Math 300

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Math 300

Before each class, your homework is to read the associated section(s) in *Book of Proof* by Richard Hammack, and do the homework problems on the associated page. Note that the homework sets for classes 6, 16 and 18 have two pages each.

Class	Day	Date	Topic	Sections
1	T	3/29	Sets, Cartesian Product	1.1-1.2
2	W	3/30	Direct Proofs	4.1-4.5, 5.3
3	F	4/1	Subsets, Power Sets	1.3-1.4
4	M	4/4	$\cup, \cap, -, \ ^{C}$	1.5-1.6
5	W	4/6	Venn Diagrams, Indexed Sets	1.7-1.8
6	F	4/8	(Conditional) Statements, Conjunctions	2.1-2.3
7	M	4/11	Truth Tables, Logical Equivalence	2.4-2.6
8	W	4/13	Quantifiers, Translation	2.7-2.12
9	F	4/15	Contrapositive	5.1-5.2
10	M	4/18	Contradiction	6.1-6.4
11	W	4/20	Non-Conditional Statements	7.1-7.4
12	F	4/22	Proofs Involving Sets	8.1-8.4
13	M	4/25	Disproof	9.1-9.3
	T	4/26	**Midterm 1 (Sections 1.1-8.4)**	
14	W	4/27	Strong Induction	10.1
15	F	4/29	Smallest Counterexamples, Fibonacci	10.2-10.3
16	M	5/2	Relations: Properties; Equivalences	11.1-11.2
17	W	5/4	Equiv. Classes, Partitions and $\mathbb{Z}/n\mathbb{Z}$	11.3-11.4
18	F	5/6	Set Relations	11.5
19	M	5/9	Functions, Injective and Surjective	12.1-12.2
20	W	5/11	Pigeonhole Principle	12.3
21	F	5/13	Composition	12.4
22	M	5/16	Inverse Functions	12.5
	T	5/17	**Midterm 2 (Sections 9.1-12.3)**	
23	W	5/18	Image and Preimage	12.6
24	F	5/20	Equal Cardinalities	13.1
25	M	5/23	Countable and Uncountable Sets	13.2
26	W	5/25	Comparing Cardinalities	13.3
27	F	5/27	CBS Theorem	13.4
	M	5/30	Memorial Day–No Class	
28	T	5/31	CBS Theorem	13.4
	F	6/10	**Final Exam, 9-11 am**	

Hand-in proofs. You must write up and turn in one proof each week. The problems that are eligible for this are marked in bold. You will revise and resubmit your proofs until they are perfect.

1. This is a good problem, which we will discuss, but it is not a proof.

2. This asks you to prove something, so you may hand it in.

3. This problem has several parts.

   (a) This part is a proof, which you may write up and hand in.

   (c) This part is a good exercise, but it is not a proof.

You must type up proofs in LaTeX, which is how mathematicians write. This is so that when you revise it, you can edit the file instead of writing out the whole thing again!

Here is an example of how I would like your handed-in proof to be.

Claim. The square root of 2 is irrational.

To show that $\sqrt{2}$ is irrational, we will show that there is no ratio p/q of natural numbers whose square is 2.

Proof. Let p and q be natural numbers with no common factors, so that the ratio p/q is in lowest terms. We will suppose (for a contradiction) that $p/q = \sqrt{2}$.

$$\begin{aligned} \text{Suppose that} \qquad \frac{p}{q} &= \sqrt{2}. \\ \text{Squaring both sides yields} \qquad \frac{p^2}{q^2} &= 2 \\ \text{and then multiplying by } q^2 \text{ yields} \qquad p^2 &= 2q^2. \end{aligned}$$

This shows that p is even, so $p = 2r$ for some natural number r.

$$\begin{aligned} \text{So we can rewrite our equation as} \qquad (2r)^2 &= 2q^2 \\ \text{and then multiply out to yield} \qquad 4r^2 &= 2q^2 \\ \text{and then simplify this to} \qquad 2r^2 &= q^2. \end{aligned}$$

This shows that q is even. But then p and q are both even, which violates our assumption that p and q have no common factors, which is a contradiction. This proves that no ratio p/q of natural numbers squares to 2, so $\sqrt{2}$ is irrational. □

Notice that this proof is mostly words, and that every part of this proof is a full English sentence. Every sentence includes verbs and proper punctuation. You should do this. When you solved the problem on your paper, your solution may have been mostly symbols, but when you write down the proof, give a clear explanation of each step, as though you are talking to the person who is reading the proof.

Discussion Skills

1. Contribute to the class every day
2. Speak to classmates, not to the instructor
3. Put up a difficult problem, even if not correct
4. Use other students' names
5. Ask questions
6. Answer other students' questions
7. Suggest an alternate solution method
8. Draw a picture
9. Connect to a similar problem
10. Summarize the discussion of a problem

The problems in this book. These problems were written by Ross Sweet, with additional input from Matthew Graham, and have been typeset and edited by Diana Davis. Problems that start with parenthetical numbers, e.g. (10.27), are taken directly from *Book of Proof* by Richard Hammack. They were used in a nine-week course that introduced students to logic and proof writing. The course met three times a week for 50 minutes, with approximately 15 students of all four class years. Students wrote up solutions on the board for the first 5-10 minutes, spent the next 30 minutes explaining their solutions to the class and discussing them, and then went to the board at the end of class in pairs or groups of three to work on the problems that they had not solved.

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1. Write each of the following sets by listing their elements between braces.
(a) $\{x \in \mathbb{R} : \cos x = 1\}$ (b) $\{5x : x \in \mathbb{Z}, |2x| \leq 8\}$

2. Write each of these sets in set-builder notation.
(a) $\{3, 6, 11, 18, 27, 38, \ldots\}$ (b) $\{\ldots, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27, \ldots\}$

3. Find the following cardinalities.
(a) $|\{\{1,4\}, a, b, \{\{3,4\}\}, \{\varnothing\}\}|$ (b) $|\{\{\{1,4\}, a, b, \{\{3,4\}\}, \{\varnothing\}\}\}|$
(c) $|\{x \in \mathbb{Z} : x^2 < 10\}|$ (d) $|\{x \in \mathbb{N} : x^2 < 10\}|$

4. Sketch the following sets of points in the x-y plane.
(a) $\{1,2\} \times [0,1]$ (b) $[0,1] \times \{1,2\}$ (c) $\left\{\left(a, \frac{a^2}{b}\right) : a \in \mathbb{R}, b \in \mathbb{Z} - \{0\}\right\}$

5. Let $A = \{1,2,3\}$ and $B = \{a,b\}$. Write the indicated sets by listing their elements between braces. (Recall Cartesian product definition.)
(a) $A \times B$ (b) $B \times B$ (c) $A \times \varnothing$ (d) $A \times \{\varnothing\}$

6. The following three sets are not the same.

	$(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$	$\mathbb{R} \times (\mathbb{R} \times \mathbb{R})$	$\mathbb{R} \times \mathbb{R} \times \mathbb{R}$
example element	1, 2, 3	1, 2, 3	1, 2, 3

(a) Place parentheses in the correct spots to come up with an example element for each set.

(b) Only using the three different example elements, describe in words why these sets are different.

(c) The last set $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$ is usually how we geometrically think about three dimensional space. Briefly, describe the geometric difference between the first two sets.

(d) The set $\mathbb{R}^3$ has a standard labeling (coordinate system) and so does $\mathbb{R}^2$. Is it possible to label the first two sets to coincide with the labeling in $\mathbb{R}^3$? If so, what choices need to be made? If not, explain why not.

6. Let C_r be the set of points that comprise a circle of radius r, and let $\mathbb{R}^* = [0, \infty)$.

(a) Consider the set $A = \mathbb{R}^* \times \{C_r : r \in \mathbb{R}^*\}$

- Sketch the cross-section $\{1\} \times \{C_r : r \in \mathbb{R}^*\}$.
- Sketch $R^* \times C_1$.

(b) Next consider the set $B = \{(r, C_r) : r \in \mathbb{R}^*\}$.

- Is B is a subset of A?
- Briefly, describe the set B geometrically.

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You may not have time to do every problem. Spend 90 *minutes doing those that most interest you.*

1. Prove that if x and y are odd, then xy is odd.

2. Let a be an integer. Prove that if $5|2a$ then $5|a$.

3. Using the statement of the Division Algorithm, prove that every integer is either odd or even.

4. State the Well Ordering Principle and use it to prove the Division Algorithm.

5.* Prove that *no* perfect square has the form $3n + 2$, where n is an integer.

6. Prove that for any integer n, 3 divides $n(n + 1)(n + 2)$.

7. Prove that if $n \in \mathbb{Z}$, then $5n^2 + 3n + 7$ is odd. (Hint: try cases.)

8. Prove that if two integers have opposite parity, then their product is even.

9. Suppose that $a, b, c \in \mathbb{Z}$. Prove that if $a|b$ and $a|c$ then $a|(b + c)$.

10.* Prove that every odd integer is the difference of two squares. For example $7 = 4^2 - 3^2$.

11.* Prove that if $n \in N$ then $\binom{2n}{n}$ is even.

12.* Let a, b and c be integers such that $a^2 + b^2 = c^2$. Prove that a is even or b is even.

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1. Decide if the following statements are true or false. Explain your answers.
(a) $\mathbb{R}^3 \subseteq \mathbb{R}^3$ (b) $\mathbb{R}^2 \subseteq \mathbb{R}^3$ (c) $\big\{(x,y) : x^2 - x = 0\big\} \subseteq \{(x,y) : x - 1 = 0\}$

2. List all of the subsets of the following sets.
(a) $\{1, 2, \varnothing\}$ (b) $\{\mathbb{R}, \mathbb{Q}, \mathbb{N}\}$ (c) $\{\mathbb{R}, \{\mathbb{Q}, \mathbb{N}\}\}$

3. Find the indicated sets and explicitly enumerate their elements.
(a) $\mathscr{P}(\mathscr{P}(\{2\}))$ (b) $\mathscr{P}(\{a, b\} \times \{0\})$
(c) $\{X \in \mathscr{P}(\{1, 2, 3\}) : |X| \leq 1\}$ (d) $\{X \subseteq \mathscr{P}(\{1, 2, 3\}) : |X| \leq 1\}$

Some problems are True or False. If it is true, write down the statement that is true and then prove it. If it is false, give a counterexample (usually) or explain why it is false (if a counterexample is not possible in that particular case).
Note: Counterexamples are generally not eligible for handing in as proofs.

4. True or False: $\mathscr{P}(\varnothing) = \varnothing$.

Some problems ask you a question. When you figure out the answer, write down the statement that is true, and prove it.

5. If a set A has n elements, how many elements does $\mathscr{P}(A)$ have?
(Check that your answer to this problem agrees with your answer to the previous problem.)

6. Let A, B, C, and D be sets with $A \subseteq C$ and $B \subseteq D$.
(a) True or False: $A \times B \subseteq C \times D$.
(b) True or False: $A \times B \subseteq D \times C$.

7. Let A and B be sets. If $\mathscr{P}(A) = \mathscr{P}(B)$, what can you conclude about the relationship between A and B?

8. Suppose that $|A| = m$ and $|B| = n$. Find the following cardinalities.

(a) $|\mathscr{P}(A \times B)|$
(b) $|\mathscr{P}(A) \times \mathscr{P}(B)|$
(c) $|\mathscr{P}(\mathscr{P}(\mathscr{P}(A)))|$
(d) $|\mathscr{P}(\mathscr{P}(\mathscr{P}(A \times \varnothing)))|$
(e) $|\{X \in \mathscr{P}(A) : |X| \leq 1\}|$
(f) $|\{X \subseteq \mathscr{P}(A) : |X| \leq 1\}|$

9. Write each set using set notation:
(a) The set of all odd integers
(b) The set of all points in the xy-plane above the line $y = x$
(c) The set of all points in the xy-plane that are inside the circle of radius 1
(d) The set of all irrational numbers (e) The set of all perfect squares

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1. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 3, 4, 7, 8\}$ have universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Find:

(a) $\bar{A}$ (d) $\bar{A} - \bar{B}$ (g) $A \cap \bar{A}$

(b) $\bar{B}$ (e) $\overline{\bar{A} \cap B}$

(c) $A - \bar{A}$ (f) $A \cup \bar{A}$ (h) $\overline{A \cup B}$

2. Find sets A, B, C, D that give a counterexample to the following claim:

Let A, B, C, and D be sets. Then $(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D)$.

3. Let A, B, and C be sets. Find another way to write the expression
$A \backslash (B \cap C) = A - (B \cap C)$ (Note: Both $\backslash$ and $-$ are used to denote set subtraction).

4. Let A, B, and C be sets. Draw a picture to illustrate the identity
$(A \backslash B) \times C = (A \times C) \backslash (B \times C)$ and then prove the identity.

5. For each $n \in \mathbb{N}$, write $n\mathbb{Z}$ for the set of integers divisible by n, i.e.

$$n\mathbb{Z} = \{m \in \mathbb{Z} : m = nk \text{ for some } k \in \mathbb{Z}\}.$$

For example, $3\mathbb{Z} = \{\dots, -6, -3, 0, 3, 6, \dots\}$.
(a) What is $3\mathbb{Z} \cap 2\mathbb{Z}$? (b) What is $4\mathbb{Z} \cap 6\mathbb{Z}$? (c) What is $n\mathbb{Z} \cap m\mathbb{Z}$ in general? Prove your answer to (c).

6. True or False: (a) $(\mathbb{R} \times \mathbb{Z}) \cap (\mathbb{Z} \times \mathbb{R}) = \mathbb{Z} \times \mathbb{Z}$ (b) $(\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}) = \mathbb{R} \times \mathbb{R}$

7. True or False: $(\mathbb{R} - \mathbb{Z}) \times \mathbb{N} = (\mathbb{R} \times \mathbb{N}) - (\mathbb{Z} \times \mathbb{N})$

8. Let B be a subset of a universal set U.
(a) Prove that $\mathscr{P}(B^C) \neq (\mathscr{P}(B))^C$, where $(\mathscr{P}(B))^C$ is in the universal set $\mathscr{P}(U)$.
To show that two sets A and B are equal, show $A \subseteq B$ and show $B \subseteq A$.
(b) Modify the statement slightly to give a true statement and prove it.

9. Consider the set $A = \{1, 2, 3, 4, 5, 6\}$. Let $B, C \subseteq A$ be subsets of A with the property that $B \cup C = A$.
(a) How many such pairs of subsets exist?
(b) How many pairs exist if we also require that $B \cap C = \emptyset$?
(c) Now suppose that $A = \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$. How do the answers to parts (a) and (b) change? Write down a true statement for part (a) and part (b), and prove each.

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1. Draw a Venn Diagram for:

(a) $B - A$

(b) $(A - B) \cap C$

(c) $(A \cup B) - C$

(d) $A \cap (B \cup C)$

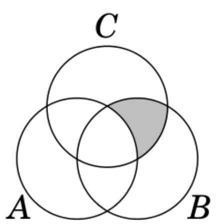
(e) $(A \cap B) \cup (A \cap C)$

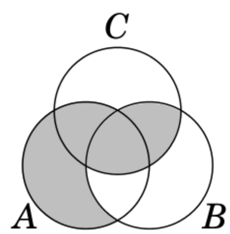
(f) $\overline{A \cap B}$

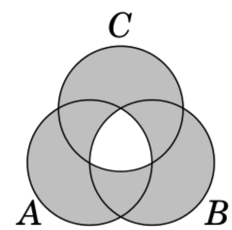
(g) $\bar{A} \cup \bar{B}$

2. Parts (d) and (e), as well as (f) and (g) suggest that a particular rule is true in general. Write it down and prove it.

3. For each figure below, write down symbolic notation for the shaded regions.







4. The *symmetric difference* of two sets A and B, denoted $A\Delta B$, is the set of elements that are in one of the sets but not both.

(a) Draw the Venn diagram for the symmetric difference of two sets A and B.

(b) True or False: $A\Delta B = (A \cup B)\backslash(A \cap B)$.

5. Let $A_n = (n-1, n)$ be the interval in $\mathbb{R}$. Rewrite $\bigcup_{n\in\mathbb{Z}} A_n$ and $\bigcap_{n\in\mathbb{Z}} A_n$ as intervals in $\mathbb{R}$.

6. In each part below, find a family of sets $\{A_n : n \in \mathbb{N}\}$, such that $A_n \subseteq \mathbb{R}$ for each $n \in \mathbb{N}$, $A_m \neq A_n$ for any $n \neq m$, and that the given conditions hold.

(a) $\bigcup_{n\in\mathbb{N}} A_n = (0, \infty)$ and $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$. (b) $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{R}$ and $\bigcap_{n\in\mathbb{N}} A_n = \{3\}$.

(c) $\bigcup_{n\in\mathbb{N}} A_n = (2, 8)$ and $\bigcap_{n\in\mathbb{N}} A_n = [3, 6]$. (d) $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{R}$ and $\bigcap_{n\in\mathbb{N}} A_n = \mathbb{N}$.

7. Let P_n be the set of polynomials in one variable with real coefficients with degree at most n. Describe the following sets:

(a) $\bigcup_{n=0}^{\infty} P_n$ (b) $\bigcap_{n=0}^{\infty} P_n$

How is $\bigcup_{n=0}^{\infty} P_n$ related to the set P of all power series in one variable with real coefficients?

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Note: This problem set has TWO pages.

1. Determine whether each of the following is a statement, an open sentence, or neither. If it is a statement, indicate its truth value. If it as an open sentence, find value(s) of the variable(s) that yield a value of true (and of false).

   (a) Every irrational number is real.
   (b) Every real number is irrational.
   (c) $x + 2$.
   (d) $x + 2 = 5$.
   (e) Go to the store.
   (f) I went to the store.
   (g) A person in Evanston went to Trader Joe's in a clown costume.
   (h) In the beginning, God created the heavens and the earth.
   (i) $\mathbb{N} \notin \mathscr{P}(\mathbb{N})$.
   (j) How are you feeling?
   (k) An unspecified person likes red shoes.
   (l) The moon is made of green cheese.
   (m) She is wearing a blue dress.
   (n) When you come to a fork in the road, take it.
   (o) Every integer is either odd or even.

2. Express each statement or open sentence in one of the following forms: $P \wedge Q, P \vee Q$, or $\neg P = {\sim}P$. For each one, say which statements P and Q stand for.

   (a) The number 8 is both even and a power of 2.
   (b) There is a quiz scheduled for Wednesday or Friday.
   (c) The number x equals zero but the number y does not.
   (d) Happy families are all alike, but each unhappy family is unhappy in its own way.
   (e) A man should look for what is, and not for what he thinks should be.
   (f) $x \in A \cap B$.
   (g) $x \in A \cup B$.
   (h) $x \in A \Delta B$.

3. Without changing their meanings, convert each of the following sentences into a sentenve have the form "P if and only if Q."

   (a) If a function has a constant derivative then it is linear, and conversely.
   (b) For a matrix A to be invertible, it is necessary and sufficient that $\det(A) \neq 0$.
   (c) For an occurrence to become an adventure, it is necessary and sufficient for one to recount it. (Jean-Paul Sarte)

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Note: This is the second page of problem set 6.

4. Given that statements (1) and (2) are both true, decide if statement (3) must be true.

   (a) The three statements are:

   (1) Everyone who loves Bill loves Sam.
   (2) I don't love Sam.
   (3) I don't love Bill.

   (b) The three statements are:

   (1) If Susie goes to the ball in the red dress, I will stay home.
   (2) Susie went to the ball in the green dress.
   (3) I did not stay home.

   (c) Let x and y be real numbers. The three statements are:

   (1) If $x > 5$, then $y < \frac{1}{5}$.
   (2) $y = 1$.
   (3) $x \leq 5$.

   (d) Let M and n be real numbers. The three statements are:

   (1) If $n > M$, then $n^2 > M^2$.
   (2) $n < M$.
   (3) $n^2 \leq M^2$.

5.* On a certain island, each inhabitant is a truth-teller or a liar (and not both, of course). A truth-teller always tells the truth and a liar always lies. Arnie and Bernie live on the island.

   (a) Suppose Arnie says, "If I am a truth-teller, then each person living on this island is a truth-teller or a liar. " Can you say whether Arnie is a truth-teller or a liar? If so, which one is he?

   (b) Suppose that Arnie had said, "If I am a truth-teller, then so is Barnie." Can you tell what Arnie and Barnie are? If so, what are they?

6. We know that each of the three statements below is correct. What can we conclude? Why?

   (a) If he was killed before noon, then his body temperature is at most 20^o C.

   (b) His body temperature is at most 20^o C and the police know who murdered him.

   (c) If the police know who murdered him, then he was killed before noon.

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1. Assume that P and Q are true statements, U and V are false statements, and W is a statement whose truth is unknown. Determine which of the following statements are true, are false, or are of unknown truth.
(a) $(P \vee Q) \vee (U \wedge V)$ (b) $(\neg P \vee \neg U) \wedge (Q \vee \neg V)$ (c) $(P \wedge \neg V) \wedge (U \vee W)$

2. Prove that an implication $P \Rightarrow Q$ is logically equivalent to $\neg P \vee Q$. Apply this to the following implication: "If the sky is always purple, then every day is Tuesday." Discuss this briefly.

3. Let P, Q, and R be statements. Prove that $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$ are logically equivalent.

4. Let P, Q, and R be statements. Prove that $(P \vee Q) \Rightarrow R$ is logically equivalent to $(P \Rightarrow R) \wedge (Q \Rightarrow R)$, and find examples of statements P, Q, and R that illustrate this.

5. Let x be a real number. Determine whether each of the statements below are true or false and briefly explain your reasoning.

(a) If $x = 3$, then $x^2 = 9$.

(b) If $x^2 = 9$, then $x = 3$.

(c) If $x^2 \neq 9$, then $x \neq 3$.

(d) If $x \neq 3$, then $x^2 \neq 9$.

(e) What can you conclude about the logical equivalences of the statements above?

(f) Is your result in part (e) an example of a more general logical equivalence?

6. Prove that $\neg(P \wedge \neg P)$ is a tautology. Give an example of a statement P that illustrates this.

1. Is it possible for the statement "For all x, $P(x)$" to be true AND for "There exists some x such that $P(x)$" to be false? If it is possible, give an example of such a $P(x)$ and x. If it is not possible, alter one (or both) of the statements above to make the first true and the second false.

2. For each of the following statements, write each as a logical statement with quantifiers. Then write the negation of each as logical statements.

(a) Insects have six legs.
(b) Some NU students are from Arizona.
(c) All fish live in water.
(d) My birthday is in July.
(e) Everyone at Northwestern wears purple.
(f) Some people like their coffee black.

3. For each of the following statements, do the following three steps:
(1) Write the statement in English without logic symbols.
(2) Write the negation of the statement in symbolic form without using the negation symbol.
(3) Write a useful negation of the statement in English that does not use logic symbols.

(a) $(\exists x \in \mathbb{Q}, x > \sqrt{2})$.
(b) $(\forall x \in \mathbb{Q}, x^2 - 2 \neq 0)$.
(c) $(\forall x \in \mathbb{Z}, x^2 \text{ is odd} \Rightarrow x \text{ is odd})$.
(d) $(\exists x \in \mathbb{R}, \cos(2x) = 2\cos x)$.

4. Consider the following statement: *For all positive integers x, there exists a real number y such that for all real numbers z, we have $y = z^x$ or $z = y^x$.*

(a) Write this statement using symbols and appropriate quantification. Use $\mathbb{R}$ for the universe of all variables.
(b) Negate the symbolic statement you obtained in the first part.

5. Let $\mathbb{R}$ be the universe for all variables.
(a) Negate the following statement using symbolic notation
(b) Determine the truth values of both the original and the negated statements.

$$\forall x, ((\exists y, x^3 = y^2) \vee (\forall z, (z^2 < 0 \Rightarrow x^3 \neq z^2)))$$

6. Consider the advertisement at right, photographed at Home Depot in 2014. Comment on the statement "One winner will get a 20V Max Impact Driver at every qualifying event." What is the intention of the advertisement, and what does the statement actually promise?

7. Write the definition of what a *perfect square* is, using quantifiers in two equivalent ways: in English and as a logical statement.

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1. Let x and y be integers. Suppose that we are trying to prove the following statement:
"If xy is even, then x is even or y is even."

(a) Write this statement symbolically, without using English words.

(b) Write the contrapositive of the statement in part (a) symbolically.

(c) Rewrite the statement in part (b) in words.

(d) Which statement is easier to prove?

2. Let P and Q be statements. Prove that $\neg(P \Leftrightarrow Q)$ is logically equivalent to
$(P \wedge \neg Q) \vee (Q \wedge \neg P)$ *Note: you don't need truth tables for this;* *see pages 49 and 50.*

3. Let a and b be integers. Suppose that we are trying to prove the following statement:
"If 3 is a factor of ab, then 3 is a factor of a or 3 is a factor of b."

(a) Write this statement symbolically, without using English words.

(b) Find another statement that is logically equivalent to the statement in part (a).

(c) Rewrite the statement in part (b) in words.

(d) Can you prove the original statement directly? If so, do it.

(e) Can you prove the statement in part (b) or part (c)? If so, do it.

4. Give a contrapositive proof of each of the following statements. You should also think about how a direct proof would work.

(a) Suppose $n \in \mathbb{Z}$. If n^2 is even, then n is even.

(b) Suppose $a, b \in Z$. If $a^2(b^2 - 2b)$ is odd, then a and b are odd.

(c) Suppose $n \in \mathbb{Z}$. If $3 \nmid n^2$, then $3 \nmid n$.

(d) Suppose $x, y, z \in \mathbb{Z}$ and $x \neq 0$. If $x \nmid yz$, then $x \nmid y$ and $x \nmid z$.

(e) Suppose $x \in \mathbb{R}$. If $x^5 + 7x^3 + 5x \geq x^4 + x^2 + 8$, then $x \geq 0$.

5. Prove each of the following statements. (We have seen (d) and (e) before.)

(a) For any $a, b \in \mathbb{Z}$, $(a+b)^3 \equiv a^3 + b^3 \pmod 3$.

(b) If $n \in \mathbb{N}$ and $2^n - 1$ is prime, then n is prime.

(c) If $n \in \mathbb{Z}$, then $4 \nmid (n^2 - 3)$.

(d) Every odd integer is the difference of two squares. (For example, $7 = 4^2 - 3^2$.)

(e) If $n \in \mathbb{N}$ then $\binom{2n}{n}$ is even.

Math 300

1. Prove that there are infinitely many prime numbers.

2. (6.5) Prove that $\sqrt{3}$ is irrational. Why does the same method of proof fail to show that $\sqrt{4}$ is irrational?

3. (6.3) Prove that $\sqrt[3]{2}$ is irrational.

4. Prove that at least one of 6.022×10^{23} and $6.022 \times 10^{23} + 1$ is **not** a perfect square.

5. (6.10) Prove that there exist no integers a and b for which $21a + 30b = 1$.

6. (6.12) Prove that for every positive $x \in \mathbb{Q}$, there is a positive $y \in \mathbb{Q}$ for which $y < x$.

7. (6.16) Prove that if a and b are positive real numbers, then $a + b \geq 2\sqrt{ab}$.

8. (6.17) Prove that for every $n \in \mathbb{Z}$, $4 \nmid (n^2 + 2)$.

9. (6.18) Prove that if $a, b \in Z$ and if $4|(a^2 + b^2)$, then a and b are not both odd.

10. Prove the following statements using any method from Chapters 4, 5, or 6.

(a) (6.19) The product of any five consecutive integers is divisible by 120. (For example, the product of 3,4,5,6, and 7 is 2520, and 2520=120 $\cdot$ 21.)

(b) (6.24) The number $\log_2 3$ is irrational.

11.*Prove that $\sqrt{3} - \sqrt{2}$ is an irrational number.

Math 300

Prove each of the following statements.

1. (7.1) Suppose that $x \in \mathbb{Z}$. Then x is even if and only if $3x + 5$ is odd.

2. (7.6) Suppose $x, y \in \mathbb{R}$. Then $x^3 + x^2y = y^2 + xy$ if and only if $y = x^2$ or $y = -x$.

3. (7.9) Suppose $a \in \mathbb{Z}$. Then $14|a$ if and only if $7|a$ and $2|a$.

4. (7.17) There is a prime number between 90 and 100.

5. (7.20) There exists an $n \in \mathbb{N}$ for which $11|(2^n - 1)$.

6. (7.21) Every real solution of $x^3 + x + 3 = 0$ is irrational.

7. (7.35) Suppose $a, b \in \mathbb{N}$. Then $a = \gcd(a, b)$ if and only if $a|b$.

—

8.* Prove the following statement, or give a counterexample: The product of any n consecutive positive integers is divisible by $n!$.

1. Let $A = \{n \in \mathbb{N} \,|\, n = m^2 \text{ for some } m \in \mathbb{N}\}$ and $B = \{p \in \mathbb{N} \,|\, p \text{ is prime}\}$.
Prove that $B \subseteq \mathbb{N} - A$.

2. Let $A = \{x \in \mathbb{R} \,|\, x^2 < 4\}$ and let $B = \{x \in \mathbb{R} \,|\, x < 2\}$.

(a) Is $A \subseteq B$? Prove or disprove.

(b) Is $B \subseteq A$? Prove or disprove.

3. If A is any set, determine whether or not the following statement is true: "$\emptyset \subseteq A$".

4.* Let A and B be sets. Prove that the following conditions are equivalent:
(a) $A \subseteq B$. (b) $A \cap B = A$. (c) $A \cup B = B$.

5. Suppose that A and B are sets and that $A \subseteq B$. Simplify the following expressions and prove that your simplification is equal to the original set.
(a) $A \setminus \emptyset$ (b) $A \setminus A$

6. Let A and B be sets. Prove that $A \subseteq B$ if and only if $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

7. Prove that $A \subseteq B$ if and only if $A \cap (B^c) = \emptyset$.

8. Prove that $B \cap \left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I}(B \cap A_i)$ for any indexing set I.

9. Let A_i be a family of sets with indexing set I. Suppose that $J \subseteq I$ is a nonempty subset.

(a) Prove that $\displaystyle\bigcup_{i \in J} A_i \subseteq \bigcup_{i \in I} A_i$.

(b) Prove that $\displaystyle\bigcap_{i \in I} A_i \subseteq \bigcap_{i \in J} A_i$.

10. How many subsets does the empty set have? Prove that the empty set is unique.

11.*Let A, B and C be sets. Show that $C - (A \cup B) = (C - A) \cap (C - B)$. What is the analogous statement for $C - (A \cap B)$? State it and prove it.

12.*Let A and B be sets. Show that $\mathscr{P}(A) \cap \mathscr{P}(B) = \mathscr{P}(A \cap B)$. Does the same hold if we take unions instead of intersections? If so, state what is true and prove it.

Math 300

Each of the following statements is either true or false. If a statement is true, prove it. If a statement is false, disprove it. *Those that are true are* **eligible for handing in**.

1. For all $x \in \mathbb{R}$, if x is irrational, then x^2 is irrational.

2. For all $x \in \mathbb{R}^+$, if x is irrational, then $\sqrt{x}$ is irrational.

3. For all $x, y \in \mathbb{R}$, if $x + y$ is irrational, then x is irrational and y is irrational.

4. For all $x, y \in \mathbb{R}$, if $x + y$ is irrational, then x is irrational or y is irrational.

5. For all sets A and B that are subsets of some universal set U, the sets $A \cap B$ and $A \backslash B$ are disjoint.

6. Let A, B, C, and D be subsets of a universal set U. If $A \subseteq B$ and $C \subseteq D$ and B and D are disjoint, then A and C are disjoint.

7. (9.1) If $x, y \in \mathbb{R}$, then $|x + y| = |x| + |y|$.

8. (9.4) For every natural number n, the number $n^2 + 17n + 17$ is prime.

9. (9.7) If A, B and C are sets, and $A \times C = B \times C$ then $A = B$.

10. (9.8) If A, B and C are sets, then $A - (B \cup C) = (A - B) \cup (A - C)$.

11. (9.14) If A and B are sets, then $\mathscr{P}(A) \cap \mathscr{P}(B) = \mathscr{P}(A \cap B)$.

12. (9.22) If p and q are prime numbers for which $p < q$, then $2p + q^2$ is odd.

13. If p and q are prime numbers for which $p > q$, then $2p + q^2$ is odd.

14. (9.32) If $n, k \in \mathbb{N}$ and $\binom{n}{k}$ is a prime number, then $k = 1$ or $k = n - 1$.

15. (9.34) If $X \subseteq A \cup B$, then $X \subseteq A$ or $X \subseteq B$.

1. Use induction to find the n^{th} derivative of $f(x) = e^{ax}$ where a is a nonzero real number.

2. Prove that for each natural number n, $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

3. (a) Explain the difference between Induction and Strong Induction.
(b) 4 Prove that Mathematical Induction implies Strong Induction.
(c) Explain why neither Induction nor Strong Induction can be used to prove a statement of the form "$\forall x \in \mathbb{R}, P(x)$", where $P(x)$ is a sentence.

4. Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be a function for which $f(1) = 1$, $f(2) = 3$, and for which the recursive formula $f(n+2) = 3f(n+1) - 2f(n)$ holds for each $n \in \mathbb{Z}^+$.
(a) Calculate $f(3)$ through $f(6)$, and use this to conjecture a general formula for $f(n)$.

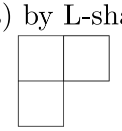
(b) Prove that your conjecture is correct.

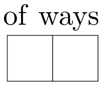
5. Follow the outline below to give a proof by induction that $\sqrt{2}$ is irrational.
(a) Prove that $\sqrt{2} \neq \frac{a}{1}$ for any $a \in \mathbb{Z}$. The rest of the proof will proceed by induction on the denominator b in $\frac{a}{b}$.
(b) Explain why every rational number can be expressed as $\frac{a}{b}$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$.
(c) Assume that $\sqrt{2} \neq \frac{a}{b}$ for all $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$ where $b \leq n$ for some $n \in \mathbb{Z}^+$. Prove that $\sqrt{2} \neq \frac{a}{n+1}$ for any $a \in \mathbb{Z}$.
(d) Explain why this proves the desired result.
(e) Explain what we took for granted in our original proof of the irrationality of $\sqrt{2}$ and why the above proof might be considered more rigorous.

6. A *polynomial with real coefficients* is a function $p : \mathbb{R} \to \mathbb{R}$ of the form $p(x) = \sum_{i=0}^{d} c_i x^i$ for some $c_i \in \mathbb{R}$. If $c_d \neq 0$, we say the *degree* of p is d. A *zero* of a polynomial is a number $a \in \mathbb{R}$ such that $p(a) = 0$. The Fundamental Theorem of Algebra states that a real polynomial of degree d has at most d real zeros. Prove this theorem by induction on d.

7. Prove that every integer $n \geq 2$ is the product of primes.

8.* Consider a grid of size $2^n \times 2^n$ for $n \geq 1$ (e.g. a chess board is a $2^3 \times 2^3$ grid) with one square missing. Prove that every such grid can be tiled (completely covered with no overlaps, where you are allowed to rotate the shape like in Tetris) by L-shaped pieces of the following form:



9.* Find the number of ways to tile a $2 \times n$ grid for $n \geq 1$ with a domino and prove that your formula holds $\forall n \geq 1$. 

10.* (a) Prove that $2^n > n^2$ for all integers $n \geq 5$.
(b) Prove that $2^n < n!$ for all integers $n \geq 4$.

Math 300

1. The following problems are about the Fibonacci sequence F_n.

   (a) (10.25) Prove that $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$.

   (b) (10.27) Prove that $F_1 + F_3 + F_5 + \ldots + F_{2n-1} = F_{2n}$.

   (c) (10.28) Prove $F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1} - 1$.

2. (10.35) Prove that if $n, k \in \mathbb{N}$, and n is even, and k is odd, then $\binom{n}{k}$ is even.

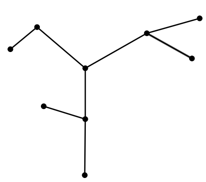
3. (10.34) Prove that $3^1 + 3^2 + 3^3 + \cdots + 3^n = \dfrac{3^{n+1} - 3}{2}$ for every $n \in \mathbb{N}$.

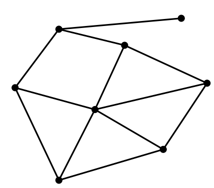
4. (10.33) Suppose n (infinitely long) straight lines lie on a plane in such a way that no two of the lines are parallel, and no three of the lines intersect at a single point. Show that this arrangement divides the plane into $\dfrac{n^2 + n + 2}{2}$ regions.

5. A *connected planar graph* consists of a collection of *vertices* (points) in a plane, which may or may not be connected by *edges* (line segments), which are not allowed to intersect. A *face* is a region that is completely enclosed by edges. We will also consider the plane surrounding the planar graph as a face. The *Euler characteristic* of a planar graph is defined by the number of vertices minus the number of edges plus the number of faces,

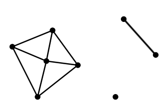
$$\chi = v - e + f.$$

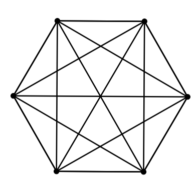
Prove by induction on e that $\chi = 2$ for every planar graph. Two examples of planar graphs are below.





The following two graphs are not allowed, as the first is not connected and the second is not planar since the edges intersect.

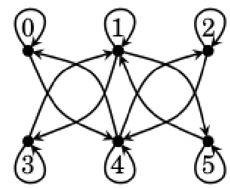




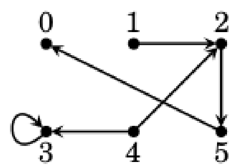
Math 300

Note: There are two pages of problems. You may not have time to do all of them, so focus on those that interest you the most.

1. (11.4)Here is a diagram for a relation R on a set A. Write the sets A and R.



2. (11.5) Here is a diagram for a relation R on a set A. Write the sets A and R.



3. Let $X = \{1, 2, 3, 4, 5\}$. If possible, define a relation on X that is

(a) reflexive, but neither symmetric nor transitive.

(b) symmetric, but neither reflexive nor transitive.

(c) transitive, but neither reflexive nor symmetric.

(d) an equivalence relation.

4. Let U be a finite nonempty set. Define a relation on $\mathscr{P}(U)$ by $A \sim B$ if and only if A and B have the same number of elements. Is $\sim$ an equivalence relation?

5. Let U be a finite nonempty set. Define a relation on $\mathscr{P}(U)$ by $A \sim B$ if and only if $A \cap B = \emptyset$. Is $\sim$ an equivalence relation?

6. A relation $\sim$ on a set X is *antisymmetric* if for all $x, y \in X$, whenever $x \sim y$ and $y \sim x$, then $x = y$. How can you see from the diagram of a relation that it is antisymmetric? Draw a nontrivial example of such a diagram on the set $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

7.* Give the definition of a partial ordering, and a simple example of a partial ordering.
(a) Is $\subset$ a partial ordering? How about $\subseteq$?
(b) Let R be the relation on $\mathbb{N}$ defined by aRb if b is a multiple of a; that is, $b = an$ for some $n \in \mathbb{N}$. Show that R is a partial ordering on $\mathbb{N}$.

8. A relation $\sim$ on a set X is called a *total order* if it is transitive, antisymmetric, and if for all $x, y \in X$, either $x \sim y$ or $y \sim x$. Total orders are usually denoted by the symbol $\leq$ rather than the usual relation symbol $\sim$.

(a) What must the diagram of a total order look like?

(b) Explain why the relation $\leq$ on the set $\mathbb{N}$ is a total order. Is $\leq$ a total order on $\mathbb{Z}$?

(c) Come up with another (not $\leq$) total order on $\mathbb{N}$, and prove that it is a total order.

9. A total order $\leq$ on a set X is called a *well-ordering* if it has the property that for every nonempty subset S of X, S has a least element by $\leq$.

(a) Explain why $\leq$ on $\mathbb{N}$ is a well-ordering.

(b) Prove that $\leq$ is **not** a well-ordering on $\mathbb{Z}$.

(c) Can you construct a well-ordering on $\mathbb{Z}$?

10. Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\} \subset \mathbb{Z}$. Define a relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ by $(n, m) \sim (n', m')$ if and only if $n + m' = n' + m$. Prove that $\sim$ is an equivalence relation and describe the equivalence classes of $\sim$. Describe a more familiar way to think about the set of equivalence classes.

11. Let S^1 be a circle. Define a relation on S^1 by $x, y \in S^1$ have $x \sim y$ if and only if x and y are antipodal (i.e. they lie on the same line passing through the center of the circle). Prove that $\sim$ is an equivalence relation. Describe the shape of the set of equivalence classes, $\{E_x : x \in S^1\}$.

12.*Consider the subset A of $\mathbb{Z} \times \mathbb{Z}$ defined by

$$A = \{(a, b) : a, b \in \mathbb{Z}, b \neq 0\}$$

and define a relation R on A by $(a, b)R(c, d)$ if $ad = bc$. Show that R is an equivalence relation on the set A and describe the equivalence classes of (a, b) for any $(a, b) \in A$.

What happens if A is defined without the restriction on b?

13.*For each $n \in \mathbb{N}$, $n \geq 3$, let S denote the set of all *symmetries* of a regular n-gon, where a symmetry is any rigid motion of the n-gon given by taking a copy of the n-gon, moving this copy in any fashion in 3-space, and then placing the copy back on the original n-gon so it exactly matches. For example, a regular 3-gon is an equilateral triangle, and its symmetries include rotating (which takes each vertex to a new one) and flipping it over (which exchanges two of the three vertices, and leaves the third fixed).

Label the vertices of the n-gon with $\{1, 2, ..., n\}$ so we can keep track of the result of each symmetry. Put a relation R on S as follows. Two symmetries f and g are related if f and g both give the same resulting position of the vertices of the n-gon.

(a) Explain why R is an equivalence relation.

(b) Find a formula for the number of equivalence classes of S, and prove your conjecture.

Math 300

1. Prove that every set A with at least two elements has at least one partition made up of proper subsets of A (i.e. where A is not an element of the partition).

2. For $n \in \mathbb{Z}^+$, let $A_n = \{m \in \mathbb{Z}^+ : m \text{ is divisible by } n\}$. Prove or disprove: the family of sets $\{A_n : n \in \mathbb{Z}^+\}$ is a partition of $\mathbb{Z}^+$.

3. Define a relation on $\mathbb{Z}$ by $x \sim y$ if and only if $x - y$ is divisible by 3. Describe the equivalence classes of the relation. Do they form a partition of $\mathbb{Z}$?

4. Let X be the set of all circles in the plane $\mathbb{R}^2$. Define an equivalence relation on X by $c \sim d$ if and only if the circles c and d have the same center. Describe the partition associated with this equivalence relation.

5. Let P be the set of all polynomials in one variable with real coefficients. Determine whether or not each of the following collections of sets forms a partition of P. If so, prove it. If not, explain which part(s) of the definition fails and give a counterexample.

(a) For $m \in \mathbb{N}$, let A_m be the set of polynomials of degree m. The collection is $\{A_m : m \in \mathbb{N}\}$.

(b) For $c \in \mathbb{R}$, let A_c be the set of polynomials p such that $p(0) = c$. The collection is $\{A_c : c \in \mathbb{R}\}$.

(c) For a polynomial q, let A_q be the set of all polynomials p such that q is a factor of p. The collection is $\{A_q : q \in P\}$.

(d) For $c \in \mathbb{R}$, let A_c be the set of polynomials p such that $p(c) = 0$. The collection is $\{A_c : c \in \mathbb{R}\}$.

6. Construct an explicit partition of $\mathbb{R}^2$ that contains an infinite number of sets. Find two other such partitions.

7. Let $S^2 \subseteq \mathbb{R}^3$ be the unit sphere centered at the origin. Define the indexed sets

$$A_r = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1 - r^2,\ z = r\},$$

where $r \in [-1, 1] = I$. Prove that $\{A_r : r \in I\}$ is a partition of S^2, and describe this partition geometrically.

Math 300

This problem set has TWO pages, and it also has some explanatory text about graphs in the middle of the first page.

1. For each of the following parts, consider the function $f(x) = x^2$, and describe the domain and range of f. Can you prove that the range is what you say it is?
(a) $f : \mathbb{R} \to \mathbb{R}$
(b) $f : \mathbb{Z} \to \mathbb{R}$
(c) $f : \mathbb{Z} \to \mathbb{Z}$

2. Let X be a nonempty set, and let A be a subset of X. We define the *characteristic function* of A in X to be $\chi_A : X \to \{0, 1\}$ defined by $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in X \backslash A \end{cases}$.
(a) Check that χ_A is actually a function.
(b) Determine the domain and range of χ_A. (Make sure you look at all possibilities for A and X).

3. For $x \in \mathbb{R}$, we define the *greatest integer function* $f : \mathbb{R} \to \mathbb{Z}$ by $f(x) = \lfloor x \rfloor = n$, where $n \in \mathbb{Z}$ and $n \leq x < n + 1$. This is often called the "floor function."
(a) Find $\lfloor \frac{1}{2} \rfloor$, $\lfloor \pi \rfloor$, $\lfloor -3.5 \rfloor$, and $\lfloor -10 \rfloor$.
(b) Prove that f is a well-defined function.
(c) Determine the range of f.
(d) Sketch the graph of f for $-5 \leq x \leq 5$.
(e) Write out a definition for the *least integer function* or *ceiling function*, denoted by $f(x) = \lceil x \rceil$.

Some notes on notation:

- Our book defines a function to be a *relation*, so we can think of a function $f : A \to B$ as a subset $f \subseteq A \times B$ (see Definition 12.1).
- In general, most mathematicians agree that a function is a *rule* $f : A \to B$ associating each element $a \in A$ to a unique element $f(a) \in B$. The subset
$$\{(a, b) \in A \times B : b = f(a)\}$$
is usually called the *graph* of f and denoted by Γ_f.
- Our book automatically identifies a function with its graph (i.e., the corresponding subset of $A \times B$), but many other books will not! We want you to be aware of this difference in notation. On the worksheets, we will frequently use Γ_f to denote the graph of f although the book will sometimes denote both by f.

4. For sets A and B, let $f : A \to B$ be a function. We define the *graph* of f to be the set $\Gamma_f = \{(a, b) \in A \times B : b = f(a), a \in A\}$.

(a) Explain how this definition of an abstract graph agrees with the familiar graphs of functions you have worked with.

(b) Let $f : A \to B$ be a function, and let $a \in A$ be fixed. Prove that the set $C_a \subseteq A \times B$ given by $C_a = \{(a, b) : b \in B\}$, intersected with Γ_f, consists of at most one point.

(c) How does the result in part (b) compare to your previous knowledge about the behavior of functions?

5. Let A and B be finite sets where A has m elements and B has n elements. Find the number of different functions $f : A \to B$.

6. Let $f, g : A \to B$ be functions with graphs Γ_f and Γ_g. Prove that $\Gamma_f \cap \Gamma_g$ is the graph of some function h. (Hint: the domain of h need not be A)

7. For any set A, a function $f : A \to A$ is said to have a *fixed point* at $a \in A$ if $f(a) = a$.

(a) Prove that the set of fixed points of f is equal to the intersection of the graph of f with the graph of the *diagonal function* $\Delta : A \to A$, $\Delta(a) = a$.
(b) Find explicit examples of functions $f : \mathbb{R} \to \mathbb{R}$, one with fixed points and one without.
(c) For a function $f \neq \Delta : \mathbb{R} \to \mathbb{R}$, what are the possibilities for the number of elements in the set of fixed points?

8.* Let $E_6 = \{[x] : x \in \mathbb{Z}\}$ denote the set of equivalence classes of the relation on $\mathbb{Z}$ given by $x \sim y$ if 6 divides $y - x$. Define $f : E_6 \to E_6$ by $f([x]) = [x^4]$. Draw an arrow diagram to represent f.

(a) Is f a function? Prove or disprove.

(b) Now let E_8 denote the set of equivalence classes of the relation on $\mathbb{Z}$ given by $x \sim y$ if 8 divides $y - x$. Define $g : E_8 \times E_8 \to E_6$ by $g([x], [y]) = [xy]$. Is g a function? Prove or disprove.

Math 300

1. Let $n \in \mathbb{N}$ and define $M_n(\mathbb{R})$ to be the set of $n \times n$ matrices with real entries. Let $F : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be the function defined by $F(A) = A^T$, the transpose of A. Prove that F is a bijection.

2. Let $f : A \to B$ be a function, and let $b \in B$ be fixed.

(a) Prove that the set $C_b \subseteq A \times B$ given by $C_b = \{(a, b) : a \in A\}$ intersected with the graph of f is a set of at most one point if and only if f is injective.

(b) How does the result in part (a) compare to your previous knowledge about the behavior of functions?

3. We define the set $C^0(\mathbb{R})$ to be the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$, and define the set $C^1(\mathbb{R})$ to be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ with continuous first derivative. Is the function $D : C^1(\mathbb{R}) \to C^0(\mathbb{R})$ given by $D(f) = f'$ injective? Is D surjective?

4. Suppose A is a nonempty finite set. Prove that there does not exist a bijective function from A to $\mathscr{P}(A)$.

5. Let A be a nonempty set and let S be the set of all functions $f : A \to \{0, 1\}$. Prove that there is a bijection between $\mathscr{P}(A)$ and S.

6. Construct an explicit bijection between the following intervals of real numbers.
(a) $[0, 1]$ and $[0, 2]$. (b) $[0, 1]$ and $[2, 5]$.

7. Construct an explicit bijection between the following sets.
(a) $\mathbb{Q}^+$ and $\mathbb{Q}^-$. (b) The set of even integers and the set of odd integers.

8*. Construct an explicit bijection $f : [0, 1) \to [0, 1]$.

9. Suppose A, B, C, and D are sets and that $A \approx C$ and $B \approx D$, where $\approx$ denotes the fact that there exists a bijection between the two sets. Prove that $A \times B \approx C \times D$.

10. Prove that if $A \approx B$, then $\mathscr{P}(A) \approx \mathscr{P}(B)$.

11.*Let A and B be sets and let $f : A \to B$ be a function. Suppose that X, $Y \subseteq A$. Prove the following statements.
(a) $f(X) \cup f(Y) = f(X \cup Y)$ (b) $f(X \cap Y) \subseteq f(X) \cap f(Y)$
(c) If f is injective, then $f(X \cap Y) = f(X) \cap f(Y)$.

Give an example to prove that it is necessary to assume that f is injective.

12.*Let A and B be sets and let $f : A \to B$ be a function. Show that f is injective if and only if $f^{-1}(f(X)) = X$ for all $X \subseteq A$. Is there a similar statement for surjectivity? If, so state it and prove it.

13.*Let A and B be sets, let $f : A \to B$ be an injective function, and let X and Y be subsets of A. Show that if $f(X) \subseteq f(Y)$, then $X \subseteq Y$. Give an example to show that the assumption of injectivity is necessary.

Math 300

1. (12.3.4) Let S be a region in the plane bounded by a square with sides of length 2. Prove that if we put five points in S, there exist (at least) two of these points that are at most a distance of $\sqrt{2}$ apart.

2. (12.3.2) Prove that if a is a natural number, then there exist two unequal natural numbers k and l for which $a^k - a^l$ is divisible by 10.

3. (12.3.6) Given a sphere S, a *great circle* of S is the intersection of S with a plane through its center. Every great circle divides S into two parts. A *hemisphere* is the union of the great circle and one of these two parts. Prove that if five points are placed arbitrarily on S, then there is a hemisphere that contains four of them.

4. (12.3.5) Prove that any set of seven distinct natural numbers contains a pair of numbers whose sum or difference is divisible by 10.

5. Let A be a finite set. Show that a function $f : A \to A$ is injective if and only if it is surjective. Is this still true if A is infinite?

6. Let A be a nonempty set and let $a \in A$.

(a) Suppose A is finite with $|A| = n$. Prove that $A \backslash \{a\}$ is finite and $|A \backslash \{a\}| = n - 1$.

(b) If A is infinite, is it possible for $A \backslash \{a\}$ to be finite? Prove or disprove.

1. Let $f : A \to B$ and $g : D \to E$ be functions.

(a) In order for the composition $g \circ f$ to be defined, what must be true?

(b) In order for the composition $f \circ g$ to be defined, what must be true?

(c) Now suppose that the conditions in (a) have been met and we are also given that both f and g are injective. Prove that $g \circ f$ is also injective.

(d) Again suppose that the conditions in (a) have been met and that both f and g are surjective. Is it true that $g \circ f$ is also surjective? (Hint: think about the case when $B \neq D$.)

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The next four questions are True or False. If it is true, write down the true statement and prove it. If it is false, give a counterexample. *True statements are eligible for handing in.* Hint: Try examples of functions, either from $\mathbb{R}$ to $\mathbb{R}$ or from a finite set to a finite set, to understand what is going on.

2. If $g \circ f$ is injective, then f is injective.

3. If $g \circ f$ is injective, then g is injective.

4. If $g \circ f$ is surjective, then f is surjective.

5. If $g \circ f$ is surjective, then g is surjective.

–

6. For the following pairs of functions, find formulas for $f \circ g$ and $g \circ f$. Decide whether $f, g, f \circ g, g \circ f$ are injective or surjective or neither. Prove your answers.

(a) (12.4.4) Suppose that $A = \{a, b, c\}$. Let $f : A \to A$ be the function $f = \{(a, c), (b, c), (c, c)\}$, and let $g : A \to A$ be the function $g = \{(a, a), (b, b), (c, c)\}$. Find $g \circ f$ and $f \circ g$.

(b) (12.4.6) Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \frac{1}{1+x^2}$ and $g(x) = 3x + 2$. Find formulas for $g \circ f$ and $f \circ g$.

(c) (12.4.7) Consider the functions $f, g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $f(m, n) = (mn, m^2)$ and $g(m, n) = (m + 1, m + n)$. Find formulas for $g \circ f$ and $f \circ g$.

(d) (12.4.9) Consider the functions $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined as $f(m, n) = m + n$ and $g : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $g(m) = (m, m)$. Find the formulas for $g \circ f$ and $f \circ g$.

1. For each of the functions below, determine whether or not the functions are invertible. If so, find the inverse (and prove that is in fact an inverse). If not, explain what fails.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.

(b) Let $f : \mathbb{N} \to \mathbb{R}$ be given by $f(x) = x^2$.

2. Prove that there is no real number k such that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin(kx)$ is invertible.

3. Let $f : A \to B$ and $g : B \to C$ be functions.

(a) Prove that if f and g are invertible, then so is $g \circ f$.

(b) Following part (a), prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

4. Let A, B, and C be nonempty sets and let $f : A \to B$, $g : B \to C$, and $h : B \to C$.

(a) Prove that if f is surjective and $g \circ f = h \circ f$, then $g = h$.

(b) Give an example where $g \circ f = h \circ f$, but $g \neq h$.

5. Define the function $D : \mathbb{N} \to \mathscr{P}(\mathbb{N})$ by setting $D(n)$ equal to the set of divisors of n.

(a) Calculate $D(n)$ for $n = 1, \ldots, 10$.

(b) Is D injective?

(c) Is D surjective? If so, find D^{-1}.

6.* A function $f : \mathbb{R} \to \mathbb{R}$ is called *continuous* at x if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $x_0 \in \mathbb{R}$ satisfies $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

(a) Explain why the above definition makes sense in reference to how you may have seen a continuous function defined before.

(b) Write the statement out symbolically and negate it.

(c) Give an example of a nowhere continuous function, i.e., a function that fails to be continuous at all $x \in \mathbb{R}$.

7.* Let A be a nonempty finite set and suppose $f : \mathscr{P}(A) \to \mathscr{P}(A)$ is a function with the property that $(f \circ f)(B) = B$ and $B \cap f(B) = \emptyset$ for all $B \in \mathscr{P}(A)$. Prove that f must be the function $f(B) = A - B$.

1. Consider the absolute value function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$. Describe each of the following sets.

(a) $f((-1, 1))$.

(b) $f(\{-1, 1\})$.

(c) $f^{-1}(\{1\})$.

(d) $f^{-1}([-1, 0))$.

(e) $f^{-1}(f([0, 1]))$.

2. Let p and q be two polynomials of degree two with real coefficients. Suppose $p^{-1}(\{0\}) = q^{-1}(\{0\})$.

(a) Give an example of such p and q, with $p \neq q$.

(b) Suppose that $p^{-1}(\{0\}) = \{0, 1\} = q^{-1}(\{0\})$. Must $p = q$? Prove or disprove.

3. Consider $D : C^1(\mathbb{R}) \to C^0(\mathbb{R})$ given by $D(f) = f'$. If $g \in C^0(\mathbb{R})$, describe $D^{-1}(\{g\})$.

4. Suppose that $f : A \to B$ is a bijection and $\mathcal{P}$ is a partition of A. Prove that $\mathcal{Q} = \{f(C) : C \in \mathcal{P}\}$ is a partition of B.

5. (12.6.7) Given a function $f : A \to B$ and subsets $W, X \subseteq A$, prove that

$$f(W \cap X) \subset f(W) \cap f(X).$$

6. (12.6.8) Given a function $f : A \to B$ and subsets $W, X \subseteq A$, the statement

$$f(W \cap X) = f(W) \cap f(X)$$

is false in general. Produce a counterexample.

7. (12.6.10) Given $f : A \to B$ and subsets $Y, Z \subseteq B$, prove that $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$. (Note that f does not have to an inverse function.)

8. Let A and B be sets and let $S \subseteq A \times B$. Define the *projection functions* $\pi_1 : S \to A$ and $\pi_2 : S \to B$ by

$$\pi_1(a, b) = a \qquad \pi_2(a, b) = b.$$

(a) Let $S = A \times B$, and describe the inverse images $\pi_1^{-1}(\{a\})$ and $\pi_2^{-1}(\{b\})$.

(b) Let $f : A \to B$ be a function and let $\Gamma_f \subseteq A \times B$ be its graph. What can you then say about the inverse images $\pi_1^{-1}(\{a\})$ and $\pi_2^{-1}(\{b\})$ for the projections $\pi_1 : \Gamma_f \to A$ and $\pi_2 : \Gamma_f \to B$?

(c) Under what conditions are π_1 and π_2 invertible?

9.* This problem wants to know if we can use power sets and functions to transfer partitions in a well-defined way.

(a) Assume that X is a nonempty set and $\mathscr{A}$ is a partition of X. Is $\{\mathscr{P}(A) : A \in \mathscr{A}\}$ a partition of $\mathscr{P}(X)$? If so, prove it and if not, give a counterexample.

(b) Suppose that $f : A \to B$ is a bijection and $\mathscr{A}$ is a partition of A. Is $\mathscr{B} = \{f(C) : C \in \mathscr{A}\}$ a partition of B? If so prove it, if not give a counterexample.

1. Show that following two sets have equal cardinality by giving a bijection from one to the other, and proving that the function you give really is a bijection.

(a) (13.1.A.1) $\mathbb{R}$ and the interval (0,1). (Suggestion: draw a picture)

(b) (13.1.A.9) $\{0,1\} \times \mathbb{N}$ and $\mathbb{N}$.

(c) (13.1.A.10) $\{0,1\} \times \mathbb{N}$ and $\mathbb{Z}$.

(d) (13.1.A.12) $\mathbb{N}$ and $\mathbb{Z}$ (Suggestion: use exercise 12.2.18.)

(e) (13.1.A.13) $\mathscr{P}(\mathbb{N})$ and $\mathscr{P}(\mathbb{Z})$ (Suggestion: use exercise 13.1.A.12 above.)

2. Let X be an infinite set, and A and B be finite subsets of X. Answer each of the following questions, and prove your results.

(a) Is $A \cap B$ finite or infinite?

(b) Is $A \backslash B$ finite or infinite?

(c) Is $X \backslash A$ finite or infinite?

(d) If $f : A \to X$ is an injective function, is $f(A)$ finite or infinite?

(e) If $|A| = m$ and $|B| = n$, what can you conclude about $|A \cup B|$?

3. Let A and B be sets. Prove that if A is infinite and $A \subseteq B$, then B is infinite.

4. Let $a, b \in \mathbb{R}$ with $a \neq b$. Prove that $\mathbb{R}$ and the interval (a, b) have the same cardinality.

5.* Prove that if $|A| = |B|$, then $|\mathscr{P}(A)| = |\mathscr{P}(B)|$. Note that there is not an assumption that the cardinality of A is finite, which means that the theorem holds for any cardinality. Your proof should reflect this.

6. Prove that $\mathbb{Q}$ is infinite by using only the definition of *finite* and the Pigeonhole Principle.

7.* Consider the function $f_c : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ defined by $f_c([x]) = [cx]$. The *orbit* of an element $[x] \in \mathbb{Z}_{10}$ is the set

$$\{[y] \in \mathbb{Z}_{10} : [y] = f_c([x]) \text{ for some } c \in \{0, 1, 2, ..., 9\}\}.$$

For example, the orbit of [1] is all of $\mathbb{Z}_{10}$.

(a) Calculate the orbits of $0, 2, 3, 4, 5, 6, 7, 8, 9, 10$.

(b) Which of these sets has 2 elements? Which have 5 elements? Which have 10 elements?

(c) Are the results of (b) explained by some divisibility statement?

Math 300

1. Suppose that the set A is countably infinite, and let $x \notin A$. Prove that $A \cup \{x\}$ is countably infinite, by constructing an explicit bijection with $\mathbb{N}$.

2. Prove that $(0, \infty)$ has the same cardinality as $\mathbb{R}$, by constructing an explicit bijection.

3. Consider the clever function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$f(m, n) = 2^{m-1}(2n - 1).$$

Prove that f is both an injection and a surjection, and use this to give the cardinality of the set $\mathbb{N} \times \mathbb{N}$.

4. Is the set of all infinite sequences of 0's and 1's finite, countably infinite, or uncountable? Prove your answer.

5. Let A be an uncountable set and let B be a nonempty subset of A.

(a) Prove that $A \times B$ is uncountable.

(b) What can you say about the cardinality of $A - B$?

6. Suppose that A is uncountable and $A \subseteq B$. What can you conclude about the cardinality of B?

7. Which of the following sets are countable, and which has the same cardinality as $\mathbb{R}$? Justify your answers, but you do not need to supply a formal proof.

(a) $\{\sqrt[n]{2} : n \in \mathbb{N}\}$.

(b) $\mathbb{Q} \cap [2, 3)$.

(c) $[0, 1] \times [0, 1]$.

(d) $\{9^x : x \in \mathbb{R}\}$.

(e) $\{S \subseteq \mathbb{N} : |S| = 7\}$.

(f) $\{[a, b] \subseteq \mathbb{R} : a, b \in \mathbb{Q}\}$.

1. Find a new (different from the proof in the text) strategy to enumerate all the elements of $\mathbb{Q}$, thus proving that $\mathbb{Q}$ is countable, using the hint below.

$$\begin{array}{cccccccccccc}
\cdots & \frac{-4}{1} & \frac{-3}{1} & \frac{-2}{1} & \frac{-1}{1} & \frac{0}{1} & \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \\
\cdots & \frac{-4}{2} & \frac{-3}{2} & \frac{-2}{2} & \frac{-1}{2} & \frac{0}{2} & \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\
\cdots & \frac{-4}{3} & \frac{-3}{3} & \frac{-2}{3} & \frac{-1}{3} & \frac{0}{3} & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\
\cdots & \frac{-4}{4} & \frac{-3}{4} & \frac{-2}{4} & \frac{-1}{4} & \frac{0}{4} & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

2. Let F be the set of all finite subsets of $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Suppose $A = \{a_1, a_2, \ldots, a_n\}$, where $a_1 < a_2 < \cdots < a_n$, is a finite subset of $\mathbb{N}$. Define a function $f : F \to \mathbb{N}$ with $f(A) = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$, where $p_1 < p_2 < \cdots < p_n$ are the first n prime numbers.

(a) Is f an injection? Is f a surjection?

(b) What can you conclude about the cardinality of F?

3.* Prove that $\mathbb{R}^2$ cannot be covered by a countable collection of lines.

4. Suppose $|A| \leq |B|$. Prove that there exists a set $C \subseteq B$ with $|A| = |C|$. Use this to prove that every nonempty set B contains subsets of all smaller cardinalities than the cardinality of B.

5. (13.3.9) Prove that if A and B are finite sets with $|A| = |B|$, then any injection $f : A \to B$ is also a surjection. Show this is not necessarily true if A and B are not finite.

6. (13.3.10) Prove that if A and B are finite sets with $|A| = |B|$, then any surjection $f : A \to B$ is also an injection. Show this is not necessarily true if A and B are not finite.

7.* Let $\mathcal{C}$ be the set of all subsets of $\mathbb{N}$ that have at most 10 elements. Show that $\mathcal{C}$ is countable.

Let $\mathcal{D}$ be the set of all finite subsets of $\mathbb{N}$. Show that $\mathcal{D}$ is countable.

8.* If n is an integer such that $n \geq 0$, then $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ is the *n-th Catalan number.*

A *lattice point* is an an element of $\mathbb{Z}\times\mathbb{Z}$ and a *lattice step* changes one coordinate by 1. A *lattice path* is a lattice walk in which each step increases one coordinate. We say that a lattice path is *monotonic* if each step increases one coordinate by 1. Prove that C_n counts the number of *monotonic lattice paths* from $(0, 0)$ to (n, n) that do not pass above the diagonal.

Math 300

1. Let $\mathscr{A}$ be a collection of sets. We define the following relation on $\mathscr{A}$. For $A, B \in \mathscr{A}$,

$$A \preceq B \text{ if } |A| \leq |B|.$$

1. Show that $\preceq$ is reflexive and transitive on $\mathscr{A}$.
2. Show that for $A, B \in \mathscr{A}$, if $A \preceq B$ and $B \preceq A$, then A and B have the same cardinality.
3. Is $\preceq$ a partial order of $\mathscr{A}$? If it is, prove it; if it isn't, give a counterexample.

2. Consider the interval $[0, 1) \subseteq \mathbb{R}$.

(a) Construct an explicit injection $f : [0, 1) \to [0, 1) \times [0, 1)$.

(b) Construct an explicit injection $g : [0, 1) \times [0, 1) \to [0, 1)$.

(c) What can you conclude from parts (a) and (b)?

(d) Use the result in part (c) to prove that $\mathbb{R}^2$ and $\mathbb{R}$ have the same cardinality.

(e) Use the result in part (d) to prove that $\mathbb{R}^n$ and $\mathbb{R}$ have the same cardinality for all $n \in \mathbb{N}$.

3.* Theorem 13.7 says that if A is a set, then $|A| < |\mathscr{P}(A)|$. This is known as Cantor's Theorem. Use this theorem to prove that the set of all infinite sequences of zeroes and ones is uncountable.

4. Use Cantor's Theorem to prove that there is no set of the form $U = \{A \mid A \text{ is a set}\}$.

5. Is it true that two uncountable sets must have the same cardinality? Why or why not?

6. Find a way to write $\mathbb{Q}$ as a countable union of countably infinite disjoint sets.

7. Find a way to write $\mathbb{R}$ as a union of countably infinite sets (you do not need to assume the sets must be disjoint). Can you write $\mathbb{R}$ as a countable union of countably infinite sets?

1.* We will play a version of the game *Battleship* with the following rules. One player is a submarine, another is a battleship. The game board is the integer lattice in the plane, i.e. $\mathbb{Z}^2 = \{(a,b) \in \mathbb{R}^2 \,|\, a, b \in \mathbb{Z}\}$. The submarine chooses a starting point on the lattice and picks a direction vector in $\mathbb{Z}^2$. Each second, the submarine moves once in the direction of this vector, to a new lattice point. The submarine then moves along the same direction vector the next second, and it repeats this process for all time. The battleship has no knowledge about the initial position or direction vector of the submarine. However, the battleship can teleport to any point on the lattice instantaneously. Each second, the battleship teleports to a lattice point and drops a depth charge, which detonates immediately. Is there a strategy the battleship can employ so it will always destroy the submarine?

2. Denote by shorthand:

$$\begin{aligned} |\mathbb{N}| = \aleph_0 &= \boldsymbol{\alpha}_0 \\ |\mathscr{P}(\mathbb{N})| &= \boldsymbol{\alpha}_1 \\ |\mathscr{P}(\mathscr{P}(\mathbb{N}))| &= \boldsymbol{\alpha}_2 \\ |\mathscr{P}(\mathscr{P}(\mathscr{P}(\mathbb{N})))| &= \boldsymbol{\alpha}_3, \text{ and so on} \ldots \end{aligned}$$

Let $X = \mathbb{N} \cup C$, where $C = \{\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots\}$.

(a) Suppose that A and B are finite disjoint sets with $|A| = n \in \mathbb{N}$ and $|B| = m \in \mathbb{N}$. What can you conclude about $|A \cup B|$?

(b) Define an addition operator, $+$ on X and determine what properties it has. Then prove that $\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_0 = \boldsymbol{\alpha}_0$.

(d) Suppose that A and B are finite disjoint sets with $|A| = n \in \mathbb{N}$ and $|B| = m \in \mathbb{N}$. What can you conclude about $|A \times B|$?

(e) Define a multiplication operation, $\cdot$ on X and determine what properties it has. Then prove that $\boldsymbol{\alpha}_0 \cdot \boldsymbol{\alpha}_0 = \boldsymbol{\alpha}_0$.

3. Define the operation of exponentiation on the set X from Problem 2 by the following: If $|A| = \kappa \in X$ and $|B| = \lambda \in X$, define $\kappa^\lambda = |\mathscr{F}(B, A)|$, the cardinality of the set of functions from B to A.

(a) Explain why this definition makes sense when $\kappa, \lambda \in \mathbb{N}$, i.e., when A and B are finite sets.

(b) Prove that $1^\kappa = 1$ for all $\kappa \in X$.

(c) Prove that $\kappa^1 = \kappa$ for all $\kappa \in X$.

(d) Prove that $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$ for all $\kappa, \lambda, \mu \in X$.

(e) Explain why the statement $2^{\aleph_0} = \boldsymbol{\alpha}_1$ makes sense.

(f) If a set A has $|A| = n$, we have shown that $|\mathscr{P}(A)| = 2^n$. Explain why the lexicographically-similar equation $|\mathscr{P}(\mathbb{N})| = 2^{\mathbb{N}}$ makes sense.