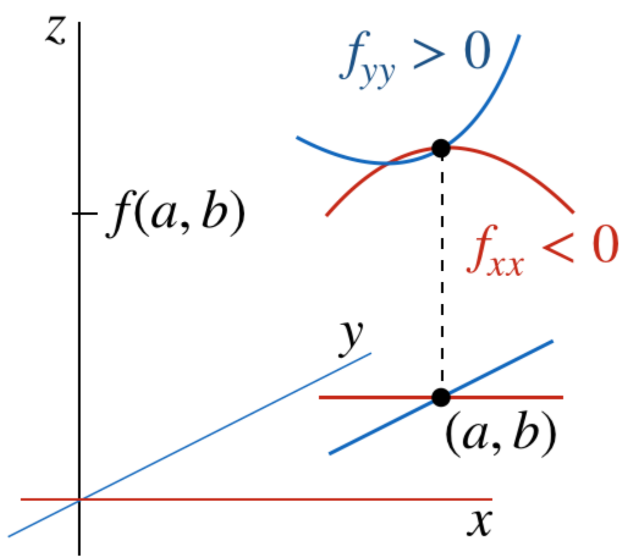
HOP / DD

3. You can *compute* a second partial derivative, but what does it *mean*? Recall from single-variable calculus that when $f''(x) > 0$, the function is *concave up*, when $f''(x) < 0$, the function is *concave down*, and when $f''(x) = 0$, the function is (at least instantaneously) flat. Second partial derivatives measure the same thing, but *in the directions* of the x- and y-axes. f_{yy}, for example, measures whether f_y is increasing, decreasing or 0 as you go in the positive y-direction.



In the picture, the red curve is a cross-section of a surface $z = f(x, y)$ in the x-direction through the point (a, b), and the blue curve is a cross-section of the same surface in the y-direction through (a, b). This information tells us that the surface $z = f(x, y)$ has a saddle / pringle shape at $(a, b, f(a, b))$.

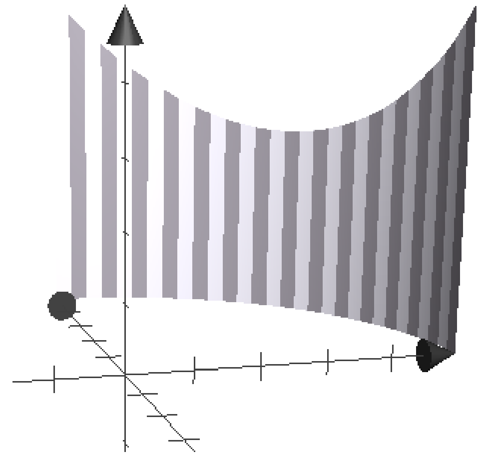
For the function $f(x, y) = 5 - x^2/3 - y^2/3$ shown in Page 3 # 5, look at the picture and say whether f_{xx} and f_{yy} should be positive, negative or 0 at $(1, 2)$. Then compute them.



DirDer / DD **4.** Suppose that you are on a landscape whose elevation can be modeled by the function $f(x, y) = e^{xy} - xy^2$, and you are standing at the point where $(x, y) = (1, 2)$.

(a) Find the rate of change of your elevation if you were to walk north (positive y-direction), or east (positive x-direction).

(b) Find the rate of change of your elevation if you were to walk south, or west.



SLI / SC **5.** Suppose that you are building a fence from $(5, 0)$ to $(0, 5)$, following the curve C that is the part of the circle of radius 5 centered at the origin. The height of the fence at the point (x, y) is $f(x, y) = 10 - x - y$. Draw a picture of this situation.

(a) Set up an integral to find the *length* of the fence.

(b) Set up, and evaluate, a single integral to find the total *area* of the fence.

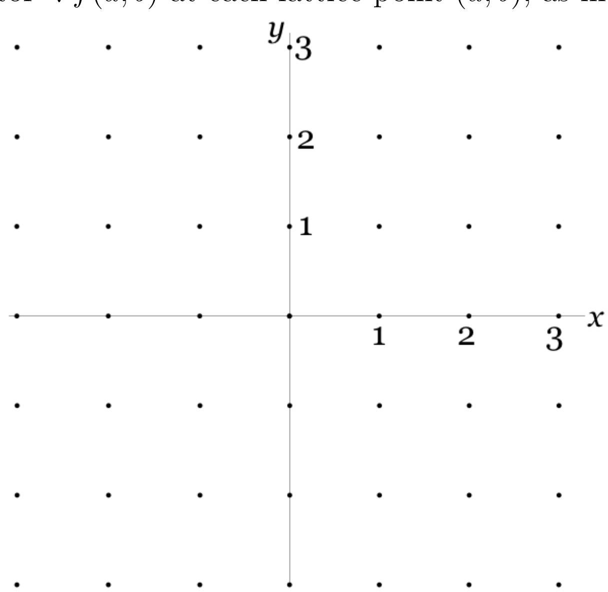
SLI / DD **6.** (Continuation) Fill in all the details for the following equation, and explain why it holds:

$$\int_C f(x, y)\, ds = \int_{t=a}^{t=b} f(\mathbf{x}(t)) \cdot |\mathbf{x}'(t)|\, dt.$$

This is called the *scalar line integral of f along C.*

On the left side, C is the curve; $f(x, y)$ is the height of the fence at the point (x, y); and ds is an infinitesimal piece of arclength along the curve. On the right side, t is the "time" parameter for the curve in the xy-plane; $\mathbf{x}(t)$ is the location of the particle that traces out the curve, at time t; $|\mathbf{x}'(t)|$ is the particle's speed at time t, and dt is an infinitesimal amount of time. When I ask you to "fill in all the details," this means you need to explain all the parts, like how C and (x, y) turned into a and b and $\vec{\mathbf{x}}(t)$, and you need to make sense of it in the context of the curve and the fence.

Grad / PEA **7.** Given a function f that is differentiable, one can form the vector $\nabla f = [f_x, f_y]$ at each point in the domain of f, to create a *gradient vector field*. Suppose that $f(x, y) = x^2 + 4y^2$. Sketch the gradient vector $\nabla f(a, b)$ at each lattice point (a, b), as in the picture below.



Hint: You may wish to scale the lengths of all of your vectors down by some factor like 1/4 so that they fit nicely on your picture.

Multivariable Calculus

DirDer / DD **1.** Find the rate of change of the function $f(x, y) = e^{xy} - xy^2$, at the point $(1, 2)$, in the direction of the vector $[5, 12]$. *Hint*: Use Page 8 # 1.

Grad / DD **2.** (Continuation) The notation for the directional derivative of the function f, at the point (a, b), in the direction of the unit vector $\mathbf{u}$, is $D_{\mathbf{u}} f(a, b)$.

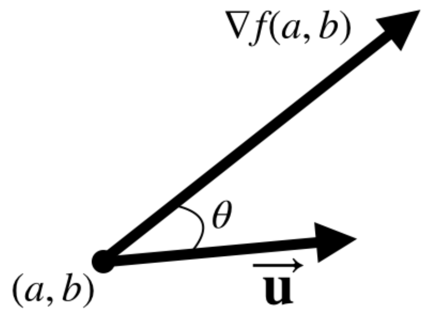
(a) Rewrite the question in problem 1, using this new notation.

(b) Explain why $D_{\mathbf{u}} f(a, b)$ is a *number*, whose meaning is a *rate of change*.

(c) Justify each of the following equalities (recall Page 8 # 1b):

$$D_{\mathbf{u}} f(a, b) = \nabla f(a, b) \bullet \mathbf{u} = |\nabla f(a, b)| \cdot |\mathbf{u}| \cdot \cos\theta = |\nabla f(a, b)| \cos\theta.$$

Here $\bullet$ denotes the dot product, $\cdot$ denotes scalar multiplication, and θ is the angle between the vectors $\nabla f(a, b)$ and $\mathbf{u}$ in the xy-plane.



(d) Suppose that you want to go in the direction of the *maximum* rate of change – because $f(x, y)$ describes your elevation on a mountain, say, and you want to ascend as quickly as possible. Which direction should your unit vector $\mathbf{u}$ point, in order to maximize the directional derivative of f at (a, b) in the direction of $\mathbf{u}$?

(e) With this in mind, explain the geometric meaning of the direction vector $\nabla f(a, b)$.

Lin / DD **3.** Find all of the points on the surface $z = 3x^2 - 4y^2$ where the tangent plane is parallel to $3x + 2y + 2z = 10$.

FSV / DD **4.** Sketch the following surfaces:

(a) $x^2 + 4y^2 - z^2 = -1$

(b) $x^2 + 4y^2 - z^2 = 0$

(c) $x^2 + 4y^2 - z^2 = 1$

(d) Explain why each of the surfaces you sketched is a *level surface* of the function

$$f(x, y, z) = x^2 + 4y^2 - z^2$$

at a different level. What does the "movie" of *all* of the level surfaces look like?

Polar / PEA **5.** The point $P = (-5, 8)$ is in the second quadrant. You are used to describing it by using the *rectangular coordinates* -5 and 8. It is also possible to accurately describe the location of P by using a different pair of coordinates: its *distance from the origin* and an *angle in standard position* (measured counter-clockwise from the positive x-axis). These numbers are called *polar coordinates*. Calculate polar coordinates for P, and explain why there is more than one correct answer.

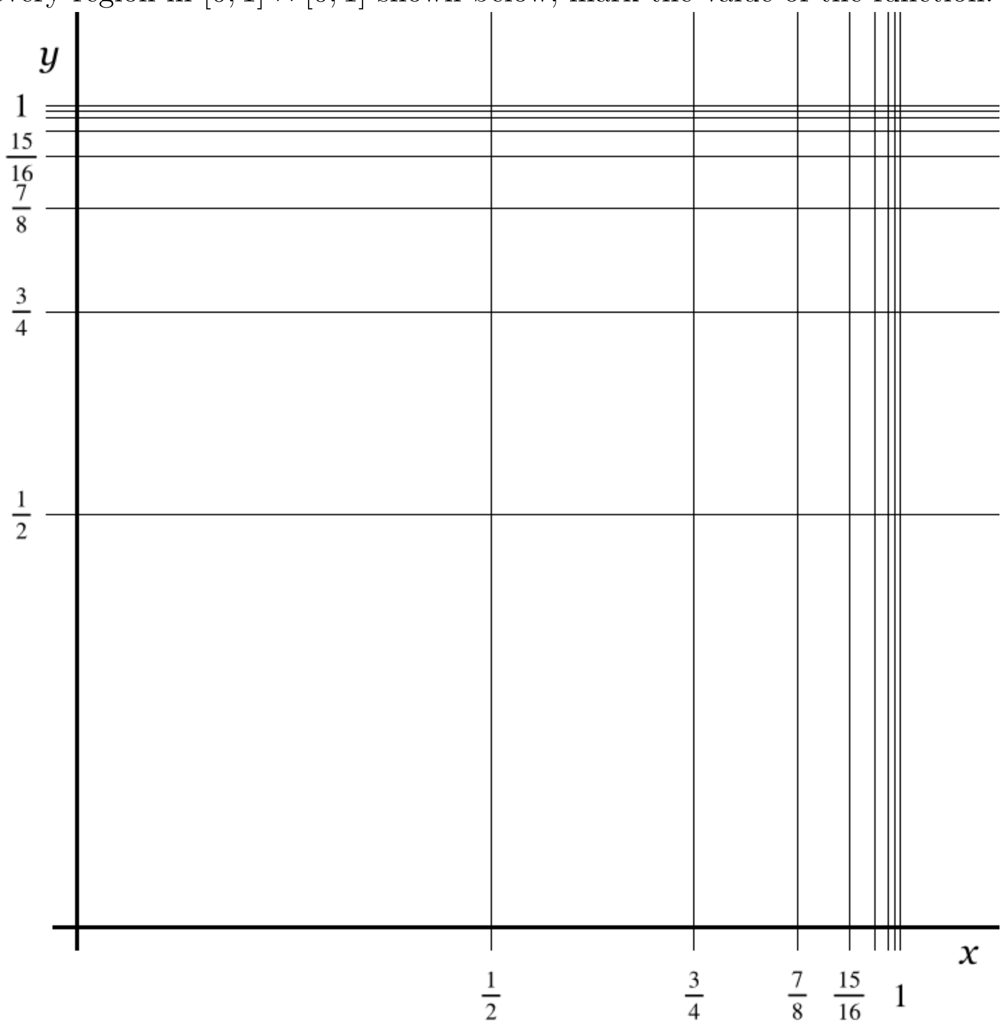
Multivariable Calculus

Interval notation: If $a \le x \le b$, we say that x is in the interval $[a, b]$. We use square brackets for "$\le$" and round brackets for "$<$," so for example if $c < y < d$, then y is in the interval (c, d). The rectangle where $a \le x < b$ and $c \le y < d$ is denoted by $[a, b) \times [c, d)$.

DbInt / DD **6.** *An example where Fubini's Theorem does not apply.* Define $f(x, y)$ on the "unit square" $[0, 1] \times [0, 1]$ as follows:

$$f(x,y) = \begin{cases} 1 & \text{on } [0,1/2) \times [0,1/2) \\ -2 & \text{on } [1/2,3/4) \times [0,1/2) \\ 4 & \text{on } [1/2,3/4) \times [1/2,3/4) \\ -8 & \text{on } [3/4,7/8) \times [1/2,3/4) \\ & \vdots \\ 0 & \text{elsewhere.} \end{cases}$$

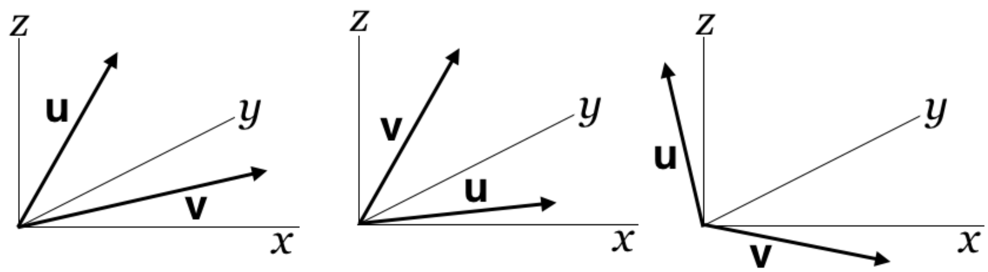
(a) For every region in $[0, 1] \times [0, 1]$ shown below, mark the value of the function.



(b) Show that $\displaystyle\int_0^1 \int_0^1 f(x, y)\ dy\ dx = \frac{1}{4}$ while $\displaystyle\int_0^1 \int_0^1 f(x, y)\ dx\ dy = 0$. Discuss.

Multivariable Calculus

CrProd / DD **1.** The direction of the cross product $\mathbf{u} \times \mathbf{v}$ is given by the *right hand rule*: Place vectors $\mathbf{u}$ and $\mathbf{v}$ tail-to-tail. Flatten your right hand, and point your fingers in the direction of $\mathbf{u}$. Now curl your fingers in the direction of $\mathbf{v}$ (you may have to flip over your hand to do this). Your thumb points in the direction of $\mathbf{u} \times \mathbf{v}$. For each set of vectors $\mathbf{u}$ and $\mathbf{v}$ below, sketch a vector in the direction of $\mathbf{u} \times \mathbf{v}$. (In these pictures, the x- and z-axes are in the plane of the page, and the y-axis extends away from you. Use your 3D imagination!)



CrProd / DD **2.** (Continuation) The orientation of the x, y and z-axes are always given by the right hand rule, so that

$$(x\text{-direction}) \times (y\text{-direction}) = (z\text{-direction}).$$

Confirm this in the pictures above. Then draw pictures of the x, y and z-axes so that:

(a) z points up and y points to the right, **(b)** z points up and y points to the left,

(c) z points up and x points to the left, **(d)** z points down.

The 200 Swiss franc bill, showing an alternative method for the right hand rule.

Curl / DD **3.** For a vector field $\mathbf{F} = [P, Q, R]$, the *curl* of $\mathbf{F}$, curl($\mathbf{F}$), is defined by the *vector*

$$[R_y - Q_z, P_z - R_x, Q_x - P_y] = \det\left(\begin{bmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{bmatrix}\right).$$

Compute the curl of each of the vector fields

(a) $\mathbf{F} = [x, y, 0]$ and **(b)** $\mathbf{G} = [-y, x, 0]$ and **(c)** $\mathbf{H} = [z, 0, -x]$.

(d) What information to you think the curl vector is intended to convey?

VF / DD **4.** Consider again $f(x,y) = x^2 + 4y^2$, whose vector field you sketched in Page 10 # 7.

(a) In one color, draw flow lines for the gradient vector field ∇f, using representative flow lines all over the picture.

(b) In a different color, add level curves to your picture, for at least 10 different levels.

(c) Explain why flow lines and level curves *always intersect perpendicularly.*

Lin / DD **5.** One reason to find the tangent plane to a surface at a point is that it gives a good *linear approximation* to the function near that point. Consider $f(x,y) = x^2y + 2x + x\sin y$.

(a) Without a calculator, compute $f(1,0)$ (in radians).

(b) Without a calculator, try to compute $f(1.1,-0.1)$. If you can't do it, explain why not.

(c) In Page 10 # 1, we found a tangent plane to $z = x^2y + 2x + x\sin y$ at the point $(1,0,2)$, which is $z = 2x + 2y$. Use this linear approximation to find a good estimate for $f(1.1,-0.1)$. Notice that you did this entire thing with a pencil and paper. Wow!

(d) Use your calculator to find $f(1.1,-0.1)$. How close was your approximation?

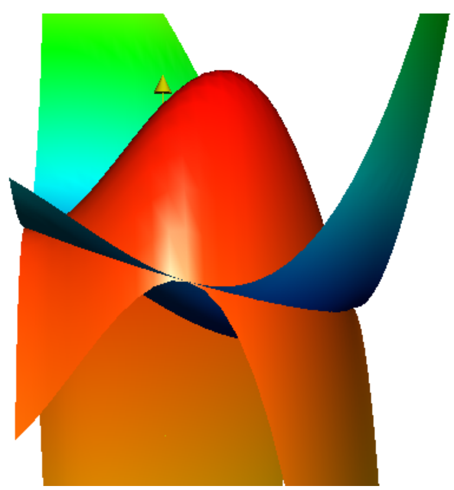
DbInt / DD **6.** A metal plate consists of the region bounded by the curves $y = x$ and $y = x^2$.

(a) Sketch this region, in a LARGE, CLEAR diagram, and set up a double integral to integrate a function $f(x,y)$ over this region.

(b) The amount of electric charge at a point (x,y) of the plate is $f(x,y) = 2xy$ coulombs per square cm. Find the total amount of charge on the plate.

DbInt / DD

7. You have found the volume under a given surface (such as x^3y or $x/4 + y/4 + 3$) over a given region. But what about the volume *between two surfaces*? For this, you have to find the region of integration in the xy-plane, and then set up the limits of integration.



(a) Consider the surfaces $z = -(x^2-y)(y-x-2)-x$ and $z = 4(x^2-y)(y-x-2)-x$, pieces of which are shown to the right. Find the curves in the xy-plane that are the shadows of the intersection curves of these surfaces. Sketch the curves in the xy-plane and shade the region between them, which is our region of integration.

(b) Write a double integral to find the volume of the solid that is enclosed between the surfaces, and then compute its value (it is tedious, so you may use a calculator).

8. *Clairaut's Theorem* says that, for a function $f(x,y)$ with continuous first and second derivatives, $f_{xy} = f_{yx}$. Make up a function $f(x,y)$ that no one else will think of, and check that the theorem holds for your example.

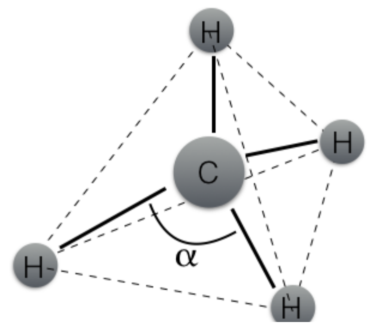
Multivariable Calculus

Review for midterm 1, which is in the evening after this class.
The following optional review problems are provided for your convenience.

0. On the course topic map on page iii, circle the topics with which you are comfortable.

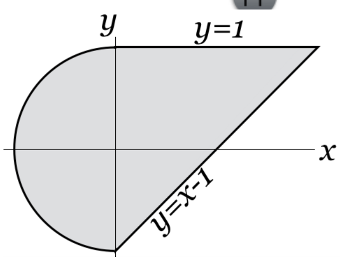
☺ / R-A

1. The methane molecule CH_4 consists of a carbon molecule bonded to four hydrogen molecules that are spaced as far apart from each other as possible. The hydrogen atoms then sit at the vertices of a tetrahedron, with the carbon atom at its center, as shown. We can model this with the carbon atom at the point $(1/2, 1/2, 1/2)$ and the hydrogen atoms at $(0, 0, 0), (1, 1, 0), (1, 0, 1)$ and $(0, 1, 1)$. Find the bond angle α formed between any two of the line segments from the carbon atom to the hydrogen atoms.



DbInt / DD

2. Set up a double integral, in both orders of integration, to integrate a function $f(x, y)$ over the shaded region $\mathcal{R}$ shown to the right, which is made from the unit circle and the two lines $y = 1$ and $y = x - 1$. Which order do you prefer?



ParEq / DD

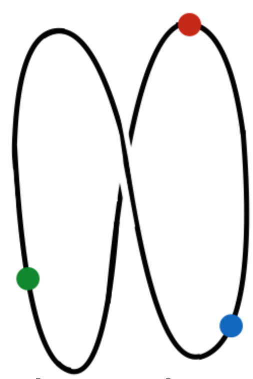
3. The figure shows the graph of the curve $\mathbf{r}(t) = [\cos t, \sin t, 2 \sin 2t]$.

(a) For which values of t, x and y do the maximum z-values occur?

(b) For which values of t, x and y do the minumum z-values occur?

(c) Use the previous parts to accurately sketch in the x, y, z-axes.

(d) Compute the velocity vector $\mathbf{r}'(t)$, and use it to find an equation for the tangent line to the curve at $t = \pi/4$. Check that your solution agrees with your sketch.



☺ / DD **4.** Find the area of the parallelogram spanned by the vectors $[1, 2, 3]$ and $[-1, 3, -6]$.

ParEq / DD **5.** A mosquito flies at a constant speed according to the equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4+t \\ 1 \\ 1-t \end{bmatrix}$. A spiderweb, with a patient spider, hangs in the position of the plane $2x + 3y + 5z = 15$. Will the mosquito get caught in the web, and if so, when and where?

Ball / DD **6.** Given the acceleration vectors $\vec{\mathbf{p}}''(t) = [6t, \cos t]$, the velocity vector $\vec{\mathbf{p}}'(0) = [1, 2]$, and the position vector $\vec{\mathbf{p}}(0) = [-\pi^3, -1]$, calculate the position vector $\vec{\mathbf{p}}(\pi)$.

7. Write an equation for a plane that passes through the point $(-1, 2, -3)$, whose normal vector is $[2, 3, -5]$.

8. Do the lines $\begin{bmatrix} 3-2t \\ -1 \\ 1+t \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1+t \\ -t \end{bmatrix}$ intersect? If so, where?

9. Make a list of problems, from any page 1–13 in this book, that you would like a classmate or the professor to explain, and any other questions you would like to ask.

Multivariable Calculus

Big picture overview.

We've already started exploring a lot of the ideas in multivariable calculus, listed below. *In italics* are topics that we have started to explore a little bit, but *not enough to test.*

0. **Setup:** Lines, planes, curves, *functions of several variables*, level curves and surfaces, cross product, quadric surfaces (paraboloids, hyperbolic paraboloids, hyperboloids)
1. **Derivatives**: Partial derivatives, tangent planes, chain rule, *higher-order partial derivatives*, directional derivatives, *the gradient*
2. **Integrals:** Double integrals, changing order of integration
3. **Calculus with vector fields:** Parametric curves, dot product, vector fields, flow lines, *divergence*, arclength, *scalar line integrals*

In single-variable calculus, you studied functions like $f(x) = x^2$, of a single variable x. How is the function value changing when you change x? Compute the slope $f'(x)$. How do we get a good linear approximation of the function? Use the slope to find a tangent line, which matches the function's value and its derivative at the point of tangency. How do we find the area under the curve, over some interval $\mathcal{I}$ of the real line? Integrate $\int_{\mathcal{I}} f(x)\ dx$.

Now in multivariable calculus, we have more dimensions, so we could have a **curve** in 3-space, like $\mathbf{r}(t) = [t, t^2, \sin t]$. Now to write an equation for a tangent line to this curve, we'd need a tangent **vector**, and it will be a **parametric equation**. To determine the length of the curve, say between time $t = 1$ and $t = 3$, we compute the **arclength**.

We can also have functions of several variables, such as the **quadric surface** $f(x, y) = 5 + x^2 - y^2$. How do we know what the surface $z = f(x, y)$ looks like? Sketch **level curves**. What if we have a function $f(x, y, z)$ of three variables? Sketch **level surfaces**.

How is the function value $f(x, y)$ changing when you change x and y? Well, it depends how much you're changing x versus y – what direction you're going. So to answer this question, we have to use a **directional derivative**. How do we get a good linear approximation to the **surface** $z = f(x, y)$ at a point? We compute the **tangent plane**, which matches the function's value and both **partial derivatives** at the point of tangency. To write down the equation for the tangent plane, we find a normal vector, using the **cross product**, and use the components of the vector as the coefficients of x, y and z as we derived in Page 6 # 2.

How do we find the volume under a surface $z = f(x, y)$, over some region $\mathcal{R}$ of the plane? Compute the **double integral** $\iint_{\mathcal{R}} f(x, y)\ dx\ dy$. What if that's difficult or impossible? **Change the order of integration** to $dy\ dx$, which requires sketching the region of integration $\mathcal{R}$.

What if we have water or wind swirling around? We can describe this motion using a **vector field**. One way to graphically represent a vector field is to draw vectors as arrows at representative points in the plane, and another way is with **flow lines**. If we want to measure how much two vectors point in the same direction, or find the angle between them, we use the **dot product**.

Multivariable Calculus

1. Let the symbol ∇ ("nabla") denote the *differential operator* in each coordinate:

$$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] \qquad \text{or} \qquad \nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right], \text{ etc.}$$

Using this notation, explain why, for a function $f(x, y)$ and a vector field $\mathbf{F}(x, y)$, we have:

(a) gradient of f is ∇f **(b)** divergence of $\mathbf{F}$ is $\nabla \bullet \mathbf{F}$ **(c)** curl of $\mathbf{F}$ is $\nabla \times \mathbf{F}$

··· / DD **2.** Let f be a function and let $\mathbf{F}$ and $\mathbf{G}$ be vector fields. Which of the following expressions make mathematical sense? If you can compute any of them, do so.

(a) curl(div($\mathbf{F}$)) **(b)** curl(∇f) **(c)** div($\mathbf{F} \bullet \mathbf{G}$) **(d)** curl(div($f$))

(e) div(curl($\mathbf{G}$)) **(f)** div(∇f) **(g)** curl(curl($\mathbf{F}$))

DirDer / DD **3.** Suppose that $f(x, y) = 5 - x^2 - y^2/2$ gives the elevation at the point (x, y) of a mountain upon which you are snowshoeing, and you are at the point $(1, 2, 2)$.

(a) Which direction should you hike, if you want to climb most steeply? *Express this direction as a unit vector, and also as an angle γ from the positive x-axis.*

(b) What is the directional derivative of elevation in that direction?

(c) Suppose you only want to climb *half* as steeply as the slope in part (b) that is given by hiking in the direction from part (a). Such a route is shown in the picture. Which direction should you go? *Hint*: Use Page 11 # 2c. *You may find it easiest to express your answer as an angle γ.*

Polar / PEA **4.** Polar coordinates for a point P in the xy-plane consist of two numbers, r and θ, where r is the distance from P to the origin O, and θ is the counter-clockwise angle between the positive x-axis and the ray OP. Find polar coordinates for each of the following points:

(a) $(0, 1)$ **(b)** $(-1, 1)$ **(c)** $(4, -3)$ **(d)** $(1, 7)$ **(e)** $(-1, -7)$

Polar / PEA **5.** Describe the configuration of all points whose polar coordinate r is 3. Describe the configuration of all points whose polar coordinate θ is 110.

☺ / DD **6.** Suppose that the flow of air in a very turbulent area is given by the vector field $[3x^2z + y^2 + x, 2xy - y, x^3 + 4z]$. You toss a plastic bag into this area and watch the wind push it around. Is it rotating?

SLI / SC **7.** Calculate the *scalar line integral* of the function $f(x, y) = 3y$ over the curve C consisting of the portion of the graph of $y = 2\sqrt{x}$ between $(1, 2)$ and $(9, 6)$.

Multivariable Calculus

Polar / PEA **1.** *More polar coordinates*

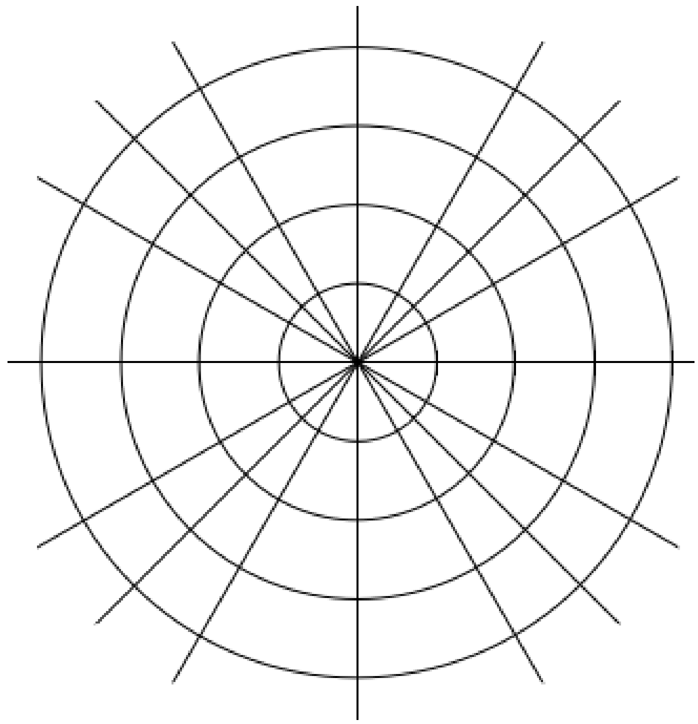
(a) Convert the polar pair $(r, \theta) = (8, 150°)$ to an equivalent Cartesian pair (x, y).

(b) Given polar coordinates r and θ for a point, how do you calculate the Cartesian coordinates x and y for the same point?

Polar / DD **2.** Figure out what the curve $r = \sin\theta$ looks like, in two different ways:

(a) Fill in the following table, and then plot the points on the "polar graph paper" below right. It has circles of radius $0.25, 0.5, 0.75$ and 1, and rays at multiples of $\theta = \pi/6$ and $\theta = \pi/4$. You will have to think about what a "negative radius" means. Connect the points to sketch the whole curve.

θ	$r = \sin\theta$
0	
$\pi/6$	
$\pi/4$	
$\pi/3$	
$\pi/2$	
$2\pi/3$	
$3\pi/4$	
$5\pi/6$	
π	
$7\pi/6$	
$5\pi/4$	
$4\pi/3$	
$3\pi/2$	
$5\pi/3$	
$7\pi/4$	
$11\pi/6$	



(b) Multiply both sides of the equation $r = \sin\theta$ by r, convert to rectangular coordinates x and y, and complete the square to put the equation into a familiar form. Does your equation agree with your sketch in part (a)?

Taylor / DD **3.** On the same set of axes, sketch the graphs of the following functions:

$$f(x) = \cos(x)$$
$$L(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$$
$$Q(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2}\left(x - \frac{\pi}{3}\right)^2$$

Explain what the functions $L(x)$ and $Q(x)$ are doing at the point $(\pi/3, 1/2)$.

Taylor / DD

4. Using a computer, on the same set of axes, graph the following functions:

$$f(x,y) = e^x \cdot \cos(y)$$
$$L(x,y) = -1 - x$$
$$Q(x,y) = -1 - x - x^2 + (y-\pi)^2$$

Explain what the functions $L(x,y)$ and $Q(x,y)$ are doing at the point $(0, \pi, -1)$.

Grad / DD

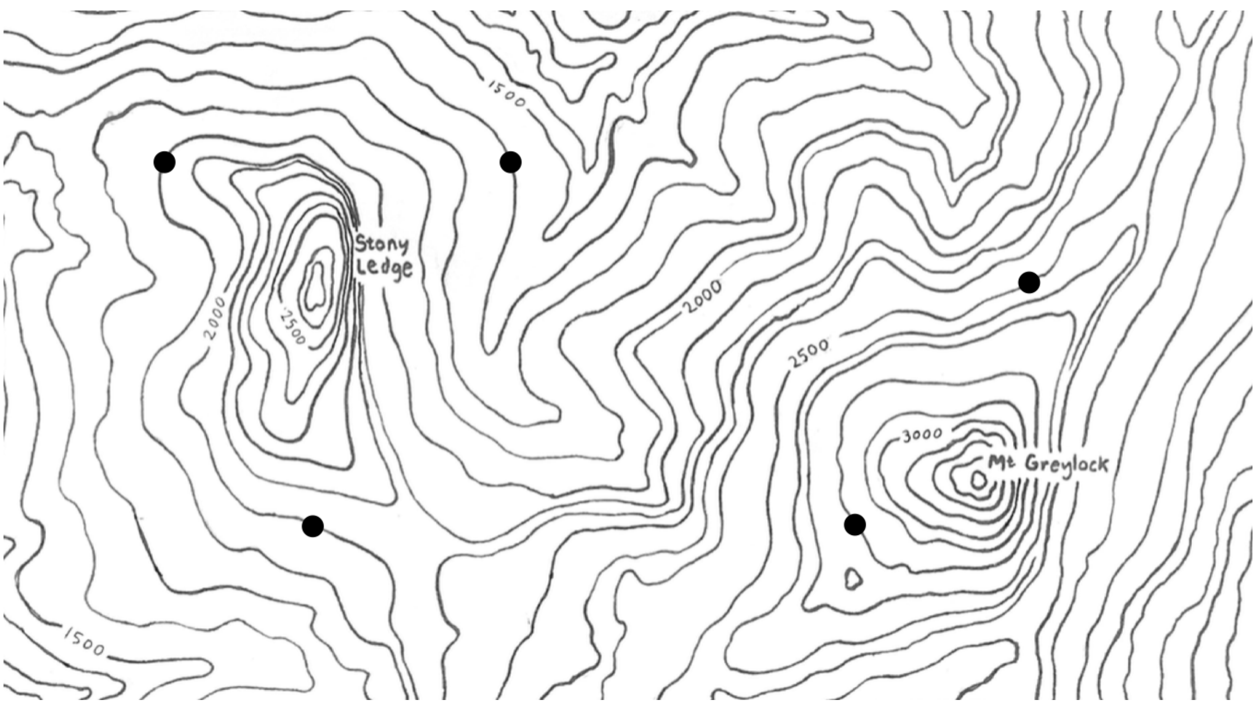
5. The topographical map below shows level curves, at 100-foot intervals, of the elevation function $f(x,y)$ for two mountains and the surrounding landscape.

(a) Locate the tops of the two mountains. Write in the level of each curve on the map, until you have a good understanding of where the higher parts are and where the lower parts are.

(b) For each of the black points in the map, sketch the gradient vector ∇f at that point.

(c) For each black point, also sketch the path ("flow line") that a rolling ball would take, if you dropped it there. Also sketch its path *up* the mountain in backwards time. Remember that your flow line should intersect *perpendicularly* with each level curve it crosses.

(d) Are any of your flow lines closed loops? If not, can you construct an example of a mountainscape that has a flow line that is a closed loop? If so, do it; if not, explain why not.



VF / DD

6. *The gradient and the curl.*

(a) Prove that the curl of a gradient vector field is always $\vec{\mathbf{0}}$.

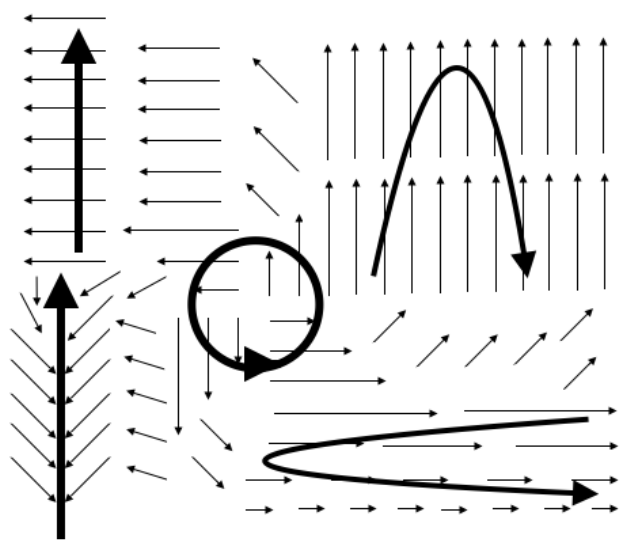
(b) Justify geometrically why a nonzero curl cannot occur for a gradient vector field. For example, you might think about when $f(x,y)$ is an elevation function as above, and consider that having a nonzero curl is similar to having a flow line that is a closed loop.

Multivariable Calculus

1. Consider the vector field **F** shown in the diagram (thin arrows), and let **T** denote a unit tangent vector to a directed curve (thick arrows). Determine whether

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds$$

is positive, negative, or 0 for each directed curve C – in other words, determine whether the *work* done by the vector field on each curve is positive, negative or 0.



SLI / DD **2.** Sketch the helix $\mathbf{h}(t) = [a\cos t, a\sin t, bt]$.

(a) Compute the direction vectors $\mathbf{h}'(t)$ and $\mathbf{h}''(t)$. Could you have anticipated their directions?

(b) Find the arclength from $t = 0$ to $t = 6\pi$. Verify that your answer is reasonable.

(c) Find the arclength from $t = 0$ to $t = T$, for any value $T > 0$. If $\mathbf{h}(t)$ represents the position of a bumblebee at time T, what does your expression in terms of T represent?

(c) Write an equation for the tangent line to $\mathbf{h}(t)$ at $t = \pi/2$. Add the line to your sketch.

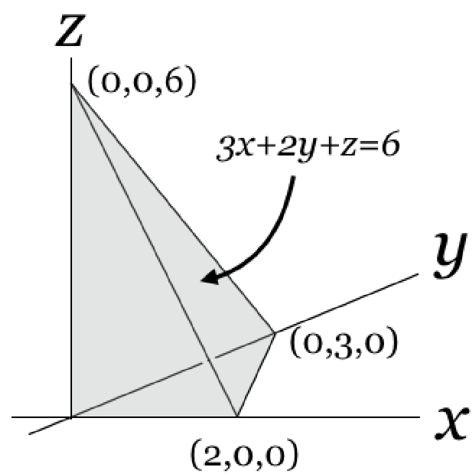
TripInt / DD **3.** The purpose of this problem is to find the volume in the first octant (the part of 3-space where x, y and z are all positive) bounded by the coordinate planes and the plane $3x + 2y + z = 6$.

(a) Find the volume of the region using basic geometry.

(b) Find the volume of the region using a double integral in the order $dy\ dx$.

When we use a double integral over a region $\mathcal{R}$ in the xy-plane to find a volume sitting over that plane, we can think of $\mathcal{R}$ as the *shadow* of the volume we want to compute. We don't have to use the shadow in the xy-plane – we can use the shadow in *any* of the three coordinate planes!

(c) Use the yz-plane as the "shadow plane," and write a double integral that finds the volume of the region using a double integral in the order $dz\ dy$.



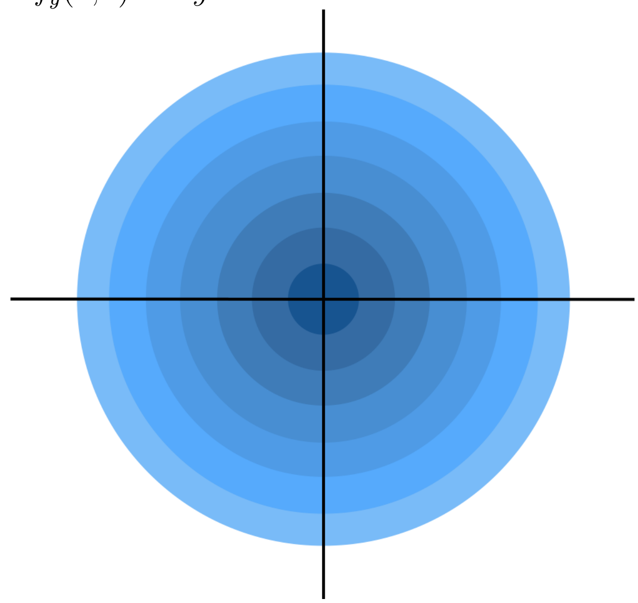
Lin / DD **4.** Show that, given a function $f(x,y)$ and a point (a,b), the tangent plane to the surface $z = f(x,y)$ at the point $(a,b,f(a,b))$ is

$$z = f(a,b) \; + \; f_x(a,b)\,(x-a) \; + \; f_y(a,b)\,(y-b).$$

This is also known as the *best linear approximation of* f *at* (a,b). Explain the terminology.

Lin / DD **5.** (Continuation) The following symbols appear in the equation above. Say which ones are variables, and which ones are numbers.

$$z \qquad f(a,b) \qquad f_x(a,b) \qquad x \qquad a \qquad f_y(a,b) \qquad y \qquad b$$



DbInt / PEA **6.** Let $V(x,y) = 1 - x^2 - y^2$ be interpreted as the speed (cm/sec) of fluid that is flowing through point (x,y) in a pipe whose cross section is the unit disk $x^2 + y^2 \le 1$. Assume that the flow is the same through every cross-section of the pipe. Notice that the flow is most rapid at the center of the pipe, and is rather sluggish near the boundary.

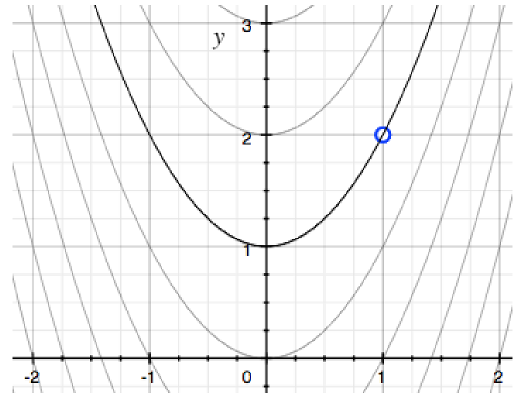
The volume of fluid that passes each second through any *small* cross-sectional box whose area is $\Delta A = \Delta x \cdot \Delta y$ is approximately $V(x,y) \cdot \Delta x \cdot \Delta y$, where (x,y) is a representative point in the small box. (Here the symbol Δ stands for a tiny distance.)

(a) Using an integral with respect to y, combine these approximations to get an approximate value for the volume of fluid that flows each second through a strip of width Δx that is parallel to the y-axis. The result will depend on the value of x representing the position of the strip.

(b) Use integration with respect to x to show that the volume of fluid that leaves the pipe (through the cross-section at the end) each second is $\pi/2 \approx 1.57$ cc.

Hint: trig substitution, $x = \sin\theta$. This requires some clever single-variable calculus, so if you get stuck at some point, it's ok; we'll later discover a better way to work this one out.

ChVar / PEA **7.** In setting up a double integral, it is customary to tile the domain of integration using little rectangles whose areas are $\Delta x \cdot \Delta y$. In some situations, however, it is better to use small tiles whose areas can be described as $r \cdot \Delta\theta \cdot \Delta r$. Sketch such a tile, and explain the formula for its area. In what situations would such tiles be useful?

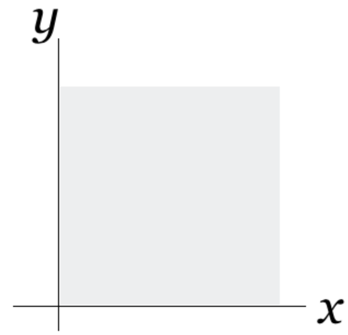


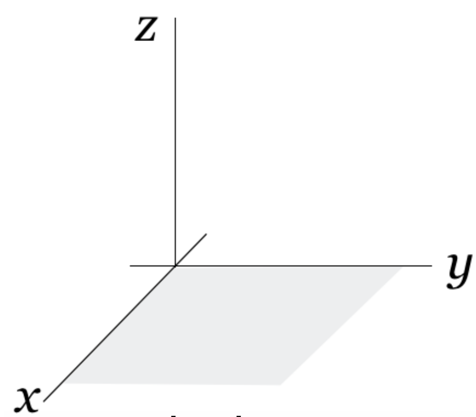
1. Suppose that you want to find the equation of the line that is *perpendicular* to the parabola $y = x^2 + 1$ at the point $(1, 2)$, as shown to the right.

(a) One way to do this is to find the slope of the curve $y = x^2 + 1$ at $x = 1$, and use it to find the equation of the perpendicular line. Do so.

(b) Another way to think about the curve $y = x^2 + 1$ is that it is a *level curve* of the function $f(x, y) = y - x^2$. At what level? Label the level of each curve shown in the picture.

(c) You know that the gradient vector $\nabla f(1, 2)$ is perpendicular to the level curve of $f(x, y)$ that passes through $(1, 2)$. Use this to find the line equation, and notice that in this case, parametric form is arguably the easiest.





2. In this problem, you will sketch the *solid* region of integration for the following integral:

$$\int_{x=0}^{x=1} \int_{y=\sqrt{x}}^{y=1} \int_{z=0}^{z=1-y} f(x, y, z)\; dz\; dy\; dx.$$

(a) First, sketch the *shadow* of the solid in the xy-plane, using the limits of integration on the outer two integrals, in the top picture on the right.

(b) Now, draw that shadow again in perspective, on the xy-plane in the bottom picture. Then imagine the surfaces $z = 0$ and $z = 1 - y$ in the 3D picture, and draw their intersections with the yz-plane. You can think about the shadow region as an infinitely tall "cookie cutter" slicing vertically through both those surfaces, and the solid region of integration is the part cut out between the surfaces. Sketch the solid in your 3D picture (or maybe just its edges).

VLI / DD **3.** In Page 16 # 1, we estimated the tendency of a vector field $\mathbf{F} = [P, Q]$ to point in the same direction as an oriented curve C, which is a *vector line integral*. We can compute its value using the integral $\int_C \mathbf{F} \bullet \mathbf{T}\; ds$ for a unit tangent vector $\mathbf{T}$.

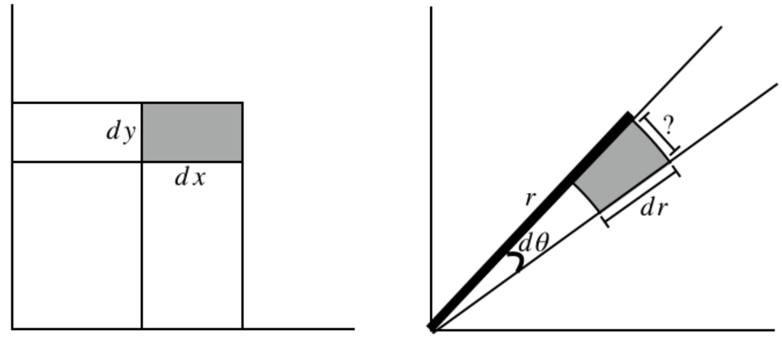
In the special case when $\mathbf{r}'(t) = [x'(t), y'(t)]$ is a *unit* tangent vector, we can use it as our unit vector $\mathbf{T}$, and integrate $[P, Q] \bullet [x'(t), y'(t)]\; dt$ over the curve C.

(a) Sketch the oriented curve D consisting of the line segment from $(0, -1)$ to $(0, 1)$, followed by the right half of the unit circle from $(0, 1)$ to $(0, -1)$. *Hint*: it should look like a "D."

(b) Let $\mathbf{F} = [-y, x]$. Estimate (positive, negative, or zero?) $\int_C \mathbf{F} \bullet \mathbf{T}\; ds$ for the two parts C_1, C_2 of D.

(c) Integrate the vector field $\mathbf{F} = [-y, x]$ over D (compute the *vector line integral*). You will need to parameterize each part, and compute the unit tangent vector $\mathbf{T}$ for each part.

5. In Page 16 # 6, you integrated $f(x,y) = 1 - x^2 - y^2$ over the unit disk $x^2 + y^2 \leq 1$. This is much easier in *polar coordinates*, replacing the Cartesian area form $dA = dx\ dy$ with the polar area form $dA = r\ dr\ d\theta$.

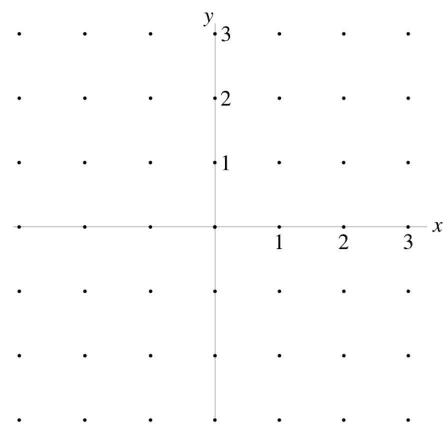


(a) Explain why the following two integrals are equal.

$$\int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} (1 - x^2 - y^2)\ dy\ dx = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (1 - r^2)\ r\ dr\ d\theta$$

(b) Compute the one on the right.

Grad / DD **6.** Let $f(x,y) = \frac{1}{4}xy$. Sketch the gradient vector field $\nabla f(x,y)$ on $[-3,3] \times [-3,3]$.

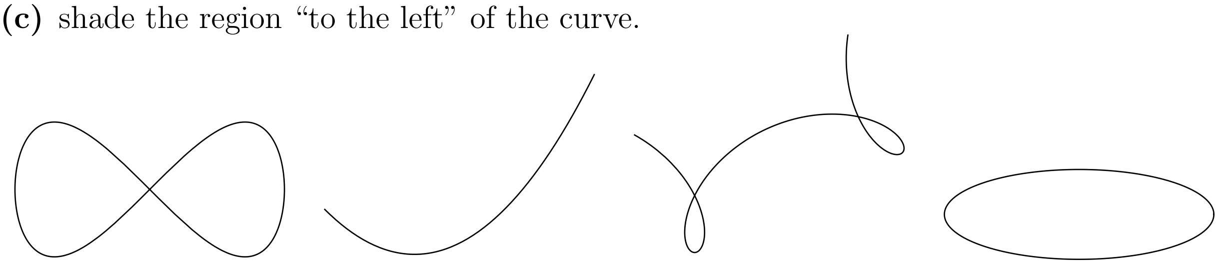


GrThm / DD **7.** A *simple* curve is one that does not intersect itself. A *closed* curve is one that ends where it starts, that "closes up." An *oriented* curve has a direction of travel. For each curve below,

(a) say whether it is simple and whether it is closed,

(b) draw an arrow on it to give it an orientation, and

(c) shade the region "to the left" of the curve.



DbInt / PEA **8.** *Using a double integral to evaluate a tricky integral.* Let $f(0) = 2$, and for nonzero values of x, let $f(x) = \dfrac{e^{-x} - e^{-3x}}{x}$.

(a) Explain why it is not possible to simply compute $\displaystyle\int_0^\infty f(x)\,dx$.

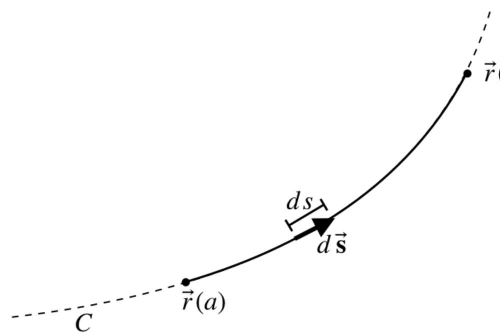
(b) Find a, b and $g(x,y)$ so that $\dfrac{e^{-x} - e^{-3x}}{x} = \displaystyle\int_a^b g(x,y)\ dy$.

(c) Evaluate the improper integral $\displaystyle\int_0^\infty f(x)\,dx$, by using the "trick" of rewriting this integral as $\displaystyle\int_0^\infty f(x)\,dx = \int_0^\infty \int_a^b g(x,y)\ dy\ dx$ and reversing the order of integration.

Multivariable Calculus

VLI / DD **1.** Consider a vector field $\mathbf{F}$, and a curve C that consists of the part of the curve $\mathbf{r}(t) = (x(t), y(t))$ from $t = a$ to $t = b$. As in Page 16 # 1 and Page 17 # 3, the notation $\mathbf{T}$ means a unit tangent vector to $\mathbf{r}(t)$, in the direction of motion. As shown below, the notation ds means a tiny distance along a curve, and the notation $d\mathbf{s}$ means a tiny *directed* distance along a curve (equivalently, a tiny tangent vector). Justify each of the equalities in the following chain of equations:

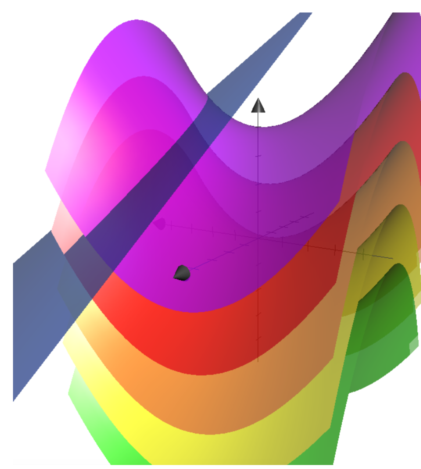
$$\int_C \mathbf{F} \bullet d\mathbf{s} = \int_C \mathbf{F} \bullet \mathbf{T}\ ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \cdot |\mathbf{r}'(t)|\ dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\ dt.$$



(a) Use the version on the far right to compute the *vector line integral* of $\mathbf{F} = \left[\frac{-y \sin x}{x^2}, \frac{\cos x}{2x}\right]$ over the piece of the parabola $y = x^2$ from $(\pi/2, \pi^2/4)$ to $(5\pi/4, 25\pi^2/16)$.

(b) The formula $\int_C \mathbf{F} \bullet d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\ dt$ gives us a way to calculate vector line integrals without having to reparameterize our curve to unit speed, or in other words when the unit tangent vector $\mathbf{T}$ is difficult to compute. Try to parameterize the curve in part **(a)** to unit speed (i.e. defining $\mathbf{r}(t)$ so that $|\mathbf{r}'(t)| = 1$), and then explain why it is difficult.

Grad / DD **2.** Consider the following problem: *Find the points on the surface $z = 3x^2 - 4y^2$ where the tangent plane is parallel to* $3x+2y+2z = 10$. Back in Page 11 # 3, you probably thought about the surface $z = 3x^2 - 4y^2$ as the graph $z = f(x, y)$ of the function $f(x, y) = 3x^2 - 4y^2$, and solved this problem either by finding a normal vector, or by setting partial derivatives equal to each other. Both are very good methods.



(a) Explain why you can think of this surface as a *level surface* of the "temperature" function $g(x, y, z) = 3x^2 - 4y^2 - z$, at level 0.

(b) Explain why, at any point (a, b, c) on the surface $g(x, y, z) = 3x^2 - 4y^2 - z = 0$, the gradient vector $\nabla g(a, b, c)$ is perpendicular to the surface at that point.

(c) Use this insight to solve the problem.

Lin / DD **3.** Suppose that you wish to compute $\sqrt{3.01^2 + 3.98^2}$ without using a calculator.

(a) Estimate the answer in your head.

(b) Find a linear approximation (think tangent plane) of the function $f(x, y) = \sqrt{x^2 + y^2}$ at a convenient point close to $(3.01, 3.98)$, and then use it to estimate the answer.

(c) Check your answer with a calculator or computer. How good was the approximation that you did *with just a pencil and paper*?

ConVF / DD **4.** Let $f(x, y)$ give the elevation of the point (x, y) for the region where you are hiking. The picture below shows the gradient vector field $\nabla f(x, y)$.

(a) Where is the highest point on the map? Where is the lowest point on the map?

(b) Identify important features such as mountaintops, valleys, streams, etc., and explain how you know where they are.

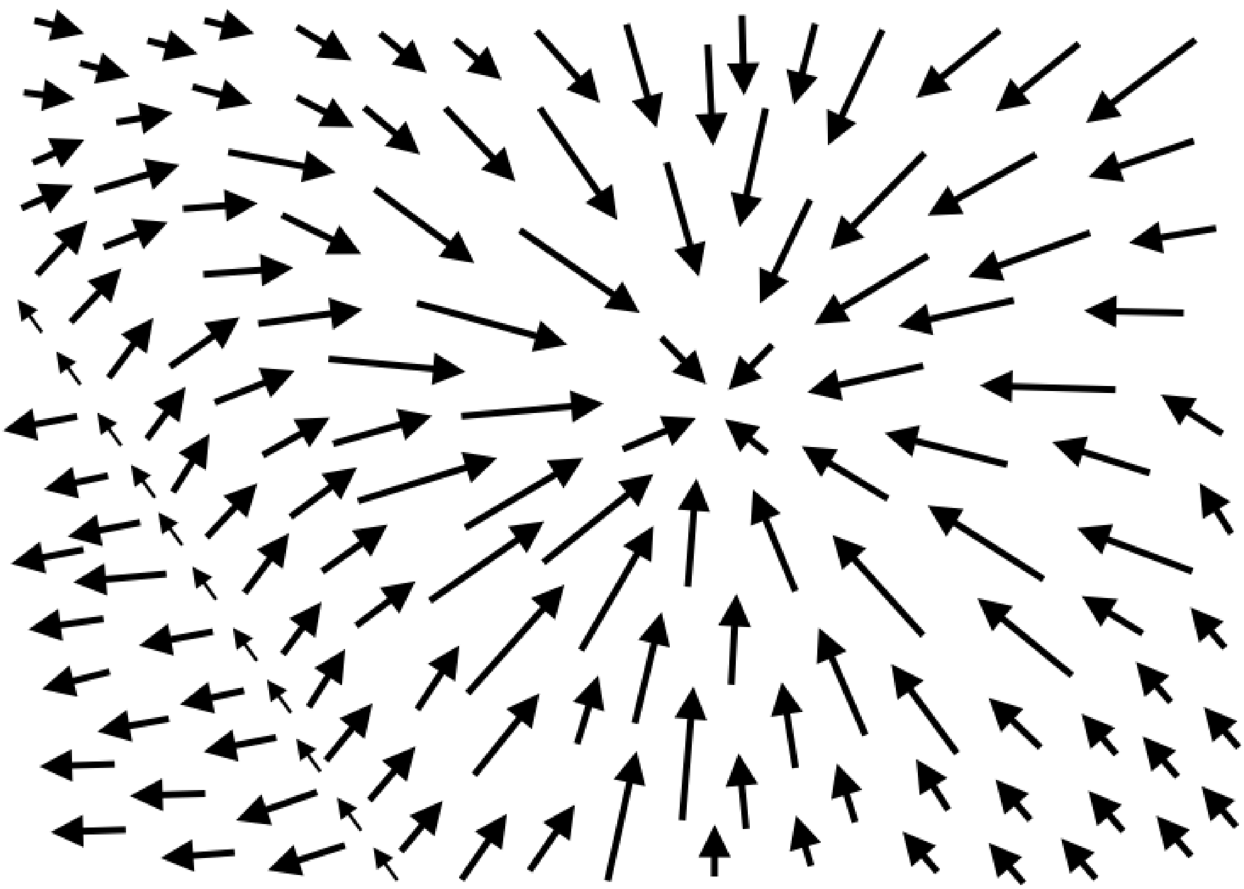
(c) Mark two points A and B of your choice, far apart. Connect A and B by a curve C_1. Estimate the value (positive? negative? big? small?) of the vector line integral

$$\int_{C_1} \nabla f \bullet d\mathbf{s}.$$

(d) Connect the same A and B by a very different curve C_2. Estimate

$$\int_{C_2} \nabla f \bullet d\mathbf{s}.$$

(e) What is the physical meaning of the integral $\int_C \nabla f \bullet d\mathbf{s}$ in this context?



VLI / PEA **5.** Let $\mathbf{F}$ be a vector field, and let the curve C be a flow line of $\mathbf{F}$. In such a situation, is it *always* true that $\int_C \mathbf{F} \bullet d\mathbf{s} > 0$? If so, explain why; if not, give a counterexample.

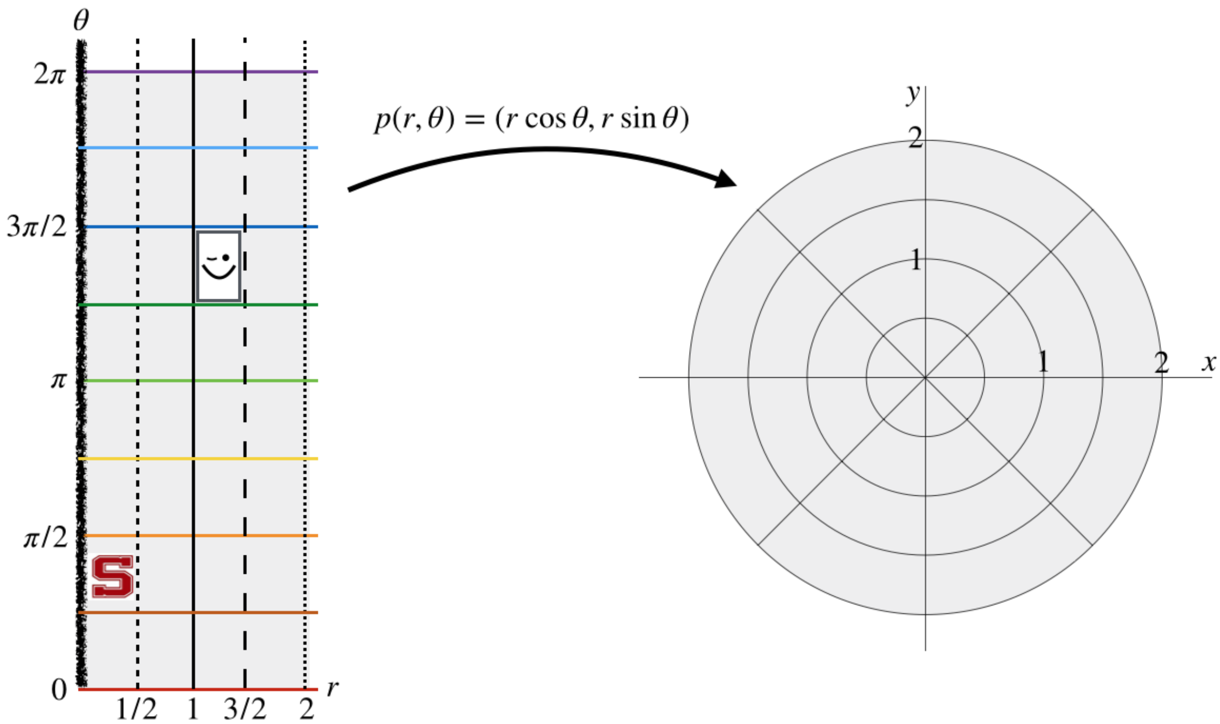
ChVar / PEA **1.** The familiar equations $x = r\cos\theta$, $y = r\sin\theta$ can be thought of as a *mapping* from the $r\theta$-plane to the xy-plane. In other words, $p(r,\theta) = (r\cos\theta, r\sin\theta)$ is a function of the type $\mathbf{R}^2 \to \mathbf{R}^2$. Point by point, p transforms regions of the $r\theta$-plane onto regions of the xy-plane. The picture below shows the effect of p on the rectangle $[0,2] \times [0,2\pi]$.

ChVar / DD **(a)** In the picture on the right, mark the image under p of each of the 5 vertical segments on the left. Use the different strokes (solid, dashed, dotted) to indicate which is which.

(b) In the picture on the right, mark the image under p of each of the 9 horizontal segments, using colors to indicate which is which.

(c) I have made little pictures in two small sub-rectangles. Sketch their images under p.

(d) A nice way to think about this mapping is that the $r\theta$-plane is a "rubber sheet," and the mapping moves and stretches it when transforming it into the xy-plane. Draw a "movie" showing two intermediary "frames" between the two pictures shown below.



ChVar / PEA **2.** (Continuation) Consider the rectangle defined by $1.9 \le r \le 2$ and $1 \le \theta \le 1.2$. Sketch it in the $r\theta$-plane, and also sketch its image under p in the xy-plane. Then find the area of each. How do the areas of these two regions compare?

ChVar / PEA **3.** (Continuation) The derivative of p at $(2,1)$, which could be denoted $p'(2,1)$, is a 2×2 matrix, and its determinant is an interesting number. Explain these statements. It may help to know that these determinant matrices are usually denoted $\dfrac{\partial(x,y)}{\partial(r,\theta)}$.

VLI / DD **4.** *Notation.* Here are some equivalent ways of writing down the vector line integral of the vector field $\mathbf{F} = [P, Q]$ over the directed curve C:

$$\int_C \mathbf{F} \bullet d\mathbf{s} = \int_C [P, Q] \bullet [dx, dy] = \int_C (P\ dx + Q\ dy).$$

(a) Justify each of the two equalities above.

(b) Sketch the curve C consisting of the line segment from $(-2, 0)$ to $(0, 0)$, followed by the line segment from $(0, 0)$ to $(0, 3)$. Let $\mathbf{F} = [2x^2 - 3y, 3x + 2y^2]$. Use the $\int_C (P\ dx + Q\ dy)$ form to compute the vector line integral of $\mathbf{F}$ over C, taking advantage of tricks to make things easier whenever you can.

ConVF / PEA **5.** Suppose that $\mathbf{F} = [P(x, y), Q(x, y)]$ is a *gradient field*, i.e. $\mathbf{F} = [P, Q] = \nabla f$ for some function $f(x, y)$, and that $\mathcal{C}$ is a piecewise differentiable path in the xy-plane. It so happens that the value of $\int_{\mathcal{C}} P\,dx + Q\,dy$ depends *only* on the endpoints of the curve traced by $\mathcal{C}$.

(a) Verify this for the field $\mathbf{F} = [xy^2, x^2y]$ by selecting at least two different piecewise differentiable paths from $(0, -1)$ to $(1, 1)$ and evaluating both integrals.

(b) A vector field that is the gradient field for a function $f(x, y)$ is called a *conservative vector field*, and f is called its *potential function*. Find a potential function f for $\mathbf{F}$, and evaluate $f(1, 1) - f(0, -1)$.

Let's call this result the *Fundamental Theorem of Line Integrals*: If $\mathbf{F}$ is a conservative vector field, and its potential function f is defined on a region containing the curve C, then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = f(\text{end point of } C) - f(\text{starting point of } C).$$

(c) Give a geometric explanation of why this is true. *Hint*: Recall Page 18 # 4.

(d) Use the Chain Rule and the Fundamental Theorem of Calculus to prove this fact.

ConVF / DD **6.** Some people like to remember, "A vector field is conservative if and only if its curl is $\mathbf{0}$." Justify this. (By the way, to apply it to a vector field $[P, Q]$ in $\mathbf{R}^2$, think of it as $[P, Q, 0]$.)

ParSurf / DD **7.** Suppose that you need to know an equation of the tangent plane to a surface S at the point $P = (0, 1, 1)$, and you know that the curves

$$\mathbf{r}_1(s) = (0, s, s) \qquad \text{and} \qquad \mathbf{r}_2(t) = (\cos t, \sin t, 1)$$

both lie on S. Find an equation for the tangent plane to S at P.

Multivariable Calculus

ParSurf / DD **1.** Consider the surface

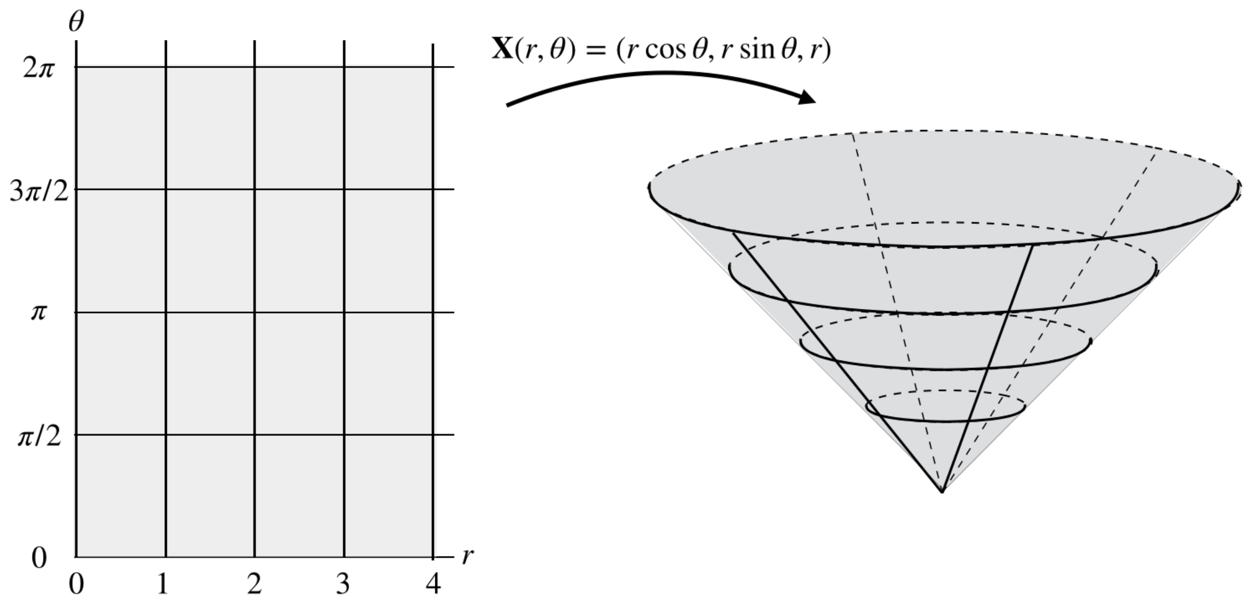
$$\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, r) \qquad \text{for} \qquad 0 \le \theta \le 2\pi, \qquad 0 \le r \le 4.$$

(a) This is called a *parameterized surface*. Explain the terminology.

(b) Explain why the curves $\mathbf{r}_1(s)$ and $\mathbf{r}_2(t)$ from Page 19 # 7 both lie on the surface $\mathbf{X}(r,\theta)$.

Terminology: We call these the r-*curve* and the θ-*curve* through a given point.

(c) We can think of $\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, r)$ as a *mapping* from the $r\theta$-plane to xyz-space. Explain the pictures below in this context, and label the one on the right.



(d) Sketch in $\mathbf{r}_1(s)$ and $\mathbf{r}_2(t)$ on the right picture above.

GrTh / PEA **2.** *Green's Theorem* says the following: If $\mathcal{R}$ is a closed, bounded region in $\mathbf{R}^2$ whose boundary C consists of finitely many simple, closed, piecewise-differentiable curves, oriented so that $\mathcal{R}$ is on the left when one traverses C, and if $\mathbf{F} = [P, Q]$ is differentiable everywhere in $\mathcal{R}$, then

$$\oint_C \mathbf{F} \bullet d\mathbf{s} = \oint_C (P\ dx + Q\ dy) = \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dx\ dy.$$

Note: The symbol $\oint$ has a circle to indicate that the line integral is over a *closed* curve.

(a) Verify the result of Green's Theorem by explicitly calculating each side of the above equation when $\mathbf{F} = [-y, x]$ and $\mathcal{R}$ is the half-disk $x^2 + y^2 \le 1, x \ge 0$. *Hint*: You've already found one side.

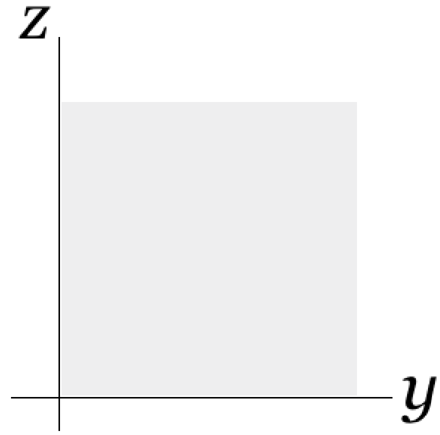
(b) Explain why $\mathbf{F}$ and $\mathcal{R}$ satisfy the requirements of Green's Theorem.

GrTh / PEA **3.** When $P(x,y) = -\frac{1}{2}y$ and $Q(x,y) = \frac{1}{2}x$, Green's Theorem is interesting. Explain.

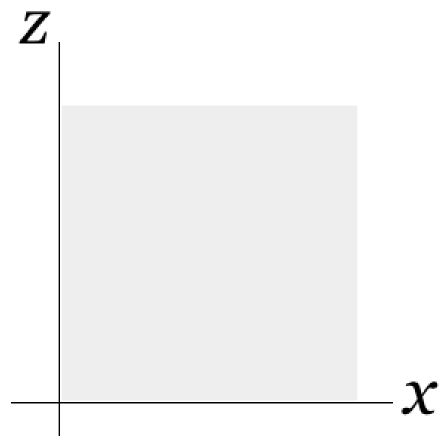
4. In Page 17 # 2, you sketched the *solid* region of integration for the integral:

$$\int_{x=0}^{x=1}\int_{y=\sqrt{x}}^{y=1}\int_{z=0}^{z=1-y} f(x,y,z)\ dz\ dy\ dx.$$

In this problem, we'll write the integral in two other orders of integration, using the other two coordinate planes as the "shadow plane," one at a time. You will need to refer to your sketch from that problem in order to do this one.



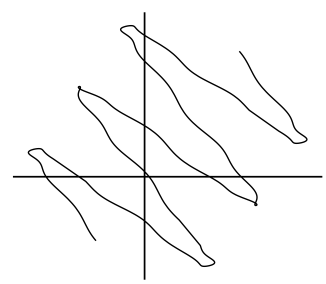
(a) Order $dx\ dy\ dz$: First, sketch the *shadow* of the solid in the yz-plane to the right, and use it to write your limits of integration for y and z. Then, for each point (y,z) in the shadow region, determine the surface through which a line, parallel to the x-axis and through the point (y,z), would enter the solid, and where it would exit the solid. Use this to write your limits of integration for x, which will be functions of y and z.



(b) Write the integral in the order $dy\ dz\ dx$: sketch the shadow of the solid in the xz-plane and use this to determine your x and z limits of integration. (You will need to find the curve of intersection of some surfaces, in terms of x and z.) Then determine the surfaces where a line parallel to the y-axis enters and exits the solid, and use this to find your y limits of integration.

ConVF / DD

5. Let $\mathbf{F} = [e^y + y^2 + 1, xe^y + 2xy + \cos y]$.

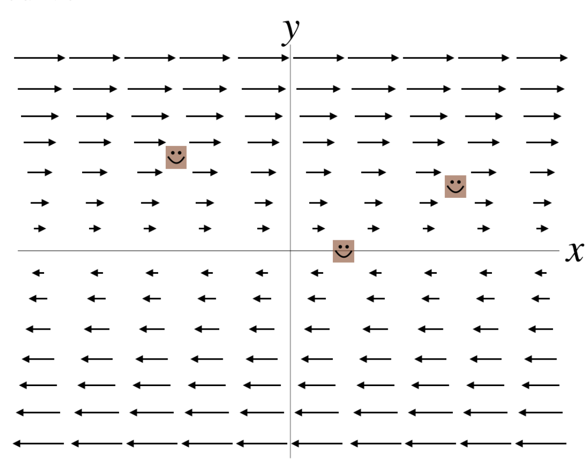


(a) Show that $\mathbf{F}$ is conservative (see Page 19 # 5), by finding a potential function $f(x,y)$ so that $\nabla f = \mathbf{F}$.

(b) Compute the line integral $\int_C \mathbf{F} \bullet d\mathbf{s}$, where C is the curve from $(-2,-2)$ to $(3,4)$ shown in the diagram.

curl / DD

6. The vector field $\mathbf{F} = [y, 0] = [y, 0, 0]$ is shown to the right.



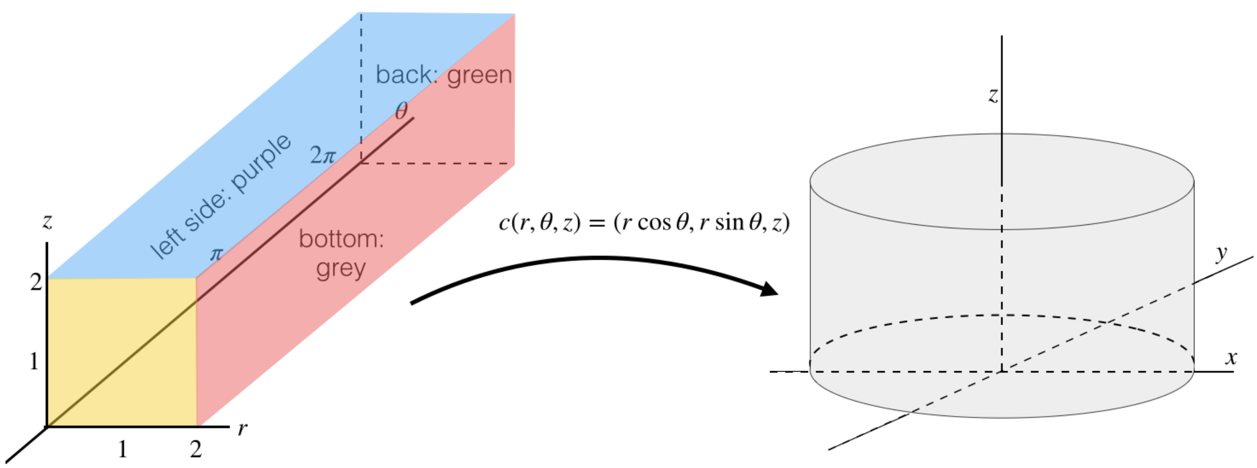
(a) Compute curl $\mathbf{F}$.

(b) There are no obvious "whirlpools" in the vector field, and yet the curl is nonzero! Imagine that $\mathbf{F}$ is the flow of water, and there are some small chunks of wood in the water, as shown. Determine whether the chunks would rotate, and in which direction, and explain how this relates to your curl calculation.

Cyl / PEA **1.** *Cylindrical coordinates* are a self-explanatory extension of polar coordinates to 3-dimensional space. The coordinate transformation is $(x, y, z) = (r\cos\theta, r\sin\theta, z)$, where $r^2 = x^2 + y^2$.

(a) The picture below shows the solid rectangular box $[0, 2] \times [0, 2\pi] \times [0, 2]$ in $r\theta z$-space. Show how the cylindrical coordinate transformation $c(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$ transforms the solid box into a solid cylinder, by coloring the images of each of its faces.

(b) The equation $x^2 + y^2 + z^2 = 1$ describes the unit sphere in rectangular coordinates. Transform it into an equation in cylindrical coordinates.



ParSf / DD **2.** The goal of this problem is to sketch the surface S defined by $\mathbf{S}(r, \theta) = (r\cos\theta, r\sin\theta, \theta)$ for $0 \le \theta \le 4\pi$ and $0 \le r \le 1$, and to learn a strategy for sketching parametric surfaces.

(a) Set $r = 1$ and sketch the curve described by $\mathbf{S}(1, \theta) = (1\cos\theta, 1\sin\theta, \theta)$ for $0 \le \theta \le 4\pi$.

(b) Repeat the previous for $r = 0$ and $r = 1/2$, and add them to your picture.

(c) Set $\theta = 0$ and sketch the curve described by $\mathbf{S}(r, 0) = (r, 0, 0)$ for $0 \le r \le 1$.

(d) Repeat the previous for $\theta = \pi/2, \pi, 3\pi/2$, etc. and add them to your picture.

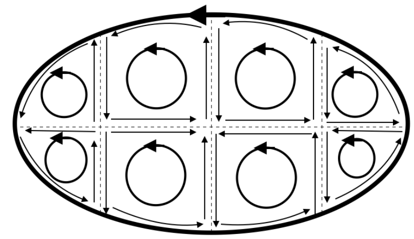
(e) Sketch the entire surface S.

3. Let C be the rectangular path from $(0, 0)$ to $(2, 0)$, to $(2, 3)$, to $(0, 3)$ to $(0, 0)$. Let

$$\mathbf{F} = [\sin x - 2y,\ y^2 + 3x].$$

Compute $\int_C \mathbf{F} \bullet d\mathbf{s}$.

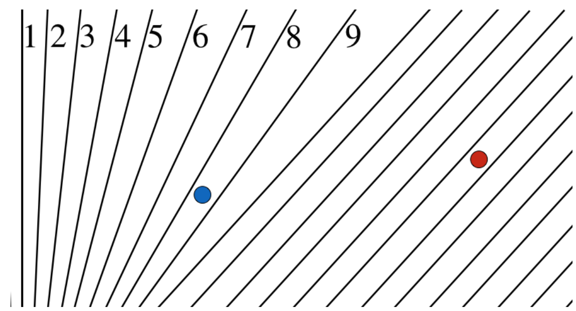
Hint: work smarter, not harder.



4. Let $\mathbf{F} = [P, Q] = [P, Q, 0]$ be a vector field. It is an abuse of notation to write the Green's Theorem equation as

$$\oint_C P\ dx + Q\ dy = \iint_D \text{curl}(\mathbf{F})\ dx\ dy,$$

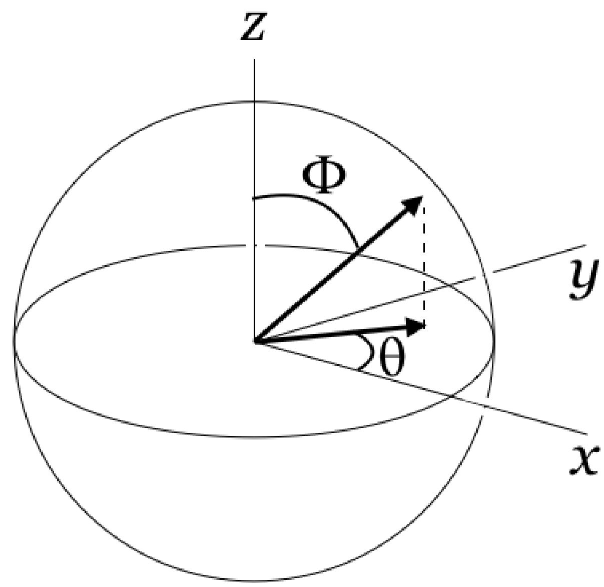
because curl($\mathbf{F}$) is a *vector*, not a scalar. But if we take this expression to mean that we are adding up the z-components of the curl vector $\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$, we can understand why Green's Theorem works: If we break our region into tiny boxes (shown in the diagram above with not-so-tiny boxes), adding up the curl at each point inside gives us the circulation (vector line integral) around the boundary, because the contributions from the interior edges cancel out. Explain this.



5. In Page 10 # 3, we explained the meaning of the second partial derivatives f_{xx} and f_{yy} as measuring concavity in the x- and y-axis directions. In this problem, we'll explore the meaning of f_{xy}, which you can think of as measuring the "twist" of a surface. The picture shows level curves of $f(x, y)$. For each part, say whether the value is positive, negative or 0, and justify your answer.

(a) $f_{xx} = (f_x)_x$ asks: how is f_x changing, as you move in the positive x-direction? Estimate f_{xx} at the red point. Then estimate $f_{yy} = (f_y)_y$ at the blue point.

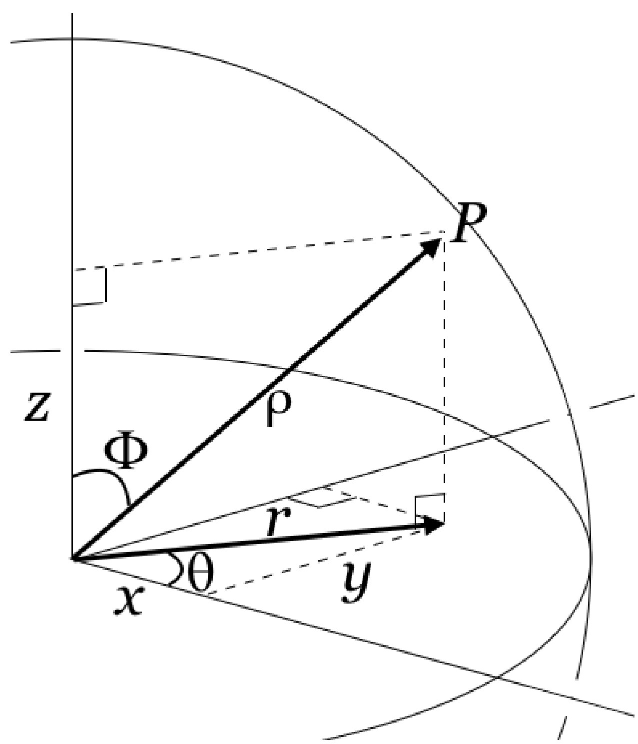
(b) $f_{xy} = (f_x)_y$ asks: how is f_x changing, as you move your (horizontal) path in the positive y-direction (shift it upwards)? Estimate f_{xy} at the blue point.



1. We have already seen Cartesian and cylindrical coordinates; *Spherical coordinates* are yet another way of using three numbers to specify a location in 3-space. Points on the unit sphere can be described parametrically by ϕ, the angle measured down from the z-axis, and θ, the angle in standard position measured from the positive x-axis. The third coordinate, ρ, measures distance from the origin.

In navigation on the Earth, θ is the angle usually called *longitude* (assuming that the Prime Meridian intersects the x-axis), and is our familiar θ from polar coordinates. The angle ϕ is the *complement* of the angle usually called *latitude*; it is the angle measured down from the North Pole. The Greek letter ϕ is called "phi," pronounced *fee*. The Greek letter ρ is called "rho" and is pronounced *roe*. We take $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$, and $\rho \ge 0$.

Look up the longitude and latitude of your hometown, and plot its location on the sphere shown. Also explain the (mathematical) difference between the r used in cylindrical coordinates, and the ρ used in spherical coordinates.



2. We would like to be able to translate back and forth between rectangular coordinates (x, y, z) and spherical coordinates (ρ, ϕ, θ). The figure on the right shows a zoomed-in version of the *first octant* in 3-space, with a point P on the surface of a sphere. Use the picture to find the coordinates x, y and z of P in terms of its coordinates ρ, ϕ and θ. *Hint*: first find the distance r in the xy-plane in terms of ρ and ϕ, and then use r to find x and y.

3. Given a point $P = (x, y, z)$, how do you find the spherical coordinate ρ?

4. Describe the configuration of all points with

(a) $r = 3$ **(b)** $\theta = 110°$ **(c)** $z = -2$

(d) $\rho = 5$ **(e)** $\phi = \pi/4$.

TripInt / DD **5.** Sketch the solid of integration corresponding to the integral $\int_0^2 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx$. Then rewrite this integral in the order $dx\ dy\ dz$.

ParSf / PEA **6.** If $\mathbf{X}(s,t)$ is a parametric surface, then $\mathbf{X}_s \times \mathbf{X}_t$ is a normal vector to the surface, and $\|\mathbf{X}_s \times \mathbf{X}_t\|$ is the area of the parallelogram spanned by $\mathbf{X}_s$ and $\mathbf{X}_t$.

(a) Let $\mathbf{X}(s,t) = [2s+t, st, s^2+t^2]$. Find $\mathbf{X}_s \times \mathbf{X}_t$.

(b) Find a parameterized surface $\mathbf{X}(s,t)$ whose normal vector at the point (s,t) is $[s-t, t-s, 0]$. How many correct answers are there? Give an answer that is different from everyone else's in the class.

☺ / PEA **7.** The integral $\iint_D \|\mathbf{X}_s \times \mathbf{X}_t\|\ ds\ dt$ is a template for what type of problem?

Multivariable Calculus

ParSf / DD **1.** In this problem, we will find the surface area of the part of the cone $x^2 + y^2 = z^2$ that lies between the planes $z = 0$ and $z = R$. Sketch this cone.

(a) Parameterize the cone $\mathbf{X}$ as a function of r and θ. Part of the job of parameterizing a surface is to specify the ranges of θ and r that give us the part of the surface that we want.

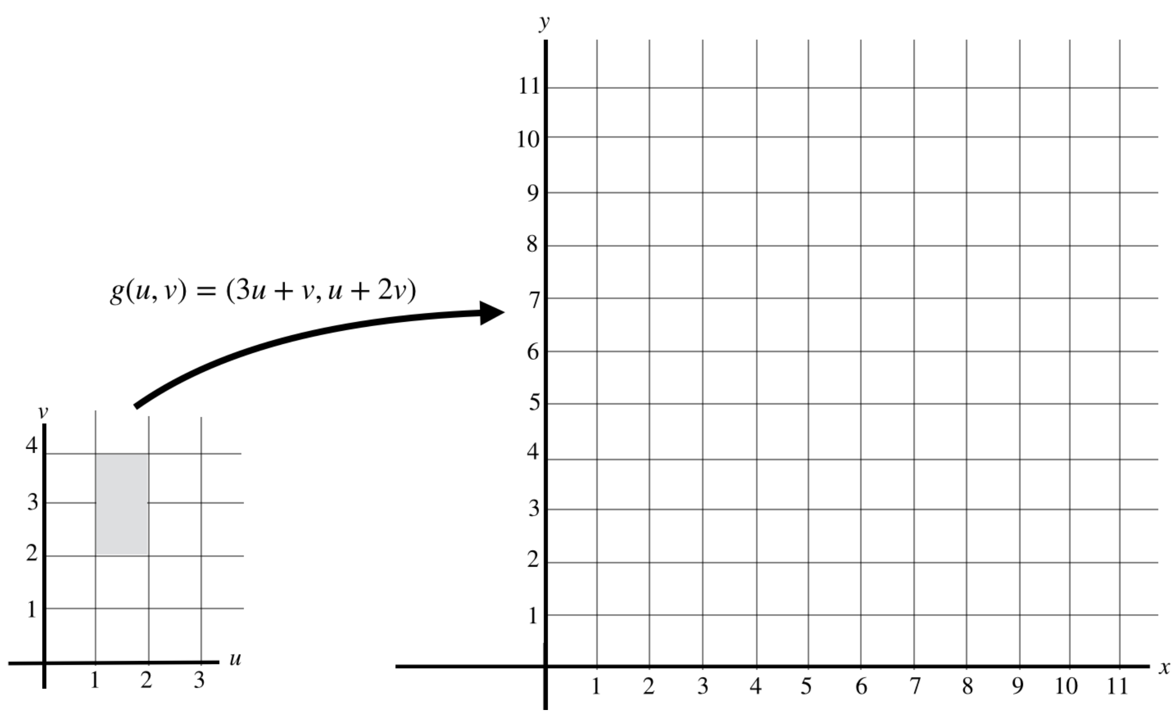
(b) Integrate $\iint_D \|\mathbf{X}_r \times \mathbf{X}_\theta\| \, dr \, d\theta$ over an appropriate region D of the $r\theta$-plane. What is the meaning of your result?

(c) Check your answer with basic geometry: slice open the cone and lay it flat as a sector of a circle, and use proportions.

ChVar / PEA **2.** Consider the linear mapping $g : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $x = 3u + v$ and $y = u + 2v$. In other words, $g(u, v) = (3u + v, u + 2v)$. Point by point, g transforms regions of the uv-plane onto regions of the xy-plane.

(a) Sketch a rectangle, such as $\mathcal{D} = [1, 2] \times [2, 4]$, in the uv-plane.

(b) Sketch the image of your rectangle $\mathcal{D}$ (which will be a simple geometric shape) in the xy-plane under the transformation g.



Hint: One possible approach is as follows. First, find the image of each of the vertices of $\mathcal{D}$, by plugging in its coordinates (u, v) into the function g and finding the output point (x, y). Then find the image of each of the edges of $\mathcal{D}$. For example, to find the image of an edge where $u = 1$, plug $(1, v)$ into g to find (parametric equations for) the image edge.

(c) Find the length of one of the rectangle's edges, and compare it to the length of the image of that edge under g. How could you calculate the local multiplier for the length from g?

(d) Find the area of the original rectangle and the area of its image, and compare them. Then calculate the determinant of $g'(0, 0)$, which is the 2×2 matrix $\begin{bmatrix} x_u(0,0) & x_v(0,0) \\ y_u(0,0) & y_v(0,0) \end{bmatrix}$.

Multivariable Calculus

TripInt / DD **3.** Find the volume between the plane $z = 0$ and the surface $z = 2x - y + 13$ over the region $\mathcal{R}$ in the xy-plane bounded by $y = x^2 - 4$ and $y = 9 - (x-1)^2$,

(a) using a double integral and **(b)** using a triple integral.

VLI / DD

4. On the table in front of you, trace the shape of a circle with your finger, going counter-clockwise. Keep your finger moving in a circle, around and around, in that same direction, and at the same time, lift your arm up so that your hand is above your head and you are looking up at it. Now which direction is the circle going, clockwise or counter-clockwise?

VLI / DD **5.** Consider the circle $z = 1, x^2 + y^2 = 1$, oriented clockwise when viewed from the origin.

(a) Sketch this circle, with its orientation.

(b) Compute the vector line integral of $\mathbf{F} = \frac{1}{2}[yz, -xz, xy]$ along the circle.

Clairaut's Theorem (Page 12 # 8) actually says that, when a function $f(x, y)$ is defined and continuous, and all of its partial derivatives exist and are continuous, *all* of the mixed partial derivatives are equal in *any* order, for example $f_{xyxyxyx} = f_{yyyxxxx}$. It also works for functions of more than two variables, so under the same differentiability and continuity assumptions about a function $f(x, y, z, w)$, we have $f_{zxwxxwzy} = f_{xwxwxyzz}$, etc.

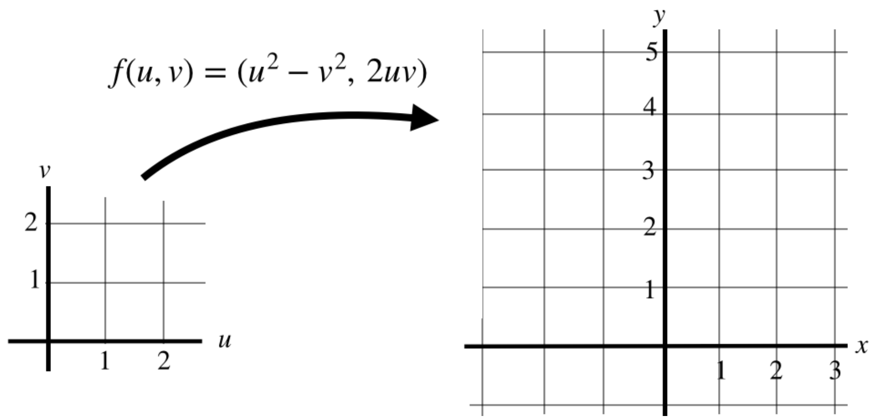
Clair / DD **6.** Compute f_{xyy} for

$$f(x, y) = ye^{\sin(1/x)} + \cos(\ln(2x^5 - 3\sin x)) + xy^2.$$

Hint: This problem is fun!

Multivariable Calculus

ChVar / PEA **1.** Consider the function $f(u,v) = (u^2 - v^2, 2uv)$. We will apply it to the rectangle $\mathcal{R}_1$ defined by $1 \le u \le 1.5$ and $1 \le v \le 1.5$.



(a) Sketch $\mathcal{R}_1$.

(b) Show that the edges of the image "quadrilateral" $\mathcal{Q}_1$ are four parabolic arcs. Sketch them, and shade $\mathcal{Q}_1$.

(c) First estimate the area of $\mathcal{Q}_1$, then calculate it exactly.

(d) What is the ratio of the area of $\mathcal{Q}_1$ to the area of $\mathcal{R}_1$?

ChVar / PEA **2.** (Continuation) Apply f to the rectangle $\mathcal{R}_2$ defined by $1 \le u \le 1.1$ and $1 \le v \le 1.1$. The image $\mathcal{Q}_2$ is enclosed by four parabolic arcs. Make a detailed sketch of $\mathcal{Q}_2$. Calculate the matrix $f'(1,1)$, and then find its determinant. You should expect the area of $\mathcal{Q}_2$ to be approximately 8 times the area of $\mathcal{R}_2$. Explain why.

ChVar / PEA **3.** (Continuation) Apply the function $g(h,k) = (2h - 2k, 2h + 2k)$ to the rectangle $\mathcal{R}_3$ defined by $0 \le h \le 0.1$ and $0 \le k \le 0.1$. Sketch the resulting quadrilateral $\mathcal{Q}_3$, and compare it to your sketch of $\mathcal{Q}_2$. Then explain what the matrix $\left[\begin{smallmatrix} 2 & -2 \\ 2 & 2 \end{smallmatrix}\right]$ reveals about the mapping f near $(u,v) = (1,1)$.

SSI / DD **4.** Let $\mathcal{S}$ be the cone surface given by the equation $z = \sqrt{x^2 + y^2}$ in cylindrical coordinates, for $0 \le x^2 + y^2 \le 16$, $0 \le z \le 4$.

(a) Parameterize this surface using just two parameters (*Hint*: r and θ). Part of the job of parameterizing a surface is to give the range of each parameter, so remember to do this.

(b) The density of electric charge at a point (x,y,z) on the cone is given by $f(x,y,z) = z$. Find the total amount of charge on the cone.

The calculation you did in this problem is called a *scalar surface integral*.

Limits / DD **5.** For some functions, it's easy to find $f(a,b)$ for any point (a,b) you want. For other functions, it's a little harder. For each of the following, find $f(0,0)$:

(a) $f(x,y) = \dfrac{\cos(\pi + x)}{y^2 - 1}$ **(b)** $f(x,y) = y + \dfrac{\sin x}{x}$ **(c)** $f(x,y) = \dfrac{x+y}{2x+y}$

Limits / DD **6.** (Continuation) You should have been able to do **(a)** easily, and **(b)** using a limit, but for **(c)** it's hard to know quite what to do.

(a) Graph the three functions on your computer as surfaces $z = f(x, y)$, and sketch the results in your notebook. Observe that some graphing programs work better than others for surfaces with vertical parts. In this case, googling `z=(x+y)/(2x+y)` works well.

Notice that the last surface is *vertical* at the origin! We will see that, for $f(x, y) = \dfrac{x+y}{2x+y}$, if you approach the origin along different lines, you get *different limits* for $f(0, 0)$.

Walk towards the origin along the line $y = 0$, coming from the positive x-axis (red path below). This means that we are considering points of the form $(x, 0)$, as $x \to 0^+$:

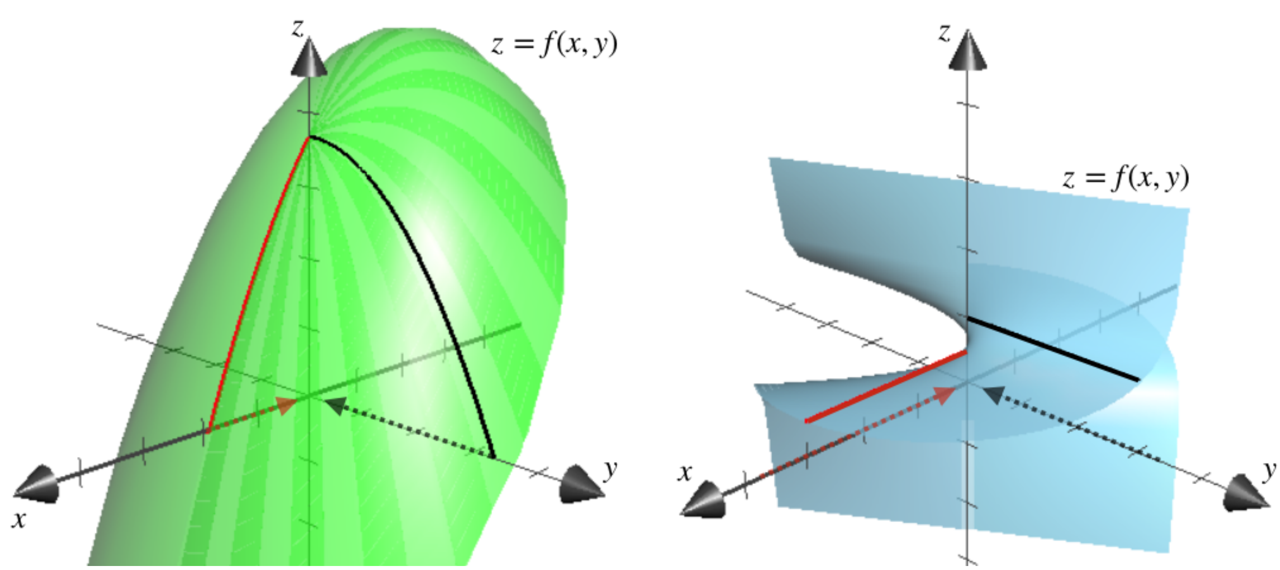
$$\lim_{(x,0)\to(0,0)} \frac{x+y}{2x+y} = \lim_{x\to 0^+} \frac{x}{2x} = \lim_{x\to 0^+} \frac{1}{2} = \frac{1}{2}.$$

(b) Explain each step of the equation above.

(c) Now do the same calculation for walking towards the origin along the y-axis (black path below). Does it matter if you are walking from the positive or negative y-axis?

(d) Repeat the calculation one more time, now using a line of the form $y = mx$, so take a limit as $(x, mx) \to (0, 0)$. Which numbers can you get as a limit?

(e) The picture below shows a well-behaved surface on the left, and the surface $z = \dfrac{x+y}{2x+y}$ on the right. Using the picture, explain the geometric reason for the different values of the limits that you found in **(b)** and **(c)**, and also **(d)**.

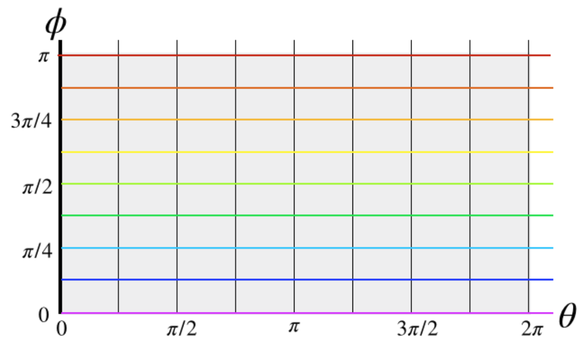


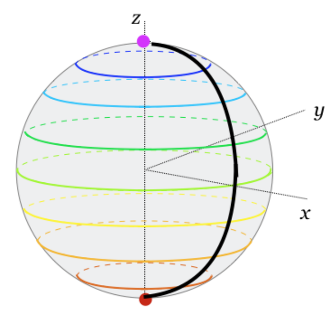
Multivariable Calculus

Limits / DD **1.** Consider the function $f(x,y)=\dfrac{x^2}{x^2+y^2}$. Does $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exist? Explain why or why not, using calculations, graphs, and any other methods of your choice.

ChVar / PEA **2.** In general, given a mapping $g:\mathbf{R}^2\to\mathbf{R}^2$, its derivative is a 2×2 matrix-valued function that provides a *local multiplier* at each point of the domain of g. Each such matrix describes how suitable domain rectangles are transformed into image "quadrilaterals," and its determinant is a multiplier that converts (approximately) the rectangular areas into the quadrilateral areas. It is customary to refer to either the matrix g' or its determinant as *the Jacobian of* g. Explain why each row of a Jacobian matrix is the gradient of a certain function.

ChVar / PEA **3.** Justify the equation $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = dx\,dy$.





4. *Volumes in spherical coordinates.* In Page 22 # 2, you showed that it is possible to translate between rectangular and spherical coordinates using the transformation

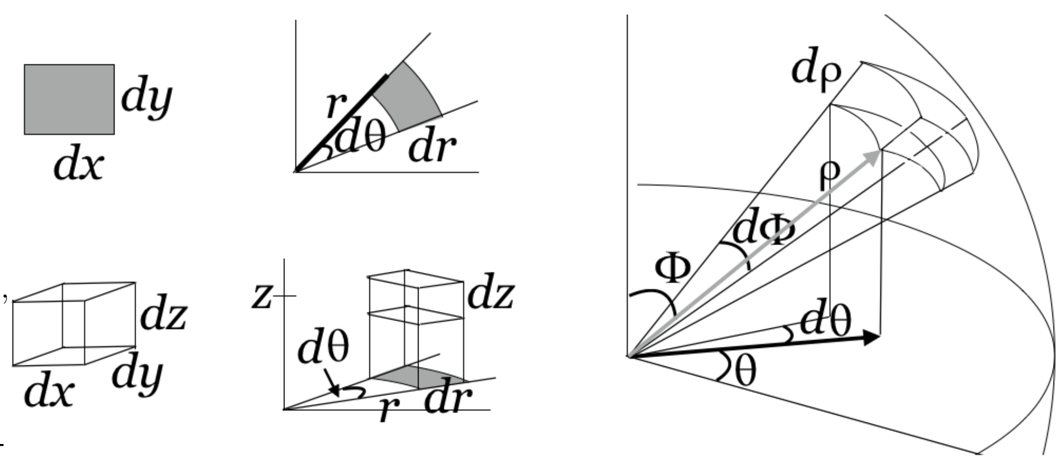
$$(x,y,z)=f(\rho,\phi,\theta)=(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi),$$

which maps the infinite prism $0\le\rho$ and $0\le\phi\le\pi$ and $0\le\theta\le 2\pi$ onto all of xyz-space.

(a) Find the 3×3 Jacobian matrix for the transformation above.

(b) Make calculations that justify the Jacobian formula $dx\,dy\,dz=\rho^2\sin\phi\cdot d\rho\,d\phi\,d\theta$.

(c) The pictures to the right show geometric representations of the rectangular area differential $dx\,dy$, the polar area differential $r\;dr\;d\theta$, the rectangular volume differential $dx\;dy\;dz$, the cylindrical volume differential



$r\;dr\;d\theta\;dz$, and the spherical volume differential $\rho^2\sin\phi\;d\rho\;d\phi\;d\theta$. Make a volume calculation that explains why the "spherical brick" in the large picture on the right has volume $\rho^2\sin\phi\;d\rho\;d\phi\;d\theta$.

SphCyl / DD **5.** In Page 6 # 3, you showed that integrating the function $f(x,y)=1$ over a *planar* region $\mathcal{R}$ gives the *area* of $\mathcal{R}$. Similarly, integrating the function $f(x,y,z)=1$ over a *solid* region gives its *volume*.

(a) Use a *cylindrical* integral to find the volume of a cylinder of radius R and height h.

(b) Use a *spherical* integral to find the volume of a sphere of radius R.

For both of these, remember to use the correct volume differentials (which appear in problem 4), and check your answer against your previous knowledge of geometry.

Opt / DD **6.** *The second derivative test, single-variable calculus.* Faced with a function like

$$f(x)=\frac{1}{4}x^3(x-2)(x+2)$$

and asked to find and classify its critical points, you have learned to do the following:

(a) Find all the *critical points* of $f(x)$, i.e. the values a for which $f'(a)=0$.

(b) Apply the *second derivative test*: Find $f''(x)$, and for each critical point a, determine if $f''(a)$ is positive, negative or 0. Then use this information to classify each critical point as a local maximum, a local minimum, or neither.

(c) Graph $f(x)$ on your graphing program, and check that your answers make sense.

(d) Repeat parts **(a)**–**(c)** for the function $g(x)=x^4$, and use this to explain why the second derivative test is sometimes inconclusive, and more information is needed.

Multivariable Calculus

Cyl / PEA **1.** Let $\mathcal{P}$ be the region in $\mathbf{R}^3$ defined by $0 \leq z \leq 4 - x^2 - y^2$. Sketch the solid. Use cylindrical coordinates to find the volume of $\mathcal{P}$.

Limits / DD **2.** Find the limit, when approaching the origin along each of the three coordinate axes (recall Page 24 # 6), and then decide whether the limit exists:

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 - y^2 + 3z^2}{x^2 + y^2 + z^2}$$

Opt / DD **3.** Consider a surface of the form $z = f(x, y)$.

(a) Explain why, if there is a local maximum or local minimum of the function f at the point (a, b), then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

(b) Graph the surface $z = -x^2 - y^4 + y^2$ on your favorite graphing program and sketch the result in your notebook. How many local maxima and local minima does it have?

Pro tip: It is sometimes easier to see a function's behavior if you scale it in the z-direction, e.g. by plotting $z = \dfrac{-x^2 - y^4 + y^2}{10}$ instead.

(c) For the function $f(x, y) = -x^2 - y^4 + y^2$, solve the system of equations $\begin{cases} f_x(x, y) = 0 \\ f_y(x, y) = 0 \end{cases}$ and find the three points (x, y) that satisfy both simultaneously. Check that your answer makes sense geometrically, using your graph from **(b)**.

(d) Is it always true that if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, then (a, b) is either a local maximum or a local minimum of $f(x, y)$? Either explain why it is always true, or give a counterexample.

Opt / DD **4.** Find the global maximum of the function $f(x, y) = 4 - x^2 + 2x - y^2 - 4y$ in two ways:

(a) By completing the square (recall Page 15 # 2(b)), identifying what kind of surface this is, and figuring out geometrically where the maximum point occurs.

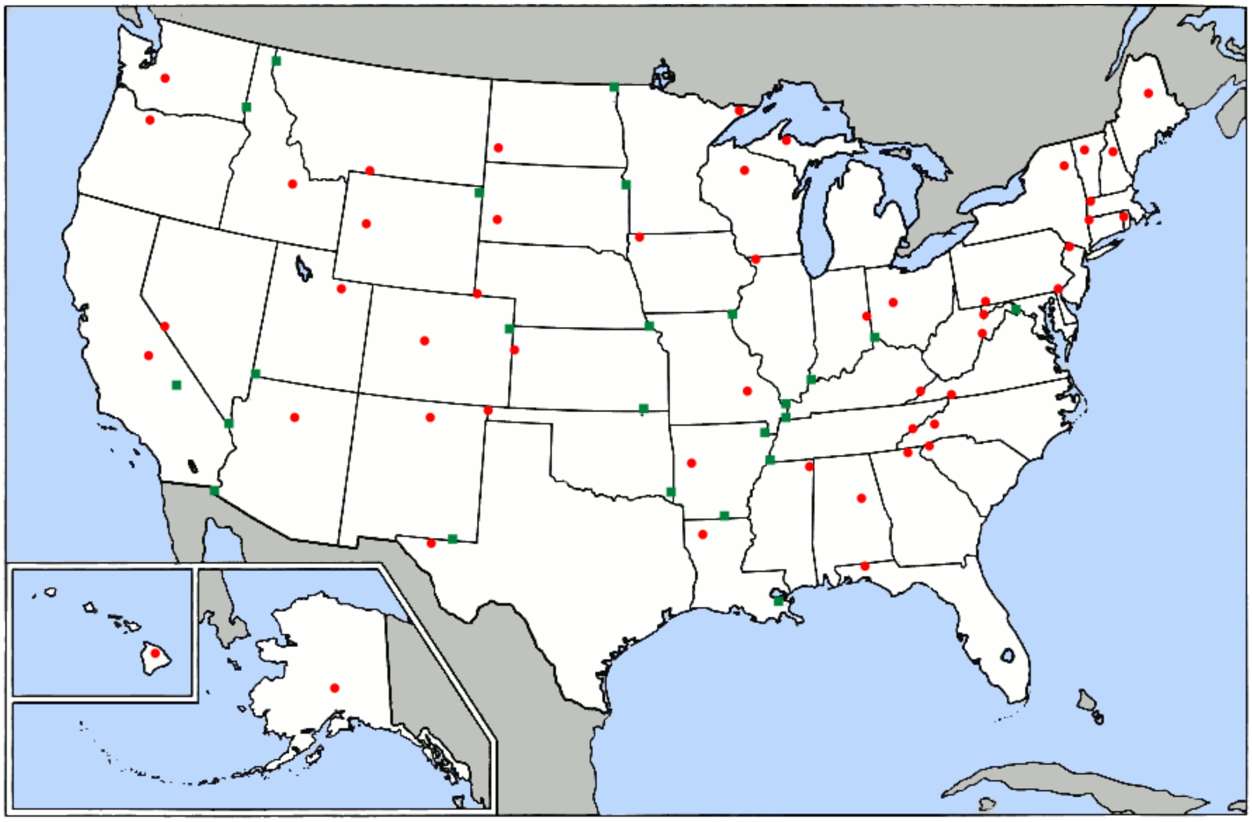
(b) By solving for the point where both partial derivatives are 0.

More problems, with pictures, on the next page!

Opt / DD **5.** The map below shows the high points (red) and low points (green) of each state. Choose your favorite 10 states, and for each one, write whether the high point occurs:

in its interior along its boundary at a corner somewhere else

For an interactive map of high points where you can zoom in for greater precision, see `https://tinyurl.com/dd50shp`.

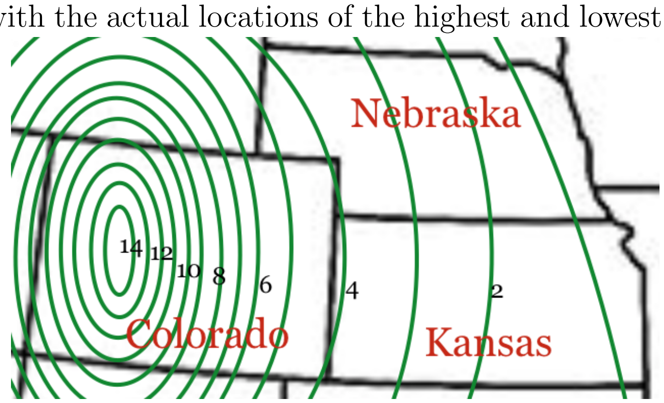


For fun: Why do many states *not* have a lowest point marked? (And yet California does!)

Opt / DD **6.** The figure below shows (fictional, but broadly plausible) topographical lines for three of the midwestern states, with elevation marked in thousands of feet.

(a) Mark the highest points in each state, and estimate their elevations.

(b) Does this agree with the actual locations of the highest and lowest points, shown above?



Opt / DD **1.** *The second derivative test, multivariable calculus.* Faced with a function like

$$f(x,y) = x^3 + 2xy - 2y^2 - 10x$$

and asked to find and classify its critical points, do the following:

(a) Find all the *critical points* of $f(x,y)$, i.e. the points (a,b) where $f_x = 0$ and $f_y = 0$.

(b) Apply the *second derivative test*: First, compute $f_{xx}(x,y), f_{xy}(x,y) = f_{yx}(x,y)$, and $f_{yy}(x,y)$. Then, for each critical point (a,b), find the eigenvalues of the *Hessian matrix*

$$\begin{bmatrix} f_{xx}(a,b) & f_{yx}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix}.$$

Then use this information to classify each critical point:

both eigenvalues are positive	$\implies f(a,b)$ is a local minimum
both eigenvalues are negative	$\implies f(a,b)$ is a local maximum
one eigenvalue is positive and one is negative	$\implies f(a,b)$ is a saddle point
some other result	$\implies$ the test is inconclusive

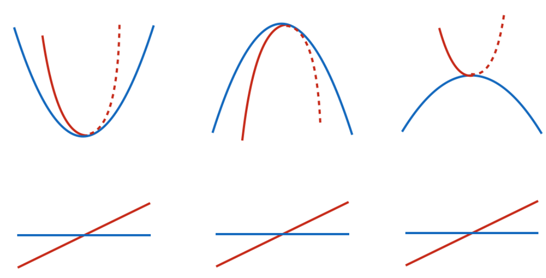
The above test is easier to understand geometrically. The following test also works.

Compute the determinant $D(a,b)$ of the Hessian matrix for each critical point (a,b). Then

$D(a,b) > 0$ and $f_{xx}(a,b) > 0$	$\implies f(a,b)$ is a local minimum
$D(a,b) > 0$ and $f_{xx}(a,b) < 0$	$\implies f(a,b)$ is a local maximum
$D(a,b) < 0$	$\implies f(a,b)$ is a saddle point
$D(a,b) = 0$	$\implies$ the test is inconclusive

(c) Graph $f(x,y)$ on your graphing program, and check that your answers make sense.

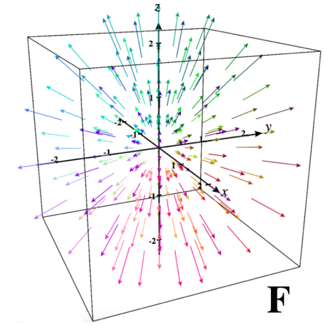
The idea is that the signs of the eigenvalues tell you whether the function is concave-up or concave-down in each of the two principal directions of the function. If both are positive, it is concave-up in both directions (left picture), so the critical point is a local minimum. The pictures for the other two cases are similar.



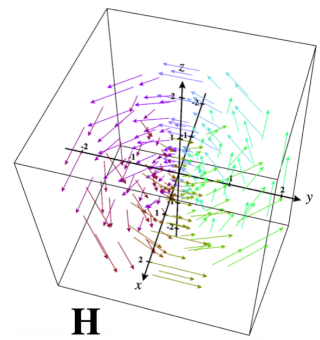
Limits / DD **2.** Consider the function $f(x,y) = \dfrac{xy^2}{x^2+y^2}$. Does $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exist? Explain why or why not. How would you prove your answer correct?

Limits / DD **3.** (Continuation) You can use lines through the origin to show that a limit *doesn't* exist, but simply showing that the limit along any line through the origin is the same *isn't* enough to prove that the limit *does* exist. In Page 29 # 1, we'll see an example of how it can go wrong. The best way to prove that a limit *does* exist is to convert to polar coordinates, and take a limit as $r \to 0$, to approach the origin from *every* direction simultaneously. Try this.

Multivariable Calculus



VSI / DD **4**. For our familiar cone surface $\mathbf{X}(r,\theta) = [r\cos\theta, r\sin\theta, r]$, the vectors $\mathbf{X}_r \times \mathbf{X}_\theta$ and $\mathbf{X}_\theta \times \mathbf{X}_r$ both give *normal vectors* to the surface. One points into the cone, while the other points out of the cone. Determine which is which, sketch the cone, and draw in these vectors.



VSI / DD **5**. (Continuation) Consider the vector fields $\mathbf{F} = [x,y,z]$, $\mathbf{G} = [x,y,0]$, and $\mathbf{H} = [-y,x,0]$. For each of these, estimate (is it positive, negative or 0?) the *vector surface integral* of the vector field over the cone, with outward-facing normal vector.

This value is denoted by $\iint_{\mathbf{X}} \mathbf{F} \bullet d\mathbf{S}$ and is called *flux*. It is the sum of the dot product of the vector field with the chosen normal vector at each point of the surface.

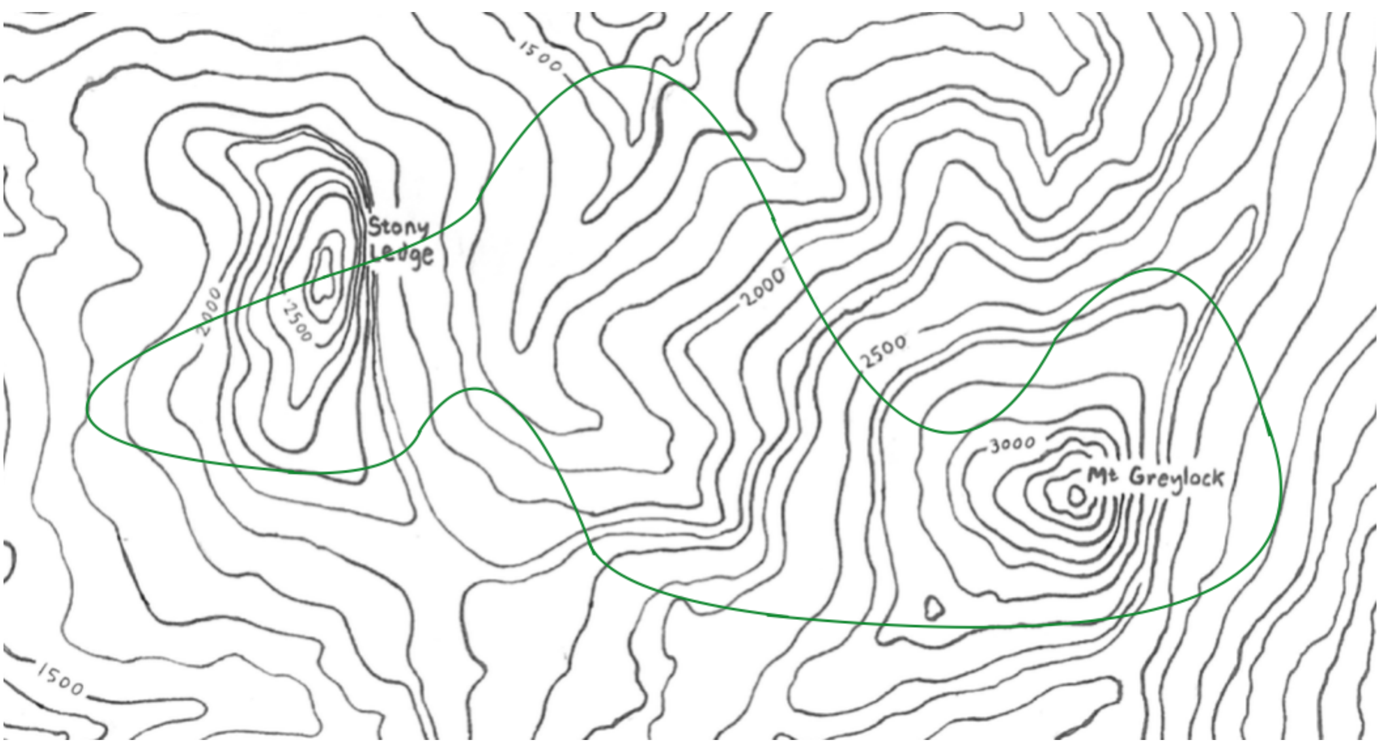
LM / DD **6.** The Outsiders club takes a hike, shown as the green curve on the topographical map below. For this hike, determine:

(a) Parts of the hike that were flat, **(b)** The steepest part of the hike,

(c) The highest elevation achieved,

(d) The lowest elevation achieved. *Hint*: The level curves are at 100-foot intervals.

(e) Mark all of the points whose elevation you would need to check, in order to be sure that you found the maximum and minimum elevation in parts **(c)** and **(d)**. What do all of these points have in common?



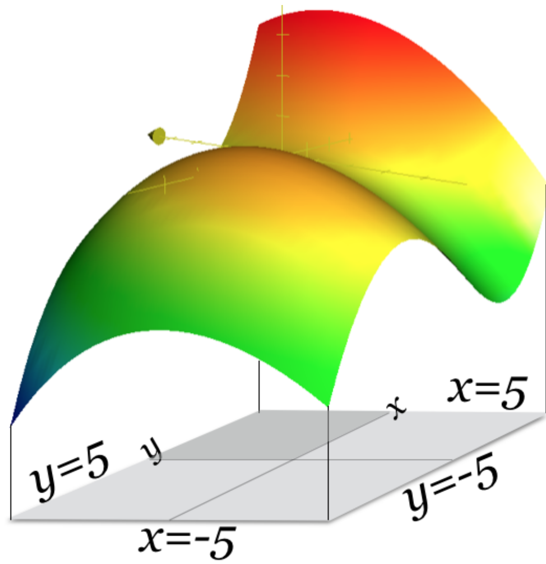
Opt / DD

1. The state high point problem, and the Outsiders hike problem, are examples of *optimizing under a constraint*. For the state high points, you are trying to *maximize* elevation, under the *constraint* that you must be in the region of the plane called "Pennsylvania."

(a) Give an example of something you are trying to maximize or minimize in your own life, and the associated constraints.

(b) Explain why, if you want the maximum and minimum values of a function on a (closed, bounded) region of the plane, you need to check the function value on all of the following points:

1. The critical points of the function that are inside the region.
2. The critical points of the boundary "cross-section" functions, which are the surface function restricted to each boundary.
3. The corners of the region.



(c) For the surface shown above, which is part of the graph of $f(x, y) = x^3 + 2xy - 2y^2 - 10x$, mark interior critical points in black, critical points of the boundary functions in blue, and the corner points in red. Based on the picture, where do you think the maximum and minimum values of the function occur, over the square region $[-5, 5] \times [-5, 5]$ shown?

2. Now we're ready to do it. For $f(x, y) = x^3 + 2xy - 2y^2 - 10x$:

(a,b)	$f(a,b)$	
		interior critical points
(5, 2.5)		critical points of boundary functions
		corners

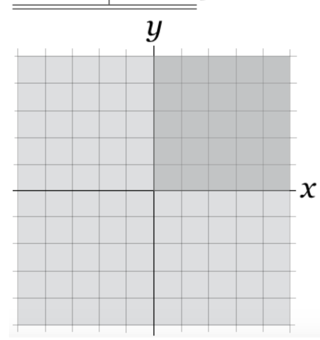
(a) Write down, on the list to the right, the critical points of f that lie inside the region where $-5 \le x \le 5$ and $-5 \le y \le 5$. *Hint*: You have already found all of the critical points.

(b) We can take a vertical cross section of f along the boundary $x = 5$, which is a function of y:

$$f(5, y) = 5^3 + 2 \cdot 5 \cdot y - 2y^2 - 10 \cdot 5 = 125 + 10y - 2y^2 - 50.$$

Find its critical points, and keep those satisfying $-5 \le y \le 5$:

$$\begin{aligned} f(y) &= 75 + 10y - 2y^2 \\ \implies f'(y) &= 10 - 4y = 0 \implies y = 2.5. \end{aligned}$$



so we have added $(5, 2.5)$ to the list. Now find the critical points along the other three boundaries and add them as well.

(c) Add the four corner points to your list, which should have 11 points listed. Also plot each point on your list on the square region, shown above.

(d) Find the value of f at each of the points on your list, and determine the maximum and minimum values of $f(x, y)$ on the square region. Check that your answer agrees with 1(c).

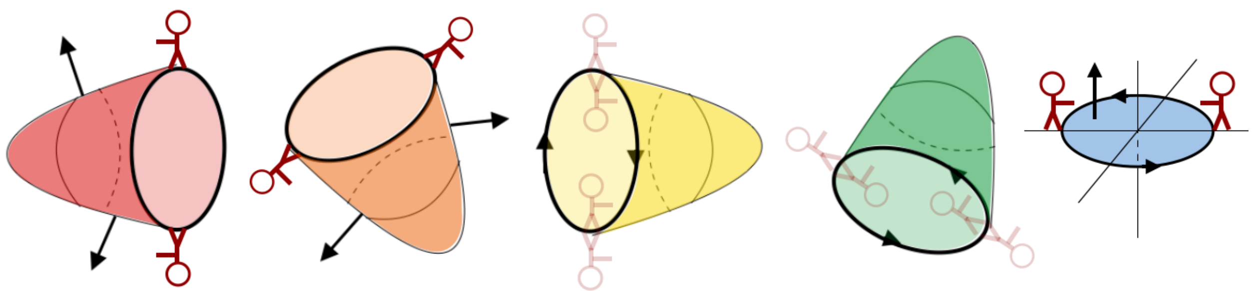
VSI / DD **3**. *Compatible orientations for a surface and its boundary*

For a vector line integral over a curve C, we integrate along an *oriented curve*: it has a direction of travel. There are two choices: the two directions that are tangent to the curve.

For vector surface integral over a surface S, we integrate over an *oriented surface*: it has an "up" direction. There are two choices: the two directions perpendicular to the surface.

We should make sure these orientations are *compatible* when an oriented curve C is the boundary of an oriented surface S. Here is the rule for compatible orientations: Imagine that you are walking along the boundary curve of the surface, in the direction of the curve's orientation, with your head pointing in the direction of the surface's orientation. *Orient the curve so that as you walk in the forward direction, your left arm is over the surface.*

(a) For the first two surfaces below, you will see that there is a person walking along the curve, with their head in the direction of the surface's orientation (indicated with an arrow), and their left arm over the surface. Given that the arm sticking out is their left arm, put a smile on their face and color in the hair on the back of their head, for both surfaces. Also draw in the appropriate orientation (direction of travel) on the boundary curves.



(b) For the third and fourth surfaces, the orientation (direction of travel) of the boundary curve is given. Given that the person is walking along the curve in the direction of its orientation, with their left arm over the surface, decide whether the inward-facing or outward-facing person is the correct one. Then draw in their smiling face and their hair, and give the surface an orientation (inward or outward) using an arrow.

(c) The rules above generalize the notion of "counter-clockwise" to 3D. Explain why these compatibility rules agree with the conditions of Green's Theorem: the "surface" is a region in the xy-plane oriented in the positive z-direction, and its boundary curve is counter-clockwise.

Cyl / DD **4.** *The weight of a wedge of cheese that gets denser as you move north.*

(a) Sketch the solid region $\mathcal{W}$ described by $x^2 + y^2 \leq 1, x \geq 0, y \geq 0, 0 \leq z \leq 2$.

(b) Calculate $\iiint_{\mathcal{W}} y \, dV$ by converting to cylindrical coordinates.

LM / DD **5.** Refer to Page 27 # 6. Explain why, if you are trying to find the maximum or minimum value of a function that occurs on a given constraint curve, you should check all the points where the constraint curve is tangent to a level curve of the function. This insight will lead us to the idea of *Lagrange multipliers*.

Multivariable Calculus

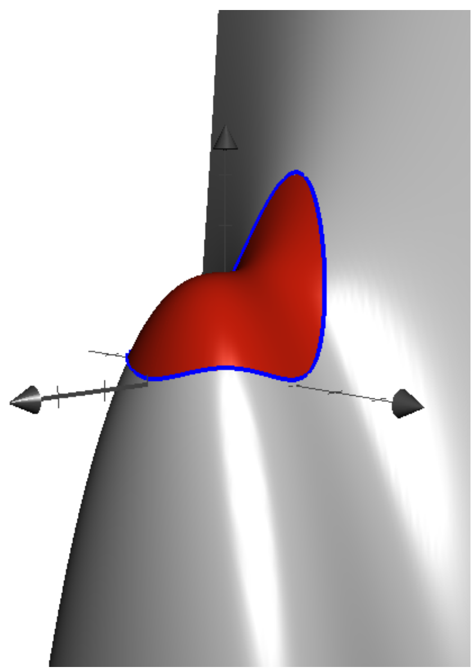
Limits / DD **1.** We've considered several different limits of the form $\lim_{(x,y)\to(0,0)} f(x,y)$. Sometimes (as in Page 24 # 6 and Page 26 # 2), approaching the origin along different lines gives different values for the limit, so we know that the limit *doesn't exist*. Other times (as in Page 27 # 3), approaching the origin along lines of the form $y = mx$ gives the same value for each m, and then we'd like to be able to say that the limit *exists*. I claimed in Page 27 # 3 that this can go wrong, which is why we need to convert to polar coordinates. I promised you an example of it going wrong, and here it is: a case where approaching the origin along different *curved* paths yields different values. Let's consider the function $f(x,y) = \dfrac{x^2y}{x^4+y^2}$.

(a) Take a limit of $f(x,y)$ as $(x,y) \to (0,0)$, approaching the origin along lines of the form $y = mx$, and show that the limit is 0 for every value of m.

(b) Now approach the origin along the parabola $y = x^2$, and show that the limit is not 0.

(c) The curve $y = x^2$ was chosen so that the numerator and denominator of the fraction have the same *degree*: the maximum total exponent of each term is 4. Looking back at previous examples, explain how you can use the degree of the numerator and denominator of a function to help you decide whether a limit is likely to exist.

Opt / DD **2.** Find the maximum and minimum values of the function $f(x,y) = 1 - x^3 - y^2$, shown as the grey surface, on the unit disk $x^2 + y^2 \leq 1$, whose image in the picture is a red disk with blue boundary, by making and checking a list as in Page 28 # 2. *Hint*: for the boundary, write $x = \cos\theta$, $y = \sin\theta$ to find f as a function of θ, and solve $f'(\theta) = 0$. Check that your answers agree with the picture.



VSI / DD **3.** The *flux* of the vector field $\mathbf{F}$ through the surface $\mathcal{S}$ is given by $\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S}$. You can think of the surface $\mathcal{S} = \mathbf{X}(s,t)$ being a net in a stream whose current is given by the vector field $\mathbf{F}$, and the integral measures how much water flows through the net. The word *flux* is from physics, measuring the amount of electric field across a surface. The $d\mathbf{S}$ is a vector quantity, and the integral adds up dot products to measure how much the vector field $\mathbf{F}$ points in the same direction as the normal vector to the tiny piece of oriented surface $d\mathbf{S}$.

Let S_1 be the cone $z = \sqrt{x^2+y^2}$ below $z = 1$, oriented outward (downward). Let $\mathbf{E} = [x, 0, -z]$ be an electric force field. Compute the electric flux of $\mathbf{E}$ across S_1, by computing

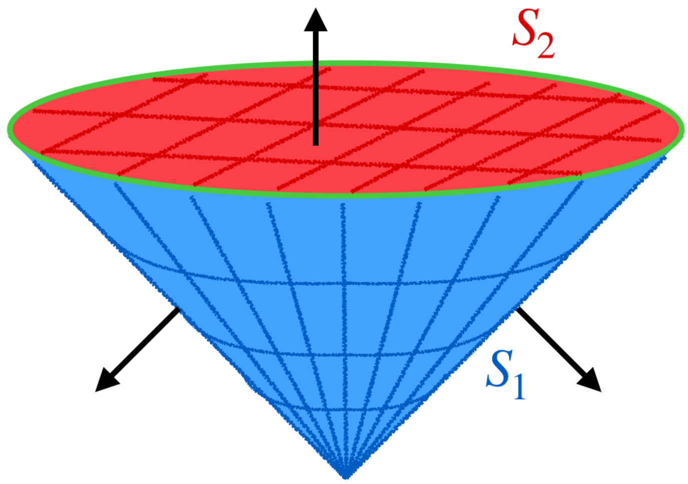
$$\iint_S \mathbf{E} \bullet d\mathbf{S} = \iint_D \mathbf{E}(\mathbf{X}(r,\theta)) \bullet (\mathbf{X}_\theta \times \mathbf{X}_r)\; dr\; d\theta$$

over an appropriate region D of the $r\theta$-plane.

4. (Continuation) We can also write

$$\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S} = \iint_{\mathcal{S}} \mathbf{F} \bullet \mathbf{n}\, dS,$$

where $\mathbf{n}$ is a *unit* normal vector and dS is a tiny bit of surface area. Explain. This form is convenient when the normal vector $\mathbf{n}$ is easy to compute. Let S_2 be the "cap" of the cone in the previous problem: the unit disk at a height of $z = 1$, with upward-facing normal vector (chosen just so that the whole closed surface, cone plus cap, is oriented outward). Use the integral on the right-hand side above to compute the electric flux of $\mathbf{E}$ over S_2.



Some integral practice.

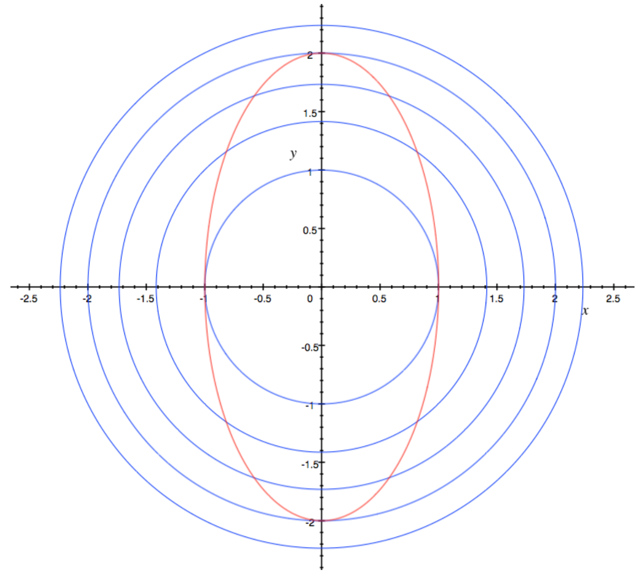
Sph / DD **5.** Sketch the solid region that is *above* the xy-plane, *outside* the cone $x^2 + y^2 = z^2$, and *inside* the unit sphere. Then compute its volume. Which coordinates are most convenient?

Sph / DD **6.** *The weight of a different block of cheese, which gets denser as you go up.*

(a) Sketch the solid region $\mathcal{W}$ described by $x^2 + y^2 + z^2 \leq 1$ and $x, y, z \geq 0$.

(b) Calculate $\iiint_{\mathcal{W}} z\, dV$.

Multivariable Calculus



LM / DD

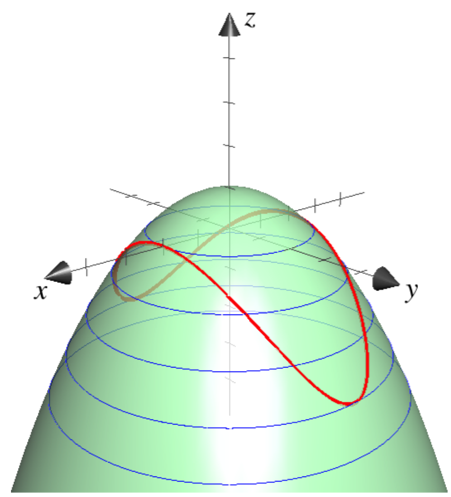
1. *Lagrange multipliers: the coolest idea in this course.* Suppose we wish to *maximize* or *minimize* the function $f(x, y) = 1 - x^2 - y^2$,

under the *constraint* $g(x, y) = x^2 + y^2/4 = 1$.

(a) The top picture to the right shows six (!) level curves in the xy-plane for $f(x, y)$ (blue). Label each one with its level.

(b) The same picture shows the constraint curve $x^2 + y^2/4 = 1$ in red. Circle the points on it where you think the maximum and minimum values of f occur.

(c) Make up a story about a hike to go along with this problem (recall Page 27 # 6, and refer to the green picture below).



(d) Express the constraint curve as a level curve of the function $g(x, y) = x^2 + y^2/4$ at level 1, and explain why you can do this.

(e) The *Lagrange multipliers equation* says that, at a maximum or minimum point (a, b) of the function $f(x, y)$ on the constraint curve $g(x, y) = c$,

$$\nabla f(a, b) = \lambda \cdot \nabla g(a, b),$$

for some number λ. Explain geometrically what the equation is saying, and why it is true.

(f) The Lagrange multipliers equation above has three variables: x, y and λ. In fact, it consists of three equations: one each from the x- and y-components of the gradient, plus one from the constraint equation. Write down and solve the Lagrange multipliers system of equations for the given function and constraint, and check that your answers agree with your guess from **(b)**.

Opt

2. Find all of the critical points of the function

$$f(x, y) = 4x - 3x^3 - 2xy^2$$

and then classify each of them using the second derivative test (recall Page 27 # 1). Check your answers by graphing the function with your favorite graphing program.

Limits / DD

3. Consider the limit $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^3 + y^3 + z^3}{x^2 + y^2 + z^2}$.

(a) Show that the limit, when approaching the origin along each of the three coordinate axes, is the same in each case.

(b) Convert to *spherical* coordinates and determine whether the limit exists.

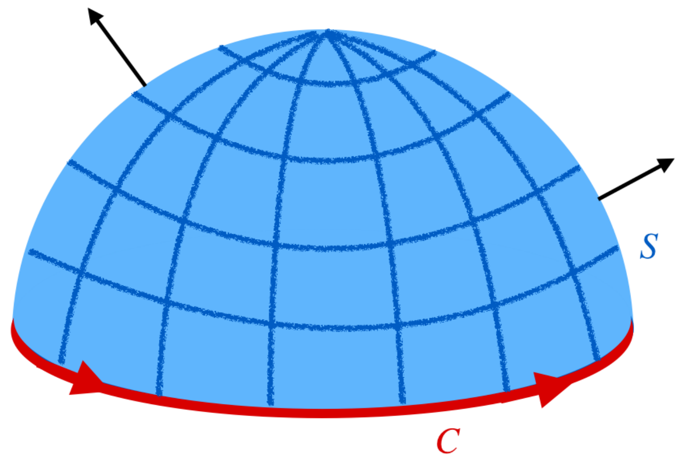
VSI / DD

4. Let $\mathbf{F} = [-y, x, z]$.

(a) Compute the vector line integral of $\mathbf{F}$ over the curve $\mathcal{C}$ consisting of the unit circle $x^2 + y^2 = 1$, $z = 0$, oriented counter-clockwise as viewed from the positive z-direction.

(b) Compute the vector surface integral of $\text{curl}\mathbf{F}$ over the surface $\mathcal{S}$ defined by $z = \sqrt{1 - x^2 - y^2}$, oriented outward.

You will need to parameterize the surface (*Hint*: spherical coordinates), calculate the vector $\mathbf{X}_\phi \times \mathbf{X}_\theta$, and compute a vector surface integral (*Hint*: Page 29 # 3).



Stokes / DD

5. You might have wondered, for the "circle in 3-space" in Page 23 # 5, whether there is a *three-dimensional* version of Green's Theorem that you could apply. Indeed there is:

Given a differentiable vector field $\mathbf{F}$ defined in $\mathbf{R}^3$, let $\mathcal{S}$ be an oriented surface, and let $\partial\mathcal{S}$ be its compatibly oriented boundary curve (as defined in Page 28 # 3).

Stokes's Theorem states that

$$\int_{\partial\mathcal{S}} \mathbf{F} \bullet d\mathbf{s} = \iint_{\mathcal{S}} \text{curl}\mathbf{F} \bullet d\mathbf{S}.$$

In words, Stokes's Theorem says the circulation of $\mathbf{F}$ around the boundary of $\mathcal{S}$ is equal to the flux of $\text{curl}\mathbf{F}$ through $\mathcal{S}$ itself.

(a) Explain why your answers to **(a)** and **(b)** of problem 4 came out the same. In particular, be sure to check that $\mathbf{F}$, $\mathcal{S}$ and $\mathcal{C} = \partial\mathcal{S}$ satisfy the requirements of the theorem.

(b) Explain how Green's Theorem is a special case of Stokes's Theorem.

The following problem is an extension of #5. If you solved #5 at home and you have time, please try #6. Otherwise, please work on #6 in your group after you are happy with #5.

Stokes / DD

6. In the problem above, the symbol "∂" means "boundary." Elsewhere it has meant "derivative." You might be wondering: why is the same symbol used for different things?

(a) The boundary of a solid disk is a circle. Differentiate πr^2 with respect to r.

(b) The boundary of a solid ball is a sphere. Differentiate $\dfrac{4}{3}\pi r^3$ with respect to r.

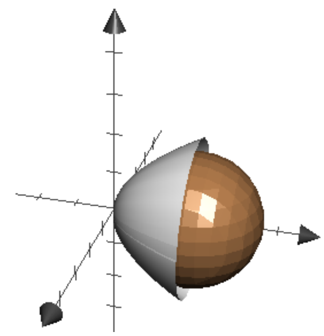
(c) Explain the relationship between "boundary" and "derivative."

Multivariable Calculus

Stokes's Theorem (Page 30 # 5) relates a vector surface integral over a surface, to a vector line integral over its boundary. Sometimes one of these is difficult or impossible to compute, and the other is easier.

Stokes / DD **1.** Let $\mathbf{F} = \left[(y-1)\sin e^{xy^z}, xyze^{xyz}, xz+y\right]$ be an electric field, and let your tinfoil hat S be the piece of the paraboloid $y = x^2 + z^2$ with $y \leq 1$, oriented with inward normal vectors. Sketch this surface and its (compatibly oriented, remember Page 28 # 3) boundary curve. Then compute the amount of curl$\mathbf{F}$ that will enter your sleeping head,

$$\iint_S \text{curl}\mathbf{F} \bullet d\mathbf{S}.$$



Stokes / DD **2.** Let $\mathbf{F} = [x \sin e^x - xz, -2xy, z^2 + y]$ give the direction and strength of the wind on Swarthmore's campus, and let C be the (oriented) triangular path from $(2,0,0)$, to $(0,2,0)$, to $(0,0,2)$, and back to $(2,0,0)$ followed by the long-range drone that you have sent to campus to cut some flowers from the flowering trees, which it will then bring to you . Sketch this path. Then compute how much the wind helps or hurts the drone on its journey,

$$\int_C \mathbf{F} \bullet d\mathbf{s}.$$

Hint: If you're having trouble computing the given integrals, go back and reread the first two sentences at the top of this page.

ChVar / PEA **3.** *Hyperbolic coordinates.* The appearance of the integral

$$\int_1^4 \int_{1/x}^{4/x} \frac{xy}{1+x^2y^2}\, dy\, dx$$

suggests that it would be helpful if xy were a single variable. With this in mind, consider the transformation of coordinates $(x,y) = (u, v/u)$.

(a) Sketch the given region of integration in the xy-plane.

(b) Show that this region is the image of a square region in the uv-plane.

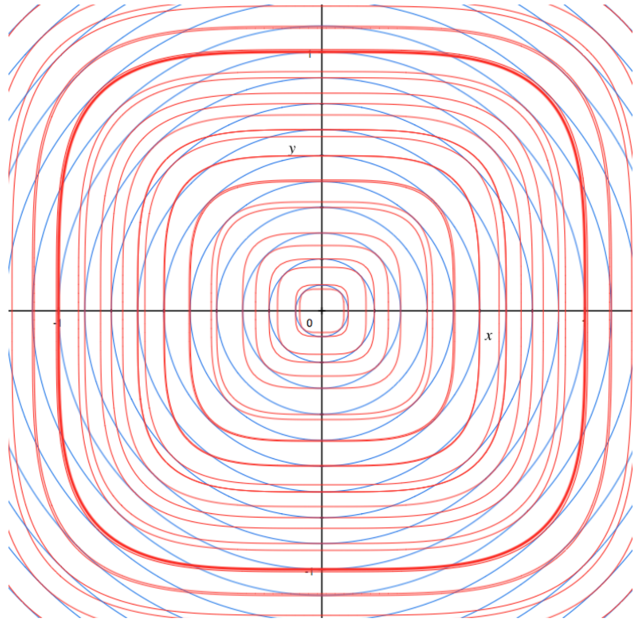
(c) Evaluate the given integral by making the indicated change of variables.

Hint: Recall Page 24 # 3 and Page 25 # 2.

More problems on the next page!

Multivariable Calculus

LM / DD **4**. Recall that the *Lagrange multipliers equation* (Page 30 # 1) says that the maximum and minimum values of a function $f(x, y)$ along a constraint curve $g(x, y) = c$ occur when the constraint curve is tangent to a level curve of the function. In the picture on the right, two families of curves are shown:

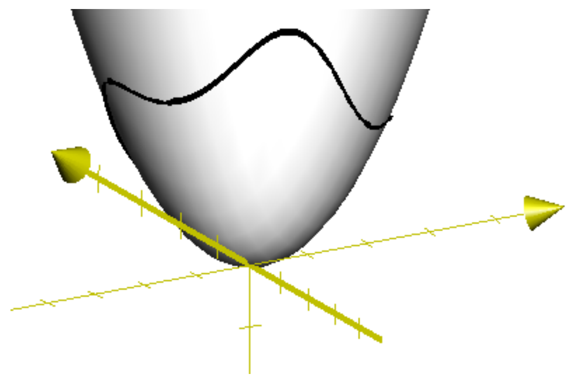


- Level curves of $f(x, y) = x^2 + y^2$ (blue)
- Level curves of $g(x, y) = x^4 + y^4$ (red)

(a) Mark all of the points (there are a lot!) where red and blue curves are tangent to each other.

(b) Only finitely many level curves are shown. If we drew all possible level curves, the points of tangency would themselves form curves. Sketch in these "tangency curves" and write down your best guess for their equations.

LM / DD **5.** Suppose that you are skateboarding in a paraboloid-shaped skate park, following a path that is like a square with rounded corners, which can be described by the equation $x^4 + y^4 = 1$. You might wonder, how high up do you get, and how low, while thusly entertained?



To answer this burning question, use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x^2+y^2$, under the constraint $x^4 + y^4 = 1$.

A view of the skate park, with the skateboarding route projected onto it, is shown above, from underneath the skate park.

LM / DD **6**. Explain the relationship between the work you did in the preceding two problems.

The idea here is that the Lagrange multipliers equation $\nabla f(x, y) = \lambda \cdot \nabla g(x, y)$ consists of two equations in three variables. Thus, it's not possible to find a single solution; it gives you an entire *curve* (or several curves, as in this case) of solutions. This tells you where *all* the solutions would occur, for different values of the constraint. Then you apply your particular constraint equation, to find the solution for your particular value of the constraint.

Multivariable Calculus

We now have several tools in our metaphorical mathematical toolbox for dealing with multivariable limits:

- Just plug in the point,
- Approach the origin along the axes, or along other special lines,
- Approach the origin along all lines of the form $y = mx$,
- Convert to polar (or spherical) coordinates and take the limit as $r \to 0$ (or $\rho \to 0$).

Limits / DD **1.** Of the above tools (methods):

(a) Which one(s) should you use when you think the limit *doesn't* exist?

(b) Which one(s) can you use to prove that the limit *does* exist?

Limits / DD **2.** For each of the following, say which method you would use, and why. Then use that method to determine whether the limit exists, and if so, what it is.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} \qquad \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}} \qquad \lim_{(x,y)\to(0,0)} \frac{x^2 + 5}{x - y + 3} \qquad \lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{xy^2}$$

LM / DD **3.** You have \$50.00 to spend on ice cream for yourself and your friends. Each scoop s costs \$1.50, and each waffle cone c costs \$1. The group's utility ("happiness") from eating s scoops and c cones is measured by $U(s, c) = \sqrt{sc}$.

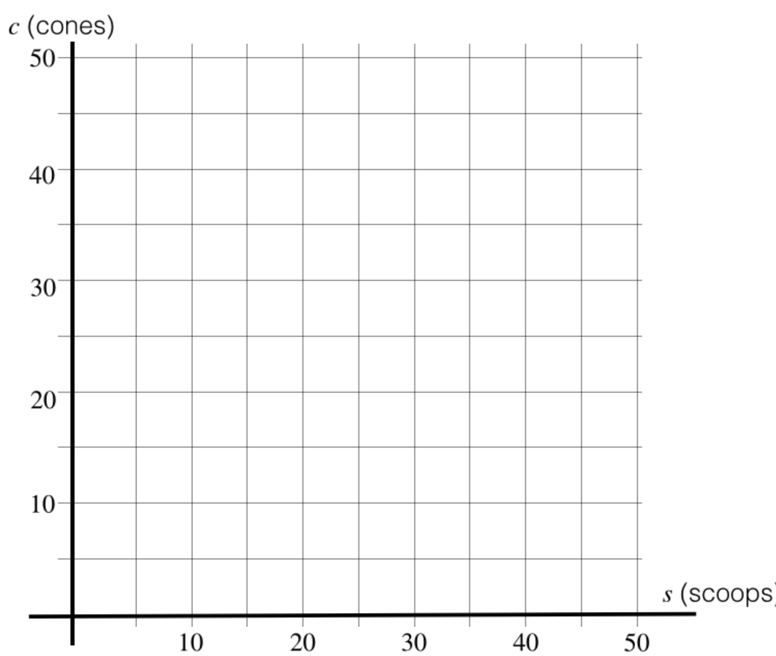
(a) In **blue** on the picture to the right, sketch level curves of the utility function for at least five different levels, and label the levels.

(b) In **red**, add in the line that is your budget constraint curve: the points representing all of the combinations of numbers (s, c) of scoops and cones that you can afford if you spend *all* of your money.

(c) Mark the approximate point on your budget constraint that maximizes total utility, and estimate its (s, c) value.

(d) Calculate: how many scoops and cones should you buy, to maximize total happiness?

Hint: Lagrange multipliers



TripInt / DD

4. Compute $\int_0^1 \int_y^1 \int_0^z \sin(z^3)\, dx\, dz\, dy$. *Hint*: use something you've learned in *this* course.

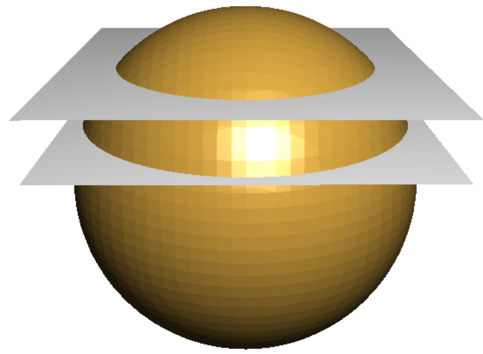
SSI / DD

5. Sketch the portion of the sphere of radius 4, centered at the origin, that is above the plane $z = 2$. Then find its surface area (note: *area* of *sphere*, not volume of solid ball).

The following problem is an extension of #5. If you solved #5 at home and you have time, please try #6. Otherwise, please work on #6 in your group after you are happy with #5.

SSI / PEA

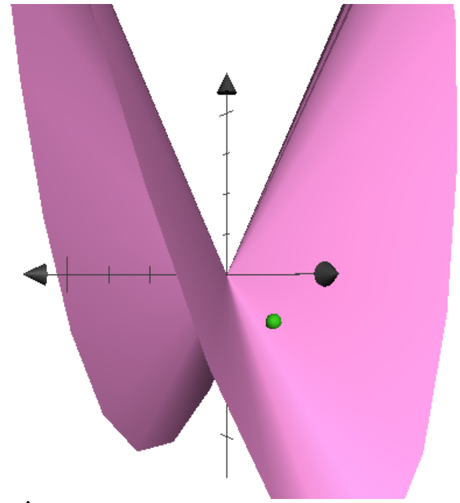
6. *The equal crust property.*



(a) Calculate the area of the part of the unit sphere (note: *area* of *sphere*, not volume of solid ball) that is found between the parallel planes $z = a$ and $z = b$, where $-1 \leq a \leq b \leq 1$.

(b) You should find that your answer depends only on the separation between the planes, not on the planes themselves. Explain why this result is named as it is.

Multivariable Calculus



Limits / DD **1**. *Continuous functions.* A function $f(x,y)$ is said to be *continuous* at (a,b) if the limit $\lim\limits_{(x,y)\to(a,b)}$ exists, and if this limit is equal to the function value $f(a,b)$.

(a) Find the limit, or show that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2-y^2}{\sqrt{x^2+y^2}}.$$

Consider the function

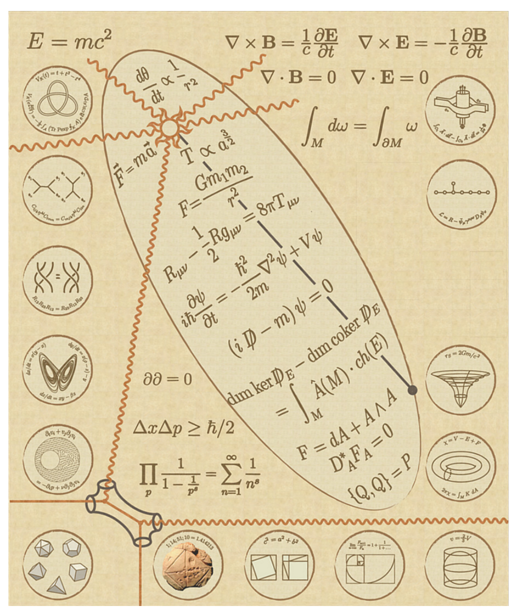
$$f(x,y) = \begin{cases} \dfrac{2x^2-y^2}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}.$$

(b) Is this function continuous at $(0,0)$? Is it continuous at $(-1,3)$?

(d) Graph the surface $z = \dfrac{2x^2-y^2}{\sqrt{x^2+y^2}}$ on your favorite graphing program to check your answer, and sketch the surface in your notebook.

Our last big theorem is *Gauss's Theorem*, also called the *Divergence Theorem*. If $\mathbf{F}$ is a vector field with continuous partial derivatives throughout a solid region E in $\mathbf{R}^3$, where the boundary surface ∂E of E has outward orientation, then Gauss's Theorem says that

$$\iint_{\partial E} \mathbf{F} \bullet d\mathbf{S} = \iiint_E \text{div}\mathbf{F}\, dV.$$



Gauss / DD **2.** Let $\mathbf{F} = [5x, 5y, 3]$, and let $\mathcal{S}$ be the unit sphere centered at the origin and oriented outward. Compute the flux of $\mathbf{F}$ over $\mathcal{S}$ in two ways:

(a) by computing the vector surface integral directly;

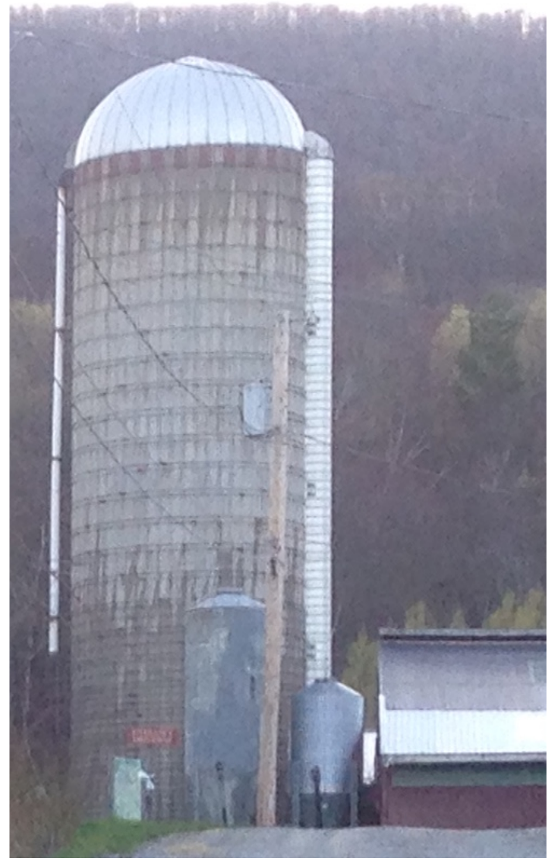
(b) by applying Gauss's Theorem.

☺ / DD **3.** (For fun) Suppose that E is a solid region in $\mathbf{R}^3$. Must its boundary surface ∂E be a *closed* surface, or can ∂E also have a boundary?

The picture to the right shows the design of the "Iconic Wall," a limestone engraving at the Simons Center for Geometry and Physics at Stony Brook University. The answer to this problem is engraved on the wall, along with many other fundamental results. Maybe you recognize some of them!

LM / DD

4. *A deep dive into silos.* A *silo* is a building used to store cattle feed (fermented *silage*). Most farms have at least one silo, and sometimes several. A typical silo is a cylindrical building with a hemispherical top, as shown to the right. Let the radius of such a silo be r, and the height of the cylindrical part be h, measured in feet.



(a) Find the total surface area in terms of r and h, *not including the floor.* (The floor is "free.")

(b) Find the volume inside the cylindrical part. Also find the volume enclosed by the hemispherical top.

(c) Typically, a silo is made of sheet metal. Suppose that you have a given amount (area) of sheet metal, and you wish to maximize the volume of the silo you construct out of it, *assuming that silage is only in the cylindrical part.* Write and solve the associated Lagrange multipliers equation $\nabla f = \lambda \nabla g$.

(d) Your solution from the previous part should be $2r = h$. Explain why you got a *curve* of solutions, rather than one exact answer, and also explain the meaning of the solution $2r = h$. Sketch a silo with this shape. Does the silo in the picture (or do other silos you have seen) have these proportions? If not, why not?

(e) Suppose that you have a given volume of silage to store in your silo, and you wish to minimize the amount of sheet metal used to construct it *assuming that silage is only in the cylindrical part.* Write and solve the associated Lagrange multipliers equation as above.

(f) You have 384π square feet of sheet metal. Find r and h to maximize the silo's volume.

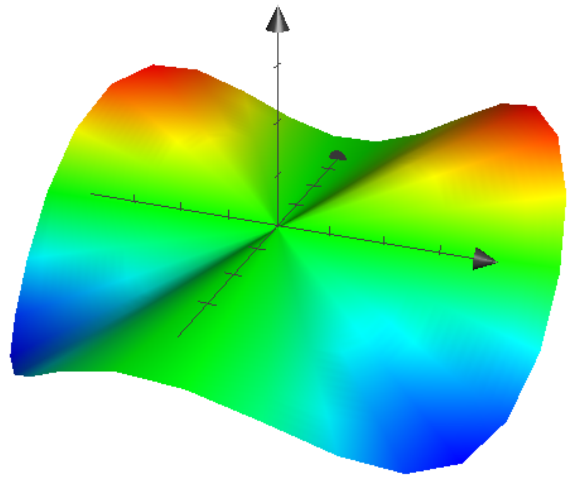
(g) You have 2000π cubic feet of silage. Find r and h to minimize the sheet metal used.

The following problem is an extension of #4. If you solved #4 at home and you have time, please try #5. Otherwise, please work on #5 in your group after you are happy with #4.

LM / DD

5. (Continuation) Write and solve the Lagrange multipliers equation, now assuming that silage can fill both the cylindrical *and the hemispherical part* of the silo. Your answer should surprise you. Explain why it makes sense. If you are not sure, try washing your table with sudsy soap for inspiration.

Multivariable Calculus



Cont / SC **1.** Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ a & \text{if } (x,y) = (0,0) \end{cases}.$$

Find a value of a that makes $f(x,y)$ continuous at $(0,0)$, or explain why this is impossible.

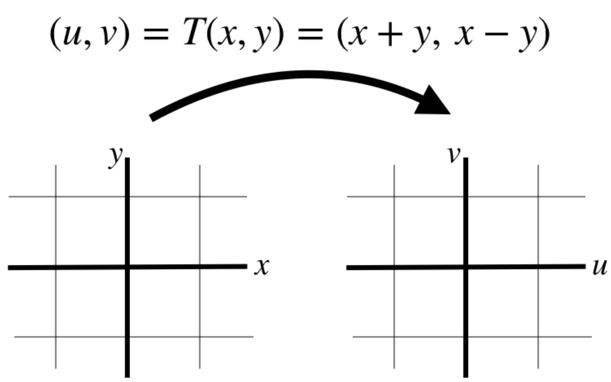
Gauss / DD **2.** Apply Gauss's Theorem to compute the flux of the vector field $\mathbf{F} = [x, 0, -z]$ over the closed surface S consisting of the cone $z = \sqrt{x^2+y^2}$ below $z = 1$, plus its circular cap, both oriented outward. Check your answer with your answers to Page 29 # 3–4.

ChVar / DD **3.** In this problem, we will compute $\iint_{\mathcal{R}} (x^2 - y^2)\, dA$, where $\mathcal{R}$ is the "diamond"-shaped region with vertices $(\pm 1, 0), (0, \pm 1)$.

(a) Sketch and shade in $\mathcal{R}$ in the xy-plane.

(b) Write down equations for the four lines that bound $\mathcal{R}$, and express each one with all of the variables on the left and the constant on the right.

(c) Consider the change of variables $u = x + y$, $v = x - y$. Explain why this is a good choice both for the function $f(x,y) = x^2 - y^2$ and also for the region $\mathcal{R}$.



(d) Sketch and shade in the corresponding region $\mathcal{R}^*$ in the uv-plane that is the image of $\mathcal{R}$ under the transformation $(u,v) = T(x,y) = (x+y, x-y)$.

(e) Find the Jacobian expansion factor $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$.

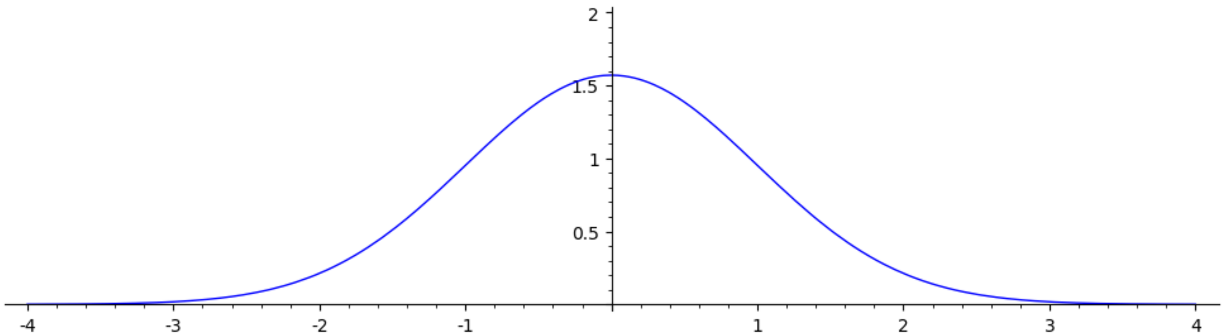
Hint: One way is to solve for x and y as functions of u and v. A clever alternate way is to compute the Jacobian $\dfrac{\partial(u,v)}{\partial(x,y)}$, and remember that the determinant of the matrix M^{-1} is the reciprocal of the determinant of M.

(f) Compute the integral from the first line. *Hint*: change of variables

Polar / DD

4. The *bell curve*, or *normal distribution*, or *Gaussian distribution*, is given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}.$$



Because bell curves describe many naturally-occurring phenomena, being able to integrate this function is very important to statisticians and many other people. For simplicity, we'll ignore the constants for now and just use $g(x) = e^{-x^2}$.

(a) Explain why it's hard to integrate $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx$.

We want this number, so let's give it a name: $A = \displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx$.

(b) Justify each of the four following equalities:

$$\begin{aligned} A^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 \\ &= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right) \cdot \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right) \\ &= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right) \cdot \left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) \\ &= \int_{y=-\infty}^{y=\infty} \int_{x=-\infty}^{x=\infty} e^{-x^2-y^2}\, dx\, dy. \end{aligned}$$

(c) Change to polar coordinates and show that $A = \sqrt{\pi}$.

(d) For the normal distribution function $f(x)$ given at the beginning, show that the total area under the curve is 1, and the inflection points occur at ± 1.

These are the reasons for the constants $1/\sqrt{2\pi}$ and 2 in $f(x)$.

☺ / DD

5. Consider a vector field $\mathbf{F}$ that is continuous on all of $\mathbf{R}^3$, and let $\mathcal{S}$ be *any* closed surface you want, with whichever orientation. Compute $\displaystyle\iint_{\mathcal{S}} \text{curl}\mathbf{F} \bullet d\mathbf{S}$.

Multivariable Calculus

Limits / DD

1. For each expression below, compute the limit or explain why it does not exist.

(a) $\displaystyle\lim_{(x,y)\to(1,2)} \frac{x^2 - xy - 2y^2}{x^2 - 4y^2}$ **(b)** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + xy - 2y^2}{x^2 + y^2}$ **(c)** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$

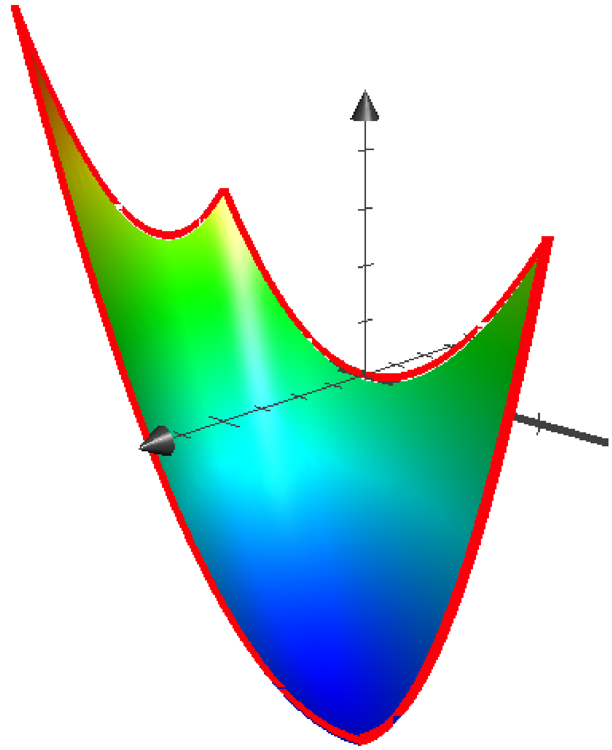
Opt / DD

2. Find the maximum and minimum values achieved by the function

$$f(x, y) = x^2 + xy + y^2 - 6y$$

over the rectangular region $[-3, 3] \times [0, 5]$.

Hint: Make and check a list as in Page 28 # 2.



ConVF / DD

3. Is the vector field $\mathbf{F} = \begin{bmatrix} -y\cos x + yz + ze^{xz} \\ z^2 - \sin x + xz \\ 2yz + xy + xe^{xz} \end{bmatrix}$ conservative?

VLI / SC

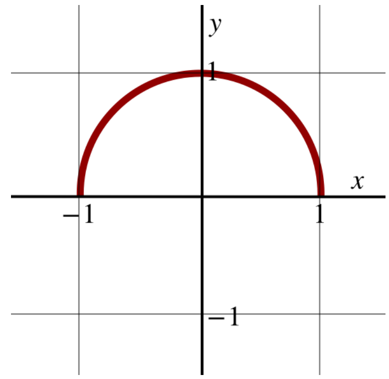
4. Swarthmore is a very windy place. By observing the paths of the leaves that are blowing around and their resulting flow lines, you are able to deduce that the wind's vector field can be described by $\mathbf{F} = [x^2 + y, y - x]$. Your path from breakfast at Sharples to multivariable calculus class in Singer is the part of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$. Does the wind help or hinder your journey to class, and by how much?

5. A 400-meter running track is made of two parallel straightaways, connected by semicircular curves, as shown. Suppose that you want to choose the dimensions of the track to maximize the area of the rectangular field at its center. How long should the straightaways be? Solve this problem using

(a) Lagrange multipliers; **(b)** single-variable calculus.

Multivariable Calculus

GrThm / DD **1.** *Closing off a curve, to apply Green's Theorem.*



Let your path C of half-circumnavigating your house be the part of the unit circle from $(1, 0)$ to $(-1, 0)$, oriented counterclockwise, and let the wind be described by

$$\mathbf{F} = \begin{bmatrix} y^2x + x^2 \\ x^2y + x - e^{y \sin y} \end{bmatrix}.$$

(a) Add the orientation of C to the picture.

(b) To figure out how hard it will be to take this walk, we would like to compute the line integral of $\mathbf{F}$ over C. We cannot do this directly, because of the $e^{y \sin y}$ term. Explain.

(c) Explain why applying Green's Theorem would make the $e^{y \sin y}$ term go away (so we would dearly love to use it), and also why we cannot apply Green's Theorem directly to C.

(d) Here is a clever trick: we will "close off" the region so that we can apply Green's Theorem. Let S be the line segment from $(-1, 0)$ to $(1, 0)$, and let D be the region now cleverly enclosed by the curves C and S.

(e) Explain why $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{s} + \int_S \mathbf{F} \cdot d\mathbf{s} = \iint_D \text{curl}(\mathbf{F})\, dA$.

(f) Use this magic trick to compute $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{s}$.

ConVF / DD **2.** *Breaking a vector field into its conservative and non-conservative parts.*

Let the vector field of difficulty of walking around Swarthmore be defined by

$$\mathbf{F} = \begin{bmatrix} -y + ye^y \\ x + xe^y + xye^y + z \\ y + 2 \end{bmatrix},$$

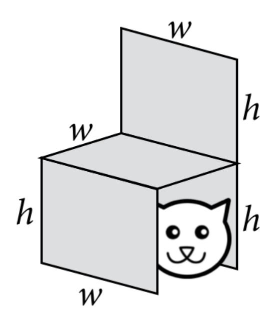
and let your path C be the left half of the unit circle in the xy-plane, oriented *clockwise* as viewed from the positive z-axis.

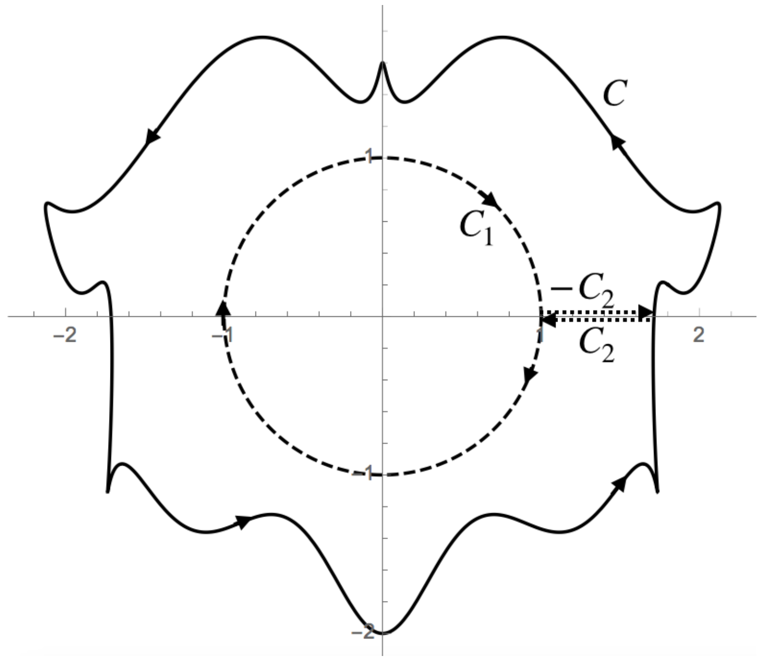
(a) Write $\mathbf{F}$ as the sum of two vector fields $\mathbf{F}_1$ ("elevation change") and $\mathbf{F}_2$ ("wind"), where $\mathbf{F}_1$ is completely conservative, and *no* part of $\mathbf{F}_2$ is conservative. *Hint*: Page 19 # 6.

(b) Explain why $\displaystyle\int_C \mathbf{F} \bullet d\mathbf{s} = \int_C \mathbf{F}_1 \bullet d\mathbf{s} + \int_C \mathbf{F}_2 \bullet d\mathbf{s}$.

(c) Use the above to compute the line integral of $\mathbf{F}$ over C.

LM / DD **3**. Ryan is making a cardboard chair in architecture class. The height of each part is the same (some length h), and the width and depth are the same, some length w. Ryan's cat likes to sit under the chair, as shown, so Ryan wishes to maximize the volume of the space under the chair. The instructor has given the students 300 in^2 of cardboard to make their chairs. What dimensions should Ryan use?





4. *Replacing a curve, using Green's Theorem.*

Suppose that we want to integrate the "unit speed circulation" vector field

$$\mathbf{F} = \left[\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right]$$

over the closed curve C, oriented counter-clockwise, shown as a solid curve in the figure. We can't compute it directly, because we don't have equations for C. We can't apply Green's Theorem, because $\mathbf{F} = [P, Q]$ isn't differentiable at the origin (it's not even defined there), which is in the region enclosed by C.

Amazingly, we can still compute the integral!

(a) Let C_1 be the unit circle (shown dashed in the figure), oriented *clockwise*. Let C_2 be an oriented line segment connecting the two curves, as shown dotted in the picture (the same curve oriented in the opposite direction, $-C_2$, is also shown). Let D be the solid region between C_1 and C. Justify the equation

$$\int_C \mathbf{F} \bullet d\mathbf{s} + \int_{C_2} \mathbf{F} \bullet d\mathbf{s} + \int_{C_1} \mathbf{F} \bullet d\mathbf{s} + \int_{-C_2} \mathbf{F} \bullet d\mathbf{s} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dx\, dy.$$

(b) Explain why this simplifies to

$$\int_C \mathbf{F} \bullet d\mathbf{s} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dx\, dy - \int_{C_1} \mathbf{F} \bullet d\mathbf{s}.$$

Compute the right side explicitly. The seemingly impossible is possible!

Grad / DD

5. *Fun with dimensions.*

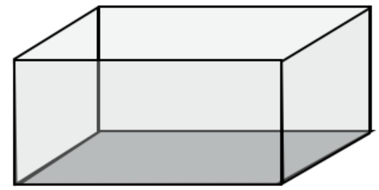
(a) A function $f(x, y)$ is a function of two variables. We write $f : \mathbf{R}^2 \to \mathbf{R}$, because the *input* is a point (x, y) in two real dimensions ($\mathbf{R}^2$), and the *output* is a real number in one dimension ($\mathbf{R}$). On the other hand, we often think of f as a surface in $\mathbf{R}^3$, which we graph as $z = f(x, y)$ to give us (x, y, z) in $\mathbf{R}^3$. Make a color-coded picture of a surface in $\mathbf{R}^3$, showing the inputs and outputs of a function $f(x, y)$ and the associated surface $z = f(x, y)$. Explain the difference between the function and the surface.

(b) Given a function $f(x, y)$, its gradient $\nabla f = [f_x, f_y]$ is a vector in $\mathbf{R}^2$. We often think of the gradient $\nabla f(a, b)$ as telling us about the steepness of the surface $z = f(x, y)$ at the point $(a,\ b,\ f(a, b)$. Make a picture showing a surface $f(x, y)$, a gradient vector in the xy-plane $\nabla f(a, b)$ at some point (a, b) in the xy-plane, and explain how it is that a 2D vector tells us something about a 3D surface.

Multivariable Calculus

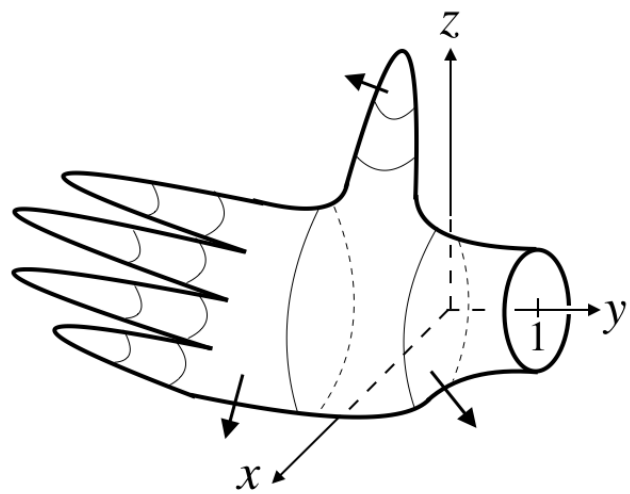
LM / DD

1. *More fishies.*



You're going to build a large aquarium in the shape of an open rectangular box without a top, which needs to hold 81 cubic feet of water. You will use slate for the rectangular base, and glass for the sides. Slate costs \$12 per square foot, and glass costs \$2 per square foot. Find the dimensions of the aquarium that minimize the cost.

2. *Replacing a surface, using Stokes's Theorem.*



Consider a vector field $\mathbf{F}$ whose curl is

$$\text{curl}\mathbf{F} = \left[y \sin e^{z^2}, (y-1)e^{\sin(x)} + 2, -ze^{\sin(x)}\right],$$

and consider the "glove" surface $\mathcal{S}$ shown in the figure, with outward normal. We wish to find the value of $\iint_{\mathcal{S}} \text{curl}\mathbf{F} \bullet d\mathbf{S}$. We cannot do this directly because curl$\mathbf{F}$ is awful and we don't have equations for $\mathcal{S}$.

(a) One option is to apply Stokes' Theorem and instead integrate $\int_C \mathbf{F} \bullet d\mathbf{s}$ over the boundary curve $C = \partial\mathcal{S}$, which in this case is the unit circle $y = 1, x^2 + z^2 = 1$. Which orientation should C have? Draw it in.

Unfortunately, we cannot do this, either, since we cannot find $\mathbf{F}$. Amazingly, we can still compute $\iint_{\mathcal{S}} \text{curl}\mathbf{F} \bullet d\mathbf{S}$!

(b) Consider the unit disk D defined by $y = 1$, $x^2 + z^2 \leq 1$, whose boundary is also C. Sketch D in the picture. By Stokes's Theorem,

$$\iint_{\mathcal{S}} \text{curl}\mathbf{F} \bullet d\mathbf{S} = \int_C \mathbf{F} \bullet d\mathbf{s} \qquad \text{and also} \qquad \int_C \mathbf{F} \bullet d\mathbf{s} = \iint_D \text{curl}\mathbf{F} \bullet d\mathbf{S},$$

as long as D has compatible orientation with C, so

$$\iint_{\mathcal{S}} \text{curl}\mathbf{F} \bullet d\mathbf{S} = \iint_D \text{curl}\mathbf{F} \bullet d\mathbf{S}.$$

Justify the three equations above.

(c) Compute the right-hand side to finish the job.

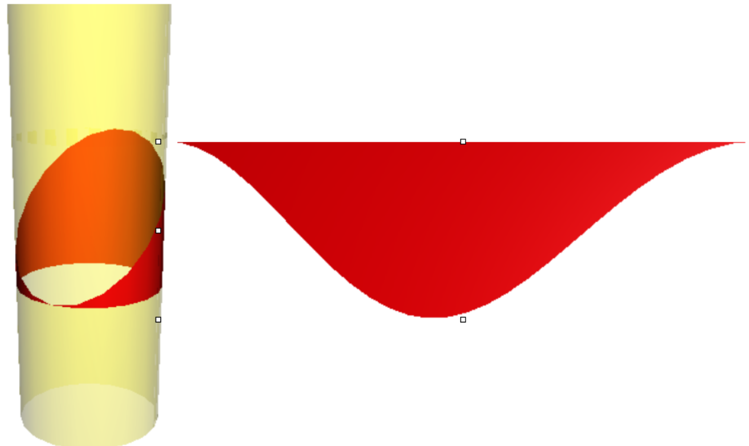
(d) Is D the only surface we could have used?

VLI / DD

3. Compute the vector line integral of $\mathbf{F} = [\sin y + 2xy^2, x\cos y + 2x^2y]$ over the curve C from $(2, 0)$ to $(0, -2)$ consisting of three-quarters of the circle of radius 2 centered at the origin, traversed counter-clockwise. *Hint*: work smarter, not harder

VSI / DD

4. *Fun with arts and crafts.*



You wrap a red strip of paper around a candle, as shown, and then use a sharp knife to cut the candle at a 45° angle, cutting the strip (and the candle) into two parts. You unwrap the top part of the strip and wonder, what is its area?

Assume that the candle is part of the solid unit cylinder $x^2 + y^2 \leq 1$, that the strip of paper originally satisfied $x^2 + y^2 = 1$ and $0 \leq z \leq 2$, and that the knife's cut was in the plane $z = x + 1$.

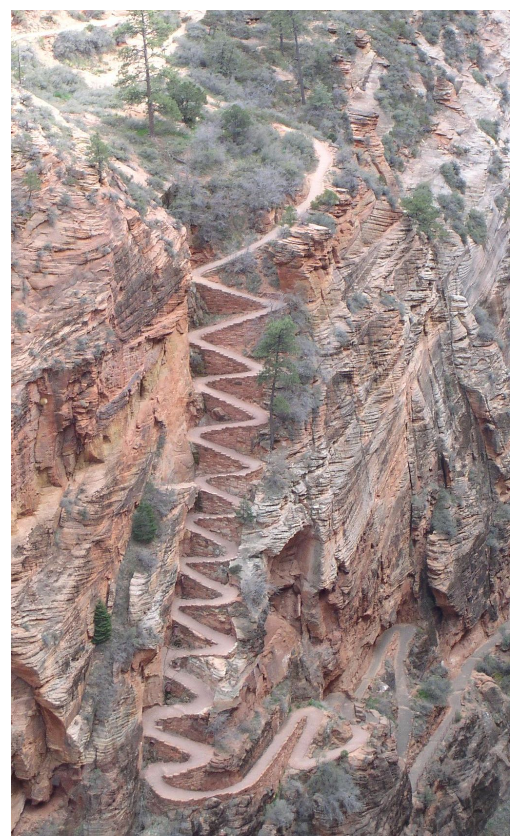
(a) Find the surface area of the unwrapped part of the paper.

Hint: The key here is parameterizing (using a surface parameterization with two variables, maybe θ and z) the part of the paper above the cut.

(b) Check your answer using geometry. *Hint*: symmetry.

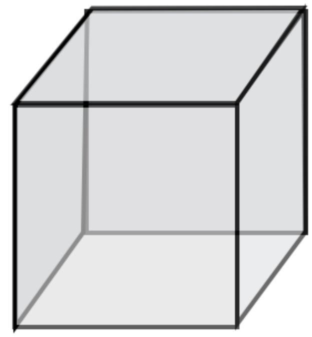
DirDer / DD

5. In Page 14 # 3, you were given an elevation function $f(x, y)$ that described a certain mountain. You used the gradient vector ∇f to find that, at the particular point where you were standing, the direction of steepest ascent up this mountain was southwest ($\gamma = 225°$). Then you used directional derivatives to find a direction of travel so that you would only ascend at *half* of the steepest rate. There turned out to be two options: either 60° to the left (counter-clockwise) or 60° to the right (clockwise) from direction γ.



Explain why it makes sense that there are two options of which direction you can hike, that give you the same steepness. Then explain how *switchbacks* on a trail (see picture) use this idea from directional derivatives, to create a hike that goes up the mountain, but at a gentler rate of incline than if you just hiked in the direction of the gradient.

Multivariable Calculus



1. *Closing off a surface, to apply Gauss's Theorem.*

Suppose we wish to compute $\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S}$, where

$$\mathbf{F} = \left[e^{\cos(yz)} \tan(z^2 y), \cos(e^{x^z}), 3z\right],$$

and $\mathcal{S}$ is the five faces of the unit cube $[0,1] \times [0,1] \times [0,1]$, all except for the bottom face on the xy-plane, as shown, with outward orientation. We do not want to compute this directly, because $\mathbf{F}$ is a mess and $\mathcal{S}$ has five parts. We would like to apply Gauss's Theorem, but we can't, because our surface $\mathcal{S}$ is not closed.

When we wanted to apply Green's Theorem to a *curve* that was not closed, we closed off the curve (Page 36 # 1), and here we can use the same strategy, by closing off the *surface*. Justify the following equation, and use it to find the value of $\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S}$:

$$\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S} + \iint_{\text{bottom face}} \mathbf{F} \bullet d\mathbf{S} = \iiint_{\text{solid cube}} \text{div}\mathbf{F}\ dV.$$

VLI / DD

2. Let C be the curve connecting $(-1,0)$ to $(1,0)$ along the top half of the unit circle, traversed clockwise. Let $\mathbf{F} = [2x+y, 3y-x]$ be a vector field in the xy-plane. We wish to compute $\displaystyle\int_C \mathbf{F} \bullet d\mathbf{s}$.

(a) You know at least three methods (tools you have) to solve this problem. List them.

(b) Solve the problem using whichever of those ways is your favorite.

Gauss / DD

3. When I was taking multivariable calculus, I wondered if there could be a way to *combine* Stokes's Theorem (top) and Gauss's Theorem (bottom):

$$\int_{\partial\mathcal{S}} \mathbf{F}_1 \bullet d\vec{\mathbf{s}} = \iint_{\mathcal{S}} \text{curl}\mathbf{F}_1 \bullet d\vec{\mathbf{S}}$$

$$\iint_{\partial E} \mathbf{F}_2 \bullet d\vec{\mathbf{S}} = \iiint_E \text{div}\mathbf{F}_2\ dV.$$

(a) Explain why, to combine them, we'd need $\mathcal{S} = \partial E$, and $\mathbf{F}_2 = \text{curl}\mathbf{F}_1$.

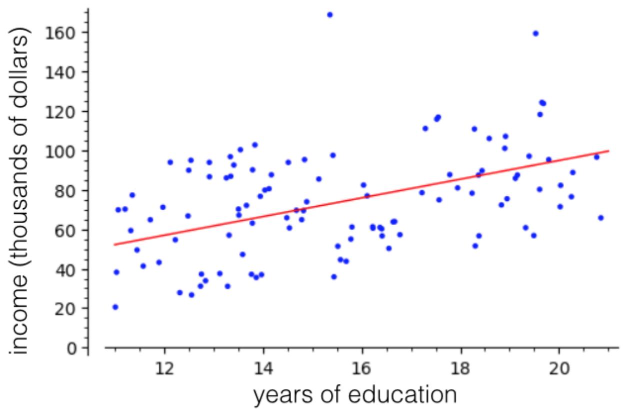
(b) Explain why, if $\mathcal{S} = \partial E$, the vector line integral (far left) would be zero.

(c) Explain why, if $\mathbf{F}_2 = \text{curl}\mathbf{F}_1$, the triple integral (far right) would be zero.

(d) Can you find a way to combine them *without* everything being zero?

The next page has a fun application of directional derivatives to politics.

4. In the United States, voters usually vote for either the *Democratic* or the *Republican* Party. Political scientists try to predict how voters will vote, based on demographics. Two factors that make a big difference in voting preferences are years of education (x) and yearly income in thousands of dollars (y). The probability r of a voter voting for a Republican candidate can be well approximated by the function



$$r(x, y) = 0.56 - 0.02x + 0.004y.$$

(a) What is the probability that a person with a college degree (16 years of education) and a \$60,000 yearly income votes Republican?

(b) How does the probability of voting Republican change as income increases?

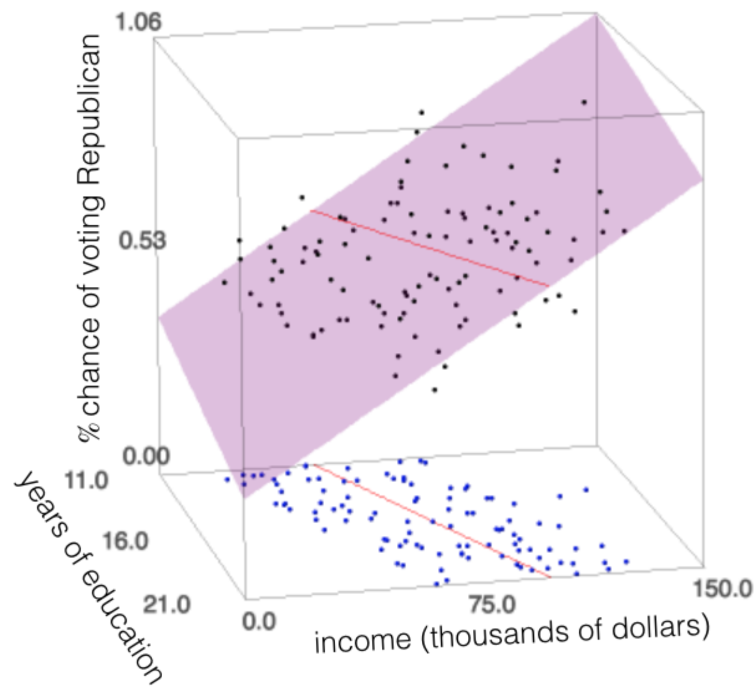
(c) How does the probability of voting Republican change as education increases?

Education (x) and income (y) are related. The relationship is well approximated by the equation $y = 5x$. This trendline is shown in red in the picture above.

(d) Find a vector in the direction of increasing education and income (in other words, a vector in the direction of the trendline).

(e) Find the directional derivative of probability of voting Republican, in the direction of increasing education and income.

(f) Interpret your result in the contexts of the picture to the right, and of politics.



While people make a big deal of the *partial* derivatives – wealthier people are more likely to vote Republican, and more educated people are more likely to vote Democratic – in fact, income and education are correlated, and the *directional* derivative shows that in the principal direction of increase of both, there is *no* change in voter preferences.

Using *directional* derivatives, rather than just *partial* derivatives, to analyze this kind of data, is a recent idea that Ella Foster-Molina, Swarthmore '07 and current Social Sciences Quantitative Laboratory Associate at Swarthmore, discovered in her Ph.D. work. No one ever thought of applying multivariable calculus ideas to statistical analysis in this way before!

5. (Challenge) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three-dimensional vectors with $\mathbf{b} \neq 0$. Show that if they satisfy $\mathbf{a} \times \mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c} = 0$, then $\mathbf{a} \bullet \mathbf{c} = 0$.

Reference

Written by the instructors at Phillips Exeter Academy

acceleration: The derivative of velocity with respect to time.

angle-addition identities: For any angles α and β, $\cos(\alpha+\beta) \equiv \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and $\sin(\alpha+\beta) \equiv \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

angle between vectors: When two vectors $\mathbf{u}$ and $\mathbf{v}$ are placed tail-to-tail, the angle θ they form can be calculated by the dot-product formula $\cos\theta = \dfrac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$. If $\mathbf{u}\bullet\mathbf{v} = 0$ then $\mathbf{u}$ is perpendicular to $\mathbf{v}$. If $\mathbf{u}\bullet\mathbf{v} < 0$ then $\mathbf{u}$ and $\mathbf{v}$ form an obtuse angle.

antiderivative: If f is the derivative of g, then g is called *an antiderivative of f.* For example, $g(x) = 2x\sqrt{x} + 5$ is an antiderivative of $f(x) = 3\sqrt{x}$, because $g' = f$.

average velocity is displacement divided by elapsed time.

bounded: Any subset of $\mathbf{R}^n$ that is contained in a suitably large *disk*.

Chain Rule: The derivative of a composite function $C(x) = f(g(x))$ is a product of derivatives, namely $C'(x) = f'(g(x))g'(x)$. The actual appearance of this rule changes from one example to another, because of the variety of function types that can be composed. For example, a curve can be traced in $\mathbf{R}^3$, on which a real-valued temperature distribution is given; the composite $\mathbf{R}^1 \longrightarrow \mathbf{R}^3 \longrightarrow \mathbf{R}^1$ simply expresses temperature as a function of time, and the derivative of this function is the dot product of two vectors.

chord: A segment that joins two points on a curve.

closed: Suppose that $\mathcal{D}$ is a set of points in $\mathbf{R}^n$, and that every convergent sequence of points in $\mathcal{D}$ actually converges to a point in $\mathcal{D}$. Then $\mathcal{D}$ is called "closed."

concavity: A graph $y = f(x)$ is *concave up* on an interval if f'' is positive throughout the interval. The graph is *concave down* on an interval if f'' is negative throughout the interval.

content: A technical term that is intended to generalize the special cases length, area, and volume, so that the word can be applied in any dimension.

continuity: A function f is *continuous at a* if $f(a) = \lim\limits_{p\to a} f(p)$. A *continuous function* is continuous at all the points in its domain.

converge (integral): An *improper integral* that has a finite value is said to *converge* to that value, which is defined using a limit of proper integrals.

critical point: A point in the domain of a function f at which f' is either zero or undefined.

cross product: Given $\mathbf{u} = [p, q, r]$ and $\mathbf{v} = [d, e, f]$, a vector that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ is $[\,qf - re, rd - pf, pe - qd\,] = \mathbf{u} \times \mathbf{v}$.

curl: A three-dimensional vector field that describes the rotational tendencies of the three-dimensional field from which it is derived.

curvature: This positive quantity is the rate at which the direction of a curve is changing, with respect to the distance traveled along it. For a circle, this is just the reciprocal of the radius. The principal *normal vector* points towards the center of curvature.

cycloid: A curve traced by a point on a wheel that rolls without slipping. Galileo named the curve, and Torricelli was the first to find its area.

cylindrical coordinates: A three-dimensional system of coordinates obtained by appending z to the usual polar-coordinate pair (r, θ).

decreasing: A function f is *decreasing* on an interval $a \le x \le b$ if $f(v) < f(u)$ holds whenever $a \le u < v \le b$ does.

derivative: Let f be a function that is defined for points $\mathbf{p}$ in $\mathbf{R}^n$, and whose values $f(\mathbf{p})$ are in $\mathbf{R}^m$. If it exists, the derivative $f'(\mathbf{a})$ is the $m \times n$ matrix that represents the best possible linear approximation to f at $\mathbf{a}$. In the case $n = 1$ (a parametrized curve in $\mathbf{R}^m$), $f'(a)$ is the $m \times 1$ matrix that is visualized as the tangent vector at $f(a)$. In the case $m = 1$, the $1 \times n$ matrix $f'(\mathbf{a})$ is visualized as the gradient vector at $\mathbf{a}$.

derivative at a point: Let f be a real-valued function that is defined for points in $\mathbf{R}^n$. Differentiability at a point $\mathbf{a}$ in the domain of f means that there is a linear function L with the property that the difference between $L(\mathbf{p})$ and $f(\mathbf{p})$ approaches 0 faster than $\mathbf{p}$ approaches $\mathbf{a}$, meaning that $0 = \lim_{\mathbf{p} \to \mathbf{a}} \dfrac{f(\mathbf{p}) - L(\mathbf{p})}{|\mathbf{p} - \mathbf{a}|}$. If such an L exists, then $f'(\mathbf{a})$ is the matrix that defines $L(\mathbf{p} - \mathbf{a})$.

determinant: A ratio that is associated with any square matrix, as follows: Except for a possible sign, the determinant of a 2×2 matrix $\mathbf{M}$ is the area of any region $\mathcal{R}$ in 2-dimensional space, divided into the area of the region that results when $\mathbf{M}$ is applied to $\mathcal{R}$. Except for a possible sign, the determinant of a 3×3 matrix $\mathbf{M}$ is the volume of any region $\mathcal{R}$ in 3-dimensional space, divided into the volume of the region that results when $\mathbf{M}$ is applied to $\mathcal{R}$.

differentiable: A function that has derivatives at all the points in its domain.

directional derivative: Given a function f defined at a point $\mathbf{p}$ in $\mathbf{R}^n$, and given a direction $\mathbf{u}$ (a unit vector) in $\mathbf{R}^n$, the derivative $D_{\mathbf{u}}f(\mathbf{p})$ is the instantaneous rate at which the values of f change when the input varies only in the direction specified by $\mathbf{u}$.

discontinuous: A function f has a *discontinuity at a* if $f(a)$ is defined but does not equal $\lim_{p \to a} f(p)$; a function is *discontinuous* if it has one or more discontinuities.

disk: Given a point $\mathbf{c}$ in $\mathbf{R}^n$, the set of all points $\mathbf{p}$ for which the distance $|\mathbf{p} - \mathbf{c}|$ is at most r is called the disk (or "ball") of radius r, centered at $\mathbf{c}$.

divergence: If $\mathbf{v}$ is a vector field, its divergence is the scalar function $\nabla \bullet \mathbf{v}$.

domain: The domain of a function consists of all the numbers for which the function returns a value. For example, the domain of a logarithm function consists of positive numbers only.

double-angle identities: Best-known are $\sin 2\theta \equiv 2\sin\theta\cos\theta$, $\cos 2\theta \equiv 2\cos^2\theta - 1$, and $\cos 2\theta \equiv 1 - 2\sin^2\theta$; special cases of the *angle-addition identities*.

double integral: A descriptive name for an integral whose domain of integration is two-dimensional. When possible, evaluation is an iterative process, whereby two single-variable integrals are evaluated instead.

e is approximately 2.71828182845904523536. This irrational number frequently appears in scientific investigations. One of the many ways of defining it is $e = \lim_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n$.

ellipsoid: A *quadric surface*, all of whose planar sections are ellipses.

extreme point: either a *local minimum* or a *local maximum*. Also called an *extremum*.

Extreme-value Theorem: Suppose that f is a continuous real-valued function that is defined throughout a *closed* and *bounded* set $\mathcal{D}$ of points. Then f attains a maximal value and a minimal value on $\mathcal{D}$. This means that there are points $\mathbf{a}$ and $\mathbf{b}$ in $\mathcal{D}$, such that $f(\mathbf{a}) \le f(\mathbf{p}) \le f(\mathbf{b})$ holds for all $\mathbf{p}$ in $\mathcal{D}$. If f is also differentiable, then $\mathbf{a}$ is either a critical point for f, or it belongs to the boundary of $\mathcal{D}$; the same is true of $\mathbf{b}$.

Fubini's Theorem: Provides conditions under which the value of an integral is independent of the iterative approach applied to it.

Fundamental Theorem of Calculus: In its narrowest sense, differentiation and integration are inverse procedures — integrating a derivative $f'(x)$ along an interval $a \le x \le b$ leads to the same value as forming the difference $f(b) - f(a)$. In multivariable calculus, this concept evolves.

gradient: This is the customary name for the *derivative* of a real-valued function, especially when the domain is multidimensional.

Greek letters: Apparently essential for doing serious math! There are 24 letters. The upper-case characters are

$$\mathrm{A}\ \mathrm{B}\ \Gamma\ \Delta\ \mathrm{E}\ \mathrm{Z}\ \mathrm{H}\ \Theta\ \mathrm{I}\ \mathrm{K}\ \Lambda\ \mathrm{M}\ \mathrm{N}\ \Xi\ \mathrm{O}\ \Pi\ \mathrm{P}\ \Sigma\ \mathrm{T}\ \Upsilon\ \Phi\ \mathrm{X}\ \Psi\ \Omega$$

and the corresponding lower-case characters are

$$\alpha\ \beta\ \gamma\ \delta\ \epsilon\ \zeta\ \eta\ \theta\ \iota\ \kappa\ \lambda\ \mu\ \nu\ \xi\ \mathrm{o}\ \pi\ \rho\ \sigma\ \tau\ \upsilon\ \phi\ \chi\ \psi\ \omega$$

Hessian: See *second derivative.*

hyperbola I: A hyperbola has two focal points, and the difference between the *focal radii* drawn to any point on the hyperbola is constant.

hyperbola II: A hyperbola is determined by a focal point, a directing line, and an eccentricity greater than 1. Measured from any point on the curve, the distance to the focus divided by the distance to the directrix is always equal to the eccentricity.

hyperboloid: One of the *quadric surfaces.* Its principal plane of reflective symmetry has a special property — every section obtained by slicing the surface perpendicular to this plane is a hyperbola.

improper integral: This is an integral $\int_{\mathcal{D}} f$ for which the domain $\mathcal{D}$ of integration is unbounded, or for which the values of the integrand f are undefined or unbounded.

increasing: A function f is *increasing* on an interval $a \le x \le b$ if $f(u) < f(v)$ holds whenever $a \le u < v \le b$ does.

integrable: Given a region $\mathcal{R}$ and a function $f(x, y)$ defined on $\mathcal{R}$, f is said to be *integrable over* $\mathcal{R}$ if the limit of Riemann sums used to define the integral of f over $\mathcal{R}$ exists.

integrand: A function whose integral is requested.

Jacobian: A traditional name for the derivative of a function f from $\mathbf{R}^n$ to $\mathbf{R}^m$. For each point $\mathbf{p}$ in the domain space, $f'(\mathbf{p})$ is an $m \times n$ matrix. When $m = n$, the matrix is square, and its determinant is also called "the Jacobian" of f. Carl Gustav Jacobi (1804-1851) was a prolific mathematician; one of his lesser accomplishments was to establish the symbol ∂ for partial differentiation.

l'Hôpital's Rule: A method for dealing with indeterminate forms: If f and g are differentiable, and $f(a) = 0 = g(a)$, then $\lim_{t \to a} \dfrac{f(t)}{g(t)}$ equals $\lim_{t \to a} \dfrac{f'(t)}{g'(t)}$, provided that the latter limit exists. The Marquis de l'Hôpital (1661-1704) wrote the first textbook on calculus.

Lagrange multipliers: A method for solving constrained extreme-value problems.

Lagrange notation: The use of primes to indicate derivatives.

level curve: The configuration of points $\mathbf{p}$ that satisfy an equation $f(\mathbf{p}) = k$, where f is a real-valued function defined for points in $\mathbf{R}^2$ and k is a constant.

level surface: The configuration of points $\mathbf{p}$ that satisfy an equation $f(\mathbf{p}) = k$, where f is a real-valued function defined for points in $\mathbf{R}^3$ and k is a constant.

linearization: A generalization of a tangent line. Given a function f, the *linearization* L of f at a is the best linear approximation of f at a, i.e. the function whose value agrees with f, $L(a) = f(a)$, and whose first derivatives or partial derivatives also all match those of f.

line integral: Given a *vector field* F and a *path* $\mathcal{C}$ (which does not have to be linear) in the domain space, a real number results from "integrating F along $\mathcal{C}$".

Mean-Value Theorem: If the curve $y = f(x)$ is continuous for $a \le x \le b$, and differentiable for $a < x < b$, then the slope of the line through $(a, f(a))$ and $(b, f(b))$ equals $f'(c)$, where c is strictly between a and b. There is also a version of this statement that applies to integrals.

normal vector: In general, this is a vector that is perpendicular to something (a line or a plane). In the analysis of parametrically defined curves, the principal normal vector (which points in the direction of the center of curvature) is the derivative of the unit tangent vector.

operator notation: A method of naming a derivative by means of a prefix, usually D, as in $D\cos x = -\sin x$, or $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$, or $D_x(u^x) = u^x(\ln u)D_x u$.

orthonormal: Describes a set of mutually perpendicular vectors of unit length.

parabola: This curve consists of all the points that are equidistant from a given point (the *focus*) and a given line (the *directrix*).

paraboloid: One of the *quadric surfaces*. Sections obtained by slicing this surface with a plane that contains the principal axis are parabolas.

partial derivative: A *directional derivative* that is obtained by allowing only one of the variables to change.

path: A parametrization for a curve.

polar coordinates: Polar coordinates for a point P in the xy-plane consist of two numbers r and θ, where r is the distance from P to the origin O, and θ is the size of an angle in standard position that has OP as its terminal ray.

polar equation: An equation written using the polar variables r and θ.

Product Rule: The derivative of $p(x) = f(x)g(x)$ is $p'(x) = f(x)g'(x) + g(x)f'(x)$. The actual appearance of this rule depends on what x, f, g, and "product" mean, however. One can multiply numbers times numbers, numbers times vectors, and vectors times vectors — in two different ways.

quadric surface: The graph of a quadratic polynomial in three variables.

Quotient Rule: The derivative of $p(x) = \dfrac{f(x)}{g(x)}$ is $p'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$. This is unchanged in multivariable calculus, because vectors cannot be used as divisors.

second derivative: The derivative of a derivative. If f is a real-valued function of $\mathbf{p}$, then $f'(\mathbf{p})$ is a vector that is usually called the *gradient* of f, and $f''(\mathbf{p})$ is a square matrix that is often called the *Hessian* of f. The entries in these arrays are *partial derivatives*.

Second-Derivative Test: When it succeeds, this theorem classifies a critical point for a differentiable function as a local maximum, a local minimum, or a saddle point (which in the one-variable case is called an inflection point). The theorem is inconclusive if the determinant of the second-derivative matrix is 0.

speed: The magnitude of *velocity.* For a parametric curve $(x, y) = (f(t), g(t))$, it is given by the formula $\sqrt{(x')^2 + (y')^2}$. Notice that that this is *not* the same as dy/dx.

spherical coordinates: Points in three-dimensional space can be described as (ρ, θ, ϕ), where ρ is the distance to the origin, θ is longitude, and ϕ is co-latitude.

triple scalar product: A formula for finding the volume of parallelepiped, in terms of its defining vectors. It is the *determinant* of a 3×3 matrix.

velocity: This n-dimensional vector is the derivative of a differentiable path in $\mathbf{R}^n$. When $n = 2$, whereby a curve $(x, y) = (f(t), g(t))$ is described parametrically, the velocity is $\left[\dfrac{df}{dt}, \dfrac{dg}{dt}\right]$ or $\left[\dfrac{dx}{dt}, \dfrac{dy}{dt}\right]$, which is tangent to the curve. Its magnitude $\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$ is the speed. The *components* of velocity are themselves derivatives.

vector field: This is a descriptive name for a function F from $\mathbf{R}^n$ to $\mathbf{R}^n$. For each $\mathbf{p}$ in the domain, $F(\mathbf{p})$ is a vector. The derivative (gradient) of a real-valued function is an example of such a field.