Math 290-3

To the Student

Contents: As you work through this book, you will discover that the various topics of multivariable calculus have been integrated into a mathematical whole. There is no Chapter 5, nor is there a section on vector fields. The curriculum is problem-centered, rather than topic-centered. Techniques and theorems will become apparent as you work through the problems, and you will need to keep appropriate notes for your records — there are no boxes containing important theorems.

Your homework: Each page of this book contains the homework assignment for one night. The first day of class, we will work on the problems on page 1, and your homework is page 2; on the second day of class, we will discuss the problems on page 2, and your homework will be page 3, and so on for the 37 class days of the quarter. You should plan to spend about 90 minutes each night solving problems for this class.

Comments on problem-solving: You should approach each problem as an exploration. Reading each question carefully is essential, especially since definitions, highlighted in italics, are routinely inserted into the problem texts. It is important to make accurate diagrams whenever appropriate. Useful strategies to keep in mind are: create an easier problem, guess and check, work backwards, and recall a similar problem. It is important that you work on each problem when assigned, since the questions you may have about a problem will likely motivate class discussion the next day.

Problem-solving requires persistence as much as it requires ingenuity. When you get stuck, or solve a problem incorrectly, back up and start over. Keep in mind that you're probably not the only one who is stuck, and that may even include your teacher. If you have taken the time to think about a problem, you should bring to class a written record of your efforts, not just a blank space in your notebook. The methods that you use to solve a problem, the corrections that you make in your approach, the means by which you test the validity of your solutions, and your ability to communicate ideas are just as important as getting the correct answer.

The problems in this text

This set of problems is based on the curriculum at Phillips Exeter Academy, a private high school in Exeter, NH. Many of the problems and figures are taken directly from the **Mathematics 5** book, written by Rick Parris and other members of the PEA Mathematics Department. The rest of the problems were written by Diana Davis (who has been both a student and a faculty member at PEA), for a multivariable calculus class at Northwestern University. Anyone is welcome to use this text, and these problems, so long as you do not sell the result for profit. If you create your own text using these problems, please give appropriate attribution, as I am doing here.

Help me help you!

Please be patient with me, as I try to be patient with you. I have spent the past month or two working on this set of problems, thinking hard about each problem and how they all connect and build the ideas, step by step. When you don't catch those connections, I will feel a little sad, maybe a little frustrated. I'm sure you can understand that.

Just remember that we are all in this together. Our goal is for each student to learn the ideas and skills of Math 290-3, really learn them — and along the way I will learn new things, too. That's the beauty of this teaching and learning method, that it recognizes the humanity in each of us, and allows us to communicate authentically, person to person.

One way of describing this method is "the student bears the laboring oar." This is a metaphor: You are rowing the boat; you are not merely along for the ride. You do the work, and in this way you do the learning. The next page gives some ideas for ways that you can do this work of moving the "boat," which is our class and your learning, forward.

You might wonder, what is my job as your teacher? Part of my job is to give you good problems to think about, which are in this book. During class, my job is to help you learn to talk about math with each other, and help you build a set of problem-solving strategies. At the beginning, I will give you lots of pointers, and as you improve your skills I won't need to help as much.

I might say things like

- "Please go up to the board and write down what you're saying."

- "Get some colored chalk and add that to the picture on the board."

- "You were confused before, and now it sounds like you understand; could you please explain what happened in your head?"

After two quarters of lecturing to you while you sit mostly silently, I am so excited to see what you can do and hear what you have to say.

Discussion Skills

1. Contribute to the class every day

2. Speak to classmates, not to the instructor

3. Put up a difficult problem, even if not correct

4. Use other students' names

5. Ask questions

6. Answer other students' questions

7. Suggest an alternate solution method

8. Draw a picture

9. Connect to a similar problem

10. Summarize the discussion of a problem

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1. A function $f(x, y)$ is said to be *continuous* at (a, b) if $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$. Most functions that we consider are continuous, but here is an odd example: Let $F(x)$ be the fractional part of x, so that $F(98.6) = 0.6, F(\pi) = 0.14159\ldots$, and $F(-87.69) = -0.31$. Now let $g(x, y) = F(x) + F(y)$. Sketch the set of points where g is *discontinuous*. What is the *area* of the set of discontinuities of g?

2. (Continuation) Sketch the surface $g(x, y)$ over the region $\mathcal{R} = [0, 1] \times [0, 1]$. Using geometry, find the volume between this surface and the xy-plane, over the region $\mathcal{R}$.

3. Given a function f that is differentiable, one can form the vector $[f_x, f_y]$ at each point in the domain of f. Any such vector is called a *gradient* vector, and the function whose values are $[f_x, f_y]$ is called a *gradient field*. Suppose that $f(x, y) = 9 - x^2 - 2y^2$. Sketch a few level curves of this function. For a few points (x, y) on these level curves, sketch the vector $[f_x, f_y]$ with its tail at (x, y). How are the gradient vectors related to the level curves?

4. If there is a number M so that $f(x, y) < M$ for all (x, y), then f is said to be *bounded above*, and the number M is called an *upper bound* for f. If there is a number N so that $N < f(x, y)$ for all (x, y), then f is said to be *bounded below*, and the number N is called a *lower bound* for f. If f is bounded both above and below, then f is said to be *bounded*.

For each of the following functions $f(x, y)$, say whether f is bounded above or bounded below (or both, or neither).

(a) xy **(b)** $e^{-x^2-y^2}$ **(c)** 2^{xy} **(d)** $-x^2 - y^2 + 2x - 6y - 8$

5. The diagram shows the graph of $y = f(x)$ for $0 \le x \le 2$. We wish to find the integral $\int_0^2 f(x)dx$. Since we do not have a formula for $f(x)$, we will estimate the value of the integral using *Riemann sums*:

**(a)** Given a suitably large value of n, the sum

$$\sum_{k=1}^{n} \frac{2}{n} f\left(\frac{2k}{n}\right)$$

is a reasonable estimate for $\int_0^2 f(x)dx$. Explain why.

(b) Sketch in the rectangles corresponding to the Riemann sum in part (a) for $n = 4$, and estimate the value of the sum.

(c) Another reasonable estimate is $\sum\limits_{k=0}^{n-1} \frac{2}{n} f\left(\frac{2k}{n}\right)$. Sketch in the rectangles corresponding to this sum for $n = 4$, and compare it to the preceding.

(d) Explain the significance of the expression $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{2}{n} f\left(\frac{2k}{n}\right)$.

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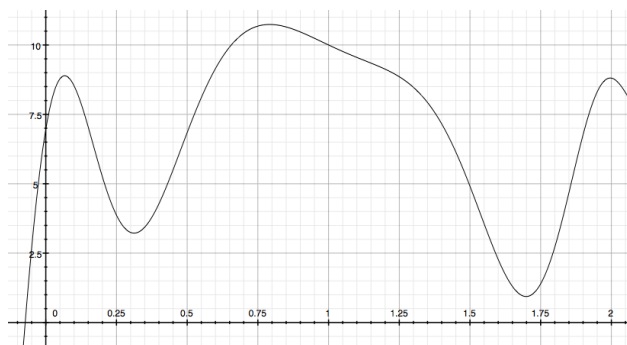
1. The diagram shows $z = (1 - x^2) \sin y$ for the rectangular domain defined by $-1 \le x \le 1$ and $0 \le y \le \pi$. This surface and the plane $z = 0$ enclose a region $\mathcal{R}$. It is possible to find the volume of $\mathcal{R}$ by integration:
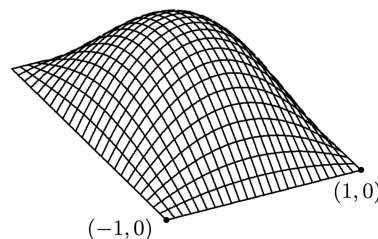
(a) Notice first that $\mathcal{R}$ can be sliced neatly into sections by cutting planes that are perpendicular to the y-axis — one for each value of y between 0 and π, inclusive. The area $A(y)$ of the slice determined by a specific value of y can be found using ordinary integration. Calculate it.

(b) Use the slice-area function $A(y)$ to find the volume of $\mathcal{R}$.

(c) Notice also that $\mathcal{R}$ can be sliced into sections by cutting planes that are perpendicular to the x-axis — one for each value of x between -1 and 1. As in (a), use ordinary integration to find the area $B(x)$ of the slice determined by a specific value of x.

(d) Integrate $B(x)$ to find the volume of $\mathcal{R}$.

2. The preceding problem illustrates how a problem can be solved using *double integration*. Justify the terminology (it does not mean that the problem was actually solved twice). Notice that the example was made especially simple because the limits on the integrals were constant — the limits on the integral used to find $A(y)$ did not depend on y, nor did the limits on the integral used to find $B(x)$ depend on x. The method of using cross-sections to find volumes can be adapted to other situations, however. For example, consider the region $\mathcal{R}$ enclosed by the surface $z = xy(6 - x - y)$ and the plane $z = 0$ for $0 \le x$, $0 \le y$, and $x + y \le 6$. Find the volume of $\mathcal{R}$.

3. Let $V(x, y) = 1 - x^2 - y^2$ be interpreted as the speed (cm/sec) of fluid that is flowing through point (x, y) in a pipe whose cross section is the unit disk $x^2 + y^2 < 1$. Assume that the flow is the same through every cross-section of the pipe. Notice that the flow is most rapid at the center of the pipe, and is rather sluggish near the boundary.

The volume of fluid that passes each second through any *small* cross-sectional box whose area is $\Delta A = \Delta x \Delta y$ is approximately $V(x, y) \Delta x \Delta y$, where (x, y) is a representative point in the small box.

(a) Using an integral with respect to y, combine these approximations to obtain an approximate value for the volume of fluid that flows each second through a strip of width Δx that is parallel to the y-axis. The result will depend on the value of x that represents the position of this strip.

(b) Use integration with respect to x to show that the volume of fluid that leaves the pipe (through the cross-section at the end) each second is $\pi/2 \approx 1.57$ cc. *Hint: trig substitution*

4. Find components for a nonzero vector that is perpendicular to both $[3, 2, 6]$ and $[8, 9, 12]$.

5. Find components for the vector that is obtained by (perpendicularly) projecting $[3, 2, 6]$ onto the direction defined by $[8, 9, 12]$.

6. Find components for the vector that is obtained by (perpendicularly) projecting $[3, 2, 6]$ onto the plane $8(x - 2) + 9(y + 1) + 12(z - 7) = 0$.

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1. Water is flowing through the square pipe whose cross-section is shown in the diagram. The speed of the flow at the point (x, y) is $f(x, y) = 1 - |x| - |y|$ cm per second. At what rate, in cc per second, is water flowing through the pipe?
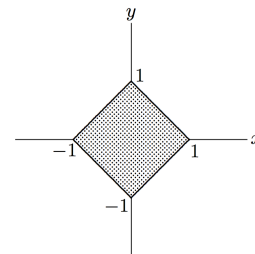
2. (Continuation) Notice that the speeds of individual water molecules vary from 0 (at the boundary) to 1 (at the center). What is the *average* speed of the water as it flows through the pipe? Explain your choice.

3. Consider the cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$. Describe their points of intersection. In particular, how would this configuration of points look to an observer stationed at $(100, 0, 0)$? How about an observer stationed at $(0, 0, 100)$?
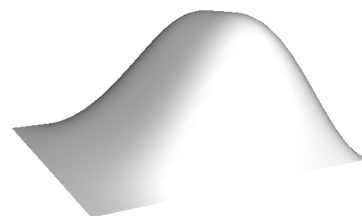
4. Consider the region of space that is common to the two solid cylinders $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$. Use the cross-sectional approach to find its volume.

5. Consider the integral

$$V = \int_0^1 \int_{-1}^1 \sin((\sqrt{\pi}x)^2) \cos((\sqrt{\pi}y)^2) dy \ dx,$$

which is the volume between the surface

$$z = f(x, y) = \sin((\sqrt{\pi}x)^2) \cos((\sqrt{\pi}y)^2),$$

shown at right, and the plane $z = 0$. Since $\sin(x^2)$ and $\cos(y^2)$ do not have antiderivatives, it is not possible to find an exact value for this integral. One solution is to use level curves of the function to estimate the value of its integral over a given region, using a Riemann sum. The plot on the next page shows level curves of f at levels $0, 0.1, 0.2, \dots, 0.9$, ordered from outer (rectangular) to inner (oval).

(a) The function has a level "curve" at level 1 in this region, consisting of a single point. Where is it?

(b) Divide the region $[0, 1] \times [-1, 1]$ into eight $1/2 \times 1/2$ squares, and choose a representative value for $f(x, y)$ within each square. Use these to estimate V.

(c) Divide the region into 32 $1/4 \times 1/4$ squares, choose a representative value for $f(x, y)$ within each square, and use these to estimate V.

(d) Is it possible to choose rectangles and representative points so that the Riemann sum estimate for V is 1.5?

(e) Explain the significance of the expression $\displaystyle\sum_{i=-3}^{4}\sum_{j=1}^{4} \frac{1}{16} f\left(\frac{i}{4}, \frac{j}{4}\right)$.

(f) Explain the significance of the expression $\displaystyle\lim_{n\to\infty} \sum_{i=-n+1}^{n}\sum_{j=1}^{n} \frac{1}{n^2} f\left(\frac{i}{n}, \frac{j}{n}\right)$.
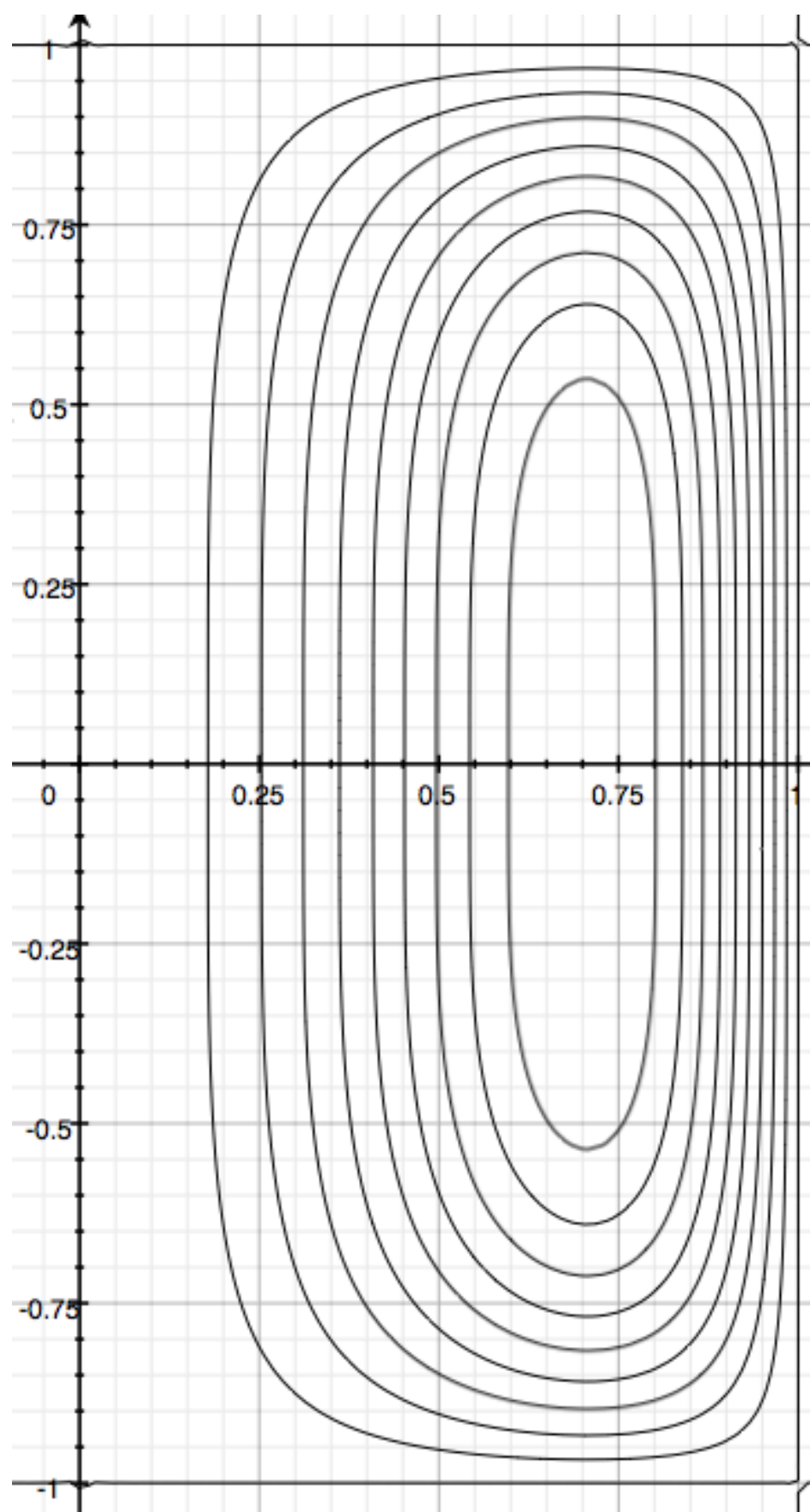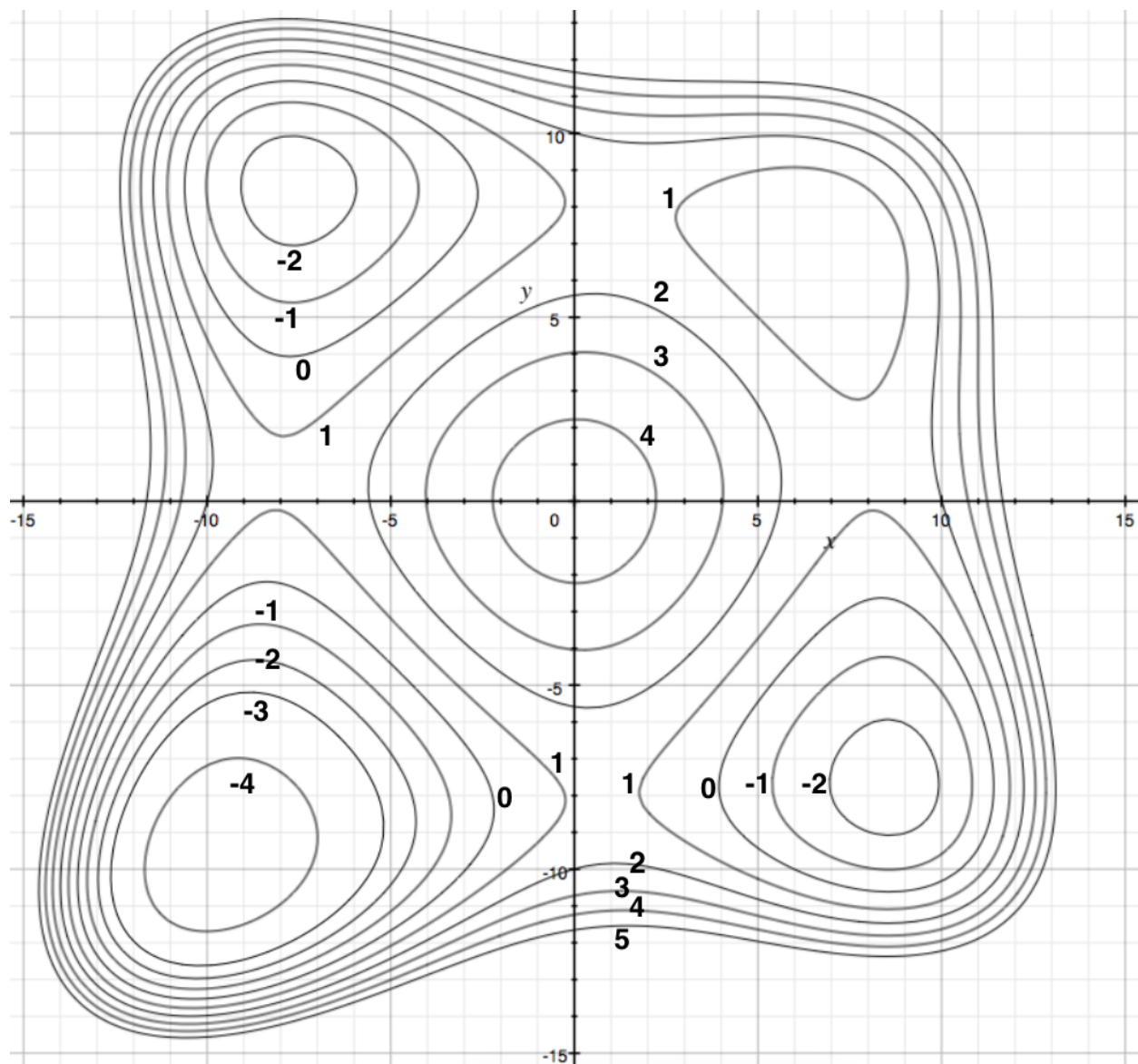
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1. Some level curves of a function $f(x, y)$ are shown in the diagram on the previous page. Sketch the vector field ∇f.

2. Consider the plane $z = 5 - 2x - 2y$ and the points $A = (0, 0, 0)$, $B = (1, 0, 0)$, $C = (1, 1, 0)$, $D = (0, 1, 0)$, $P = (0, 0, 5)$, $Q = (1, 0, 3)$, $R = (1, 1, 1)$, and $S = (0, 1, 3)$. Notice that $PQRS$ is a quadrilateral that lies on the plane, and that $ABCD$ is its projection onto the xy-plane. Compare the areas of these quadrilaterals. Then choose a different example of the same sort — a quadrilateral on the plane $z = 5 - 2x - 2y$ and its projection onto the xy-plane — and compare their areas. Look for a pattern and explain it.

3. You are of course familiar with the formulas $x = r \cos \theta$ and $y = r \sin \theta$ (which convert polar coordinates into Cartesian coordinates), and also with the formulas $r^2 = x^2 + y^2$ and $\tan \theta = y/x$ (which convert Cartesian coordinates into polar coordinates). Use these to calculate $\dfrac{\partial x}{\partial \theta}$ (which is a function of r and θ) and $\dfrac{\partial \theta}{\partial x}$ (which is a function of x and y). Also compute $\dfrac{\partial x}{\partial \theta}$ as a function of x and y. Are the results what you expected?

4. Given a point inside the unit circle, the distance to the origin is some number between 0 and 1. What is the *average* of all these distances? It should also be a number between 0 and 1. Justify your approach. *Hint: To evaluate your integral, sketch the integrand as a surface and compute a volume using basic geometry.*

5. Consider the triangle $\mathcal{T}$ on the xy-plane formed by points $A = (0, 0, 0)$, $B = (1, 0, 0)$, and $C = (1, 1, 0)$. The surface $z = y \cos \left(\frac{\pi}{2} x^3 \right)$, along with $\mathcal{T}$ and the plane $y = x$, enclose a region of space. Make a sketch of this region, then use the cross-sectional method to find its volume. You might notice that it makes a difference whether you begin by slicing the region perpendicular to the x-axis (the $dy\,dx$ approach) or by slicing perpendicular to the y-axis (the $dx\,dy$ approach).

6. The paraboloid $z = 9 - x^2 - y^2$ is cut by the plane $z = 6 - 2x$. The intersection curve is an ellipse, most of which is showing in the diagram. Use double integration to find the volume of the region that is enclosed by these two surfaces. In other words, the region is above the plane and below the paraboloid.

7. Find the area between the curves $y = x^2$ and $y = 2 - x$ in two ways: using a single integral, and using a double integral.

8. Suppose that the temperature at point (x, y) of a metal plate is $T(x, y) = 100e^{-x} \sin y$, for $0 \le x \le 1$ and $0 \le y \le \pi$. The temperatures in this plate therefore range between 0 and 100 degrees, inclusive. What is the average of all these temperatures?
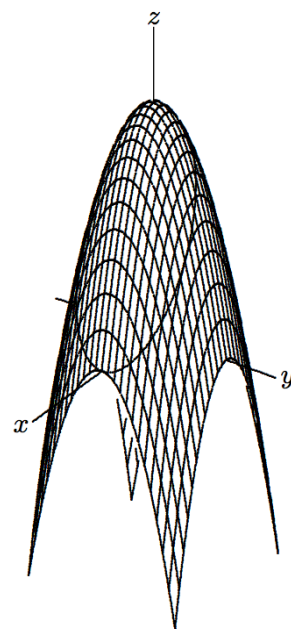
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1. As we have seen, a double integral is defined as the *limit* of a Riemann sum: in words, if we are breaking up a region $\mathcal{R}$ into tiny rectangles,

$$\iint_{\mathcal{R}} f(x,y) = \lim_{\text{area of rectangles}\to 0} \sum_{\text{tiny rectangles}} f(\text{sample point}) \cdot (\text{area of rectangle}).$$

Since this definition depends on a limit, we should make sure that the limit exists.

(a) Consider the function $f(x,y) = \begin{cases} \dfrac{1}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ and consider the region $\mathcal{R} = [-1,1] \times [-1,1]$. Find a point (x_0, y_0) in $\mathcal{R}$ so that $f(x_0, y_0) = 10000$.

(b) Explain why the limit associated to the integral $\iint_{\mathcal{R}} f(x,y)$ does not exist.

(c) Replace $x^2 + y^2$ with $\sqrt{x^2+y^2}$ in the definition of $f(x,y)$ above. Does the integral of $\iint_{\mathcal{R}} f(x,y)$ exist for this definition of f?

2. Use the "tiny rectangles" and "sample point" limiting Riemann sum definition of a double integral to justify the following claim: If $f(x,y) \leq g(x,y)$ for each point (x,y), then $\iint_{\mathcal{R}} f(x,y)\,dA \leq \iint_{\mathcal{R}} g(x,y)\,dA$ for every region $\mathcal{R}$.

3. Evaluate the double integral $\displaystyle\int_0^1 \int_x^1 \cos\left(y^2\right)\,dy\,dx$ without using a calculator. You need to describe the domain of the integration in a way that is different from the given description. This is called *reversing the order of integration*.

4. *Fubini's Theorem* states that

$$\int_c^d \int_a^b f(x,y)\,dx\,dy \;=\; \int_a^b \int_c^d f(x,y)\,dy\,dx$$

is true whenever f is a function that is continuous at all points in the rectangle $a \leq x \leq b$ and $c \leq y \leq d$. Despite the intuitive content of this statement, a proof is not easy, and this will be left for a later course. It suffices to do examples that illustrate its non-trivial content. Verify the conclusion of the theorem using $f(x,y) = x\sin(xy)$ and the rectangle $1 \leq x \leq 2$ and $0 \leq y \leq \pi$.

5. *An example where Fubini's Theorem fails.* Define $f(x,y)$ on $\mathcal{R} = [0,1] \times [0,1]$ as follows:

$$f(x,y) = \begin{cases} 1 & \text{on } [0,1/2) \times [0,1/2) \\ -2 & \text{on } [1/2, 1/2+1/4) \times [0,1/2) \\ 4 & \text{on } [1/2, 1/2+1/4) \times [1/2, 1/2+1/4) \\ -8 & \text{on } [1/2+1/4, 1/2+1/4+1/8) \times [1/2, 1/2+1/4) \\ \vdots \\ 0 & \text{elsewhere.} \end{cases}$$

Show that $\displaystyle\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \frac{1}{4}$ while $\displaystyle\int_0^1 \int_0^1 f(x,y)\,dx\,dy = 0$. Discuss.

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1. Evaluate the integral $\int_0^1 \int_0^1 \dfrac{x-y}{(x+y)^3}\, dy\, dx$. Then reverse the order of integration and evaluate again. Explain why you should not be surprised by the result.

2. *The cycloid.* A wheel of radius 1 rolls along the x-axis without slipping. A mark on the rim follows a path that starts at $(0,0)$, as shown in the figure below.

(a) Find the x-coordinate of the point P where the mark first returns to the x-axis.

(b) Find both coordinates of the center after the wheel makes a quarter-turn.

(c) Find both coordinates of the mark after the wheel makes a quarter-turn.

(d) Find both coordinates of the mark after the wheel rolls a distance t, where $t < \dfrac{1}{2}\pi$.

(e) Check your formulas to see whether they are also correct for $\dfrac{1}{2}\pi \le t$.
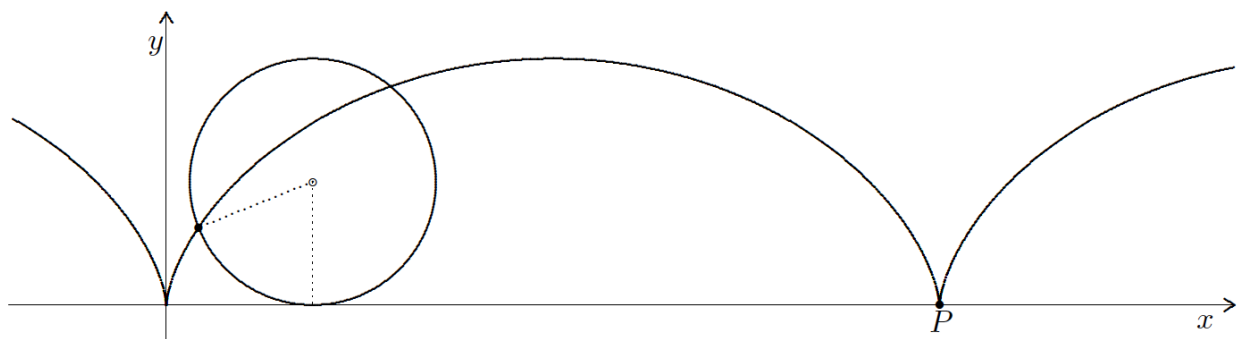
3. If a random point were chosen in the square defined by $0 \le x \le 1$ and $0 \le y \le 1$, its distance from the origin would be somewhere between 0 and $\sqrt{2}$. What is the average of all these distances?

4. In setting up a double integral, it is customary to tile the domain of integration using little rectangles whose areas are $\Delta x \Delta y$. In some situations, however, it is better to use small tiles whose areas can be described as $r\Delta r\Delta\theta$. Sketch such a tile, and explain the formula for its area. In what situations would such tiles be useful?

5. Verify that the point $P = (2, 6, 3)$ is on the sphere $x^2 + y^2 + z^2 = 49$. Consider a *small* piece $\mathcal{S}$ of the spherical surface that includes P, and let $\mathcal{R}$ be the projection of $\mathcal{S}$ onto the xy-plane.

(a) Explain why the area of $\mathcal{R}$ is approximately $3/7$ times the area of $\mathcal{S}$, and why the approximation gets better and better as the dimensions of $\mathcal{S}$ decrease to zero.

(b) By considering the angle between two vectors, explain where the ratio $3/7$ comes from.

(c) What would the ratio have been if $P = (0, 0, 7)$ had been selected instead?

6. Find the volume between the plane $z = 0$ and the surface $z = 2x - y + 13$ over the region $\mathcal{R}$ in the xy-plane bounded by $y = x^2 - 4$ and $y = 9 - (x - 1)^2$, using a double integral and then by using a triple integral.

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1. Find the volume of the solid region enclosed by the xy-plane, the cylinders $r = 1$ and $r = 2$, the planes $\theta = 0$ and $\theta = \pi/2$, and the plane $z = x + y$.

2. Reverse the order of integration in $\displaystyle\int_0^3 \int_{9-3x}^{9-x^2} f(x, y)\, dy\, dx$. In other words, rewrite the integral using $dx\, dy$ as the area differential.

3. Prove that for any constant c and any region $\mathcal{R}$, $\displaystyle\iint_{\mathcal{R}} cf(x, y)\, dA = c \iint_{\mathcal{R}} f(x, y)\, dA$.

4. Explain why $\displaystyle 2\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\sqrt{1 - x^2 - y^2}}\, dx\, dy$ equals the surface area of the unit sphere. Explain why this is an improper integral, then describe how to deal with it properly.

5. (Continuation) In problems like the preceding, a different coordinate system works better. Show how to re-express the problem using polar variables r and θ instead of x and y. This requires that you replace the differential element of area $dx\, dy$ by something polar, and that you put new limits on the integrals. Evaluate the resulting double integral. You should not need your calculator for this version of the question.

6. Convert $\displaystyle\int_0^1 \int_y^1 x\, dx\, dy$ into polar form. Evaluate both versions of the double integral and interpret the result.

7. The purpose of this problem is to find the volume in the first quadrant bounded by the coordinate planes and the plane $3x + 2y + z = 6$.

(a) Find the volume of the region using basic geometry.

(b) Find the volume of the region using a triple integral. For practice, set up the triple integral in several different orders (you need only compute one of them).

(c) Set up a double integral to find the volume. You need not compute it.

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1. Discuss the definition

$$\int_0^\infty \int_0^\infty f(x,y)\,dx\,dy = \lim_{a\to\infty} \int_0^a \int_0^a f(x,y)\,dx\,dy\,.$$

2. Explain why

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dx\,dy = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right).$$

3. Explain why

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dx\,dy = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r\,dr\,d\theta.$$

4. Explain why

$$\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}\,.$$

First proved by Laplace, this is a significant result for statistical work.

5. The familiar equations $x = r\cos\theta$, $y = r\sin\theta$ can be thought of as a *mapping* from the $r\theta$-plane to the xy-plane. In other words, $p(r,\theta) = (r\cos\theta, r\sin\theta)$ is a function of the type $\mathbf{R}^2 \to \mathbf{R}^2$. Point by point, p transforms regions of the $r\theta$-plane onto regions of the xy-plane. In particular, consider the rectangle defined by $2 \le r \le 2.1$ and $1 \le \theta \le 1.2$. What is its image in the xy-plane? How do the areas of these two regions compare?

6. (Continuation) The derivative of p at $(2,1)$, which could be denoted $p'(2,1)$, is a 2×2 matrix, and its determinant is an interesting number. Explain these statements. It may help to know that these determinant matrices are usually denoted $\dfrac{\partial(x,y)}{\partial(r,\theta)}$.

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1. Consider the linear mapping $g : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $x = 3u + v$ and $y = u + 2v$. In other words, $g(u, v) = (3u + v, u + 2v)$. Point by point, g transforms regions of the uv-plane onto regions of the xy-plane. Select any uv-rectangle and calculate its g-image (which is a simple geometric shape).

(a) Find the length of one of the rectangle's edges, and compare it to the length of the image of that edge under f. How could you calculate the local multiplier for the length from f?

(b) Compare the area of the image with the area of the rectangle, and then calculate the determinant of $g'(0,0)$, which is the 2×2 matrix $\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$.

2. Consider the function $f(u, v) = (u^2 - v^2, 2uv)$. Apply it to the rectangle $\mathcal{R}$ defined by $1 \le u \le 1.5$ and $1 \le v \le 1.5$. Show that the image "quadrilateral" $\mathcal{Q}$ is enclosed by four parabolic arcs. First estimate the area of $\mathcal{Q}$, then calculate it exactly. What is the ratio of this area to the area of $\mathcal{R}$?

3. (Continuation) Apply f to the rectangle $\mathcal{R}$ defined by $1 \le u \le 1.1$ and $1 \le v \le 1.1$. The image $\mathcal{Q}$ is enclosed by four parabolic arcs. Make a detailed sketch of $\mathcal{Q}$. Calculate the matrix $f'(1, 1)$, and then find its determinant. You should expect the area of $\mathcal{Q}$ to be approximately 8 times the area of $\mathcal{R}$. Explain why.

4. (Continuation) Apply the function $g(h, k) = (2h - 2k, 2h + 2k)$ to the rectangle defined by $0 \le h \le 0.1$ and $0 \le k \le 0.1$. Compare the result with the $\mathcal{Q}$ calculated in the preceding item. Then explain what the matrix $\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$ reveals about the mapping f in the vicinity of $(u, v) = (1, 1)$.

5. In general, given a mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$, its derivative is a 2×2 matrix-valued function that provides a *local multiplier* at each point of the domain of f. Each such matrix describes how suitable domain rectangles are transformed into image quadrilaterals, and its determinant is a multiplier that converts (approximately) the rectangular areas into the quadrilateral areas. Explain the words "local" and "suitable", and make use of the limit concept in your answer. It is customary to refer to either the matrix f' or its determinant as *the Jacobian of f*.

6. Explain why each row of a Jacobian matrix is the gradient of a certain function.

7. Justify the equation $\dfrac{\partial(x, y)}{\partial(u, v)} \, du \, dv = dx \, dy$.

8. Consider a rectangular box defined by $0 \le x \le 4, 3 \le y \le 5$, and $2 \le z \le 4$, and suppose that the temperature at any point in the box is given by $T(x, y, z) = (x^2)/y + 3$. Find the average temperature in the box.

Math 290-3

1. Given a region $\mathcal{R}$ and a function $f(x,y)$ defined on $\mathcal{R}$, f is said to be *integrable over* $\mathcal{R}$ if the limit of Riemann sums used to define the integral of f over $\mathcal{R}$ exists. Make up a function $f(x,y)$ that is *not* integrable over the region $[1,2] \times [3,4]$.

2. Suppose that $f(x,y)$ is bounded over a region $\mathcal{R}$, and the set of points in $\mathcal{R}$ where f is discontinuous has area 0. Explain why f must be integrable over $\mathcal{R}$.

3. Evaluate $\displaystyle\int_{-1}^{2}\int_{2-x}^{4-x^2} dy\, dx$. Explain why your answer can be considered as an area or as a volume.

4. Is it feasible to reverse the order of integration in the preceding?

5. Evaluate $\displaystyle\int_{0}^{2\pi}\int_{0}^{1+\cos\theta} r\, dr\, d\theta$. Explain why your answer can be considered as an area or as a volume.

6. Is it feasible to reverse the order of integration in the preceding?

7. Let $\mathcal{R}$ be the rectangular region defined by $0 \leq u \leq 2$ and $1 \leq v \leq 2$. Let $\mathcal{Q}$ be the region obtained by applying the mapping $(x,y) = (u^2 - v^2, 2uv)$ to $\mathcal{R}$.

(a) Sketch the four-sided region $\mathcal{Q}$.

(b) Find the area of $\mathcal{Q}$.

8. Let $\mathcal{R}$ be the triangular region whose vertices are $(0,0)$, $(1,0)$, and $(1,\sqrt{3})$. Let $f(x,y) = \dfrac{1}{(1+x^2+y^2)^2}$. Evaluate the integral of f over $\mathcal{R}$.

9. Sketch the first-quadrant region $\mathcal{R}$ defined by $225 \leq 9x^2 + 25y^2 \leq 900$. Integrate the function $f(x,y) = xy$ over $\mathcal{R}$. *Hint*: Consider the quasi-polar change of variables $(x,y) = (5u\cos t, 3u\sin t)$.

Math 290-3

Quiz at beginning of class

1. _Spherical coordinates I._ Points on the unit sphere $x^2 + y^2 + z^2 = 1$ can be described parametrically by

$$x = \sin\phi\cos\theta$$
$$y = \sin\phi\sin\theta$$
$$z = \cos\phi,$$

where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. The angle θ is the angle usually called _longitude_, and the angle $\pi/2 - \phi$ is the angle usually called _latitude_. This defines a mapping $f : \mathbf{R}^2 \to \mathbf{R}^3$, which "wraps" the rectangle $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$ around the sphere. The $\phi\theta$ grid is transformed into the familiar latitude-longitude grid on the sphere.

(a) Write the 3×2 matrix $f'(\phi, \theta)$.

(b) Explain why the columns of $f'(\phi, \theta)$ are vectors tangent to the sphere at $f(\phi, \theta)$.

(c) Calculate $\mathbf{n}(\phi, \theta)$, the cross product of these column vectors.

(d) Show that the length of $\mathbf{n}(\phi, \theta)$ is $\sin\phi$. Use this _Jacobian_ in a double integral to confirm that the surface area of the unit sphere is indeed 4π.

2. Calculate the area of the unit sphere that is found between the parallel planes $z = a$ and $z = b$, where $-1 \le a \le b \le 1$. You should find that your answer depends only on the separation between the planes, not on the planes themselves. This is called the _equal crust property._

3. _Spherical coordinates II._ By using spheres of varying radius in addition to the angles ϕ and θ, every point of xyz-space can be given new coordinates. The coordinate map

$$f(\rho, \phi, \theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$$

is obtained by simply inserting ρ into each of the equations in item 1 above. The symbol ρ (the Greek r) stands for $\sqrt{x^2 + y^2 + z^2}$, the distance to the origin. The infinite prism $0 \le \rho$ and $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$ is mapped by f onto all of xyz-space.

(a) Make calculations that justify the Jacobian formula $dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

(b) Make a sketch of a small "spherical brick" whose volume is $\rho^2\sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta$.

Math 290-3

1. Given a 3×3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, its *determinant* is, except for sign, the volume of the parallelepiped defined by the three column vectors $\mathbf{u} = [a, d, g]$, $\mathbf{v} = [b, e, h]$, and $\mathbf{w} = [c, f, i]$. Show that it can be evaluated as a *triple scalar product* in any of the equivalent forms $\mathbf{u}{\cdot}(\mathbf{v} \times \mathbf{w})$, $\mathbf{v}{\cdot}(\mathbf{w} \times \mathbf{u})$, or $\mathbf{w}{\cdot}(\mathbf{u} \times \mathbf{v})$, all of which equal $aei + bfg + cdh - afh - bdi - ceg$. The determinant is positive if and only if $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ form a right-handed coordinate system.

2. Use the preceding to obtain the spherical-coordinate Jacobian $\rho^2 \sin \phi$.

3. *Cylindrical coordinates* are a self-explanatory extension of polar coordinates to 3-dimensional space. The coordinate transformation is $(x, y, z) = (r \cos \theta, r \sin \theta, z)$, where $r^2 = x^2 + y^2$. Notice the distinction between the polar variable r and the spherical variable ρ. Use a determinant to justify the equation $dx \, dy \, dz = r \, dr \, d\theta \, dz$.

4. Let $\mathcal{P}$ be the region in $\mathbf{R}^3$ defined by $0 \leq z \leq 4 - x^2 - y^2$. Use cylindrical coordinates to find the volume of $\mathcal{P}$.

5. Sketch the solid of integration corresponding to the integral $\displaystyle\int_0^2 \int_0^x \int_0^y f(x, y, z) \, dz \, dy \, dx$. Then rewrite this integral in the orders $dx \, dy \, dz$ and $dy \, dz \, dx$.

6. Consider the region $\mathcal{S}$ in $\mathbf{R}^3$ that is enclosed by the cone $\phi \leq \kappa$ and the concentric spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, where $a < b$. Find the volume of $\mathcal{S}$, expressed in terms of a, b, and κ.

7. Consider the integral
$$\int_0^1 \int_{x^{20}}^{x^{1/20}} \sin(x + y) \, dy \, dx,$$
whose value is some number v. It is not possible to find an exact value for this integral, so we will estimate it.

(a) Show that $0 < v < 1$.

(b) Improve the estimate: find $a > 0$ and $b < 1$ so that $a < v < b$. Explain your reasoning.

8. Consider the solid region bounded by the planes $y = 0$ and $y = 1$, and the surfaces $z = 4 - x^2$ and $z + y = 0$. Suppose that the density of the solid at a point (x, y, z) in this region is given by $x + y + z$. Set up an integral to find the total mass of the solid.

Math 290-3

1. If $(x(t), y(t))$ is a parametric curve, then $\left[\dfrac{dx}{dt}, \dfrac{dy}{dt}\right]$ is its velocity and $\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$ is its speed. Find at least two parameterized curves whose speed is $\sqrt{t^4 - 2t^2 + 1 + 4t^2}$.

2. Let $f(u, v) = \left(\sqrt{2u}\cos v, \sqrt{2u}\sin v\right)$.

(a) Calculate the matrix $f'(u, v)$.

(b) Show that f is an *area-preserving* transformation of coordinates.

3. Consider the region $\mathcal{E}$ enclosed by the ellipse $x^2 - 2xy + 2y^2 = 25$. Find a linear change of coordinates that sends a circle in the tu-plane to $\mathcal{E}$, and use this to find the area of $\mathcal{E}$.

4. Rewrite $\displaystyle\int_0^1 \int_0^x \int_{x^2+y^2}^2 f(x, y, z)\, dz\, dy\, dx$ in the order $dy\, dz\, dx$.

5. Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} \int_1^2 e^x \sin\frac{y}{x}\, dx\, dy$.

6. The integral $\displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ is a template for what type of problem?

7. Show that the function $p(t) = \dfrac{1}{\sqrt{2\pi}}\, e^{-t^2/2}$ has the following two properties:

(a) $1 = \displaystyle\int_{-\infty}^{\infty} p(t)\, dt$

(b) $1 = \displaystyle\int_{-\infty}^{\infty} t^2 p(t)\, dt$

This is the so-called *standard normal distribution of probability*.

Math 290-3

1. Consider that part of the unit 3-ball $\rho \leq 1$ that lies inside the circular cylinder $r = \cos\theta$ and above the plane $z = 0$. Find the volume of this region.

2. The appearance of the integral $\int_1^4 \int_{1/x}^{4/x} \dfrac{xy}{1 + x^2 y^2}\, dy\, dx$ suggests that it would be helpful if xy were a single variable. With this in mind, consider the transformation of coordinates $(x, y) = (u, v/u)$.

(a) Sketch the given region of integration in the xy-plane.

(b) Show that this region is the image of a square region in the uv-plane.

(c) Evaluate the given integral by making the indicated change of variables.

3. Consider the triple integral $\int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x, y, z) dz\, dy\, dx$.

(a) Sketch the solid region of integration.

(b) Rewrite the integral in the orders $dy\, dx\, dz$ and $dx\, dz\, dy$.

4. The cycloid $(x, y) = (t - \sin t, 1 - \cos t)$ is the path followed by a point on the edge of a wheel of unit radius that is rolling along the x-axis. The point begins its journey at the origin (when $t = 0$) and returns to the x-axis at $x = 2\pi$ (when $t = 2\pi$), after the wheel has made one complete turn. What is the length of the cycloidal path that joins these x-intercepts?

5. On different axes, sketch the vector field $\mathbf{F}(x, y) = [x, y]$ and $\mathbf{G}(x, y) = [-y, x]$. "Sketch the vector field" means "at each point (x, y), draw the vector $[x, y]$ with its tail at (x, y)." On each, also sketch the unit circle $\mathcal{C}$ oriented counter-clockwise, and the segment $\mathcal{S}$ from $(0, 0)$ to $(1, 0)$, oriented in the positive direction. *Circulation* measures how much a vector field points in the same direction as an oriented curve, as though there are beads on a wire and the vector field is pushing them along. Estimate (is it positive, negative, or zero?) the total circulation of $\mathbf{F}$ over $\mathcal{C}$ and over $\mathcal{S}$, and the total circulation of $\mathbf{G}$ over $\mathcal{C}$ and over $\mathcal{S}$.

6. The cone $\sin\phi = \frac{5}{13}$ is sliced by the plane $4x + 4y + 7z = 112$. One of the intersection points is $P = (3, 4, 12)$. The intersection curve is an ellipse. If this cone were cut along the ray from the origin through P, it could be "unrolled", thus forming an infinite sector.

(a) What is the angular size of this sector?

(b) The ellipse has one point for each value of the longitude θ, for $0 \leq \theta < 2\pi$, so it should be possible to express all the other variables in terms of θ. Try it.

(c) What is the range of ρ-values on the ellipse?

(d) When the cone is "unrolled", the ellipse becomes a curve that connects two points on the radial edges of the sector. Explain why this curve cannot be a (straight) segment.

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Math 290-3

1. Find the volume of the region $\mathcal{R}$ bounded by the planes $z = 0$ and $z = 2y$ and by the parabolic cylinder $x^2 + y = 4$.

2. Consider the integral $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-1+x^2+2y^2}^{1-2x^2-y^2} e^{\cos(x^2+y^2)} \, dz \, dy \, dx$, whose value is some number v. It is not possible to find an exact value for this integral, so we will estimate it.

(a) Sketch the solid region of integration and explain why its volume is less than $4\pi/3$.

(b) Show that $0 < v < 4e\pi/3$. Improve this estimate if you can.

3. *Using a double integral to evaluate a tricky integral.* Let $f(0) = 2$, and for nonzero values of x, let $f(x) = \dfrac{e^{-x} - e^{-3x}}{x}$. Let's integrate this function from 0 to ∞.

(a) Show that f is differentiable at $x = 0$.

(b) Find a, b and g so that $\dfrac{e^{-x} - e^{-3x}}{x} = \int_a^b g(x, y) \, dy$.

(c) Evaluate the improper integral $\int_0^\infty f(x) \, dx$, by using the "trick" of rewriting this integral as $\int_0^\infty f(x) \, dx = \int_0^\infty \int_a^b g(x, y) \, dy \, dx$ and reversing the order of integration.

4. Compute $\iint_D xy^2 \, dA$, where D is the region in the first quadrant bounded by the curves $xy = 1, xy = 4, xy^2 = 1$, and $xy^2 = 4$. *Hint*: After you make your change of variables, instead of solving for the Jacobian $\left| \dfrac{\partial(x,y)}{\partial(u,v)} \right|$, use your knowledge of determinants and inverses and use the easier-to-compute $\left| \dfrac{\partial(u,v)}{\partial(x,y)} \right|$.

5. Consider the position function $\mathbf{r}(t) = (x(t), y(t))$ of a particle at time t.

(a) Explain why the function $\mathbf{r}'(t)$ gives the particle's velocity vector at time t.

(b) Explain why the function $\mathbf{r}''(t)$ gives the particle's acceleration vector at time t.

(c) Explain how to find the length of the path $\mathbf{r}(t)$ from $t = a$ to $t = b$.

6. A vector field can represent a force acting on a particle. If a particle is moving in a straight line of length s, and is acted on by a vector field that has constant length F and is parallel to the line in the same direction as the particle is moving, then the work done is Fs. A vector field that is perpendicular to the path of the particle does zero work. Finally, the work done by a vector field is linear: multiplying a vector field by a scalar c multiplies the work done by c, and adding two vector fields adds together the work done.

**(a)** Explain why the calculation of work should involve a dot product.

(b) Set up an integral to find the work done by the vector field $\mathbf{F}(x, y) = [y, 0]$ in moving a particle from $(-1, 0)$ to $(1, 0)$ across the top half of the unit circle.

Math 290-3

1. The diagram shows the region $\mathcal{S}$ bounded by the spiral $r\theta = 1$, the circle $r = 1$, the positive x-axis, and the circle $r = 2$.

(a) Find coordinates x and y for the point A where the spiral intersects the circle $r = 2$.

(b) Find the area of $\mathcal{S}$.
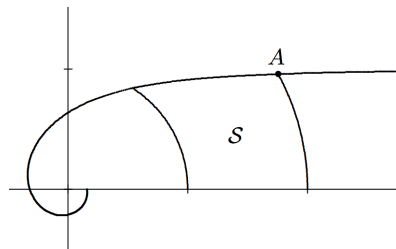
2. (From Colley, *Vector Calculus, 4th edition*): Consider, but do not evaluate, the integral

$$\int_{-2}^{2} \int_{0}^{\frac{1}{2}\sqrt{4-x^2}} \int_{x^2+3y^2}^{4-y^2} (x^3 + y^3) \, dz \, dy \, dx.$$

(a) Describe and sketch the solid region of integration.

(b) Rewrite the integral in the orders $dz\ dx\ dy, dy\ dz\ dx$, and $dx\ dy\ dz$.

3. Estimate the value of the integral $v = \int_{0}^{1} \int_{x}^{1} \int_{0}^{y^2} x^2\, e^{z^{3/2}} \, dz \, dy \, dx$. In other words, find numbers a and b such that $a \le v \le b$, using whatever methods you can.

4. The cylinder $x^2 + y^2 = 25$ is cut by the plane $2x + 6y + 3z = 42$. The intersection curve is an ellipse. Find its area.

5. Sketch the helical arc $\mathbf{h}(t) = [a \cos t, a \sin t, bt]$.

(a) Compute the direction vectors $\mathbf{h}'(t)$ and $\mathbf{h}''(t)$. Could you have anticipated their directions?

(b) Find the length of the arc from $t = 0$ to $t = 4\pi$.

(c) Write an equation for the tangent line to $\mathbf{h}(t)$ at $t = \pi/2$. Add the line to your sketch.

6. Consider the figure-eight curve obtained by intersecting the unit sphere $\rho = 1$ and the cylinder $r = \cos \theta$. Find a parametrization for the entire curve.

Math 290-3

1. Evaluate $\displaystyle\int_0^3 \int_{y^2}^9 y e^{-x^2}\, dx\, dy$.

2. Using spherical coordinates, find the volume of the solid above the xy-plane, outside the cone $x^2 + y^2 = z^2$, and inside the unit sphere.

3. Rewrite $\displaystyle\int_1^2 \int_1^z \int_{1/y}^2 f(x,y,z)\, dx\, dy\, dz$ in the order $dy\, dz\, dx$.

4. Is the function $f(x,y) = \begin{cases} \dfrac{1}{1 - x^2 - y^2} & \text{everywhere except the unit circle} \\ 0 & \text{on the unit circle} \end{cases}$

integrable over the square $[-2, 2] \times [-2, 2]$? Explain why or why not.

5. Show that the linear transformation $(x, y) = \left(\dfrac{u + v\sqrt{3}}{2}, \dfrac{v - u\sqrt{3}}{2} \right)$ is area-preserving.

6. (Continuation) What condition on the coefficients a, b, c, k, m, and n ensures that the generic linear transformation $(x, y) = (a + bu + cv, k + mu + nv)$ is area-preserving?

7. Consider the linear transformation that transforms the circle circumscribing the square whose vertices are $(0,0)$, $(2,0)$, $(2,2)$, and $(0,2)$ into the ellipse circumscribing the parallelogram whose vertices are $(0,0)$, $(6,2)$, $(8,6)$, and $(2,4)$.

(a) Is $(8,6)$ a vertex of the ellipse?

(b) The image of the center of the circle is $(4,3)$. Is this the center of the ellipse?

8. Given the acceleration vectors $\mathbf{p}''(t) = [6t, \cos t]$, the velocity vector $\mathbf{p}'(0) = [1,2]$, and the position vector $\mathbf{p}(0) = [-\pi^3, -1]$, calculate the position vector $\mathbf{p}(\pi)$.

9. As you have seen, the nonlinear transformation $(x, y) = (u^2 - v^2, 2uv)$ distorts regions and alters their areas. It does have a special property, however; at every point except the origin, this transformation is *conformal*, which means that it preserves the sizes of angles.

(a) Find an example to illustrate this statement.

(b) Prove the general assertion.

Math 290-3

Practice midterm 1 - 2016

1. Say whether the following are true or false, and justify your answers.

(a) For any function $f : \mathbf{R}^2 \to \mathbf{R}$, the following equality holds:

$\int_{-\pi/4}^{\pi/4} \int_0^2 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta = \int_0^{\sqrt{2}} \int_{-x}^x f(x,y)\, dy\, dx + \int_{\sqrt{2}}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\, dy\, dx.$

(b) For any function $f : \mathbf{R}^2 \to \mathbf{R}$, the following equality holds:

$$\int_0^{\pi/4} \int_{\sec\theta}^2 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta = \int_0^1 \int_1^{2-y} f(x,y)\, dx\, dy.$$

(c) Ice cream in an ice cream cone fills the space that is both inside the cone $z = r\sqrt{3}$ and inside the sphere $\rho = 5$. Since someone's hand is on the cone warming it up, the temperature in degrees Celsius of the ice cream at point (ρ, ϕ, θ) is θ, where $0 \le \theta < 2\pi$. True or False: the average temperature of ice cream in the cone is above $4°$.

(d) The transformation $T(x,y) = (4x^2 - x + 25y^2 - 2y + 20xy, 2x + 5y)$ is area-preserving, in the sense that $Area(T(\Omega)) = Area(\Omega)$ for any region $\Omega \subset \mathbf{R}^2$.

(e) The function $f(x,y) = \begin{cases} \dfrac{1}{\sqrt{x^2 + y^2}} & (x,y) \ne \mathbf{0} \\ 0 & (x,y) = \mathbf{0} \end{cases}$ is integrable over the unit disk.

2. Show that $\pi \le \iint_D (\sqrt{x^2 + y^2} + 1)^4 dA \le 16\pi$, where D is the unit disk $x^2 + y^2 \le 1$.

3. Show that $\iint_{[0,1]\times[0,1]} (x^2 + 1)e^{y^2 - x^2} dA \le 2e$.

4. Rewrite as a single integral: $\int_{-4}^0 \int_{-\sqrt{x+4}}^{\sqrt{x+4}} f(x,y)\, dy\, dx + \int_0^5 \int_{x-2}^{\sqrt{x+4}} f(x,y)\, dy\, dx.$

5. Rewrite $\int_0^2 \int_0^{4-2y} \int_0^{\sqrt{x}} f(x,y,z)\, dz\, dx\, dy$ in the orders $dy\, dz\, dx$ and $dx\, dy\, dz$.

6. Compute the volume of the region in $\mathbf{R}^3$ that is above the paraboloid $z = 1 - x^2 - y^2$ and below the upper half of the sphere $x^2 + y^2 + z^2 = 1$.

7. Evaluate $\iint_R xy\, dA$, where R is the region in the first quadrant bounded by $y = x$, $y = 3x$, $xy = 1$, and $xy = 3$.

8. Evaluate the integral $\int_0^1 \int_x^1 e^{y^2}\, dy\, dx$.

Math 290-3

Midterm 1 in evening - 2015 exam

1. Say whether the following are true or false, and justify your answers:

(a) Suppose that R is a rectangle in $\mathbf{R}^2$ and that $f : \mathbf{R}^2 \to \mathbf{R}$ is continuous. If $\iint_R f(x,y)\, dA = 0$, then every Riemann sum of f over R has the value 0.

(b) $\displaystyle\int_0^1 \int_0^x dy\, dx + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} dy\, dx = \frac{\pi}{4}.$

(c) For every number k, $\displaystyle\int_{10}^{20} \int_0^{e^x} \left[x^2 + y^2 + (k^2+1)^3 \right]\, dy\, dx \geq 0.$

Optional, not part of exam: Improve this bound.

(d) For every continuous function $f : \mathbf{R}^2 \to \mathbf{R}$, $\displaystyle\int_0^1 \int_{-x}^x f(x,y)\, dy\, dx = \int_{-1}^1 \int_0^1 v\, f(v, uv)\, dv\, du.$

2. Find the value of $\displaystyle\int_{-1}^0 \int_{-x}^1 e^{y^2}\, dy\, dx + \int_0^1 \int_x^1 e^{y^2}\, dy\, dx.$

3. Let $f : \mathbf{R}^3 \to \mathbf{R}$ be continuous. Rewrite the following integral in the order $dy\, dx\, dz$:
$$\int_0^1 \int_0^{1-y} \int_0^{1-y} f(x,y,z)\, dx\, dz\, dy.$$

4. Find a change of variables for the double integral $\iint_D (6y+3)\, dA$, where D is the region in the xy-plane bounded by the curves $y = -x, y = 1-x, x = y^2$, and $x = y^2 - 1$, so that the region of integration is a rectangle in the uv-plane. Set up the new integral, but do not evaluate it. *Since this is not a timed exam for you, you are welcome to try to evaluate it.*

5. Let E be the region above the xy-plane that lies outside the sphere $x^2 + y^2 + z^2 = 2$ and inside the sphere $x^2 + y^2 + z^2 = 2z$. Find the value of $\displaystyle\iiint_E \frac{1}{x^2 + y^2 + z^2}\, dV.$

6. Consider the equation $\displaystyle\int_0^2 \int_0^x ke^{kx^{10}}\, dy\, dx = \int_0^1 \int_y^{2-y} e^{kx^{10}}\, dx\, dy + \int_1^2 \int_{2-x}^x e^{kx^{10}} dy\, dx.$
Find a value of k for which this equation holds, and a value of k for which it does not.

Math 290-3

1. Let $\mathcal{P}$ be the surface $x^2 + 4y^2 + z = 16$, called a *paraboloid*. Set up a double integral whose value is the *area* of that part of $\mathcal{P}$ that lies above the plane $z = 0$. You do *not* need to evaluate your integral.

2. Interpret the three-dimensional arclength differential

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

in terms of local multipliers.

3. The semicircular sector defined by $x^2 + y^2 \leq 36$ and $0 \leq y$ can be rolled up to form a cone, which can be placed so that its vertex is at the origin and its axis of symmetry is the positive z-axis. In the process, the segment that joins $(0, 6)$ to $(6, 0)$ is transformed into a curve $\mathcal{C}$ on the cone.

(a) Describe the cone using spherical coordinates.

(b) Is $\mathcal{C}$ a planar curve? Give your reasons.

4. Given functions $P(x, y)$ and $Q(x, y)$, and a path $\mathcal{C} : (x, y) = (x(t), y(t))$ parametrized for $a \leq t \leq b$, the integral formula $\int_a^b \left(P(x, y)\frac{dx}{dt} + Q(x, y)\frac{dy}{dt} \right) dt$, usually abbreviated to just $\int_{\mathcal{C}} P\, dx + Q\, dy$, is called a *line integral*. How is the value of a line integral affected if the path $\mathcal{C}$ is replaced by tracing its curve in the opposite direction? Explain.

Math 290-3

1. For the vector field **F** shown in the diagram, determine whether $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$ is positive, negative, or 0 for each directed curve C − in other words, determine whether the *work* done by the vector field on each curve is positive, negative or 0.

2. An object that moves in response to a central force must have a *planar* orbit. In other words, if it is given that $\mathbf{p}''(t)$ and $\mathbf{p}(t)$ are always parallel, then the curve traced by $\mathbf{p}(t)$ must lie in a plane. Prove this statement.

3. Given continuous functions P and Q, and a continuously differentiable path $\mathcal{C}$, the line integral $\int_C P \, dx + Q \, dy$ is defined. Notice that the integrand is a dot product:

$$P \, dx + Q \, dy = \left(P \frac{dx}{dt} + Q \frac{dy}{dt} \right) dt = [P, Q] \bullet \left[\frac{dx}{dt}, \frac{dy}{dt} \right] dt$$

This construction has many applications, especially in physics. When $\mathcal{C}$ is a closed path, this integral is sometimes called the *circulation* of the vector field $[P, Q]$ around $\mathcal{C}$. A positive value indicates a tendency for the vector field to point in the same direction as the motion along the path. Explain.

4. Consider the gradient field $\mathbf{F} = \nabla f$ that you sketched for page 4#1, for the function

$$f(x, y) = \frac{1}{1000} \left(4500 - 105x^2 - 105y^2 + 3y^2x + 3x^2y + 0.8x^4 + 0.8y^4 \right).$$

The vector field **F** has four *sinks* and one *source*. Explain the terminology. Find a point that is neither a source nor a sink, and explain your reasoning. *Hint*: If you didn't sketch enough vectors in your vector field the first time, add more now! It should be covered in little arrows.

5. Suppose that $\mathbf{r}(t)$ is a curve in $\mathbf{R}^3$ with the property that its direction of motion is always perpendicular to its position vector, i.e. $\mathbf{r}(t) \bullet \mathbf{r}'(t) = 0$. Show that $\mathbf{r}(t)$ lies completely on a sphere. *Hint*: differentiate both sides of the equation $\|\mathbf{r}(t)\|^2 = \mathbf{r}(t) \bullet \mathbf{r}(t)$. You will have to decide how to differentiate a dot product.
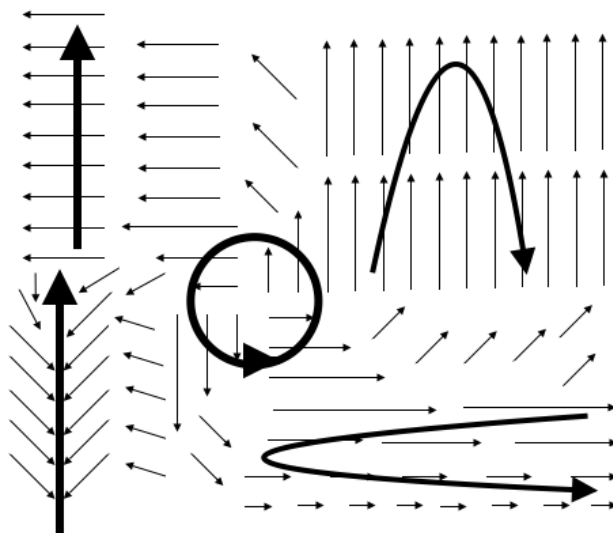
Math 290-3

1. The familiar gradient vector $\nabla \mathbf{f} = \left[\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z} \right]$ can be written as ∇f, where ∇ is the "differential operator" defined (for $\mathbf{R}^3$) by

$$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

We now define the *divergence* and *curl* of a vector field $\mathbf{F} = [P, Q, R]$ by

$$\mathrm{div}(\mathbf{F}) = \nabla \bullet \mathbf{F} \quad \text{and} \quad \mathrm{curl}(\mathbf{F}) = \nabla \times \mathbf{F}.$$

(a) Write out the formula for $\mathrm{div}(\mathbf{F})$ in terms of P, Q and R. Then compute the divergence of the vector fields $\mathbf{F}_1 = [x, y, 0]$ and $\mathbf{F}_2 = [-y, x, 0]$ and $\mathbf{F}_3 = [1, 2, 3]$. What does this number measure?

(b) Write out the formula for $\mathrm{curl}(\mathbf{F})$ in terms of P, Q and R. Then compute the curl of the vector fields $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{F}_4 = [-z, 0, x]$. What does the *direction* of the curl measure?

2. Suppose that you are building a fence from $(5, 0)$ to $(0, 5)$, following the circle of radius 5 centered at the origin. The height of the fence at the point (x, y) is $10 - x - y$.

(a) Set up a line integral to find the length of the fence.

(b) Set up, and evaluate, a line integral to find the total area of the fence.

3. (Continuation) Fill in all the details for the following equation, and explain why it holds:

$$\int_C f(x, y) \, ds = \int_a^b f(\mathbf{x}(t)) \, \|\mathbf{x}'(t)\| dt.$$

"Fill in all the details" means you need to explain all the parts, like how C and (x, y) turned into a and b and $\mathbf{x}(t)$. This is called the *scalar line integral of f along C*.

4. We have seen that the *circulation* of a vector field $\mathbf{F} = [P, Q, R]$ over a directed curve C is $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$, where $\mathbf{T}$ is the unit tangent vector to the curve. Explain each equality:

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_C P \, dx + Q \, dy + R \, dz.$$

Note that the $\mathbf{s}$ is bold in the second integral.

5. (Continuation) Let C be the intersection of the cylinder $x^2 + z^2 = 1$ with the plane $y + z = 1$, with counter-clockwise orientation when viewed from the negative y-axis.

(a) Find parametric equations to describe C.

(b) Compute $\displaystyle\int_C z \, dx + (1 - y) \, dy + (z - x) \, dz$.

Math 290-3

1. Consider the vector field $\mathbf{F} = [xy \sin(z) + y, y + xe^z, xyz]$.

(a) Compute $\text{div}(\mathbf{F})$ and find the value of the divergence at $(1, 1, 0)$, $(-1, 1, 0)$ and $(1, -2, 0)$. What does the vector field look like near each of these points?

(b) Compute $\text{curl}(\mathbf{F})$ and find the direction of the curl at $(0, 0, 0)$ and at $(-1, 1, 0)$. What does the vector field look like near these points?

2. As we have seen, we can measure the tendency of a vector field $\mathbf{F} = [P, Q]$ to point in the same direction as an oriented curve C by integrating $[P, Q] \bullet \left[\dfrac{dx}{dt}, \dfrac{dy}{dt}\right] dt$ over the curve C. When $\left[\dfrac{dx}{dt}, \dfrac{dy}{dt}\right]$ is a *unit* tangent vector $\mathbf{T}$, we can write the integral as $\int_C \mathbf{F} \cdot \mathbf{T} \; ds$. Explain. Then integrate the vector field $\mathbf{F} = [x^2 + y, -(x + 1)y]$ over the curve consisting of the line segment from $(0, 0)$ to $(2, 0)$ followed by the line segment from $(2, 0)$ to $(2, 2)$.

3. (Continuation) Consider a vector field $\mathbf{F}$, and a curve C that consists of the part of the curve $\mathbf{x}(t) = (x(t), y(t))$ from $t = a$ to $t = b$. Explain why

$$\int_C \mathbf{F} \cdot \mathbf{T} \; ds = \int_a^b \mathbf{F}(\mathbf{x}(t)) \bullet \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} \; \|\mathbf{x}'(t)\| \; dt = \int_a^b \mathbf{F}(\mathbf{x}(t)) \bullet \mathbf{x}'(t) \; dt.$$

Then use this formula to compute the line integral of $\mathbf{F} = \left[\dfrac{-y \sin x}{x^2}, \dfrac{\cos x}{2x}\right]$ over the piece of the parabola $y = x^2$ from $(\pi/2, \pi^2/4)$ to $(5\pi/4, 25\pi^2/16)$. This formula gives us a way to calculate vector line integrals without having to reparameterize our curve to unit speed, or in other words when the unit tangent vector $\mathbf{T}$ is difficult to compute.

4. Let C be the curve consisting of the line segment from $(0, -1)$ to $(0, 1)$, followed by the right half of the unit circle from $(0, 1)$ to $(0, -1)$. Compute the line integral of $\mathbf{F} = [-y, x]$ over C. Soon we will have an easier way to compute this.

5. A *simple* curve is one that does not intersect itself A *closed* curve is one that ends where it starts, that "closes up." An *oriented* curve has a direction of travel. For each curve below, say whether it is simple and whether it is closed, and draw an arrow on it to give it an orientation.
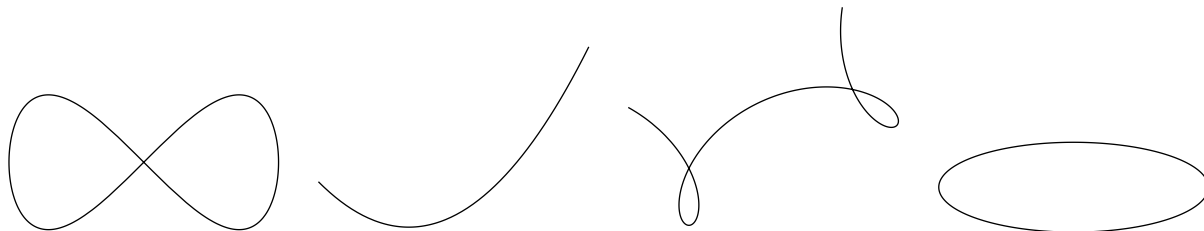
Math 290-3

1. Let f be a function and let $\mathbf{F}$ and $\mathbf{G}$ be vector fields. Which of the following expressions make mathematical sense? If you can compute any of them, do so.

(a) $\mathrm{curl}(\mathrm{div}(\mathbf{F}))$

(b) $\mathrm{curl}(\nabla f)$

(c) $\mathrm{div}(\mathbf{F} \bullet \mathbf{G})$

(d) $\mathrm{curl}(\mathrm{div}(f))$

(e) $\mathrm{div}(\mathrm{curl}(\mathbf{G}))$

(f) $\mathrm{div}(\nabla f)$

2. *Green's Theorem* says the following: If D is a closed, bounded region in $\mathbf{R}^2$ whose boundary C consists of finitely many simple, closed, piecewise-differentiable curves, oriented so that D is on the left when one traverses C, and if $\mathbf{F} = [P, Q]$ is differentiable everywhere in D, then

$$\oint_C P\ dx + Q\ dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dx\ dy.$$

Note: The symbol $\oint$ has a circle to indicate that the line integral is over a *closed* curve.

Verify the result of Green's Theorem by explicitly calculating each side of the above equation when $\mathbf{F} = [-y, x]$ and D is the half-disk $x^2 + y^2 \le 1, x \ge 0$. *Hint*: You calculated one side on the previous page. Also, explain why $\mathbf{F}$ and D satisfy the requirements of Green's Theorem.

3. When $P(x, y) = -\frac{1}{2}y$ and $Q(x, y) = \frac{1}{2}x$, Green's Theorem is interesting. Explain.

4. A curved wire connects the points $(1, 0, 0)$ and $(1, 0, 4\pi)$ according to the parametric equation $\mathbf{x}(t) = (\cos t, \sin t, t)$ for $0 \le t \le 4\pi$. The electric charge of the wire at the point (x, y, z) is given by xy. Find the total charge of the wire. Can you explain, geometrically, why this answer (0) makes sense?

5. (From Math 290-2, Salon 7, slightly edited) Suppose that you need to know an equation of the tangent plane to a surface S at the point $P = (7, 1, 3)$, and you know that the curves

$$\mathbf{r}_1(t) = (2t + 3, t - 1, t^2 - 1)$$
$$\mathbf{r}_2(t) = (3t + 4, 2 - t^2, 7 - 4t)$$

both lie on S. Find the equation for the tangent plane to S at P.

6. (Continuation) Show that $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ both lie on the surface

$$\mathbf{S}(t, u) = (2t + 3u, t - u^2, 3 - 4u + t^2).$$

This is called a *parameterized surface*. Explain the terminology. Using $\mathbf{S}(t, u)$, express the curves $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$.

Notation: We call these the *t-curve* and the *u-curve* through a given point.

Math 290-3

1. Recall the differential operator $\nabla = \left[\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right]$ and suppose that the function f and the vector field $\mathbf{F}$ are differentiable everywhere. Show that:

(a) $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f.$

(b) $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}.$

2. Sketch the curve given by $\mathbf{r}(t) = (t^3 - t, 1 - t^2)$. Then find the area inside the closed loop of the curve.

3. Find the circulation of the vector field $\left[0, \dfrac{1}{1 + x^2}\right]$ around the piecewise-linear path that goes from $(2, 0)$ to $(2, 1)$ to $(1, 1)$ to $(1, 0)$ to $(2, 0)$.

4. To represent a vector field $\mathbf{F}$ on $\mathbf{R}^2$, we can draw little arrows at many representative points, to show the direction and magnitude of the vector field at each point. Another way is to sketch the *flow lines*, which show trajectories of particles under the effect of the vector field, as though you drop a piece of wood into the water and see where it goes. Then the vector field arrows are tangent vectors to the flow lines. Draw the flow lines for the vector fields $\mathbf{F} = [x, y]$ and $\mathbf{G} = [-y, x]$ and $\mathbf{H} = [1, 2]$. You can also add flow lines to the vector field you sketched for page 4#1.

Math 290-3

1. Suppose that $[P(x, y), Q(x, y)]$ is a *gradient field*, i.e. $[P, Q] = \nabla f$ for some function $f(x, y)$, and that $\mathcal{C}$ is a piecewise differentiable path in the xy-plane. It so happens that the value of $\int_{\mathcal{C}} P\, dx + Q\, dy$ depends *only* on the endpoints of the curve traced by $\mathcal{C}$.

(a) Verify this for the field $\mathbf{F} = [xy^2, x^2y]$ by selecting at least two different piecewise differentiable paths from $(0, -1)$ to $(1, 1)$ and evaluating both integrals.

(b) A vector field that is the gradient field for a function $f(x, y)$ is called a *conservative vector field*, and f is called its *potential function*. Find a potential function f for $\mathbf{F}$, and evaluate $f(1, 1) - f(0, -1)$.

Let's call this result the *Fundamental Theorem of Line Integrals*: If $\mathbf{F}$ is a conservative vector field, and its potential function f is defined on a region containing the curve C, then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = f(\text{end point of } C) - f(\text{starting point of } C).$$

(c) Use the Chain Rule and the Fundamental Theorem of Calculus to prove this fact.

2. Some people like to remember, "A vector field is conservative if and only if its curl is $\mathbf{0}$." Justify this. (By the way, to apply it to a vector field $[P, Q]$ in $\mathbf{R}^2$, think of it as $[P, Q, 0]$.)

3. Let $\mathbf{F} = [P, Q] = \left[\dfrac{-y}{x^2 + y^2}, \dfrac{x}{x^2 + y^2}\right]$, and sketch $\mathbf{F}$ (*Hint*: separately consider the direction and magnitude at each point.) Let C be the unit circle, oriented counter-clockwise. Estimate: is $\oint_C \mathbf{F} \bullet d\mathbf{s}$ positive, negative or 0? Now compute the line integral to verify your prediction. Finally, show that $Q_x - P_y = 0$ at each point in $\mathbf{R}^2 - \{(0, 0)\}$, and therefore $\iint_{x^2+y^2 \leq 1}(Q_x - P_y)\, dA = 0$. Does this contradict Green's Theorem?

4. Spherical coordinates give us the parameterization $\mathbf{S}(\phi, \theta) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)$ for the sphere of radius R centered at the origin. This transformation maps the (ϕ, θ) plane $\mathbf{R}^2$ to the sphere in $\mathbf{R}^3$. Sketch the curves $\mathbf{S}_\phi$ and $\mathbf{S}_\theta$ on the sphere for a given choice of ϕ and θ. Find the normal vector to $\mathbf{S}$ at $\mathbf{S}(\phi_0, \theta_0)$ and check that your answer is reasonable.

5. Consider the vector field $\mathbf{F}$, defined throughout $\mathbf{R}^2$ by $\mathbf{F} = [2xy - y, x^2 + x - y^2]$. The diagram at right shows the flow lines of this field, rather than the vectors themselves.

(a) The diagram shows that the field $\mathbf{F}$ has four critical points, where $\mathbf{F} = \mathbf{0}$. Find and label these points.

(b) Show that this field has zero divergence (which means that it has no *sources* or *sinks*).

**(c)** Put arrows on the trajectories, to show the directions of the field vectors.

(d) Estimate the circulation of $\mathbf{F}$ around the positively directed circle $x^2 + y^2 = 2$. Is it positive or negative?

Math 290-3
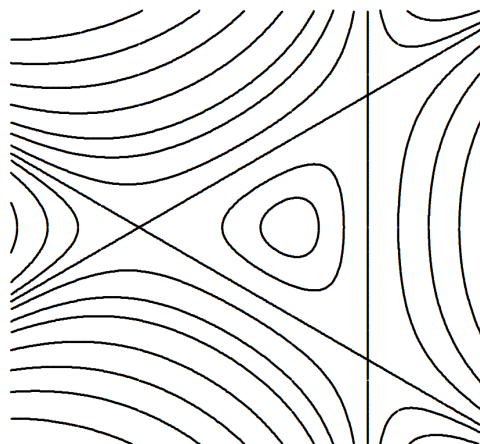
1. Let C be the part of the unit circle from $(1,0)$ to $(-1,0)$, oriented counter-clockwise, and let $\mathbf{F} = [y^2 x + x^2, x^2 y + x - e^{y \sin y}]$. We would like to compute the line integral of $\mathbf{F}$ over C. We cannot do this directly, because of the $e^{y \sin y}$ term, and the curve is not closed, so it seems we cannot apply Green's Theorem. Here is a clever trick: "close off" the region so that we can apply Green's Theorem. Let S be the line segment from $(-1,0)$ to $(1,0)$, and let D be the region now cleverly enclosed by the curves C and S.

(a) Explain why $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{s} + \int_S \mathbf{F} \cdot d\mathbf{s} = \iint_D \text{curl}(\mathbf{F}) \, dA$.

(b) Compute $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{s}$.
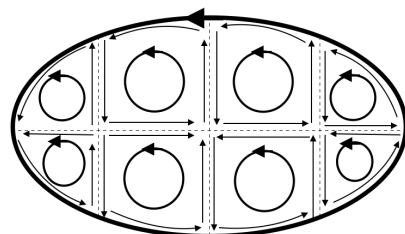
2. (Continuation) It is an abuse of notation to write the Green's Theorem equation as $\displaystyle\oint_C P \, dx + Q \, dy = \iint_D \text{curl}(\mathbf{F}) \, dx \, dy$ as is implied above, because $\text{curl}(\mathbf{F})$ is a *vector*, not a scalar. But if we take this expression to mean that we are adding up the z-components of the curl vector $\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$, we can understand why Green's Theorem works: If we break our region into tiny boxes (shown in the diagram above with not-so-tiny boxes), adding up the curl at each point inside gives us the circulation around the boundary, because the contributions from the interior edges cancel out. Explain this.

3. For any field $\mathbf{F}$ or function f defined on $\mathbf{R}^3$, show that $\nabla \bullet (\nabla \times \mathbf{F}) = 0$ and $\nabla \times (\nabla f) = \mathbf{0}$.

4. Let C be the line segment from $(0,0,0)$ to $(1,1,0)$, followed by the line segment from $(1,1,0)$ to $(2,3,1)$. Show that the following line integral is path-independent. Then evaluate it in two ways: by computing the line integral along C, and by applying the Fundamental Theorem of Line Integrals.

$$\int_C (y + 2z) \, dx + (x - 3z) \, dy + (2x - 3y) \, dz$$

5. Let $\mathbf{F}$ be a vector field, and suppose that the curve C is a flow line of $\mathbf{F}$. In such a situation, is it *always* true that $\displaystyle\int_C \mathbf{F} \bullet d\mathbf{s} > 0$? If so, explain why; if not, give a counterexample.

6. Find all functions $P(x,y)$ such that the vector field $\mathbf{F} = [P(x,y), x \sin y - y \cos x]$ is conservative.

7. Sketch the surface defined by $\mathbf{S}(r, \theta) = (r \cos \theta, r \sin \theta, \theta)$ for $0 \leq \theta \leq 4\pi$ and $0 \leq r \leq 1$. *Hint*: Recall the helix described by $\mathbf{x}(t) = (\cos t, \sin t, t)$.

Math 290-3

1. Suppose that the flow of air in a very turbulent area is given by the vector field $[3x^2z + y^2 + x, 2xy - y, x^3 + 4z]$. You toss a plastic bag into this area and watch the wind push it around. Is it rotating?

2. Show that if C is the boundary of any rectangular region in $\mathbf{R}^2$, then $\oint_C (xy^2 + 5y)dx + x^2y \, dy$ depends only on the area of the rectangle, and not on its location in $\mathbf{R}^2$.

3. Let $\mathbf{F} = [e^y + y^2 + 1, xe^y + 2xy + \cos y]$.

(a) Show that $\mathbf{F}$ is conservative.

(b) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$, where C is the curve from $(-2, 2)$ to $(3, 4)$ shown in the diagram.

4. Let $\mathbf{F} = [-y + ye^y, x + xe^y + xye^y + z, y + 2]$. Compute the line integral of $\mathbf{F}$ over the left half of the unit circle in the xy-plane, oriented clockwise as viewed from the positive z-axis. *Hint*: $\mathbf{F}$ is *almost* conservative.
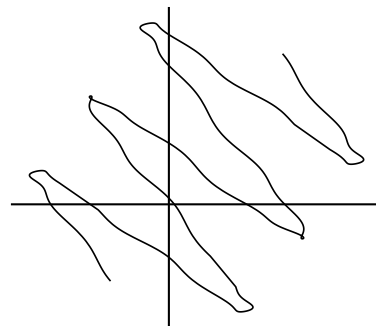
5. Sketch the surface described by $\mathbf{C}(r, \theta) = (r\cos\theta, r\sin\theta, r)$. Also sketch the r-curve and the θ-curve through the point $P = (0, 1, 1)$, and the vectors $\mathbf{C}_r(P)$ and $\mathbf{C}_\theta(P)$. Find the normal vector to the surface at P. Check that your answer makes sense geometrically.

6. Calculate the *scalar line integral* of the function $f(x, y) = 3y$ over the curve C consisting of the portion of the graph of $y = 2\sqrt{x}$ between $(1, 2)$ and $(9, 6)$.

Math 290-3

1. Suppose that we want to integrate the vector field $\mathbf{F} = \left[\dfrac{-y}{x^2 + y^2}, \dfrac{x}{x^2 + y^2}\right]$ over the closed curve C, oriented counter-clockwise, shown as a solid curve in the figure. We can't compute it directly, because we don't have equations for C. We can't apply Green's Theorem, because $\mathbf{F} = [P, Q]$ isn't differentiable at the origin (it's not even defined there), which is in the region enclosed by C. Amazingly, we can still compute the integral!
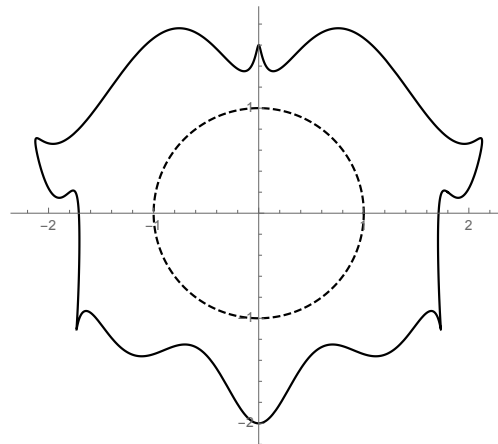
(a) Let C_1 be the unit circle (shown in the figure), oriented *clockwise*, and let C_2 be the line segment connecting the two curves along the x-axis, oriented in the negative x-direction (sketch this in). Let D be the solid region between C_1 and C. Justify the equation

$$\int_C \mathbf{F} \cdot d\mathbf{s} + \int_{C_2} \mathbf{F} \cdot d\mathbf{s} + \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{s} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy.$$

(b) Explain why this simplifies to $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy - \int_{C_1} \mathbf{F} \cdot d\mathbf{s}$. Compute the right side explicitly. The seemingly impossible is possible!

2. (Continuation) Find $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{s}$ for the curve C from the previous problem, where $\mathbf{F} = [e^{-y} - y\sin(xy), -(xe^{-y} + x\sin(xy))]$.

3. A spiraling wire connects the points $(1, \pi)$ and $(2, 4\pi)$ along the parabola $y = \pi x^2$. The density of the wire at the point (x, y) is given by $8x$. Find the total mass of the wire.

4. Let $\mathbf{F} = [xy - \cos x, x^2 - ye^y]$ and let C be the path that starts at $(1, 0)$, follows the curve $x = 1 - y^2$ up to $(0, 1)$, and then follows the curve $x = y^2 - 1$ down to $(-1, 0)$. Compute the line integral of $\mathbf{F}$ over C.

5. If $\mathbf{X}(s, t)$ is a parametric surface, then $\mathbf{X}_s \times \mathbf{X}_t$ is a normal vector to the surface, and $\|\mathbf{X}_s \times \mathbf{X}_t\|$ is the area of the parallelogram spanned by $\mathbf{X}_s$ and $\mathbf{X}_t$. Find at least two parameterized surfaces whose normal vector at the point (s, t) is $[s, t, s + t]$.

6. The integral $\displaystyle\iint_D \|\mathbf{X}_s \times \mathbf{X}_t\|\; ds\, dt$ is a template for what type of problem?

Math 290-3

Practice exam for Midterm 2 - 2016 - TWO pages long

1. True or False:

   (a) Let $\mathbf{F} = [-y, x]$ and let C be the line $x = 1$, oriented in the positive y-direction. Then $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$.

   (b) The arclength of the curve with parametric equation $\mathbf{r}(t) = (t^4, \cos^3 t)$, $0 \le t \le 1$ is less than 5.

   (c) If C is a piecewise $\mathcal{C}^1$ curve then $\int_{-C} f(x, y)\, ds = -\int_C f(x, y)\, ds$.

   (d) Suppose that $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is defined on $\mathbf{R}^2 \backslash \{\mathbf{a}, \mathbf{b}\}$ and satisfies $N_x = M_y$ at each point in its domain. Let C_1, C_2, and C_3 be as pictured below left:
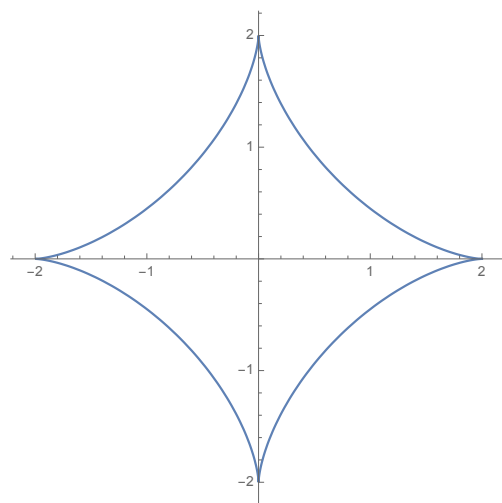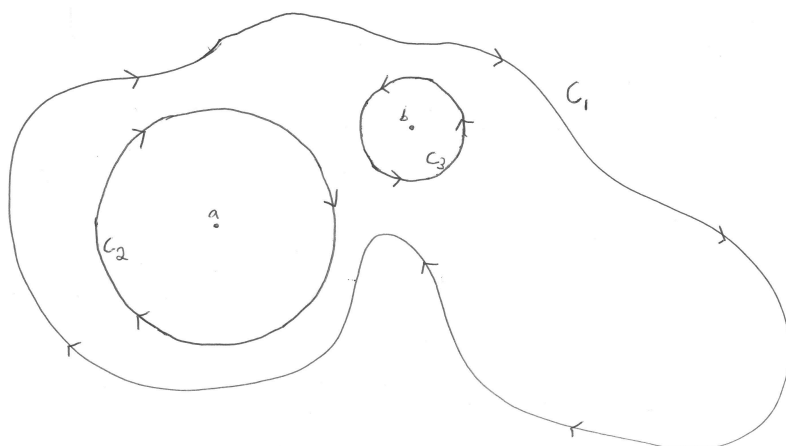
   If $\oint_{C_1} \mathbf{F} \bullet d\mathbf{s} = 5$ and $\oint_{C_2} \mathbf{F} \bullet d\mathbf{s} = 3$, then $\oint_{C_3} \mathbf{F} \bullet d\mathbf{s} = 2$.

2. For each vector field $\mathbf{F}$ below, determine whether or not $\mathbf{F}$ is conservative on $\mathbb{R}^3$. If it is, then find a potential for $\mathbf{F}$. If it is not, then explain how you know that $\mathbf{F}$ has no potential function.

   (a) $\mathbf{F} = xy\mathbf{i} + xy\mathbf{j}$

   (b) $\mathbf{F} = (2xyz + \sin(x))\mathbf{i} + x^2 z\mathbf{j} + x^2 y\mathbf{k}$

3. Consider the curve $\mathbf{r}(t) = (3\sin t, 4t, 3\cos t)$. If you start at point $(0, 0, 3)$ and move 5 units in the positive direction, where are you now?

4. For $a > 0$, let D by the region enclosed by the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$. The hypocycloid with $a = 2$ is shown in the figure above right.

   (a) Find a vector field $\mathbf{F}$ so that $\oint_{bD} \mathbf{F} \bullet d\mathbf{s} = \text{Area}(D)$.

   (b) Compute $\text{Area}(D)$.

Math 290-3

5. Suppose that f is harmonic on $\mathbb{R}^2$ (i.e. $f_{xx} + f_{yy} = 0$ at each point). Prove that, for every simple closed curve C, $\oint_C f_y dx - f_x dy = 0$.

6. Prove that curl $(f\mathbf{F}) = f\,\text{curl}\,\mathbf{F} + (\nabla f) \times \mathbf{F}$ where $\mathbf{F}$ is a $\mathcal{C}^1$ vector field in $\mathbf{R}^3$ and f is a $\mathcal{C}^1$ scalar valued function.

7. Sketch the surfaces described by the following equations:

   (a) $\mathbf{B}(s,t) = \begin{bmatrix} s^2+t^2 \\ t \\ s \end{bmatrix}, 0 \le s \le 5, \;\; 0 \le t \le 5$

   (b) $\mathbf{A}(s,t) = \begin{bmatrix} s \\ t \\ \sqrt{1-s^2-t^2} \end{bmatrix}, 0 \le s \le 1, \;\; -\sqrt{1-s^2} \le t \le \sqrt{1-s^2}$

   (c) $\mathbf{E}(s,t) = \begin{bmatrix} 1 \sin s \cos t \\ 2 \sin s \sin t \\ 3 \cos s \end{bmatrix}, 0 \le s \le \pi, \;\; 0 \le t \le 2\pi$

   (d) $\mathbf{P}(s,t) = \begin{bmatrix} 3s \\ t^2+10 \\ 1-2s+2t^2 \end{bmatrix}, 0 \le s \le 4, \;\; 0 \le t \le 2$

   (e) $\mathbf{T}(s,t) = \begin{bmatrix} (5+2\cos s)\cos t \\ (5+2\cos s)\sin t \\ 2 \sin s \end{bmatrix}, 0 \le s,t \le 2\pi$

8. Morty Schapiro (Northwestern's president) is driving in a snowstorm around the southern part of Lake Michigan, which we can model as the bottom half of a circle of radius 50 centered at the origin, from $(-50,0)$ to $(50,0)$. The amount of snowfall accumulating on top of his car depends on where he is, and is described by the function $S(x,y) = 1 - x + y$ millimeters of snow per mile driven. How much snow will have accumulated on his car when he reaches the end of the journey?

9. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{F} = 2xy^2\mathbf{i} + 2x^2y\mathbf{j}$ and C is the portion of the graph $y = \sqrt{x}e^{x^2-4}$ from $(0,0)$ to $(2,\sqrt{2})$.

10. Compute $\displaystyle\int_C \mathbf{F} \bullet d\mathbf{s}$, where $\mathbf{F} = (\cos(xy^2) - xy^2 \sin(xy^2))\mathbf{i} - 2x^2y\sin(xy^2)\mathbf{j}$ and where C is the curve formed by the line segment connecting $(0,0)$ to $(0,2)$, followed by the right-half of the circle $x^2 + y^2 = 4$ connecting $(0,2)$ to $(0,-2)$, followed by the line segment connecting $(0,-2)$ to $(1,-1)$.

11. Let C be the curve consisting of the portion of the parabola $x = 1 - y^2$ connecting $(-3,-2)$ to $(0,-1)$, followed by the line segment connecting $(0,-1)$ to $(0,1)$, followed by the portion of the parabola $x = 1 - y^2$ connecting $(0,1)$ to $(-3,2)$. Compute
$$\int_C (e^{x^2} - 4)dx + (y^3x + 3y\cos(xy))dy.$$

12. Let C be the curve consisting of the line segment connecting $(3,0)$ to $(0,1)$, followed by the portion of the parabola $y = (x-1)^2$ connecting $(0,1)$ to $(1,0)$. Compute $\displaystyle\int_C \mathbf{F} \bullet d\mathbf{s}$, where $\mathbf{F} = 2y\mathbf{i} + x^2\mathbf{j}$.

Math 290-3

Midterm 2 from 2015 - exam in evening

1. Say whether the following are true or false, and justify your answers:

(a) Let $\mathbf{F}(x, y) = [0, x]$ and let C be the ellipse $x^2 + y^2/4 = 1$, oriented counter-clockwise. Then $\int_C \mathbf{F} \cdot d\mathbf{s} > 0$.

(b) Let $\mathbf{F}(x, y) = [2xy + \sin y, x^2 + x \cos y]$, and let C be the curve with parametric equations $\mathbf{x}(t) = (\sin(\pi t), t^2)$ for $-1 \le t \le 1$. Then $\int_C \mathbf{F} \cdot d\mathbf{s} > 0$.

(c) Let D be the region in $\mathbf{R}^2$ obtained by removing the origin. If $\mathbf{F}$ is a C^1 vector field on D such that curl$\mathbf{F} = \mathbf{0}$ at each point of D, then $\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$ for every closed curve C in D.

(d) Let $\mathbf{F}(x, y) = [P(x, y), Q(x, y)]$, let D be the rectangle $[0, 1] \times [0, 2]$, and let C be the curve consisting of the line segment from $(0, 0)$ to $(1, 0)$, followed by the line segment from $(1, 0)$ to $(1, 2)$, followed by the line segment from $(1, 2)$ to $(0, 2)$. Assume that $\mathbf{F}$ has continuous partial derivatives in D. Then $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \int_0^2 \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy \; dx + \int_0^2 Q(0, t) \; dt$.

(e) Let C be the portion of the curve defined by $1 - ye^{-x^2}$ that starts at $(-1, e)$ and ends at $(1, e)$. There is a real number k such that $\displaystyle\int_C -2xye^{-x^2} \; dx + \left(e^{-x^2} + k^2\right) dy > 0$.

2. For each of the following, find an example of a nonzero vector field $\mathbf{F}(x, y, z)$ on $\mathbf{R}^3$ for which the statement is true, and one for which it is false.

(a) div$(\mathbf{F}) = 0$ everywhere, and $(\text{curl}\mathbf{F})(p) \ne \mathbf{0}$ for every point p in $\mathbf{R}^3$.

(b) curl $\mathbf{F} = \nabla(\text{div}\mathbf{F})$.

3. Determine the value of the scalar line integral $\displaystyle\int_C (2xy - yz) \; ds$, where C is the intersection of the cylinder $y^2 + z^2 = 1$ and the plane $z = x$.

4. Compute the vector line integral $\displaystyle\int_C \left(y + \sin y + e^{x^4}\right) dx + (y + (x - 1) \cos y) dy$, where C is the left half of the circle $(x - 1)^2 + y^2 = 1$, oriented clockwise.

5. Let $\mathbf{F}(x, y) = [ye^y + y^2 - y\pi \sin(xy\pi), xe^y + xye^y + 2yx - x\pi \sin(xy\pi)]$.

(a) Show that $\mathbf{F}$ is conservative on $\mathbf{R}^2$.

(b) Find the value of $\displaystyle\int_C (y + ye^y + y^2 - y\pi \sin(xy\pi))dx + (-x_x e^y + 2yx - x\pi \sin(xy\pi))dy$, where C is the piece of the parabola $x = y^2 - 1$ that starts at $(0, -1)$ and ends at $(0, 1)$.

6. Compute $\displaystyle\int_C (z + y \sin^2(z + 1))dx - x \cos^2(z + 1)dy + z^{100} e^{x \cos y} dz$, where C is the boundary of the square $[0, 1] \times [0, 1]$ in the xy-plane, oriented counter-clockwise when viewed from the positive z-direction.

Math 290-3

In class - midterm the previous evening

1. In this problem, we will find the surface area of the part of the cone $x^2 + y^2 = z^2$ that lies between the planes $z = 0$ and $z = R$. Sketch this cone.

(a) Parameterize the surface $\mathbf{X}$ as a function of r and θ and integrate $\iint_D \|\mathbf{X}_r \times \mathbf{X}_\theta\| \, dr \, d\theta$ over an appropriate region D of the $r\theta$-plane.

(b) Check your answer with basic geometry: slice open the cone and lay it flat.

2. (Continuation) Suppose that the temperature at a point (x, y, z) is given by z. Find the average temperature of the cone. By the way, the integral you use to compute the "total temperature" is called a *scalar surface integral*.

3. (Continuation) The vector $\mathbf{X}_r \times \mathbf{X}_\theta$ gives a normal vector to the surface $\mathbf{X}(r, \theta)$, as does the vector $\mathbf{X}_\theta \times \mathbf{X}_r$. One points into the cone, while the other points out of the cone. Determine which is which, and add them to your sketch. For a surface in $\mathbf{R}^3$, we choose an *orientation*, which you can think of as choosing "which way is up."

Consider the vector fields $\mathbf{F} = [x, y, z]$, $\mathbf{G} = [x, y, 0]$, and $\mathbf{H} = [-y, x, 0]$. For each of these, estimate (is it positive, negative or 0?) the *vector surface integral* of the vector field over the cone, with outward-facing normal vector. This is denoted by $\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$ and is called *flux*. Is the integral used to compute surface area the flux of a vector field?

4. (Continuation) When we did line integrals, we oriented our curves (by choosing one of the two tangent directions), and now that we are doing surface integrals, we are orienting our surfaces (by choosing one of the two normal vectors). We want to make sure that these orientations are *compatible* when an oriented curve is the boundary of an oriented surface. Here is how to orient them compatibly: Imagine that you are walking along the boundary of the surface, in such a way that your head points in the direction of the chosen normal vector. Orient the curve so that when you walk, your left arm is over the surface.

(a) Give an orientation to the boundary of the cone, compatible with the outward-facing normal vector, and add it to your sketch. Check that the opposite orientation would be compatible with an inward-facing normal vector.

(b) Explain why this agrees with the counter-clockwise orientation in Green's Theorem, when the "surface" is a region in the xy-plane whose boundary is a simple closed curve.

FOR FUN. The cone has a notion of "inward" and "outward;" a plane does not, but it has a notion of "upward" and "downward," so in each case you can make a consistent choice of a normal vector, to give the surface an orientation. Another way to say this is that you could paint one side of the surface red, and the other side blue. You may be surprised to learn that not every surface has this property. Cut two narrow strips of paper. With one of the strips, tape the ends together in the usual way, like an admission bracelet to a concert. With your pencil, shade one side and leave the other side white. With the second strip, tape the ends together, but *twist* one end a half turn before taping it to the other end. This is called a *Möbius strip*. With your pencil, shade one side and leave the other side white. Hmm . . .

Math 290-3

1. The *flux* of the vector field $\mathbf{F}$ through the surface S is given by $\iint_S \mathbf{F} \cdot d\mathbf{S}$. You can think of this as the surface $S = \mathbf{X}(s,t)$ being a net in a stream whose current is given by the vector field $\mathbf{F}$, and the integral measures how much water flows through the net. The word *flux* is from physics, measuring the amount of electric field across a surface. The $d\mathbf{S}$ is a vector quantity, and the integral adds up dot products to measure how much the vector field $\mathbf{F}$ points in the same direction as the normal vector to the tiny piece of oriented surface $d\mathbf{S}$.

Let S be the cone $z = \sqrt{x^2 + y^2}$ below $z = 1$, oriented outward (downward). Let $\mathbf{E} = [x, 0, -z]$ be an electric force field. Compute the electric flux of $\mathbf{E}$ across S, by computing

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \iint_D \mathbf{E}(\mathbf{X}(r, \theta)) \cdot (\mathbf{X}_\theta \times \mathbf{X}_r) \ dr \ d\theta$$

over an appropriate region D of the $r\theta$-plane.

2. (Continuation) We can also write

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \ dS,$$

where $\mathbf{n}$ is a *unit* normal vector and dS is a tiny bit of surface area. Explain. This form is convenient when the normal vector $\mathbf{n}$ is easy to compute. Let C be the "cap" of the cone in the previous problem: the unit disk at a height of $z = 1$, with upward-facing normal vector (chosen just so that the whole closed surface, cone plus cap, is oriented outward). Use the integral on the right-hand side above to compute the electric flux of $\mathbf{E}$ over C.

3. Given a differentiable vector field $\mathbf{F}$ defined in $\mathbf{R}^3$, let $\mathcal{S}$ be an (oriented) surface and $\partial \mathcal{S}$ be its (compatibly oriented) boundary. *Stokes's Theorem* states that

$$\int_{\partial \mathcal{S}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \text{curl}\mathbf{F} \cdot d\mathbf{S}.$$

That is, the circulation of $\mathbf{F}$ around the boundary of $\mathcal{S}$ is equal to the flux of $\text{curl}\mathbf{F}$ through $\mathcal{S}$ itself. Explain how Green's Theorem is a special case of Stokes's Theorem.

4. Compute both sides of Stokes's Theorem for the surface S defined by $z = \sqrt{1 - x^2 - y^2}$, with outward normal, and the vector field $\mathbf{F} = [-y, x, z]$.

Hint: For the left side, parameterize the boundary curve and compute the vector line integral. For the right side, parameterize the surface (perhaps using the surface parameterization $\mathbf{X}(\theta, \phi)$ given in 11#1), compute $\mathbf{X}_\phi \times \mathbf{X}_\theta$, and compute the vector surface integral as in #1 above, now with the vector field $\text{curl}\mathbf{F}$ instead of $\mathbf{F}$.

Math 290-3

1. Let $\mathbf{F} = \left[(y-1)\sin e^{xy^z}, xyze^{xyz}, xz+y\right]$, and let S be the piece of the paraboloid $y = x^2 + z^2$ with $y \leq 1$, oriented with outward normal vectors. Sketch this surface and its (oriented) boundary curve. Then compute $\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$.

2. Let $\mathbf{F}$ be the field $[x \sin e^x - xz, -2xy, z^2 + y]$, and let C be the triangular path from $(2, 0, 0)$, to $(0, 2, 0)$, to $(0, 0, 2)$, and back to $(2, 0, 0)$. Sketch this (oriented) path. Then compute $\int_C \mathbf{F} \cdot d\mathbf{s}$.

3. Our last big theorem is *Gauss's Theorem*, also called the *Divergence Theorem*, which says that if $\mathbf{F}$ is a vector field with continuous partial derivatives throughout a solid region E in $\mathbf{R}^3$, where the boundary surface ∂E of E has outward orientation, then

$$\iint_{\partial E} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div}\mathbf{F} \; dV.$$

Use this theorem to help you calculate the flux of the field $\mathbf{F} = [5x^2, 4y, 3]$ through the unit sphere $\mathcal{S}$, using an outward-pointing normal for $\mathcal{S}$.

4. (Continuation) If E is a solid region in $\mathbf{R}^3$, and ∂E is its boundary surface, must ∂E be a *closed* surface, or can it also have a boundary?

5. Compute the line integral of $\mathbf{F} = \frac{1}{2}[yz, -xz, xy]$ along the circle $z = 1, x^2 + y^2 = 1$, oriented clockwise when viewed from the origin.

6. Let $b > a > 0$. Take the circle of radius a in the xy-plane centered at the point $(b, 0, 0)$, and revolve it around the y-axis. The surface obtained is called a *torus*. Parameterize this torus, and find its surface area.

7. Is the integral $\iint_D \|\mathbf{X}_s \times \mathbf{X}_t\| \; ds \; dt$ that is used to compute surface area a flux integral for some vector field?

Math 290-3

1. Consider a vector field $\mathbf{F}$ whose curl is

$$\text{curl}\mathbf{F} = \left[y^{y^y} \sin e^{z^2}, (y-1)e^{x^{x^x}} + 2, -ze^{x^{x^x}} \right],$$

and the weird surface $\mathcal{S}$ shown in the figure, with outward normal. We wish to find the value of $\iint_{\mathcal{S}} \text{curl}\mathbf{F} \cdot d\mathbf{S}$. We cannot do this directly because $\text{curl}\mathbf{F}$ is awful and we don't have equations for $\mathcal{S}$. One option is to apply Stokes' Theorem and instead integrate $\int_C \mathbf{F} \cdot d\mathbf{s}$ over the boundary curve $C = \partial\mathcal{S}$ that is the unit circle $y = 1, x^2 + z^2 = 1$. Which orientation should C have? Draw it in. Unfortunately, we cannot do this, either, since we cannot find $\mathbf{F}$. Amazingly, we can still compute $\iint_{\mathcal{S}} \text{curl}\mathbf{F} \cdot d\mathbf{S}$!
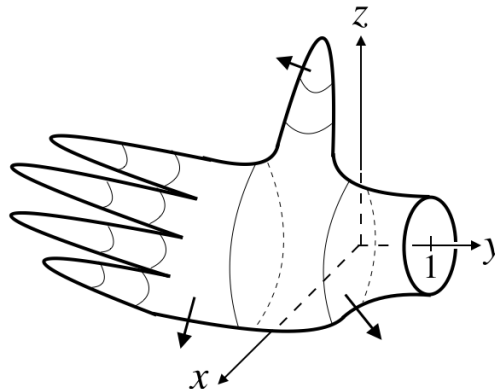
Replacing a surface using Stokes's Theorem. Consider the unit disk D defined by $y = 1$, $x^2 + z^2 \leq 1$, whose boundary is also C (and sketch it in). By Stokes's Theorem,

$$\iint_{\mathcal{S}} \text{curl}\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{s} \quad \text{and also} \quad \int_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \text{curl}\mathbf{F} \cdot d\mathbf{S},$$

as long as D has compatible orientation with C, so $\iint_{\mathcal{S}} \text{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_D \text{curl}\mathbf{F} \cdot d\mathbf{S}$. Compute the right-hand side to finish the job. Is D the only surface we could have used?

2. (Continuation) Write down the rule: If S_1 and S_2

$$\text{then} \iint_{S_1} \text{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \text{curl}\mathbf{F} \cdot d\mathbf{S}.$$

3. (Continuation) Is it true that there is a vector field $\mathbf{F}$ whose curl is given in #1?

4. Let S be the cylinder of radius 1, centered at the origin, whose central axis is the z-axis, between height $z = 0$ and $z = 3$, with top and bottom disks attached (so that it is like a closed aluminum can), with inward orientation. Sketch this cylinder. Compute

$$\iint_S \left[y^{123} e^{\sin\cos(yz)}, y - x^{z^x}, z^2 - z \right] \cdot d\mathbf{S}.$$

5. Find the surface area of the portion of the sphere of radius 4 centered at the origin that is above the plane $z = 2$.

6. True or False: if S is the sphere of radius a centered at the origin, then

$$\iint_S (z^3 - z + 2) \, dS = \iint_S (x - y^5 + 2) \, dS.$$

Math 290-3

1. Suppose we wish to compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \left[e^{\cos(yz)} \tan(z^2 y), \cos(e^{x^z}), 3z \right]$ and S is the five faces of the unit cube $[0, 1] \times [0, 1] \times [0, 1]$, all except for the bottom face on the xy-plane, with outward orientation. We would like to apply Gauss's Theorem, but we can't, because our surface S is not closed.

When we wanted to apply Green's Theorem to a *curve* that was not closed, we closed off the curve, and here we can use the same strategy, by closing off the *surface*. Justify the following equation, and use it to find the value of $\iint_S \mathbf{F} \cdot d\mathbf{S}$:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} + \iint_{\text{bottom face}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\text{cube}} \operatorname{div}\mathbf{F} \ dV.$$

2. Apply Gauss's Theorem to compute the flux of the vector field $\mathbf{F} = [x, 0, -z]$ over the closed surface S consisting of the cone $z = \sqrt{x^2 + y^2}$ below $z = 1$, plus its circular cap. Check your answer with your answers to $33\#1 - 2$.

3. Consider a vector field $\mathbf{F}$ that is continuous on all of $\mathbf{R}^3$, and let S be *any* closed surface you want, with whichever orientation. Compute $\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$.

4. Suppose that $\mathbf{F}$ is a vector field with $\operatorname{curl}\mathbf{F} = [ye^z, xz\sin(2^z + x^2), 2y + 1]$. Compute the flux of $\operatorname{curl}\mathbf{F}$ across the piece of the paraboloid $z = x^2 + y^2, z \le 1$ with outward normal.

5. Find the surface area of the part of the cylinder $x^2 + y^2 = 4$ that lies between the planes $z = 2hx$ and $z = 4h - 2hx$. You sketched this surface on Midterm 1. Check your answer using geometry.

6. One way to do problem #4 is to use Stokes's Theorem and replace the surface with an easier surface. Another way is to close off the surface and apply Gauss's Theorem. Whichever of these strategies you didn't use above, try it now.

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1. Compute $\iint_S [ye^{\cos\sin z}, x^{203}e^{\tan z}, x - z^2] \cdot d\mathbf{S}$, where S is the part of the hyperboloid $x^2 + y^2 - z^2 = 1$ between $z = -2$ and $z = 2$.

2. In the st-plane, the vector $[1,0,0]$ in the s-direction, $[0,1,0]$ in the t-direction, and $[1,0,0] \times [0,1,0]$ follow the right-hand rule: You can imagine these three vectors forming the x, y, z axes. When you parameterize a surface $\mathbf{X}(s,t)$, the parameterization lifts and bends the st-plane up into $\mathbf{R}^3$, but preserves the right-hand rule: on the surface $\mathbf{X}(s,t)$, the vectors $\mathbf{X}_s, \mathbf{X}_t, \mathbf{X}_s \times \mathbf{X}_t$ still follow the right-hand rule. The direction of the vector $\mathbf{X}_s \times \mathbf{X}_t$ is the *induced* orientation (inward, outward, upward, etc.) of the surface. For each surface below, without explicitly computing the cross product, determine the induced orientation by determining the directions of $\mathbf{X}_s$ and $\mathbf{X}_t$ (analogously for r and θ, etc.) and using the right-hand rule. *Hint*: sketch the surfaces

(a) $\mathbf{C}(r,\theta) = (r\cos\theta, r\sin\theta, r)$

(b) $\mathbf{P}(x,y) = (x, y, 2 - x - y)$

(c) $\mathbf{T}(t,h) = (2\cos t, 2\sin t, h)$

3. For each surface below, orient the surface and boundary curve compatibly.
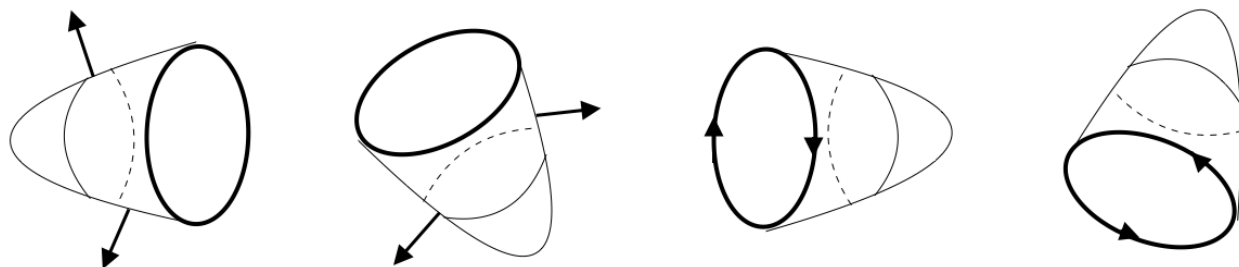
4. Compute the flux of $\mathbf{F} = [\sin(yz), 2x + e^z, 1 - xy]$ over the top half of the unit sphere.

5. Let C be the circle of radius 2 centered at the origin. Which of the following integrals is zero by symmetry?

(a) $\oint_C (x^3 + y^3)\, ds$

(b) $\oint_C (x^2 + y^2 + xy)\, ds$

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Reference

acceleration: The derivative of velocity with respect to time.

angle-addition identities: For any angles α and β, $\cos(\alpha + \beta) \equiv \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and $\sin(\alpha + \beta) \equiv \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

angle between vectors: When two vectors $\mathbf{u}$ and $\mathbf{v}$ are placed tail-to-tail, the angle θ they form can be calculated by the dot-product formula $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$. If $\mathbf{u}\cdot\mathbf{v} = 0$ then $\mathbf{u}$ is perpendicular to $\mathbf{v}$. If $\mathbf{u}\cdot\mathbf{v} < 0$ then $\mathbf{u}$ and $\mathbf{v}$ form an obtuse angle.

antiderivative: If f is the derivative of g, then g is called *an antiderivative of f*. For example, $g(x) = 2x\sqrt{x} + 5$ is an antiderivative of $f(x) = 3\sqrt{x}$, because $g' = f$.

average velocity is displacement divided by elapsed time.

bounded: Any subset of $\mathbf{R}^n$ that is contained in a suitably large *disk*.

Chain Rule: The derivative of a composite function $C(x) = f(g(x))$ is a product of derivatives, namely $C'(x) = f'(g(x))g'(x)$. The actual appearance of this rule changes from one example to another, because of the variety of function types that can be composed. For example, a curve can be traced in $\mathbf{R}^3$, on which a real-valued temperature distribution is given; the composite $\mathbf{R}^1 \longrightarrow \mathbf{R}^3 \longrightarrow \mathbf{R}^1$ simply expresses temperature as a function of time, and the derivative of this function is the dot product of two vectors.

chord: A segment that joins two points on a curve.

closed: Suppose that $\mathcal{D}$ is a set of points in $\mathbf{R}^n$, and that every convergent sequence of points in $\mathcal{D}$ actually converges to a point in $\mathcal{D}$. Then $\mathcal{D}$ is called "closed."

comparison of series: Given two infinite series Σa_n and Σb_n, about which $0 < a_n \leq b_n$ is known to be true for all n, the convergence of Σb_n implies the convergence of Σa_n, and the divergence of Σa_n implies the divergence of Σb_n.

concavity: A graph $y = f(x)$ is *concave up* on an interval if f'' is positive throughout the interval. The graph is *concave down* on an interval if f'' is negative throughout the interval.

content: A technical term that is intended to generalize the special cases length, area, and volume, so that the word can be applied in any dimension.

continuity: A function f is *continuous at a* if $f(a) = \lim_{p\to a} f(p)$. A *continuous function* is continuous at all the points in its domain.

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converge (integral): An *improper integral* that has a finite value is said to *converge* to that value, which is defined using a limit of proper integrals.

cross product: Given $\mathbf{u} = [p, q, r]$ and $\mathbf{v} = [d, e, f]$, a vector that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ is $[qf - re, rd - pf, pe - qd] = \mathbf{u} \times \mathbf{v}$.

curl: A three-dimensional vector field that describes the rotational tendencies of the three-dimensional field from which it is derived.

cycloid: A curve traced by a point on a wheel that rolls without slipping. Galileo named the curve, and Torricelli was the first to find its area.

cylindrical coordinates: A three-dimensional system of coordinates obtained by appending z to the usual polar-coordinate pair (r, θ).

derivative: Let f be a function that is defined for points $\mathbf{p}$ in $\mathbf{R}^n$, and whose values $f(\mathbf{p})$ are in $\mathbf{R}^m$. If it exists, the derivative $f'(\mathbf{a})$ is the $m \times n$ matrix that represents the best possible linear approximation to f at $\mathbf{a}$. In the case $n = 1$ (a parametrized curve in $\mathbf{R}^m$), $f'(a)$ is the $m \times 1$ matrix that is visualized as the tangent vector at $f(a)$. In the case $m = 1$, the $1 \times n$ matrix $f'(\mathbf{a})$ is visualized as the gradient vector at $\mathbf{a}$.

derivative at a point: Let f be a real-valued function that is defined for points in $\mathbf{R}^n$. Differentiability at a point $\mathbf{a}$ in the domain of f means that there is a linear function L with the property that the difference between $L(\mathbf{p})$ and $f(\mathbf{p})$ approaches 0 faster than $\mathbf{p}$ approaches $\mathbf{a}$, meaning that $0 = \lim\limits_{\mathbf{p} \to \mathbf{a}} \dfrac{f(\mathbf{p}) - L(\mathbf{p})}{|\mathbf{p} - \mathbf{a}|}$. If such an L exists, then $f'(\mathbf{a})$ is the matrix that defines $L(\mathbf{p} - \mathbf{a})$.

determinant: A ratio that is associated with any square matrix, as follows: Except for a possible sign, the determinant of a 2×2 matrix $\mathbf{M}$ is the area of any region $\mathcal{R}$ in 2-dimensional space, divided into the area of the region that results when $\mathbf{M}$ is applied to $\mathcal{R}$. Except for a possible sign, the determinant of a 3×3 matrix $\mathbf{M}$ is the volume of any region $\mathcal{R}$ in 3-dimensional space, divided into the volume of the region that results when $\mathbf{M}$ is applied to $\mathcal{R}$.

discontinuous: A function f has a *discontinuity at a* if $f(a)$ is defined but does not equal $\lim\limits_{p \to a} f(p)$; a function is *discontinuous* if it has one or more discontinuities.

disk: Given a point $\mathbf{c}$ in $\mathbf{R}^n$, the set of all points $\mathbf{p}$ for which the distance $|\mathbf{p} - \mathbf{c}|$ is at most r is called the disk (or "ball") of radius r, centered at $\mathbf{c}$.

divergence: If $\mathbf{v}$ is a vector field, its divergence is the scalar function $\nabla \bullet \mathbf{v}$.

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domain: The domain of a function consists of all the numbers for which the function returns a value. For example, the domain of a logarithm function consists of positive numbers only.

double-angle identities: Best-known are $\sin 2\theta \equiv 2 \sin \theta \cos \theta$, $\cos 2\theta \equiv 2 \cos^2 \theta - 1$, and $\cos 2\theta \equiv 1 - 2 \sin^2 \theta$; special cases of the *angle-addition identities*.

double integral: A descriptive name for an integral whose domain of integration is two-dimensional. When possible, evaluation is an iterative process, whereby two single-variable integrals are evaluated instead.

e is approximately 2.71828182845904523536. This irrational number frequently appears in scientific investigations. One of the many ways of defining it is $e = \lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$.

ellipsoid: A *quadric surface*, all of whose planar sections are ellipses.

Fubini's Theorem: Provides conditions under which the value of an integral is independent of the iterative approach applied to it.

Fundamental Theorem of Calculus: In its narrowest sense, differentiation and integration are inverse procedures — integrating a derivative $f'(x)$ along an interval $a \leq x \leq b$ leads to the same value as forming the difference $f(b) - f(a)$. In multivariable calculus, this concept evolves.

gradient: This is the customary name for the *derivative* of a real-valued function, especially when the domain is multidimensional.

Greek letters: Apparently essential for doing serious math! There are 24 letters. The upper-case characters are

$$\text{A B } \Gamma \; \Delta \text{ E Z H } \Theta \text{ I K } \Lambda \text{ M N } \Xi \text{ O } \Pi \text{ P } \Sigma \text{ T } \Upsilon \; \Phi \text{ X } \Psi \; \Omega$$

and the corresponding lower-case characters are

$$\alpha \; \beta \; \gamma \; \delta \; \epsilon \; \zeta \; \eta \; \theta \; \iota \; \kappa \; \lambda \; \mu \; \nu \; \xi \; o \; \pi \; \rho \; \sigma \; \tau \; \upsilon \; \phi \; \chi \; \psi \; \omega$$

Green's Theorem: Equates a given *line integral* to a special double integral over the region enclosed by the given contour. The self-taught George Green (1793-1841) developed a mathematical theory of electricity and magnetism.

Hessian: See *second derivative*.

improper integral: This is an integral $\int_{\mathcal{D}} f$ for which the domain $\mathcal{D}$ of integration is unbounded, or for which the values of the integrand f are undefined or unbounded.

integrable: Given a region $\mathcal{R}$ and a function $f(x, y)$ defined on $\mathcal{R}$, f is said to be *integrable over* $\mathcal{R}$ if the limit of Riemann sums used to define the integral of f over $\mathcal{R}$ exists.

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integrand: A function whose integral is requested.

Jacobian: A traditional name for the derivative of a function f from $\mathbf{R}^n$ to $\mathbf{R}^m$. For each point $\mathbf{p}$ in the domain space, $f'(\mathbf{p})$ is an $m \times n$ matrix. When $m = n$, the matrix is square, and its determinant is also called "the Jacobian" of f. Carl Gustav Jacobi (1804-1851) was a prolific mathematician; one of his lesser accomplishments was to establish the symbol ∂ for partial differentiation.

l'Hôpital's Rule: A method for dealing with indeterminate forms: If f and g are differentiable, and $f(a) = 0 = g(a)$, then $\lim\limits_{t \to a} \dfrac{f(t)}{g(t)}$ equals $\lim\limits_{t \to a} \dfrac{f'(t)}{g'(t)}$, provided that the latter limit exists. The Marquis de l'Hôpital (1661-1704) wrote the first textbook on calculus.

Lagrange notation: The use of primes to indicate derivatives.

level curve: The configuration of points $\mathbf{p}$ that satisfy an equation $f(\mathbf{p}) = k$, where f is a real-valued function defined for points in $\mathbf{R}^2$ and k is a constant.

level surface: The configuration of points $\mathbf{p}$ that satisfy an equation $f(\mathbf{p}) = k$, where f is a real-valued function defined for points in $\mathbf{R}^3$ and k is a constant.

line integral: Given a *vector field F* and a *path $\mathcal{C}$* (which does not have to be linear) in the domain space, a real number results from "integrating F along $\mathcal{C}$".

normal vector: In general, this is a vector that is perpendicular to something (a line or a plane). In the analysis of parametrically defined curves, the principal normal vector (which points in the direction of the center of curvature) is the derivative of the unit tangent vector.

operator notation: A method of naming a derivative by means of a prefix, usually D, as in $D \cos x = -\sin x$, or $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$, or $D_x(u^x) = u^x(\ln u)D_x u$.

orthonormal: Describes a set of mutually perpendicular vectors of unit length.

paraboloid: One of the *quadric surfaces*. Sections obtained by slicing this surface with a plane that contains the principal axis are parabolas.

partial derivative: A *directional derivative* that is obtained by allowing only one of the variables to change.

path: A parametrization for a curve.

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polar coordinates: Polar coordinates for a point P in the xy-plane consist of two numbers r and θ, where r is the distance from P to the origin O, and θ is the size of an angle in standard position that has OP as its terminal ray.

polar equation: An equation written using the polar variables r and θ.

potential function: An antiderivative for a *vector field*.

Product Rule: The derivative of $p(x) = f(x)g(x)$ is $p'(x) = f(x)g'(x) + g(x)f'(x)$. The actual appearance of this rule depends on what x, f, g, and "product" mean, however. One can multiply numbers times numbers, numbers times vectors, and vectors times vectors — in two different ways.

Quotient Rule: The derivative of $p(x) = \dfrac{f(x)}{g(x)}$ is $p'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$. This is unchanged in multivariable calculus, because vectors cannot be used as divisors.

second derivative: The derivative of a derivative. If f is a real-valued function of $\mathbf{p}$, then $f'(\mathbf{p})$ is a vector that is usually called the *gradient* of f, and $f''(\mathbf{p})$ is a square matrix that is often called the *Hessian* of f. The entries in these arrays are *partial derivatives*.

speed: The magnitude of *velocity*. For a parametric curve $(x, y) = (f(t), g(t))$, it is given by the formula $\sqrt{(x')^2 + (y')^2}$. Notice that that this is *not* the same as dy/dx.

spherical coordinates: Points in three-dimensional space can be described as (ρ, θ, ϕ), where ρ is the distance to the origin, θ is longitude, and ϕ is co-latitude.

stereographic projection: Establishes a one-to-one correspondence between the points of a plane and the points of a punctured sphere, or between the points of a line and the points of a punctured circle.

triple scalar product: A formula for finding the volume of parallelepiped, in terms of its defining vectors. It is the *determinant* of a 3×3 matrix.

velocity: This n-dimensional vector is the derivative of a differentiable path in $\mathbf{R}^n$. When $n = 2$, whereby a curve $(x, y) = (f(t), g(t))$ is described parametrically, the velocity is $\left[\dfrac{df}{dt}, \dfrac{dg}{dt}\right]$ or $\left[\dfrac{dx}{dt}, \dfrac{dy}{dt}\right]$, which is tangent to the curve. Its magnitude $\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$ is the speed. The *components* of velocity are themselves derivatives.

vector field: This is a descriptive name for a function F from $\mathbf{R}^n$ to $\mathbf{R}^n$. For each $\mathbf{p}$ in the domain, $F(\mathbf{p})$ is a vector. The derivative (gradient) of a real-valued function is an example of such a field.