Discrete Mathematics

Diana Davis
Swarthmore, PA
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The problems in this text

The method of instruction used with these problems is based on the curriculum at Phillips Exeter Academy, a private high school in Exeter, NH. Most of the problems and figures, especially those about graph theory and apportionment, are taken directly from the *Discrete Mathematics* book, written by Rick Parris and other members of the Phillips Exeter Academy Mathematics Department (cited in the margin as PEA). Some problems about gerrymandering and districting are taken from the Metric Geometry and Gerrymandering Group (MGGG). Many of the logic and set theory problems are from Frank Morgan's *Real Analysis* (FM), or written by Ross Sweet for Math 300 at Northwestern University (RS). Many Euler characteristic and 4/5/6-Color Theorem problems are from a book published by the AMS whose title and author I can't find right now (AMS). The rest were written by Diana Davis specifically for this course (DD). If you create your own text using these problems, please give appropriate attribution, as I am doing here.

About the course

This course meets twice a week for 75 minutes for a total of 26 classes, for which the homework is the numbered pages, usually two pages per class, numbered with a and b.

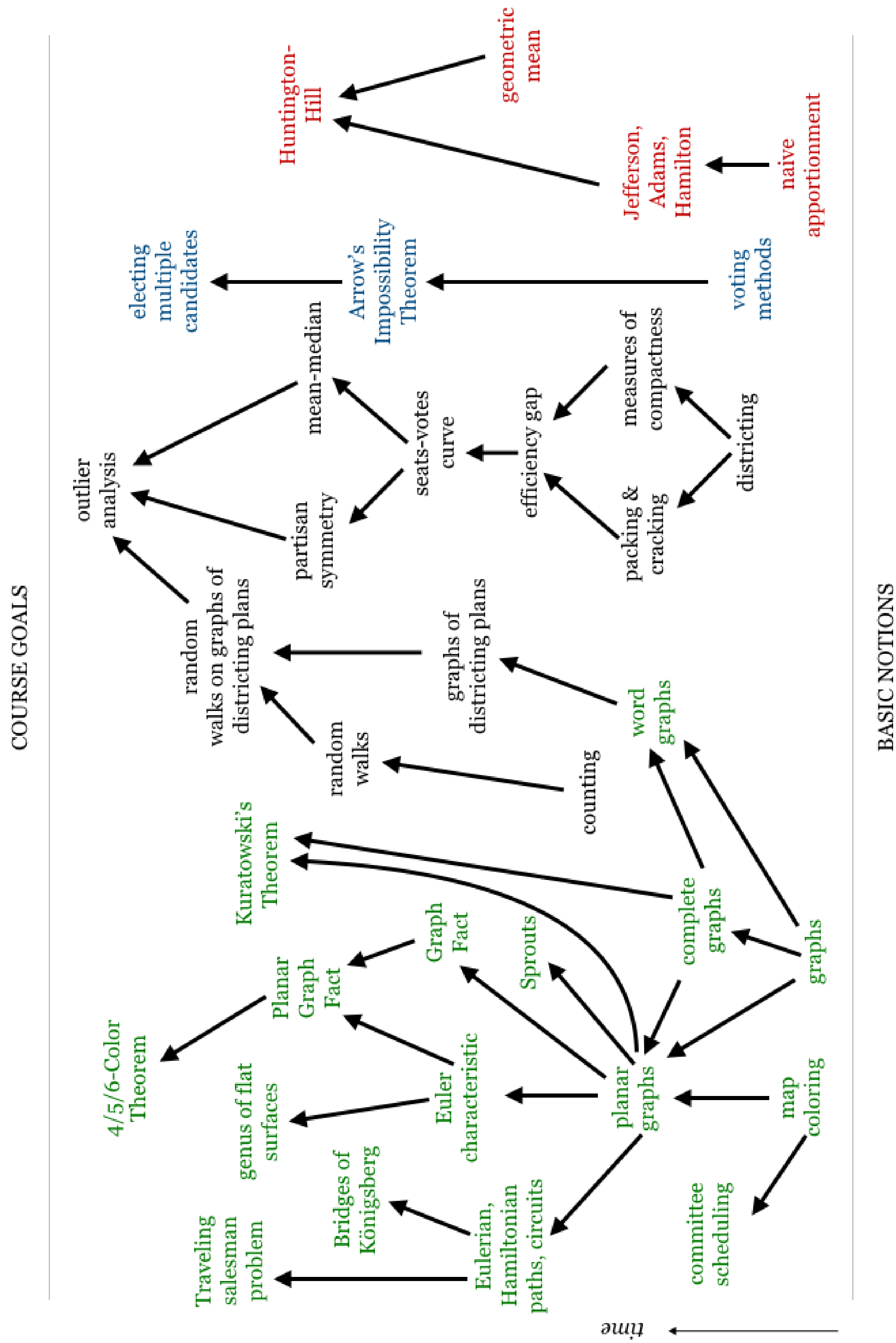
To the Student

Contents: As you work through this book, you will discover that the various topics of discrete mathematics have been integrated into a mathematical whole. There is no Chapter 5, nor is there a section on coloring graphs. The curriculum is problem-centered, rather than topic-centered. Techniques and theorems will become apparent as you work through the problems, and you will need to keep appropriate notes for your records — there are no boxes containing important ideas. Key words are defined in the problems, where they appear italicized. The definitions also appear in the index at the end of the book.

Your homework: Each page of this book contains the homework assignment for one night. The first day of class, we will work on the problems on page 1, and your homework is page 2 (or 2a and 2b); on the second day of class, we will discuss the problems on page 2, and your homework will be page 3, and so on for each day of the semester. You should plan to spend two to three hours solving problems for each class meeting.

Comments on problem-solving: Approach each problem as an exploration! Draw a picture whenever you can. Useful strategies to keep in mind are: create an easier problem, guess and check, work backwards, and recall a similar problem. It is important that you work on each problem when assigned, since the questions you may have about a problem will likely motivate class discussion the next day. Problem-solving requires persistence as much as it requires ingenuity. When you get stuck, or solve a problem incorrectly, back up and start over. Keep in mind that you're probably not the only one who is stuck, and that may even include your teacher. If you have taken the time to think about a problem, you should bring to class a written record of your efforts, not just a blank space in your notebook. The methods that you use to solve a problem, the corrections that you make in your approach, the means by which you test the validity of your solutions, and your ability to communicate ideas are just as important as getting the correct answer.

The following is an approximate map of how the concepts in this course build on each other, starting with basic notions at the bottom and building in sophistication towards the top.

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Hand-in proofs. You must write up and turn in two proofs each week. The problems that are eligible for this are marked in **bold**. You will revise and resubmit your proofs until they are perfect. You must type up proofs in LaTeX, where it will be clear and beautiful. This is so that when you revise it, you can edit the file instead of writing out the whole thing again!

1. This is a good problem, which we will discuss, but it is not a proof.

2. This asks you to prove something, so you may hand it in.

3. This problem has several parts.

(a) This part is a proof, which you may write up and hand in.

(b) This part is a good exercise, but it is not a proof.

Here is an example of how I would like your handed-in proof to be.

Claim. The square root of 2 is irrational.

Proof. To show that $\sqrt{2}$ is irrational, we will show that there is no ratio p/q of natural numbers whose square is 2.

Let p and q be natural numbers with no common factors, so that the ratio p/q is in lowest terms. We will suppose (for a contradiction) that $p/q = \sqrt{2}$.

$$\begin{aligned} \text{Suppose that} \qquad & \frac{p}{q} = \sqrt{2}. \\ \text{Squaring both sides yields} \qquad & \frac{p^2}{q^2} = 2 \\ \text{and then multiplying by } q^2 \text{ yields} \qquad & p^2 = 2q^2. \end{aligned}$$

This shows that p is even, so $p = 2r$ for some natural number r.

$$\begin{aligned} \text{So we can rewrite our equation as} \qquad & (2r)^2 = 2q^2 \\ \text{and then multiply out to yield} \qquad & 4r^2 = 2q^2 \\ \text{and then simplify this to} \qquad & 2r^2 = q^2. \end{aligned}$$

This shows that q is even. But then p and q are both even, which violates our assumption that p and q have no common factors, which is a contradiction. This proves that no ratio p/q of natural numbers squares to 2, so $\sqrt{2}$ is irrational. □

Notice that this proof is mostly words, and that every part of this proof is a full English sentence. Every sentence includes verbs and proper punctuation. You should do this. When you solved the problem on your paper, your solution may have been mostly symbols, but when you write down the proof, give a clear explanation of each step, as though you are talking to the person who is reading the proof.

Discussion Skills

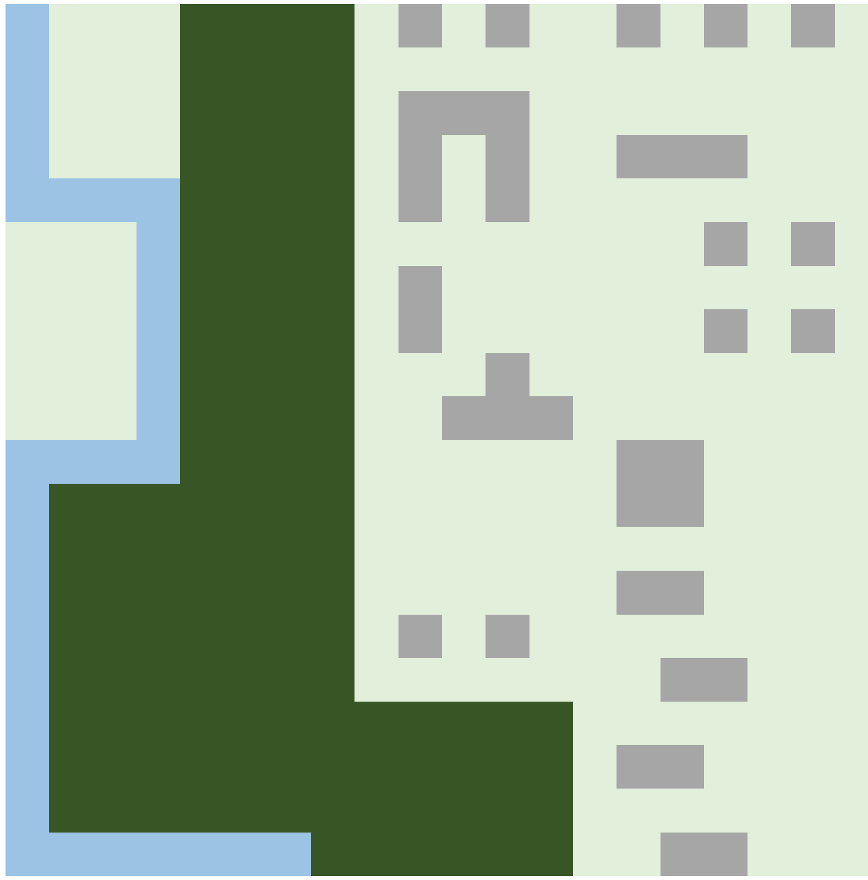
1. Contribute to the class every day
2. Speak to classmates, not to the instructor
3. Put up a difficult problem, even if not correct
4. Use other students' names
5. Ask questions
6. Answer other students' questions
7. Suggest an alternate solution method
8. Draw a picture
9. Connect to a similar problem
10. Summarize the discussion of a problem

Discrete Mathematics

First day - in class

DD

1. Due to increasing hostilities over the issues of divestment and hummus, Swarthmore has decided to split into Swarth College and More College. In dividing the land holdings of the College shown to the right, the Trustees wish to give both colleges the same amount of land, they want each college's land to be a single piece, they want the pieces to each be as "compact" as possible, and they want the division to be "fair" and "logical" with respect to the Crum Creek, the Crum Woods, the buildings, and the green space. Help them do so.



DD

2. A 20-member student advisory group represents all of the students majoring in STEM fields at Swarthmore. 39 students per year major in biology, 13 in chemistry, 70 in computer science, 40 in engineering, 41 in mathematics, 12 in physics, and 29 in special majors in science. How do you think the 20 seats in the group should be distributed to the departments?

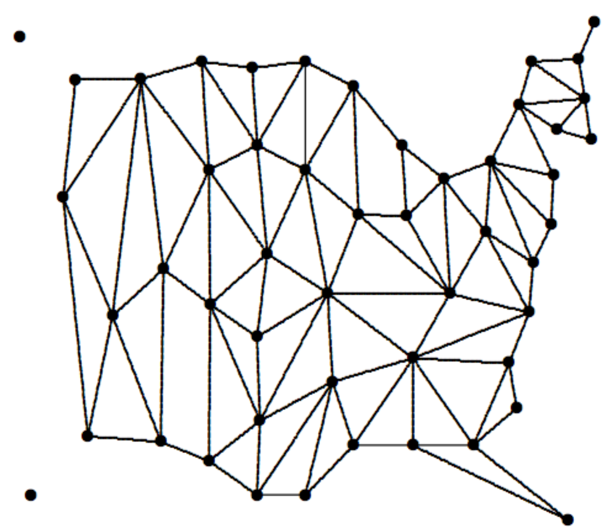
PEA

3. A *graph* is a network of dots and lines.

(a) What is the meaning of the graph shown at right?

(b) What do the dots represent?

(c) Why are some dots joined by segments and others are not?



Discrete Mathematics

PEA

1. Twelve students meet on the first day of class. As they leave the classroom, one of them conducts a small mathematical experiment, asking each student how many times they shook hands with another student. The twelve reported values were 3, 5, 6, 4, 7, 5, 4, 6, 5, 8, 4, and 6. What do you think of this data? Is it believable?

PEA

2. In the United States, the *census*, taken every 10 years, counts how many people live in each state. This data is used for many purposes, including deciding how many representatives each state will get in the national House of Representatives that meets in Washington D.C.

The table to the right shows the results of the 1790 census. If you were President, how would you distribute the 120 seats in the House of Representatives to the fifteen states?

State	**1790 Population**
Connecticut	236841
Delaware	55540
Georgia	70835
Kentucky	68705
Maryland	278514
Massachusetts	475327
New Hampshire	141822
New Jersey	179570
New York	331589
North Carolina	353523
Pennsylvania	432879
Rhode Island	68446
South Carolina	206236
Vermont	85533
Virginia	630560
Total	**3615920**

PEA

3. A third-grade teacher is arranging a field trip for eight of the students in the class. These students do not always behave well together. Adam gets along with Ben, Frank, and Harry; Ben gets along with Adam, Chris, David, and Harry; Chris gets along with Ben and Frank; David gets along with Ben, Eric, and Frank; Eric gets along with David, Frank, and George; Frank gets along with everyone except Ben; George gets along only with Eric and Frank; Harry gets along with Adam, Ben, and Frank. Draw a graph that models this situation, and say what the vertices and edges represent.

FM

Sets.

- A *set* is a notion that we won't define, because any definition would end up using a word like "collection," which we'd then need to define. We'll just assume that we understand what is meant by a *set*, and let this notion of a set be fundamental.
- We use a capital letter to denote a set, e.g. "Let S be the set of even numbers."
- The symbol $\in$ means "is/be an element of," and $\notin$ means "is not an element of."
- We use a lower-case letter to denote an element of a set, e.g. "Let $s \in S$."
- To describe the elements of a set, use curly braces $\{\}$. For example, $S = \{\dots, -4, -2, 0, 2, 4, \dots\}$ or $S = \{x : x \text{ is an even number}\}$. The colon ":" means "such that," so that the latter set is read aloud as "S is the set of x such that x is an even number."

FM

4. Let $A = \{1, 2, 3, 4\}$. Which of the following are true statements?

(a) $5 \in A$ (b) $6 \notin A$ (d) $7 \notin a$ (d) $\{4, 3, 2\} \in A$

MORE PROBLEMS ON THE NEXT PAGE!

FM **Talking about sets.**

- $X \subset Y$ is read "X is a subset of Y," and means that every x in X is also in Y.
- An equivalent notation to $X \subset Y$ is $X \subseteq Y$. If one wants to specify that $X \neq Y$, one can write $X \subsetneq Y$. Otherwise, $X \subset Y$ allows for the possibility that $X = Y$.

FM 5. Let S and A be as above, and let $B = \{1, 2, 3, 4, 5, 6\}$. Which are true? Explain.

(a) $A \subset B$ (b) $B \subset S$ (c) $S \subset B$ (d) $\{3\} \subset A$ (e) $3 \subset B$

PEA 6. Show that it is possible to color a map of the United States using only four colors, so that adjacent states do not receive the same color. For this you may wish to use the map at the beginning of this problem book, or the graph in Page 1 # 3, which represents the fifty states and their borders — two states (vertices) are joined when they share a border.

Convention. When a problem is labeled (Continuation), it means that it retains everything about the previous problem, but is labeled as a separate problem for convenience.

PEA **7**. (Continuation) Is it possible to do the coloring job described in the preceding item with only three colors? Either show how to do it, or prove (give a complete and convincing explanation) that it is impossible.

Discrete Mathematics

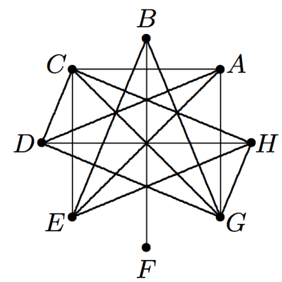
PEA 1. When two vertices of a graph are joined by an edge, the vertices are called *adjacent.* A graph in which every pair of vertices is adjacent is called a *complete graph.*

(a) How many edges are needed for a 5-vertex graph to be complete?

(b) How many edges are needed for a 24-vertex graph to be complete?

PEA 2. A graph can be *colored with n colors* if each of its vertices can be assigned one of the n colors in such a way that adjacent vertices are not assigned the same color. What does coloring a graph have to do with coloring a map?

PEA 3. The smallest value of n for which a graph can be colored with n colors is called its *chromatic number.* Find the chromatic number of the graph shown at right. Each vertex represents a student, and vertices are joined by an edge if the students do *not* get along. Why might you want to color this graph with a minimal number of colors?



The Apportionment Problem

In every method of apportioning the (currently 435) delegates in the House of Representatives, the *ideal quota* for a state is calculated by the formula 435·(state population/total population). Because this is not likely to be an integer, it is necessary to either round up to the *upper quota* or round down to the *lower quota* to obtain a meaningful result. A method of apportionment must specify exactly how this rounding is to be done.

The simplest method of apportionment was proposed in 1790 by Alexander Hamilton, and it is so intuitively appealing that you may have thought of it yourself already: Calculate each state's share of the total number of available seats, based on population proportions, and round down to give each state as many seats as prescribed by the integer part of its ideal quota. Award the remaining seats to those states that have the *largest fractional parts.*

PEA 4. Apply the Hamilton method to the following small, three-state examples, and determine how many delegates to assign to each state. The names of the states are A, B, and C.

(a) The populations are A = 453000, B = 442000, and C = 105000, with 100 delegates.

(b) The populations are A = 453000, B = 442000, and C = 105000, with 101 delegates.

(c) The populations are A = 647000, B = 247000, and C = 106000, with 100 delegates.

(d) The populations are A = 650000, B = 255000, and C = 105000, with 101 delegates.

(e) Discuss the interesting anomalies that arise in these examples. They are called the *Alabama paradox* (for historical reasons that we will see later) and the *population paradox.*

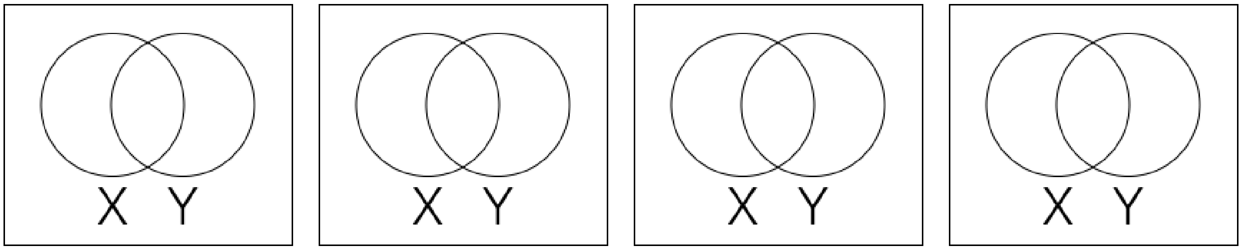
Working with sets.

FM

- The *intersection* $X \cap Y$ of two sets X and Y is the set of all elements that are in X *and* in Y: $X \cap Y = \{x : x \in X \text{ and } x \in Y\}$.
- The *union* $X \cup Y$ of two sets X and Y is the set of all elements that are in X *or* in Y: $X \cap Y = \{x : x \in X \text{ or } x \in Y\}$.
- The *complement* X^C of a set X is the set of points *not* in X: $X^C = \{x : x \notin X\}$. For this to make sense, the "universal set" that X lives in must be understood.
- The set $X - Y$, or $X \setminus Y$, is the set of all points *in* X that are *not* in Y.

RS

5. Shade the regions corresponding to $X \cap Y$, $X \cup Y$, X^C, and $X - Y$, respectively.



DD

6. Show that it is possible to color a map of Europe (as shown below) using only four colors, so that adjacent countries do not receive the same color.

DD

7. (Continuation) Is it possible to color this map with only three colors? Either show how to do it, or explain why it is impossible.

Discrete Mathematics

DD

1. Use the Hamilton method (page 3) to assign the 20 advisory board seats from Page 1 # 2 to the STEM departments. How does this compare to the apportionment you suggested?

PEA

2. What is the chromatic number of a complete graph (Page 3 # 1) that has

(a) 7 vertices? **(b)** n vertices?

PEA

3. A *connected* graph has the property that any two of its vertices can be joined by a chain of edges (also called a *path*), so that each edge in the chain has a vertex in common with the edge that follows. For example, the USA graph on page 1 is *not* connected. Give an example of a connected graph with twelve vertices that can be colored with only two colors.

RS

4. Let X be a subset of a universal set U, and let X and Y be subsets of U. Simplify:

(a) $(X \cup Y) \cap (U - X)$ (b) $X \cup (Y \cap X^C)$ (c) $(X \cap Y) \cup (X \cap Y^C)$

FM

Implication. There are many ways to say that one statement A implies another statement B. The following all mean exactly the same thing:

- If A, then B.
- A implies B (written $A \implies B$).
- A only if B.
- B is necessary for A.
- B if A (written $B \Longleftarrow A$).
- not B implies not A. (This is the *contrapositive*.)

FM

5. Let statement A be "the number n is divisible by four," and let statement B be "the number n is even." Write out the six implications above, using these statements, as full and reasonable sentences in English (read them aloud to make sure). Considering this example, do you agree that they are all logically equivalent?

PEA

6. A map-coloring algorithm, applied to the USA map: Color the states in alphabetical order, always using the first available color. For instance, Alabama gets color number 1, as do Alaska, Arizona, and Arkansas. California gets color number 2, Colorado gets color number 1, and so on. How close does this algorithm come to doing the best possible job (using only four colors)?

PEA

7. Compare the configuration formed by the states Nevada, California, Oregon, Idaho, Utah, and Arizona with the configuration formed by the states West Virginia, Ohio, Pennsylvania, Maryland, Virginia, and Kentucky. What do they have in common?

PEA

8. The *valence* of a vertex is the number of attached edges.

(a) What is the largest valence that occurs in the USA graph on page 1?

(b) What is the smallest valence?

PEA

Divisor Methods of Apportionment

According to the Hamilton method, quota-rounding decisions are made only after the *entire* list of quotas has been examined (and ranked in order of decreasing fractional parts). There are several other methods for apportionment, each one characterized by a rounding rule that is meant to be applied to individual states, without specific reference to the quotas of other states. These methods are described below.

First, each state's population is divided by an *arbitrary* district size, giving a fractional number of representatives for each state. This number is rounded up or down to give a whole number, according to which method (below) is in effect. Add up the whole numbers for each state, and see if you have hit your target number of representatives. If not, change the arbitrary district size that you used.

The Jefferson method: All adjusted quotas are rounded down. Because all the fractional parts are being discarded, the divisor must be smaller than the ideal district size (total population / number of representatives), if the target number of representatives (currently 435) is to be hit exactly. This method, proposed by Thomas Jefferson, was approved by Washington and applied to the 1790 census, with a House size of 105 and a divisor of 33,000.

A significant amount of trial and error is necessary to carry out this divisor method (or any of the others). If the divisor is too small, the total number of assigned representatives will exceed 435; if the divisor is too large, the total will fall short of 435. For a project of this size (each trial divisor must be divided into *all fifty* state populations), it is desirable to **use a computer to carry out the numerical work, perhaps with a spreadsheet**.

The Adams method: Adjusted quotas are rounded *up*. An acceptable divisor must be larger than the ideal district size. This method was proposed by John Quincy Adams. It has never been adopted.

The Webster method: Adjusted quotas are rounded *in the usual way* — to the nearest whole number. This method was proposed in 1831 by Daniel Webster, and used in the 1840 census.

PEA

1. Students at Pascal High School have a 25-member student council, which represents the 2000 members of the student body. The class-by-class sizes are: 581 Seniors, 506 Juniors, 486 Sophomores, and 427 Freshmen. Apportion the seats on the council using the:

(a) Jefferson method, (b) Adams method, (c) Webster method.

(d) What divisors did you end up using? Make a note of these three numbers.

Hint: **Use a spreadsheet** to make calculations, especially finding a suitable divisor, quicker.

Discrete Mathematics

PEA 2. Is it possible for a graph to have exactly one vertex whose valence (see Page 4 # 8) is odd, and for the rest of the valences to be even?

PEA 3. Given a graph, its valences can be listed. Conversely, given a list of nonnegative integers, one wonders whether there is a graph with precisely these valences. Which of the following lists correspond to actual graphs?

(a) 4,4,4,4,4 (b) 4,4,2,2,2 (c) 4,4,3,3,2 (d) 4,4,4,3,2

PEA 4. A graph has the following valences: 8,6,6,5,5,3,3,3,3.

(a) How many edges does the graph have?

(b) Draw such a graph.

(c) Is there more than one example?

DD **5**. Prove that any graph that represents a map (vertices of the graph are states/countries, with edges joining those that share a border) can be drawn without any edge crossings.

DD **6**. For each of the following, give an example of how to do it, or prove that it is impossible. (Proofs of impossibility are eligible for handing in; examples are not.) Draw a map whose associated graph contains a complete graph on:

(a) 3 vertices (b) 4 vertices (c) 5 vertices (c) 6 vertices.

PEA 7. Invent an algorithm for coloring a graph, trying to use as few colors as possible.

PEA 8. Twelve members of the mathematics department have been assigned to eight committees, which need to pursue some important end-of-term business. Each committee has to meet for an entire day to do its job. The committees:

A: Davis, Wang, Schofield
B: Wang, Grood, Mavinga
C: Bergstrand, Mavinga, Black
D: Chen, Black, Davis
E: Johnson, Chen, Hunter
F: Talvacchia, Bergstrand, Johnson
G: Hunter, Drellich, Grood
H: Drellich, Schofield, Talvacchia

Notice that each member is on two of the committees. Is it possible for the committee work to be done in fewer than eight days? If so, how many?

RS 9. Draw a Venn Diagram for:

(a) $B - A$ (b) $(A - B) \cap C$ (c) $(A \cup B) - C$ (d) $A \cap (B \cup C)$

(e) $(A \cap B) \cup (A \cap C)$ (f) $(A \cap B)^C$ (g) $A^C \cup B^C$

Set equivalence. To prove that two sets S_1 and S_2 are equal, you must prove set inclusion in both directions: $S_1 \subset S_2$ and $S_2 \subset S_1$. To do this, first suppose that $x \in S_1$ and show that x must also be in S_2. Then, suppose that $x \in S_2$ and show that x must also be in S_1.

RS 10. Parts (d) and (e), and (f) and (g), each suggest that a particular rule is true in general.

(a) Write them down and **(b)** prove one and **(c)** prove the other.

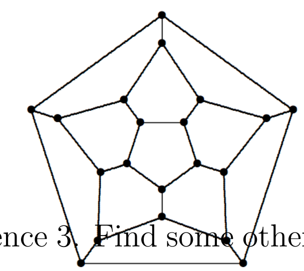
Discrete Mathematics

For the problems on apportionment, you may use the spreadsheet you received electronically. Alternatively, you are welcome to create your own tool to do apportionment calculations.

PEA **Every state gets a representative**: This is mandated by the Constitution. Because the Hamilton, Jefferson, and Webster methods do round down at least some of the time, their rounding rules must be amended to prohibit rounding down to zero. In the spreadsheet, this is done with the `MAX` command, e.g. `MAX(ROUNDDOWN(population/quota,0),1)`. Here the `ROUNDDOWN(population/quota,0)` part implements the Jefferson method of rounding down (the 0 means we round with 0 decimal places) and the `MAX([Jefferson],1)` part means we take the maximum of: what the Jefferson method gave us, or 1.

PEA 1. Apply the methods of Jefferson, Adams, and Webster to the 2010 USA census, and compare the results with the Hamilton apportionment. You will need to use trial and error to find three acceptable divisors, of course. To compare apportionments side-by-side, copy the resulting column after each apportionment has been successfully completed, and then `Paste > Paste Special... > Values`. For each divisor method, make note of how its result differs from that of the Hamilton method.

PEA 2. Repeat the preceding comparison for the 1790 USA census. There were only fifteen states then, and the total number of representatives was 105. The method actually used was Jefferson's. Rediscover his divisor.



PEA **3**. What is the chromatic number of the graph shown at right?

PEA 4. The graph shown at right has been borrowed from the realm of geometry. Can you identify the source of this diagram?

PEA 5. The graph at right is called *trivalent* because every vertex has valence 3. Find some other examples of trivalent graphs.

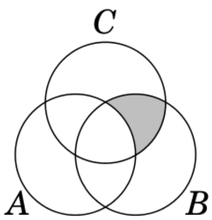
PEA 6. Graphs provide a means of summarizing information. To add to the display, it is sometimes appropriate to label the edges of the network with meaningful numerical data. Give an example of how this might be done.

FM **Converse and logical equivalence.** The *converse* of the statement "A implies B" is the statement "B implies A." If a statement and its converse are both true, we say A and B are *logically equivalent*, or in other words $A \iff B$, or in other words "A if and only if B."

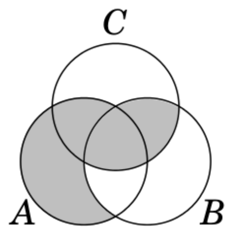
FM 7. Let statement A be "x is a Swarthmore student" and let statement B be "x is a human being." Write out (a) the implication $A \implies B$, (b) its contrapositive (Page 4 # 4), and (c) its converse. Which of these implications are true?

DD 8. Repeat the preceding problem, this time using statements A and B that you make up.

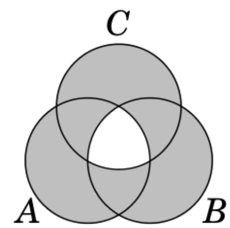
RS 9. For each figure below, write down symbolic notation for the shaded regions.



(a)



(b)



(c)

Useful sets.

- The *empty set* ∅, the set consisting of no elements.
- The *natural numbers* $\mathbf{N} = \{1, 2, 3, \ldots\}$. In Europe, $\mathbf{N}$ starts with 0.

RS 10. Find infinitely many nonempty sets $S_1, S_2, \ldots$ of natural numbers such that

$$\mathbf{N} \supset S_1 \supset S_2 \supset S_3 \cdots$$

and so that the infinite intersection of *all* of the sets $S_1 \cap S_2 \cap \cdots$ is ∅.

Discrete Mathematics

PEA

How are elections decided? The goal of an election is to determine *the will of the voters* — in effect, to produce a *group preference list* based on the individual ballots (individual preference lists). This is an elusive goal: the various voting methods do not always agree on the winner, and each one is susceptible to paradoxes and manipulation.

When there are only two candidates, the winner is necessarily whomever gets a *majority* of the votes (more than 50%). When more than two candidates are on the ballot, however, there is no simple, satisfactory answer. Here are some of the traditional approaches:

The Plurality Method: The election is won by the candidate with the most first-place votes, even though the winner might fail to receive a majority of votes (used in most U.S. elections).

Point Count: Each voter ranks all the candidates from first to last, and candidates receive points accordingly: The *Borda system* is to give 0 points for last place, 1 point for next-to-last, 2 points for next-to-next-to-last, and so on. Other systems of awarding points are also possible – 5 for first, 3 for second, 1 for third, for example (used at some U.S. universities for student and faculty assemblies).

Condorcet Method: Each candidate meets every other candidate in a head-to-head majority contest. A candidate who emerges *undefeated* is the *Condorcet winner*. It is possible to be the tournament champion (have the best won-lost record) without being undefeated, however. A candidate who goes winless is the *Condorcet loser*. Pronounced "con-door-say."

Winners Runoff: The top two finishers in a plurality contest meet in a head-to-head majority contest (used in France).

Instant Runoff: Each voter votes for their favorite candidate. The candidate with the least votes is eliminated. Repeat until there is only one candidate left (used in Australia, and also just recently voted into law in Maine, where it is called *ranked choice voting*).

To analyze these methods, it is convenient to assume that each voter is willing and able to rank the candidates from most preferred to least preferred, and that the ballots collected contain all this information. This enables the election to be evaluated as the result of a single ballot, no mater what method is being used. For example, if a voter prefers candidate P to candidate Q, and candidate Q to candidate R, the ballot essentially consists of the list PQR, which provides all the information necessary to carry out any of the above methods.

PEA

1. In the 1912 Presidential election, the three candidates were Wilson, Roosevelt, and Taft. Polls showed the following preferences: WRT 45%, RTW 30%, and TRW 25% (the most liked candidate listed first).

(a) Who won the plurality contest?

(b) Was there a Condorcet winner? a Condorcet loser?

(c) Who would have won with the scoring system that assigns the points 2,1,0 (the simple Borda count)? Are there point systems that give different outcomes?

(d) Who would have won if a winners' runoff had been held between the top two finishers in the plurality contest?

DD

2. *If a graph has a circuit (closed path) of odd length, then the chromatic number of the graph is at least three.* Prove this statement or give a counterexample.

PEA 3. The *edge-skeleton* of a polyhedron is the graph that consists of the vertices and the edges of the polyhedron. Show that the edge-skeleton of a cube has chromatic number 2.

FM **True and false.** An implication is true if B is true, or if A is false (in which case we say that the implication is "vacuously true.") For example, the statement "If 5 is even, then 15 is prime" is vacuously true. An implication is false *only* if A is true and B is false.

DD 4. Explain each of the following hilarious jokes by rewriting it as a vacuously true statement:

- Every car I own is a Maserati.
- Swarthmore Football, undefeated since 2001.

DD 5. Write the contrapositive of each of the following implications. *Hint*: first write them in the form $A \implies B$, then write the contrapositive, then translate back into everyday English.

(a) The early bird gets the worm. (b) No good deed goes unpunished.

(c) If it ain't broke, don't fix it.

(d) People who live in glass houses shouldn't throw stones.

PEA 6. After each census, the number of representatives per state is re-calculated. Frequently, the number of representatives is increased, so that the number of citizens represented by each one is not too big. The Hamilton method, which was used from 1850 to 1890, has the undesirable property that *increasing* the size of the House can cause a *decrease* in representation for some state. This became known as the *Alabama Paradox* when it appeared after the 1880 census, because representatives were not being assigned to Alabama in a monotonic fashion for House sizes between 298 and 302. This phenomenon had actually occurred in 1870 with Rhode Island, and it happened again in 1900 with both Colorado and Maine in fluctuation. A Congressional bill was proposed, which would have enlarged the House to from 356 to 357. The resulting debate was bitter and partisan, leading Congress to scrap the Hamilton method in favor of the Webster method, with a House size of 386. This ended the controversy. Compare the Webster apportionment for House size 386 with the Hamilton apportionments for House sizes 356, 357, 358, 385, 386, and 387 and then explain what the fuss was about.

PEA 7. The current method of apportionment (used since the 1940 census) is the invention of two mathematicians, Edward Huntington and Joseph Hill. It is a divisor method, whose rounding rule is a bit mysterious. At first glance, one might think that the normal Webster rounding was in effect. To see that this is not the case, consider the 1990 census: Select the Huntington-Hill method, for which an acceptable divisor is $575,000$. Look at the adjusted quota entries for Oklahoma and Mississippi; how are they rounded?

PEA **8**. Show that the *geometric mean* of two different positive numbers is always between the numbers, and is smaller than the *arithmetic mean*. In other words, explain why, if x and y are positive numbers with $x < y$, the inequality $x < \sqrt{xy} < \frac{1}{2}(x + y) < y$ holds.

PEA **9**. If a graph is connected and has 12 vertices, what is the smallest number of edges it can have? What if the graph is connected and has n vertices? Prove your answer correct.

PEA **10**. A graph that is connected, with as few edges as its vertex count allows (see the previous problem), is called a *tree*. Prove that a tree cannot contain any circuits.

DD **11**. If in a group of n people, everyone shakes hands with everyone else exactly once, how many total handshakes are there? Prove your answer correct.

Discrete Mathematics

DD 1. Every year in the spring, the Mathematics and Statistics department makes a department T-shirt. Every math major or minor who votes in the T-shirt contest receives a free T-shirt. Each student ranks all of the designs (here, colors) in the order of their preference. Use the "instant runoff" method to determine what color the department shirt should be.

student	garnet	black	white	grey	red	green
Amy	5	1	3	2	6	4
Brandon	5	6	1	2	4	3
Claire	1	4	5	2	3	6
Daniel	3	6	2	1	5	4
Ellora	3	4	6	2	5	1
Francisco	2	6	4	1	3	5
Gillie	2	1	5	3	4	6

PEA 2. You have seen that the Hamilton method of apportionment can produce the Alabama Paradox, which is one of the reasons that this method fell into disuse during the nineteenth century. Perhaps the divisor methods (Jefferson, Adams, Webster) that have taken its place are susceptible to the same flaw, however. What do you think?

PEA 3. What happens to the adjusted quota for a state when the trial divisor is made slightly larger? when the trial divisor is made slightly smaller?

PEA 4. Show that there are two inequivalent graphs that share the valence list 3, 3, 2, 2, 2.

PEA 5. Is it possible for a tournament director to set up a schedule that has each of the nine entered teams playing exactly five games against the others?

PEA 6. Is the six-state system consisting of Illinois and its five neighbors equivalent to the six-state system consisting of Nevada and its five neighbors?

PEA 7. The *Welsh-Powell algorithm* for coloring a graph: Arrange the vertices of a graph by valence, in non-increasing order, breaking ties randomly. Color the vertices in this order, *always using the first available and allowable color*. For example, when applied to the USA map, either Missouri or Tennessee gets color 1, the other gets color 2, Kentucky gets color 3, and so forth. Try out this algorithm on at least three simple graphs of your choice, to see whether it always gives a coloring with the smallest possible number of colors.

PEA **8**. Let k be the largest valence in a graph. Prove that the chromatic number of this graph is at most $k+1$. *Note:* Do not assume that the graph in question is a complete graph.

DD 9. Write the *negation* of each of the following statements.

The way I like to think about negation is that someone says the statement X, and then you say, "no, you're wrong, [negation of X]." Another thing to keep in mind is that the negation of a *generalization* statement, as in part (a) below, is an *existence* statement, as in part (b) below, and the negation of an existence statement is a generalization.

(a) Everyone in the class is named John. (b) Someone in the class is named Jane.

(c) All elements of set S are even. (d) This graph has a vertex of valence more than 4.

FM 10. Let $S \subset \mathbf{N}$. Consider the statements:

A: All elements of S are even. (For all $x \in S$, x is even.)

B: Some element of S is even. (There exists an $x \in S$ such that x is even.)

(a) Does $B \implies A$? (b) Does $A \implies B$?

PEA

Which method is fairest?

The most obvious source of dissatisfaction with any apportionment is the *state-to-state variation in district size*. In other words, the number of citizens per representative is not the same for all states. There are methods of apportionment that seek to *minimize* this variation.

Here is an example: If the Hamilton method were used to apportion the House for the 1990 census, it would give Mississippi 4 representatives (its lower ideal quota) and New Jersey would receive 14 (its upper ideal quota). Why might an apportionment method reverse this lower/upper decision and give 5 to Mississippi and 13 to New Jersey? Let us look at district sizes. Under the Hamilton method, there would be 646611 citizens for each representative in Mississippi, and 553474 for each representative in New Jersey; the difference 93137 is a measure of *inequity* in this two-state apportionment. On the other hand, if New Jersey gave a seat to Mississippi, the New Jersey district size would rise to 596049 while the Mississippi district size would drop to 517289, creating an inequity of only $596049 - 517289 = 78760$. The improvement justifies the transfer.

Here is a contrasting example: According to the Huntington-Hill method, Montana gets one representative, creating a district size of 803655; meanwhile, North Carolina gets twelve representatives and a district size of 554802. The difference in sizes is 248853. If North Carolina were to surrender one of its seats to Montana, the district sizes would become 605239 (N Carolina) and 401828 (Montana); the difference has dropped to 203411. It seems that the transfer ought to be made. What justifies *not* making it?

The Huntington-Hill theory equates inequity with the *ratio* of district sizes. The Montana-North Carolina inequity is therefore $803655/554802 = 1.449$, meaning that the representative in Montana has 44.9% more constituents than each representative in North Carolina has. If North Carolina were to surrender a representative to Montana, the inequity would become $605239/401828 = 1.506$, meaning that each representative in North Carolina would have 50.6% more constituents than each representative in Montana. Because switching a seat from North Carolina to Montana would therefore *increase* the inequity, it is not done.

The inescapable conflict centers on the choice of *how* one chooses to measure inequity. There are even more possibilities than the two mentioned above. One could instead focus on the fractional part of a representative per person (the reciprocal of the district size, that is), and seek to minimize the state-to-state variation in this quantity. *This is what the Webster method does*. In fact, *any* divisor method can be interpreted (and thus recommended) as a method to minimize state-to-state variation according to some calculated sense of inequity.

Measuring inequity by the *differences* in district size (as in the first example) was first proposed in 1831 by James Dean, a mathematics professor at the University of Vermont. It has never been used, but Montana sued for its adoption after the 1990 census. The second example explains why. The suit was denied by the Supreme Court in 1991.

PEA

1. If the Webster method were applied to the 1990 census, it would give Massachusetts 11 representatives and Oklahoma 5. The Huntington-Hill method, on the other hand, gives Massachusetts 10 representatives and Oklahoma 6. Give a district-size explanation that justifies this transfer of a representative from Massachusetts to Oklahoma.

Discrete Mathematics

PEA 2. The Huntington-Hill method of apportionment makes its rounding decisions according to whether the adjusted quota is less than or greater than the geometric mean of the *lower adjusted quota* (L) and *upper adjusted quota* (U). The geometric mean formula is $\sqrt{L \cdot U}$. Explain why this method automatically gives every state at least one representative.

PEA 3. The Senior Dance Committee is deciding the format of this year's big event; the options are Caribbean Cruise, Hawaiian Holiday, or Riviera Romance. The nine Committee members rank them as follows: CHR, CHR, HRC, HRC, HRC, RCH, RCH, RCH, RCH.

(a) On the basis of plurality, which theme would prevail?

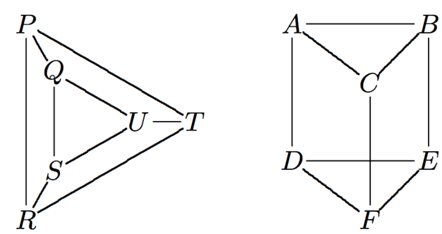
(b) Suppose that the chairperson favors the Riviera theme; what method (or methods) might the Committee find itself using?

(c) If the two members who rank the Riviera theme last were clever, how might they gain consolation through strategic (insincere) voting?

PEA **4**. Prove that the chromatic number of any tree is 2.

PEA 5. Each of us is associated with a graph that is commonly called an *ancestral tree.* I am at the root of my graph, with my parents adjacent to me, their parents adjacent to them, and so on. What do you think of this terminology? Draw and label a few vertices of your graph, then try estimating its size.

PEA 6. The graphs shown at right are *isomorphic.* This means that the labels on one figure can be matched with the labels on the other so that adjacent vertices correspond to adjacent vertices. One correspondence that works matches P with A, Q with D, R with C, S with F, T with B, and U with E. Verify that this correspondence works. Then find *another* way of matching the vertices of the two figures that gives an isomorphism.



PEA 7. (Continuation) In the second graph, notice that edge CF intersects edge DE. This creates an *accidental crossing*: the intersection does not mean anything; it is a unintended consequence of the way the graph was drawn. Notice that the first graph is drawn so that there are no accidental crossings. Graphs that do not have accidental crossings are called *planar.* Graphs that are isomorphic to planar graphs are also called planar, because any accidental crossings are avoidable. Thus the second graph shown above is planar. Make up your own example of a planar graph that has accidental crossings.

DD 8. Write the negation of each of the following statements.

(a) Insects have six legs. (b) Some Swarthmore students are from Arizona.

(c) All fish live in water. (d) My birthday is in July.

(e) Everyone at Swarthmore wears garnet. (f) Some people like their coffee black.

RS 9. Let x and y be integers. Suppose that we are trying to prove the following statement: *"If xy is even, then x is even or y is even."*
(a) Write the contrapositive of the statement in part (a).
(b) Which statement is easier to prove? Prove that one.

Discrete Mathematics

Review for midterm 1, which is in the evening after this class. The following review problems are provided for your convenience; you are welcome to spend your time working on other problems if you prefer.

0. Make a list of problems, from any page 1-10 in this book, that you would like a classmate or the professor to explain, and any other questions you would like to ask.

PEA 1. What is the conventional name for a graph that has *no circuits* of any size?

PEA 2. Is it possible for four countries to each border on all the others? Why or why not?

PEA 3. Is it possible for five countries to each border on all the others? Why or why not?

PEA 4. Is it possible to find four of the United States whose border graph is isomorphic to the complete graph on four vertices?

DD 5. Is it possible to color the map of South America (below) with three colors? If so, do so; if not, prove that it is impossible.

DD 6. Write the converse of each of the following implications. Which of the converses are true? *Hint*: First write it in the form $A \implies B$, then write out the statement $B \implies A$, and then write the latter down as a sentence in English (read it aloud to make sure).

(a) Every even number is divisible by 2.

(c) If it's a day that ends in 'y,' then I have Discrete Math class.

(b) Every human is a Swarthmore student.

(d) A number that is greater than 20 is greater than 10.

RS 7.

(a) Prove that for any sets A, B, if $A \subset B$, then $A \cap B^C = \emptyset$.

(b) Is the converse true? Prove it or give a counterexample.

DD 8. Use the Adams, Webster and Hamilton methods to apportion 23 representatives to Swarthmore's faculty council from the STEM departments, which have the following numbers of faculty: Mathematics and Statistics 19, Biology 18, Chemistry and Biochemistry 14, Computer Science 13, Physics and Astronomy 10, Engineering 7.

DD 9. For the review dinner at DD's house, 12 students voted on what they wanted to eat. The results of the vote are below. Use the various voting methods to determine what should be prepared for dinner, and be prepared to justify your conclusion.

student	1st choice	2nd choice	3rd choice	4th choice
Andi	sandwiches	salad	soup	toast
Brendan	sandwiches	salad	toast	soup
Catherine	sandwiches	salad	toast	soup
David	sandwiches	toast	soup	salad
Effie	sandwiches	soup	toast	salad
Ferdinand	soup	salad	sandwiches	toast
Grace	soup	salad	toast	sandwiches
Hillary	soup	salad	sandwiches	toast
Ira	soup	toast	sandwiches	salad
J.C.	salad	soup	toast	sandwiches
Kyle	salad	soup	sandwiches	toast
Laura	salad	toast	sandwiches	soup

n. Revisit problem 0 and make sure you have a list of everything you'd like to discuss in the review.

Discrete Mathematics

PEA 1. Fourteen members of the Swarthmore Board of Managers are in town to conduct their annual business, which is done in eight subcommittees that each need to hold a day-long meeting. It is your job to schedule the meetings so that they can get back as soon as possible to their homes, families, and regular jobs. The eight subcommittees are:

Alumni affairs: Jones, Bengali, Singleton, Poor
Budget review: Bengali, Poor, Turner, Rosado
Capital giving: Rosado, Martinez, Harnett, Lovelace
Dining halls: Turner, Chen, Kemp, Lovelace
Environment: Harnett, Economy, Wolf
Faculty life: Scott, Chen, Wolf
Grand plans: Jones, Economy, Kemp, Lovelace
Health and happiness: Scott, Martinez, Poor, Jones

PEA 2. Suppose that the following medal count occurs in the next Olympic Games: the USA wins 7 Gold, 15 Silver, and 6 Bronze; Japan wins 8 Gold, 12 Silver, and 7 Bronze. Devise point-counting systems that make each team the overall champion.

Gerrymandering.[1] We have explored how to do *apportionment* of representatives to each state. Once we have decided how many representatives a state gets – for instance, Washington has 10 – we then need to do *districting* of each state. For example, Washington is cut up into 10 geographic regions, which each elect a representative. How should we cut it up?

With apportionment, we can propose algorithms (Webster, Adams. . .) for how best to divide an integer (such as 435) into 50 integer parts. There is no such algorithm for districting. However, there are some laws for how the the geometry of the districting must be done:

(1) Districts should have equal population.

(2) Districts should be as compact as possible.

(3) Each district should consist of a single contiguous piece.

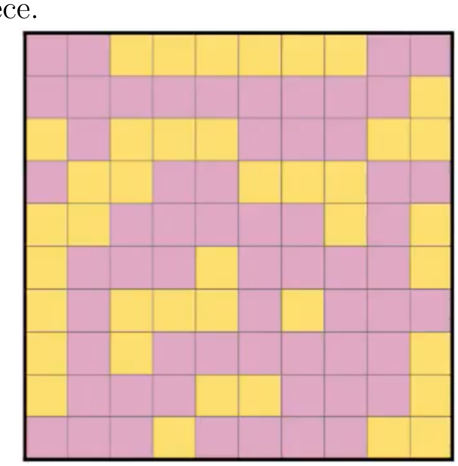
There are also non-geometric rules:

(4) Districts should respect county and city boundaries.

(5) Districts should respect communities of interest.

(6) Districts must comply with the Voting Rights Act.

We will begin with the geometric rules.

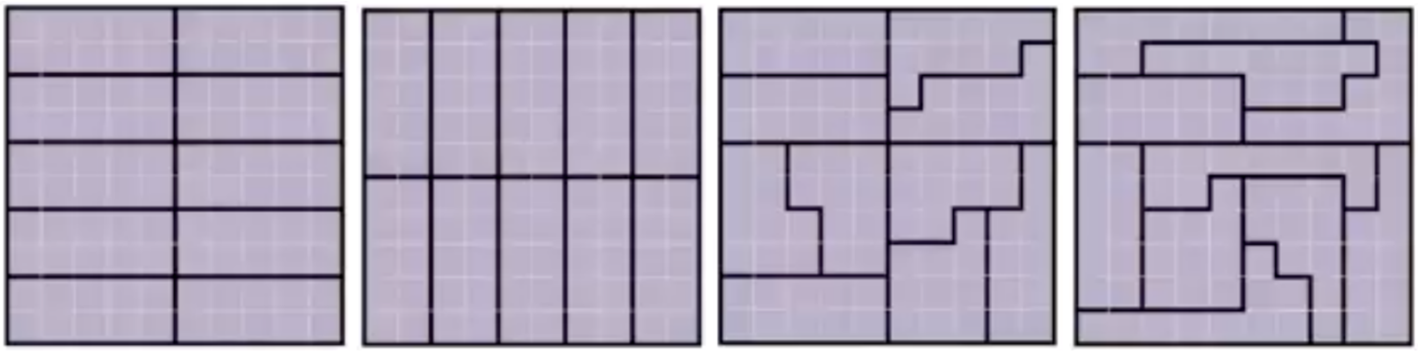


3. The 10×10 grid to the right represents 100 towns in a state, each of which supports either the pink (60 towns) or the yellow (40 towns) party. This state gets 10 representatives, each of whom belongs to one of the two parties.

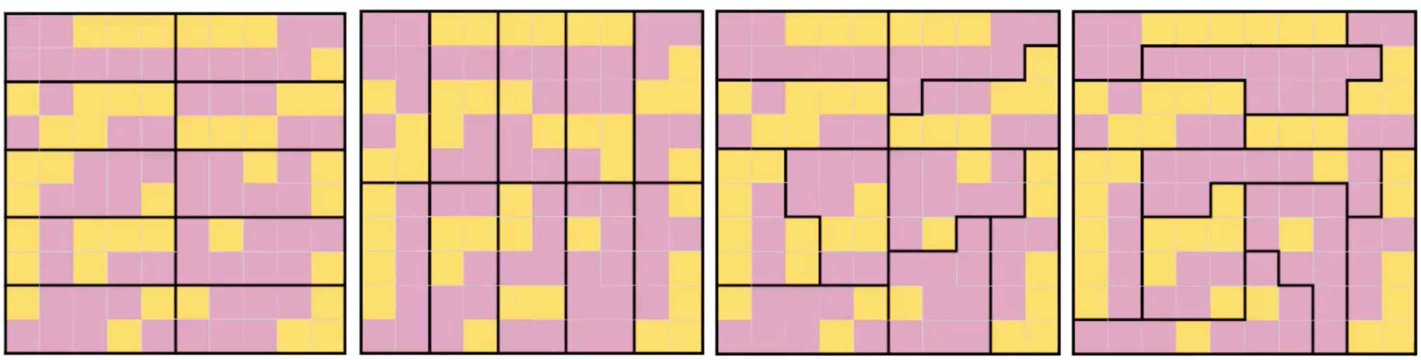
(a) How many representatives should each party get? What outcomes would you accept as "fair"?

[1]This section, including the figures, is taken from Moon Duchin's lecture "Political Geometry" at the 2018 Joint Mathematics Meetings, available at `https://www.youtube.com/watch?v=VddL0evo7QY`

(b) Below are four proposed districting plans, for dividing the state into 10 districts. For each map, explain why it satisfies rules (1), (2) and (3) listed on the previous page. Which one seems the most "fair," based on the district shapes?



(c) Below, each of the four districting plans has been overlaid onto the state map. A party *wins* a district if it has the most votes (squares of its color) in that district. Determine how many representatives each party wins, using each of the four maps.



(d) Do any of the maps seem to be designed to yield a particular outcome? If so, explain how the map has been engineered, or *gerrymandered*, to make this outcome happen.

PEA

4. *A handshake puzzle*: My wife and I recently attended a party with four other married couples. Various handshakes took place. No one shook hands with one's own spouse, and no two people shook hands more than once. After all the handshakes were over, I asked the nine other people, including my wife, how many times they had shaken hands. To my surprise, each person gave a different answer. How many times did my wife shake hands?

Discrete Mathematics

DD

1. In 2016, voters in the state of Maine approved the use of "ranked-choice voting" (instant runoff). In 2017, the Maine legislature voted to delay the change to 2022. "It's the most horrific thing in the world," said Maine's Governor Paul LePage at the time. In June 2018 (see campaign sign to the right), the voters of Maine voted to maintain the use of ranked-choice voting, effective immediately.

(a) Give some reasons why citizens might want to use this voting system.

(b) Give some reasons why a politician might oppose this voting system.

In pondering your answers, I encourage you to look up articles about this vote.

DD

Proof by induction. So far, we have proved statements in two ways: *directly*, and by proving the *contrapositive*. The proof technique of *mathematical induction* works like knocking down a line of dominoes that are all set up in a row. It has two steps:

(1) *Base case*: you can knock down domino number 1.

(2) *Inductive hypothesis*: knocking down domino number k knocks down domino $k+1$.

2. Fill in the missing steps (write in the blanks) in the following proof.

Claim. For any natural number n, $1+2+\cdots+(n-1)+n = n(n+1)/2$.

Proof. We will prove this by induction.

Base case: When $n=1$, the sum of 1 through n is just 1, and $n(n+1)/2 = 1(1+1)/2 =$ (a)________. Since these are the same, the base case holds.

Inductive hypothesis: We will assume that $1+2+\cdots+(n-1)+n = n(n+1)/2$ holds for the natural number k. We will show that the formula also holds for $k+1$.

Inductive step:

$$\begin{aligned}1+2+\cdots+(k-1)+k+(k+1) &= \frac{1}{2}k(k+1)+(k+1), \text{ because (b)\_\_\_\_\_\_\_\_} \\ &= \frac{1}{2}\Big(k(k+1)+2k+2\Big), \text{ because (c)\_\_\_\_\_\_\_\_} \\ &= \frac{1}{2}\Big(k^2+3k+2\Big), \text{ because (d)\_\_\_\_\_\_\_\_} \\ &= \frac{1}{2}(k+1)(k+2), \text{ because (e)\_\_\_\_\_\_\_\_}.\end{aligned}$$

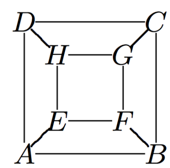
This proves that the formula holds for $k+1$, because (f)________________. Thus, $1+2+\cdots+(n-1)+n = n(n+1)/2$ for all natural numbers n.

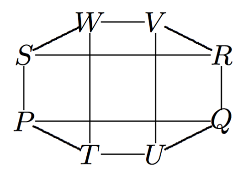
PEA

3. If the Hamilton method were applied to the 2000 census, it would give California 52 representatives and Utah four representatives. The Huntington-Hill method, on the other hand, gives California 53 representatives and Utah receives only three. Give a district-size explanation that justifies this transfer of a representative from Utah to California.

PEA 4. The Jefferson method of apportionment, which was used from 1790 through 1820, produced some troubling results. This was the basis for a speech by Daniel Webster (a Senator from Massachusetts) in 1831, when he argued for adoption of his apportionment method. At the urging of former President John Quincy Adams (who proposed his own method as well), Webster was trying to do something about the diminishing representation of the New England states, as well as remind Congress about the specific wording of the apportionment section of the Constitution. Despite the Senator's eloquent defense of the quota concept, the Jeffersonian forces won the debate. Apply the three methods (Jefferson, Webster, Adams) to the 1830 census and explain what the issue was.

PEA 5. Swarthmore has 1600 students, each of whom is enrolled in four academic courses.





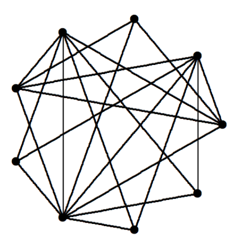
(a) The average class size is 20 students. Estimate how many classes are being taught.

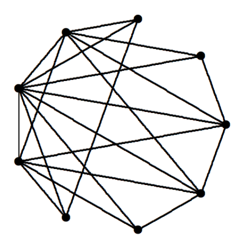
(b) These sections have to be scheduled into eight time blocks. Explain how this scheduling problem can be viewed as a coloring question about a suitable graph.

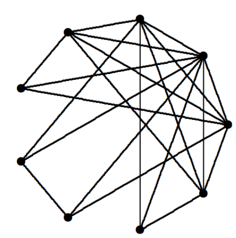
PEA 6. By writing out an explicit correspondence between their vertices, explain why the two graphs shown above are isomorphic.

PEA 7. By definition, a *planar* graph is isomorphic to a graph that can be drawn without accidental crossings. For example, graph $PQRSTUVW$ shown above is planar, even though it has four accidental crossings, because it is isomorphic to $ABCDEFGH$, which has none. It is often difficult to decide whether or not a graph is planar. Consider the edge-skeleton of the icosahedron, shown at right − is it planar?[2]

PEA 8. Two of the three graphs shown below are isomorphic, meaning that they are connected in essentially the same way. Can you tell *which two* ?







PEA **9**. Is it possible for the entries in the valence list of a graph to be all different? Explain.

PEA 10. I am considering my ancestral graph (Page 9 # 5) again.

(a) What are the possible valences for this graph?

(b) Suggest a meaningful way of attaching numerical labels to the edges.

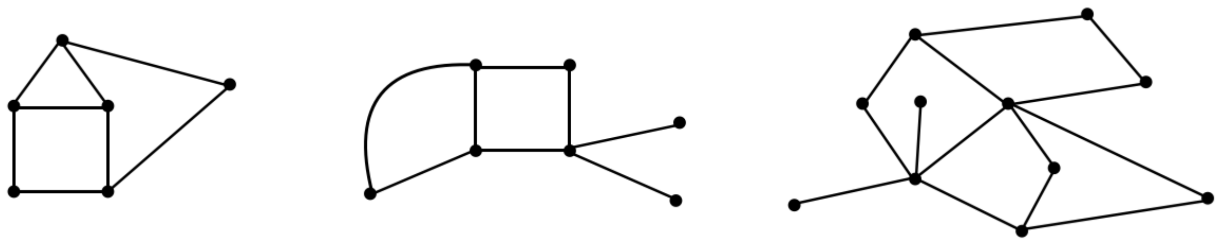
(c) Think of this graph as being *directed*, with arrows leading back in time (in other words, arrows point from children to their parents). Suppose that vertices A and B initiate paths that both lead to a vertex P. If both paths consist of two edges, what does that say about vertices A and B? What if the path from B to P had three edges?

[2]To decide if a graph is planar, you move the vertices around in your head or on your paper. Several free apps make a fun game out of this; for example, I enjoy playing *Tangled*.

Discrete Mathematics

AMS

1. A planar graph divides the plane into separate regions, which we'll call *faces.* The big region outside the graph also counts as a face. For example, the left graph below has four faces. For each of the three graphs below, calculate the following four numbers: the number of vertices v, the number of edges e, the number of faces f, and the *Euler characteristic* $v - e + f$. Come up with a conjecture about the relationships between these quantities.



MGGG

2. In Page 11 # 3, it seemed that the fourth map had been *gerrymandered* in favor of the yellow party. It would be nice to have a mathematical measurement to support this claim. People have argued that "bizarre shapes," or *non-compact* districts, indicate gerrymandering. Here are some ways of measuring *compactness* that have been proposed. For each state, you calculate the measure for each district, and then take the average over all the districts.

(1) *Total perimeter.*

(2) *Skew:* W/L, where W is the diameter of the smallest circle containing the district, and L is the diameter of the largest circle that fits inside.

(3) *Isoperimetric:* $16A/P^2$, where A is the district's area and P is its perimeter.

(4) *Square Reock:* A/S, where A is the district's area and S is the area of the smallest square that contains the district.

Using the maps from Page 11 # 3, calculate each of these numbers for the

(a) first and second (which are the same), (b) third, and (c) fourth map in Page 11 # 3.

(d) Which measure do you think is the best for measuring compactness and lack thereof?

PEA

3. Pascal High School has a 25-member student council to represent its 2000 students. There are 581 Seniors, 506 Juniors, 486 Sophomores, and 427 Freshmen. The Webster method assigns seven representatives to both the Senior and Junior classes. This leaves the Seniors disadvantaged relative to the Juniors, producing a *representational deficiency*, whose formula is $D_{sj} = r_j(p_s/p_j) - r_s$, where $r_j = 7$, $p_s = 581$, $p_j = 506$, and $r_s = 7$.

(a) What is the meaning of the number $r_j(p_s/p_j)$?

(b) Interpret the formula for D_{sj}, and calculate its value.

(c) The Jefferson method transfers one representative from the Juniors to the Seniors, thereby disadvantaging the Juniors. Show that the size of the representational deficiency is decreased by this transfer, which is why the Jefferson method does it.

PEA

4. A graph whose vertices all have valence 3 is called *trivalent.* It turns out that, except for the complete 4-vertex graph, all trivalent graphs can all be colored with (at most) three colors. (We showed in Page __ # __ that at most four colors are needed.) Verify this result experimentally: Invent your own trivalent, ten-vertex graph, then color it minimally.

PEA

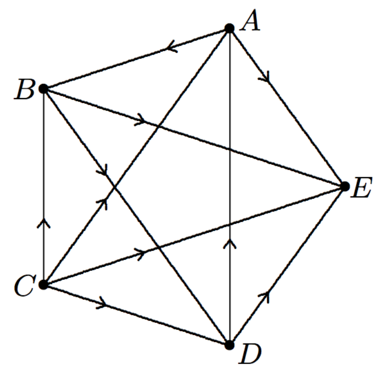
5. The most famous statement about graph coloring is the following *Four-Color Theorem*:

If a graph is planar, then its chromatic number is at most four.

Although it was proposed as a theorem in 1852, this was not proved until 1976. The proof was remarkable, because a computer (at the University of Illinois) was needed to carry out a mind-boggling amount of case-by-case checking. Many mathematicians are still hopeful that a simpler and more elegant proof will be found someday. What is the *converse* of the Four-Color Theorem? Is it a true statement?

PEA

6. The results of a 5-candidate preference poll are shown in the graph at right. Each vertex represents a candidate, and each pair of vertices is joined by an edge, representing a one-on-one comparison. On each edge, the arrow is directed from the winner toward the loser. Because of the arrows, this is an example of a *directed graph* (usually called a *digraph* for short). You can see that there is both a Condorcet winner and a Condorcet loser in this preference poll. Which candidates are these?



PEA

7. (Continuation) Produce a similar digraph that represents the results of the 5-candidate preference poll: BEDAC (21%), EBCAD (17%), CAEBD (30%), DCAEB (32%)

DD

8. *Real voting is messy.* Each year, the mathematics and statistics department at Swarthmore has two distinguished lectures: the *Kitao* and the *Dresden*. Members of the department vote for their top three choices of speakers, *or as many as they wish to rank.* Fictional results of the spring 2018 vote are as shown.

John Baez	2, 2, 2, 4
Eugenia Chang	1, 2, 3, 8
Laura DeMarco	1, 3, 4, 4
Edward Frenkel	1, 99, 99
Piper Harron	2, 2, 2, 3, 4
Nancy Kopell	2, 2, 3
Emily Riehl	1, 1, 1, 1, 6
John Rinzel	3
Richard Schwartz	3, 4, 7
Francis Su	1, 1, 1, 2, 3, 5
Suzanne Weekes	1, 3, 3, 5
Amie Wilkinson	2, 2, 3, 3, 5
Melanie Wood	1, 2, 4, 4, 6

(a) Which two people should the department have invited to give the 2018-2019 lectures? Justify your answer in a way that would convince the faculty of the department.

(b) Distinguished mathematicians are busy people, and they frequently decline lecture invitations. Make a ranked list of *all* of the people that the department should invite, so that as soon as one declines, we can invite the next one.

PEA

9. Just as a planar graph can be drawn without accidental crossings on a plane surface, a *spherical* graph can be drawn without accidental crossings on a spherical surface, and a *toroidal* graph can be drawn without accidental crossings on a *torus*, which is the surface of a doughnut, or an inner tube.

(a) Explain why — despite the terminology — there is no essential difference between the terms *planar* and *spherical* when they are applied to graphs.

(b) Provide a simple example of a graph that is toroidal but not planar.

Discrete Mathematics

PEA **The geometric mean and Huntington-Hill apportionment: Part I**

In the following, p stands for a state's population, r stands for its number of representatives, and thus p/r stands for its average district size (citizens per representative).

It is common sense that State 1 is *disadvantaged* compared to State 2 if the district size p_1/r_1 is greater than the district size p_2/r_2. The Huntington-Hill method of measuring this inequity is to calculate the *ratio* of p_1/r_1 to p_2/r_2, which can be written as $\dfrac{p_1 r_2}{p_2 r_1}$.

PEA 1. *State 2 should give up one of its representatives to State 1 if it reduces the inequity.* Justify this statement. Also explain why State 2 must have at least two representatives for this to make sense.

PEA 2. Show that, after such a transfer, State 2 would have $r_2 - 1$ representatives and district size $p_2/(r_2-1)$, while State 1 would have r_1+1 representatives and district size $p_1/(r_1+1)$.

If State 1 were *still* disadvantaged after such a move, then it is obvious that this transfer must be made. The ambiguous case occurs when State 2 becomes disadvantaged by the move, meaning that the district size $p_2/(r_2-1)$ is greater than the district size $p_1/(r_1+1)$. The proposed transfer must be rejected if it would increase the inequity. In other words, if

$$\frac{p_1 r_2}{p_2 r_1} < \frac{p_2(r_1+1)}{p_1(r_2-1)},$$

then the Huntington-Hill method leaves the apportionment as is, because *State 2 would be worse off after the transfer than State 1 was before it.*

PEA 3. Explain why the left side of the above comparison represents the inequity before State 2 transfers a representative to State 1, and the right side the inequity after the transfer.

Further algebraic simplification

$$\frac{p_1^2}{r_1(r_1+1)} < \frac{p_2^2}{(r_2-1)r_2}$$
$$\frac{p_1}{\sqrt{r_1(r_1+1)}} < \frac{p_2}{\sqrt{(r_2-1)r_2}}$$

puts all of State 1's data on one side of the equation, and all of State 2's data on the other.

PEA 4. Justify and explain each of the following claims about the last inequality above:

(a) The geometric-mean formula $\sqrt{L \cdot U}$ (see Page 9 # 2) has finally appeared.

(b) Each side of this inequality can be viewed as a district size, because each is roughly p/r.

Recall the definitions of *vertices*, *edges* and *faces* of a planar graph (Page 13 # 1).

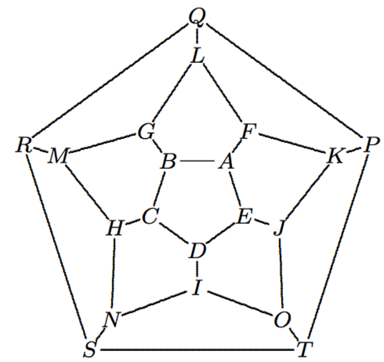
PEA 5. Compute v, e, f and the *Euler characteristic* $\chi(g) = v - e + f$ for the edge skeletons of the cube, dodecahedron, and icosahedron. Why do you think that in the definition of faces of a planar graph, the big region outside of a planar graph is defined as a face?

AMS 6. **Graph fact.** For a planar graph, $3f \le 2e$. We will prove this later. For now:

(a) Draw three planar graphs of your choice, and check that the relationship holds for them.

(b) What graphs have $3f = 2e$?

PEA 7. In 1859, the Irish mathematician Sir William Rowan Hamilton marketed a puzzle that consisted of a wooden regular dodecahedron (having twelve pentagonal faces), with each of the twenty vertices labeled to represent a famous city. The puzzle was to find a route that traveled along the edges of the solid, visiting each city exactly once and returning to the start. A small nail was driven into each vertex so that the path could be traced with string. Use the graphical representation at right to find a solution to Hamilton's puzzle.



PEA 8. A circuit that includes every vertex of a graph exactly once is called a *Hamiltonian circuit.* (Look above for the origin of this terminology.)

(a) Does every graph have a Hamiltonian circuit?

(b) How many different Hamiltonian circuits does the complete five-vertex graph have?

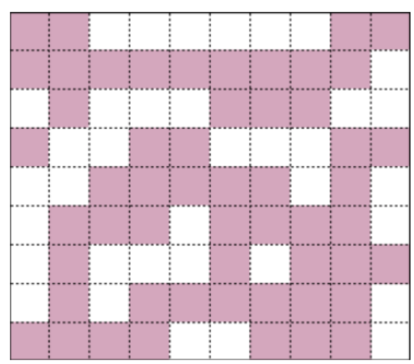
(c) How many different Hamiltonian circuits does the complete n-vertex graph have?

MGGG 9. **Gerrymandering in Squaretopia.**The state of Squaretopia consists of 90 towns: 36 vote for the "White" party and 54 vote for the "Pink" party. Squaretopia needs to be districted into 10 regions of equal area.

For each part, also note down how many districts Pink wins.

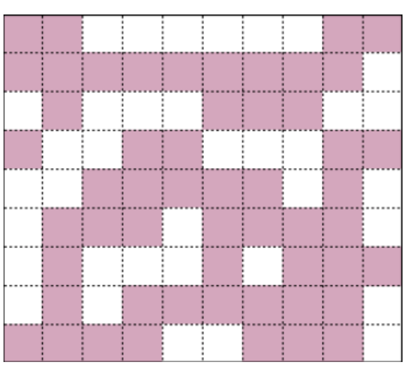
(a) **Compactness.** Divide Squaretopia into 10 districts that are as "compact" as possible.

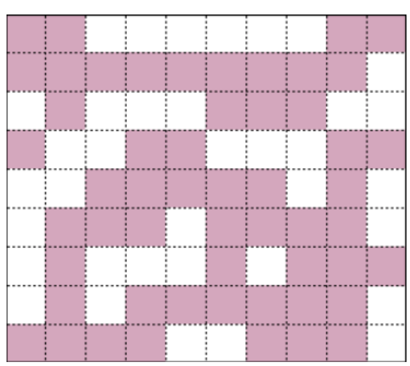
(b) **Proportionality.** Divide Squaretopia into 10 districts so that Pink's seat share (the proportion of districts it wins) is equal to its vote share (its overall proportion of voters).

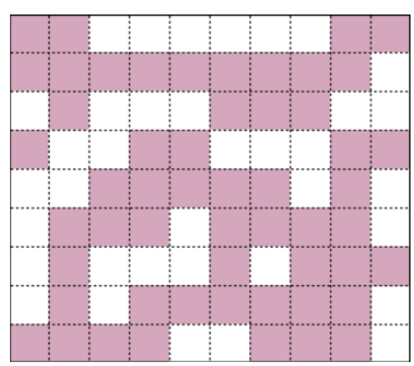


(c) **Gerrymander for Pink.** Divide Squaretopia into 10 districts so that Pink wins as *many* districts as possible. Keep your districts as compact as you can.

(d) **Gerrymander for White.** Divide Squaretopia into 10 districts so that Pink wins as *few* districts as possible. Keep your districts as compact as you can.





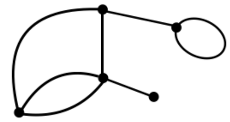


Discrete Mathematics

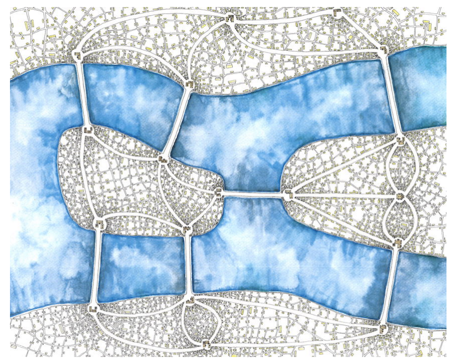
AMS **1**. Use induction to prove that the Euler characteristic of a nontrivial connected planar graph is 2. (You will have to decide whether to induct on vertices, edges or faces.)

AMS **2**. The *order* of a face is the number of edges bordering it. Prove the following statement: *The sum of all of the orders of the faces of a graph is* $2e$. Note that if a face touches the same edge on two sides, it counts as two edges.

AMS **3**. Use the result of the previous problem to prove the Graph Fact (Page 14 # 6): *For a planar graph without multi-edges or loops*, $3f \leq 2e$. (Examples of a multi-edge and a loop are shown to the right.)



DD 4. *The seven bridges of Königsberg.* The German city of Königsberg sat on two banks of a river, plus two islands in the river. There were seven bridges between the parts of the city, as shown to the right. The citizens of Königsberg wondered if it was possible to take a walk around the city, crossing every bridge exactly once. What do you think? *Historical notes:* The great mathematician Leonhard Euler invented graph theory to solve this problem. Königsberg is now Kalingrad, Russia.

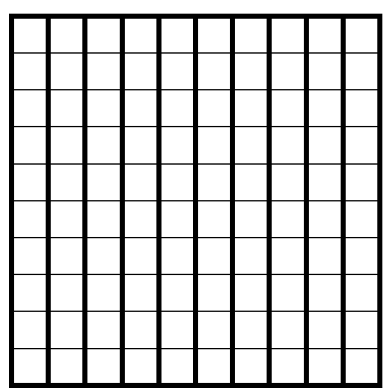


DD In many countries, the land is divided into districts, in which the citizens vote for candidates who will represent them. However, the United States is the only country where the lines are frequently re-drawn *and* the people who are in power decide where to draw the lines:

- The U.S. Constitution mandates that, within each state, districts have equal population. Thus, after each Census (which happens every 10 years; the next one is in 2020) the lines are re-drawn to adjust for changing population.
- Generally, each state legislature (which consists of representatives of each town, who meet in the state capital) votes to approve the new state district map.

This creates an incentive for whichever party (Democrat or Republican) has a majority in the state legislature to draw districts so that members of their party are more likely to be elected to the House of Representatives. Drawing lines to disadvantage voters based on race is *illegal* (by the Voting Rights Act). Drawing lines to disadvantage voters based on political party is *legal*, as long as it can be shown that this effect is not persistent over time.

DD 5. You, a member of the Gray party, are dividing your 100-town state into 10 districts. 44 of the towns vote Gray and 56 vote White. Ignoring geography, we can represent the towns by 100 squares, and divide them into vertical districts.

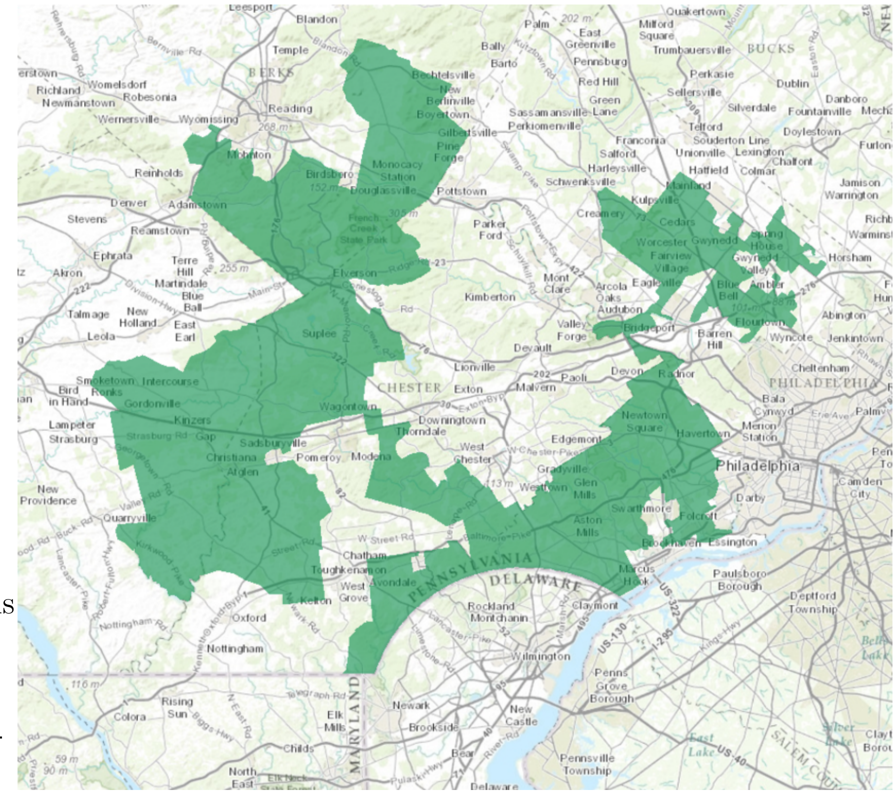


(a) Shade 44 of the squares gray so that the Gray party wins as many districts as possible.

(b) The two classic techniques for *gerrymandering* (districting to benefit your political party) are *packing* your opponents into a few districts that they win by a lot, and *cracking* the remaining opponents so that they don't have quite enough in any other district to win. Of the 10 districts you drew in (a), which ones are packed and which are cracked?

DD 6. So far, we have been assuming that, within each town, all of the voters who live there vote for the same political party. This is obviously unreasonable. In Massachusetts, which has 9 districts each electing a representative, 65% of citizens vote Democrat and 35% vote Republican. Each town has roughly this same proportion of voters. How many representatives should Massachusetts elect of each political party? How should we draw the lines to make this happen?

DD 7. *Pennsylvania.* Our state has been the center of a lot of recent events in gerrymandering news, so we will study it in detail. To the right (in green) is Pennsylvania's seventh Congressional district, which was one of the three most-gerrymandered districts in the whole country, until the entire districting map was declared unconstitutional earlier this year and was redrawn.

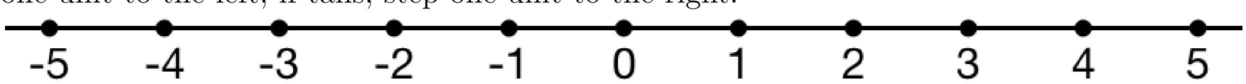


(a) This district has been called "Goofy kicking Donald Duck." Explain. (If you are unfamiliar with these characters, search for images of them.)

(b) Find Swarthmore. Why do you think the designers of PA-7 cut Swarthmore out of it?

(c) Do you think PA-7 was designed to elect a Republican or a Democratic representative?

DD 8. *A random walk.* Stand at 0 on a number line and flip a coin: if it comes up heads, step one unit to the left; if tails, step one unit to the right.



(a) Using an actual coin, do a 10-step random walk and write down where you land: ______

(b) Repeat this exercise 9 more times, yielding a total of 10 numbers between -10 and 10. Record the outcomes of your 10 random walks (we will pool everyone's data in class).

DD 9. On Page 7, five voting methods are defined. Here are two more:

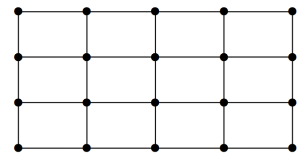
Dictatorship: Designate a voter. This voter's favorite candidate wins.
Fractional votes: Each voter gets some number of votes (maybe 10) and can assign them to candidates however they like: all 10 points to one candidate, or 1 point to each of 10 different candidates, etc. (*For fun:* How many different ways are there to partition 10 votes?)

Come up with your own voting method that is different from any that we have studied so far, and write down some of its strengths and weaknesses.

Discrete Mathematics

AMS **1**. Use the Euler characteristic, and the Graph Fact, to show that the complete graph on 5 vertices is not planar.

PEA/DD 2. The graph shown at right is called a *4-by-5 grid*, because there are 4 rows of vertices, and 5 vertices in each row.



(a) Suppose that this graph represents the streets and intersections in a small town. There is a street lamp at each intersection (vertex) and in the evening, the town lamplighter must light every lamp. Is there a way to visit every lamp once, without visiting any lamp more than once?

(b) Restate the above question using mathematical terminology.

(c) In general, for what values of m and n does an m-by-n grid have a Hamiltonian circuit?

(d) It snowed! Is there a way for the town snowplower to plow every street, without driving down any street more than once? If not, can you make it possible by adding new streets?

PEA 3. Here is an example from sports, which shows how difficult it is to devise a system of ranking that can be counted on to obey common sense. The problem is to choose the winner of a cross-country race between two teams of seven runners each. The traditional method is fairly simple: A runner who finishes in nth place receives n points, and a team's score is the sum of the points of its first *five* finishers. Low score wins, of course.

(a) For example, suppose that the runners in a race between team A and team B finish in the order AAABBBBBAABABA. Show that team A wins, 25 to 30.

(b) Which team wins the race if the order of finish is ABBAABABBAABAB?

(c) Suppose that three teams take part in a race, and that the twenty-one runners finish in the order CAAABBBBCCCCBCAABABA. This single race can be scored as three separate 2-way races, by simply ignoring the presence of one team at a time. Verify that the race between team A and team B has already been dealt with in (a). Score the race between team B and team C (ignore A), then score the race between team C and team A (ignore B).

(d) Draw a Condorcet diagram representing these outcomes. Are the results of these three calculations conclusive in determining a ranking of the three teams?

DD 4. (Continuation) The outcome above is a *cycle*, which makes it difficult to determine which team is truly the best. Cycles are also problematic when determining a group preference list for candidates in an election.

(a) Make up an example of an election, with at least three candidates and at least three voters with their preference lists, where it is not possible to make a ranked list of group preferences using the Condorcet method, because its outcome is a cycle (not a tie).

(b) Explain why you might want to choose an election method that guarantees that it will output a ranked group preference list with no cycles.

DD 5. Your friend did a 100-step random walk by flipping a coin 100 times.

(a) "I ended on 11," your friend says. What is your response?

(b) "I ended on 96," your friend says. What is your response?

Discrete Mathematics

PEA **The geometric mean and Huntington-Hill apportionment: Part II**

In **Part I**, we showed that, for states State 1, State 2 with populations p_1, p_2, and initial numbers of representatives r_1, r_2, if
$$\frac{p_1}{\sqrt{r_1(r_1+1)}} < \frac{p_2}{\sqrt{(r_2-1)r_2}}$$
then State 2 should *not* give up a representative to State 1. We also showed that each side of this inequality can be viewed as a district size, because each is roughly p/r. Suppose that d is a number such that
$$\frac{p_1}{\sqrt{r_1(r_1+1)}} < d < \frac{p_2}{\sqrt{(r_2-1)r_2}}\,.$$

PEA **6**. Prove that such a number d always exists.

PEA 7. Explain why the above inequality is equivalent to
$$\frac{p_1}{d} < \sqrt{r_1(r_1+1)} \quad \text{and} \quad \sqrt{(r_2-1)r_2} < \frac{p_2}{d}\,.$$

PEA **8**. Use the above inequality to prove that, for any Huntington-Hill divisor d, if you round p_1/d *down* you get the integer r_1, and if you round p_2/d *up* you get the integer r_2.

This reasoning for States 1 and 2 above must of course apply to *every* pair of states. Suppose that the Huntington-Hill method assigns r_1, r_2, r_3, … representatives to the states whose populations are p_1, p_2, p_3, … . In order that no state-to-state transfers be necessary,
$$\frac{p_n}{\sqrt{r_n(r_n+1)}} < \frac{p_m}{\sqrt{(r_m-1)r_m}}$$
must hold for any states m and n. In particular, the *largest* of the numbers $p_n/\sqrt{r_n(r_n+1)}$ is smaller than the *smallest* of the numbers $p_m/\sqrt{(r_m-1)r_m}$. Let d be any number between these extremes, so that
$$\frac{p}{\sqrt{r(r+1)}} < d < \frac{p}{\sqrt{(r-1)r}}$$
holds for every state.

PEA 9. Show how to rewrite the pair of inequalities above as
$$\sqrt{(r-1)r} < \frac{p}{d} < \sqrt{r(r+1)}.$$

Any such number d serves as a suitable Huntington-Hill divisor, and the inequalities define the Huntington-Hill rounding process. The apportionment for a state with population p is determined by where the value of p/d falls in relation to the list of square roots
$$\sqrt{0}\,, \sqrt{2}\,, \sqrt{6}\,, \sqrt{12}\,, \sqrt{20}\,, \sqrt{30}\,, \sqrt{42}\,, \sqrt{56}\,, \sqrt{72}\,, \ldots$$
In other words, p/d must fall between two successive terms of this sequence — say $\sqrt{(r-1)r}$ and $\sqrt{r(r+1)}$ — and the state is therefore awarded r representatives.

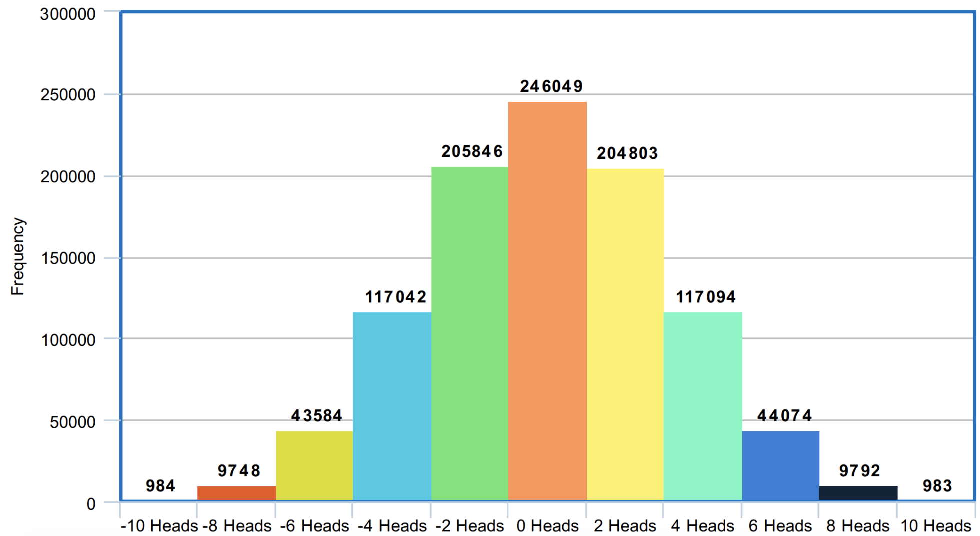
PEA 10. It was tacitly assumed above that equalities of the type
$$\frac{p_n}{\sqrt{r_n(r_n+1)}} = \frac{p_m}{\sqrt{(r_m-1)r_m}}$$
cannot occur. Is this a reasonable assumption to make?

DD 11. Make up an example of an apportionment problem where the Huntington-Hill and Webster methods give different results.

Discrete Mathematics

DD **1**. Prove that, for an n-step random walk on the integers starting at 0, the ending position is odd if n is odd, and even if n is even.

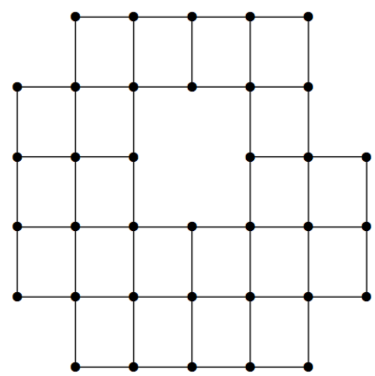
DD 2. (Continuation) Compute the probabilities of landing on $-2, -1, 0, 1$ and 2, for a 2-step random walk. What are the probabilities of landing points for a 3-step random walk? What about for a 10-step random walk?



For this last part, check that your answers agree with the histogram to the right, for a million trials of a 10-step random walk (picture by Sam Yan).

AMS **3**. Prove that a connected graph has an Eulerian circuit if and only if every valence is even.

PEA 4. If a connected graph has vertices of odd valence, then an Eulerian circuit is out of the question. There is still the possibility of an Eulerian path, however (which finishes at a vertex different from the starting vertex). Suppose that a graph has two vertices of odd valence. Introducing an extra edge that joins these two vertices is called *Eulerizing the graph*. The graph shown at right does not have an Eulerian circuit. If it is *Eulerized*, however, a circuit can be found. When an extra edge duplicates an existing edge (joining edges already adjacent), think of it as a *reused edge*. Show that this graph can be Eulerized by means of reused edges, and do it using as few duplicates as possible.



AMS **5**. As discussed in Page 13 # 5, the *Four-Color Theorem* was very difficult to prove, and so far, only a computer-aided proof is known. Instead of proving that, we'll prove the *Six-Color Theorem.* To do so, we'll need the following result:

Planar graph fact: Every planar graph has at least one vertex of valence 5 or less.

Prove this. *Hint*: use the Graph Fact and the Euler Characteristic.

PEA 6. *A word graph.* Consider a graph whose vertices are English words, with an edge connecting two vertices if it is possible to transform one into the other by changing one letter, as in MATH to PATH. Create the graph for the words EAT, PAN, PAT, RAN, RAT, RUN, RUT. Is it connected? Is it planar?

Discrete Mathematics

DD *Arrow's Impossibility Theorem.* Consider the following requirements for a *voting system*, which is any algorithm that takes as input the voters' ranked lists of preferences of candidates, and outputs a ranked group preference list (possibly with ties) of all of the candidates:

(A) *Universal admissibility:* Each voter can rank the candidates however they like.

(B) *Unanimity:* If every voter prefers P to Q, the voting system ranks P over Q.

(C) *Irrelevant alternatives:* If, when the voting systems ranks only candidates P and Q (as in the Condorcet method), P is preferred to Q, then introducing a third option, R, must not make Q ranked above P.

(D) *No cycles:* The system always produces a ranked list, with no cycles.

(E) There are no dictators.

Arrow's Impossibility Theorem: No voting system exists that satisfies all five requirements. In particular, insisting on requirements A, B, C and D necessitates a dictatorship.

DD 7. Show by means of an example that the Condorcet method violates the *no cycles* criterion.

DD **8**. Show that the Condorcet method satisfies the *irrelevant alternatives* criterion.

DD 9. In the 1992 U.S. presidential election, the candidates were Bill Clinton (C), George H.W. Bush (B), and Ross Perot (P), who won 43, 38, and 19 percent of the popular vote, respectively. Suppose that voters' actual preference lists were as follows: CBP (43%), BPC (38%), PBC (19%).

(a) Show that, in a race between Clinton and Bush (removing Perot), Bush wins.

(b) Show that with the *Plurality* method (as was actually used), Clinton wins.

(c) Explain why this violates the *irrelevant alternatives* criterion.

THIS ASSIGNMENT HAS A THIRD PAGE
DON'T MISS PROBLEM 10
IT IS SUPER

DD Geometric ways of measuring partisan gerrymandering, such as the Skew, Isoperimetric and Square Reock measures (Page 13 # 2), tend to actually measure districts' *compactness*. It is arguably more relevant to directly assess whether a districting plan favors one political party over the other. Many ways of measuring this have been proposed; we will study two of them: *partisan symmetry* and *efficiency gap*.

Partisan symmetry. This measure uses the *seats-votes* curve, which has the *vote share* along the x-axis and the *seat share* along the y-axis. We'll follow convention, and use the Republican vote share. One election gives you exactly one data point on this curve. To construct the rest of it, political scientists use the assumption of *uniform partisan swing.*

DD 10. In 2016, the percentages of Republican voters in New Mexico's three congressional districts were 34.9%, 62.7%, and 37.6%. Refer to the picture at the bottom of the page.[3]

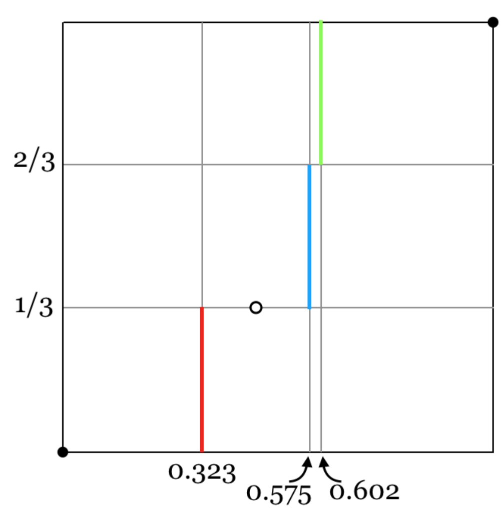
- Assuming an equal number of voters in each district, the Republican vote share was $(34.9 + 62.7 + 37.6)/3 = 45\%$, and the Republican seat share was $1/3$, so the outcome of this election was $(0.45, 0.33)$. This is the white dot in the picture.
- Assuming "uniform partisan swing" means that we add an equal number of percentage points to each district until one of them goes over (or under) 50%. First, we add 12.4% so that the third district hits 50%. This yields the simulated election $(47.3, 75.1, 50)$ for a vote share of $(47.3 + 75.1 + 50)/3 = 57.5\%$ and a seat share of $2/3$, meaning that the seat share jumps from $1/3$ to $2/3$ at a vote share of 0.575 (the blue line).
- Adding 2.7 more percentage points so the first district goes over 50% yields the simulated election $(50, 77.8, 52.7)$ for a vote share of 60.2% and a seat share of $3/3$, meaning that the seat share jumps from $2/3$ to 1 at a vote share of 0.602 (the green line).
- Going the other way from the true election results until all of the districts are under 50% requires subtracting 12.7 percentage points, yielding the simulated election results $(22.2, 50, 24.7)$, for vote share 32.3% and seat share $0/3$, meaning that the vote share jumps from 0 to $1/3$ at a vote share of 0.323 (the red line).

(a) Explain why the points $(0, 0)$ and $(1, 1)$ must both be on *every* seats-votes curve.

(b) Explain why the seats-votes curve is horizontal with 0 seat share between votes shares of 0 and 0.323, and horizontal with $1/3$ seat share between vote shares of 0.323 and 0.575. Fill in and explain the other two horizontal parts of the seats-votes curve.

(c) Based on the seats-votes curve, do you think that New Mexico's districting plan favors Republicans, Democrats, or neither? Explain.

(d) Do you think that the assumption of uniform partisan swing is plausible? Explain.

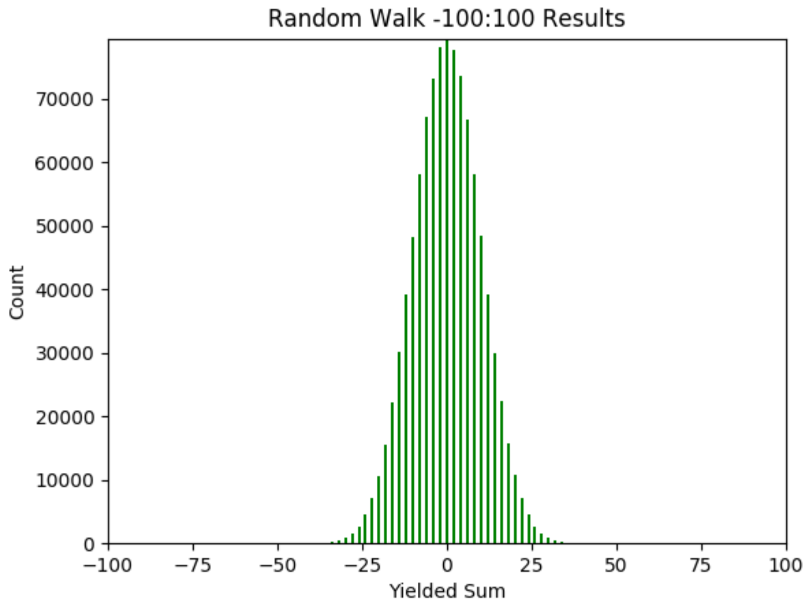


[3]This problem is from Moon Duchin's course Math of Social Choice, Math 19-02 at Tufts University.

Discrete Mathematics

DD **1**. Prove that a connected graph has an Eulerian path if and only if it has 0 or 2 vertices of odd valence.

PEA 2. Eulerizing a graph requires that the vertices of odd valence be paired up. It therefore seems that this technique will not work on a graph that has an *odd* number of odd vertices. What do you think?



DD 3. The results of a million 100-step random walks are at right (picture by Salima Bourguiba). Using the graph (or your data, if you have it), estimate the probability of a result *greater than*
(a) 0 (b) 10
(c) 25 (d) 50.

DD 4. Explain what it might mean to go for a random walk on a *graph*. Choose your favorite graph, devise a method of random walking on it, and record the outcome of a 10-step random walk.

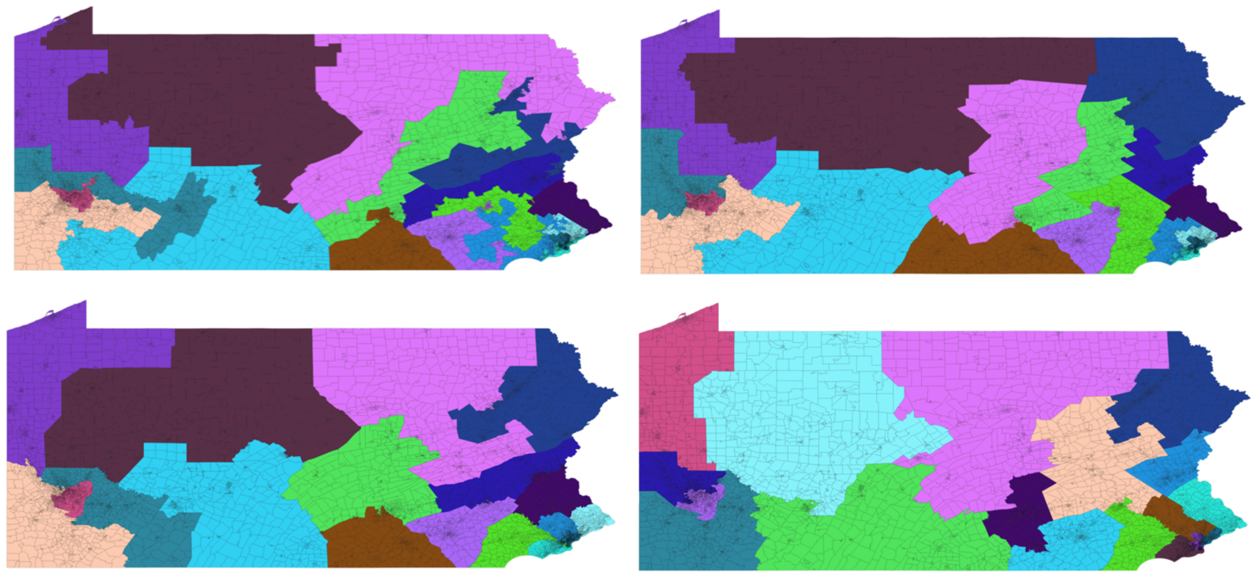
DD 5. *Shape isn't everything.* Historically, in order to create districting plans that favored a particular political party, the people who drew the lines used unusual shapes, such as "Goofy kicking Donald Duck." With modern mapping software, such geometric gymnastics have become unnecessary, and it is possible to create maps with "compact" districts that nonetheless strongly favor one political party.

The pictures on the next page show four plans for dividing Pennsylvania into 18 districts with equal population.[4] Pennsylvania has about 50% Democratic and 50% Republican voters. The plan that contains "Goofy kicking Donald Duck" (GKDD), which was enacted by the (Republican) legislature in 2011, has resulted in electing 13 Republican representatives and 5 Democrats. Of the plans below, two of them result in (actual or predicted) results of 13R-5D, and the other two are close to 9R-9D. Which ones do you think are which?

To be more specific, the maps are (in chronological order of invention) GKDD, the map proposed by the (Republican) legislature to replace GKDD, the map proposed by the (Democratic) governor to replace GDKK, and the map created by an impartial "special master."

DD 6. (Continuation) Indeed, one of the maps on the next page is the current districting plan, enacted earlier this year, under which Tuesday's election (November 6, 2018) was conducted. Look up the results of the election: After Tuesday's elections, how many of PA's 18 representatives are Republicans and how many are Democrats? Who was elected as the congressional representative for district 7, which contains Swarthmore?

[4]Pictures from Moon Duchin's talk "Random Walks and Gerrymandering," available at `https://vimeo.com/293465324`



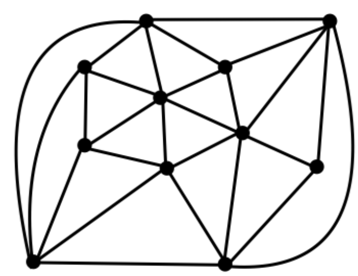
AMS 7. **The Six-Color Theorem.** Every planar graph can be colored with at most six colors.

Proof. To 6-color any planar graph, follow these four steps:

1. Locate a vertex of order 5 or less. This exists because (a) ______________________.
2. Delete that vertex and all edges connected to it.

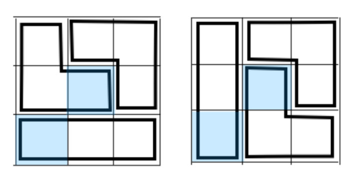
   Keep repeating steps 1 and 2 until only 5 vertices are left. Keep track of the order in which you deleted the vertices.
3. Color the five remaining vertices with the colors 1, 2, 3, 4, 5.
4. Put back the last vertex and edges you deleted. Color that vertex a different color than the vertices adjacent to it. This is possible because (b) ______________________.

Repeat step 4, replacing vertices in the reverse of the order they were deleted. There is always an available color for the replaced vertex, because (c) ______________________. After you put all the vertices back, you will have reconstructed the original graph, and each vertex will be colored with one of the 6 colors.



(d) Run the algorithm from the theorem to color the graph to the right. Record all the steps of the process.

DD 8. Consider a state consisting of 9 towns in a 3×3 block. Your job is to divide them into 3 districts, each consisting of 3 towns, where, as usual, towns within a district must meet along an edge. Two possible ways of doing this are shown. Draw pictures of all possible ways of doing this. How many are there?



DD 9. (Continuation) By flipping which districts the two blue squares are assigned to, we can turn the districting plan on the left into the one on the right. Create a graph, where the vertices are districting plans of this 3×3 block, and an edge connects two plans if, as in this example, a swap of two squares (any two) transforms one into the other. Is it connected? Is it trivalent? Is it planar? DON'T MISS THE 10TH PROBLEM

DD 10. The table below summarizes the calculations from Page 17 # 10. The actual election results are in bold, and the "tipping points" for each district are listed.

New Mexico's 2012 Congressional election

R seats	District 1	District 2	District 3	R vote share
0 or 1	0.222	0.5	0.249	0.323
1	**0.349**	**0.627**	**0.376**	**0.45**
1 or 2	0.473	0.751	0.5	0.575
2 or 3	0.5	0.778	0.527	0.602

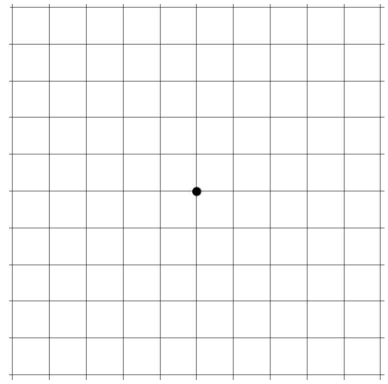
(a) Fill in the table below for Nevada's 2012 Congressional election results, and then produce a seats-votes curve. Do you think that New Mexico's districting plan favors Republicans, Democrats, or neither? Explain.

Nevada's 2012 Congressional election

R seats	District 1	District 2	District 3	District 4	R vote share
0 or 1					
1 or 2					
2	**0.332**	**0.612**	**0.540**	**0.457**	**0.485**
2 or 3					
3 or 4					

(b) Explain why the point $(0.5, 0.5)$ should be on every seats-votes curve (for a state with an even number of representatives), if the map favors neither Republicans nor Democrats.

Discrete Mathematics

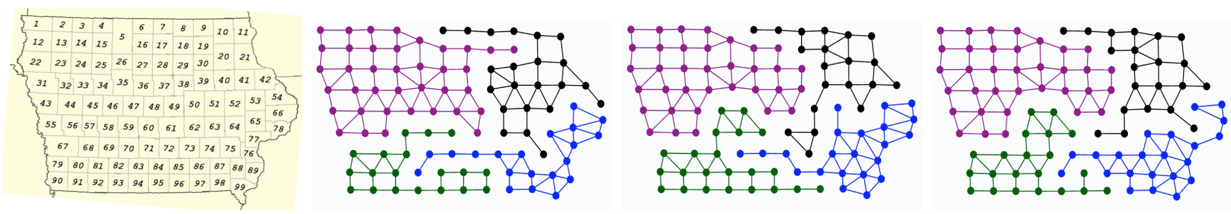


DD 1. *A random walk in the plane.* Start at $(0,0)$ on the integer grid in the plane, and move one unit up, down, right, or left, each with probability 1/4. You can do this by assigning each number 1,2,3,4 to a direction, and asking a random number generator to give you a list of random numbers from 1 to 4 (search for this online). Do a 100-step random walk in the plane, and plot the entire path on the grid (or on a larger grid if necessary).

DD 2. (Continuation) Suppose you did a random walk *forever.* What is the probability that you will return, at least once, to your starting point, for a random walk on

(a) the integers, (b) the integer grid in the plane? (c) the integer grid in 3-space?

DD 3. *Graphs of districting plans.* The laws of the state of Iowa say that each of its 4 congressional districts must be made up of whole counties. A map of Iowa's 99 counties is on the left, and three possible districting plans follow.[5]



(a) Which counties switched districts to get from the first to the second, and to get from the second to the third, plans?

(b) What do you think the *graph* of all possible (connected, equal-population, reasonably compact) districting plans looks like?

(c) What would it mean to go for a *random walk* on this graph?

DD 4. *Notation:* K_n and $K_{m,n}$

(a) The *complete graph on n vertices* is denoted by K_n. Draw pictures of K_4 and K_5.

(b) A two-colorable graph is called *bipartite*. A *complete bipartite graph* on m and n vertices contains all possible edges from the m red vertices to the n blue vertices. Draw in $K_{2,4}$ and $K_{3,3}$ to the right.

(c) Of K_4, K_5, $K_{2,4}$ and $K_{3,3}$, which ones are planar?

DD **5**. A *subgraph* is obtained from a graph by doing any of the following moves:

- Delete an edge.
- Delete a vertex, and all of the edges connected to it.
- Delete a vertex of valence 2, combining the two "dangling" edges into one edge.

Prove that, for any (positive integer) n, K_{n-1} is a subgraph of K_n.

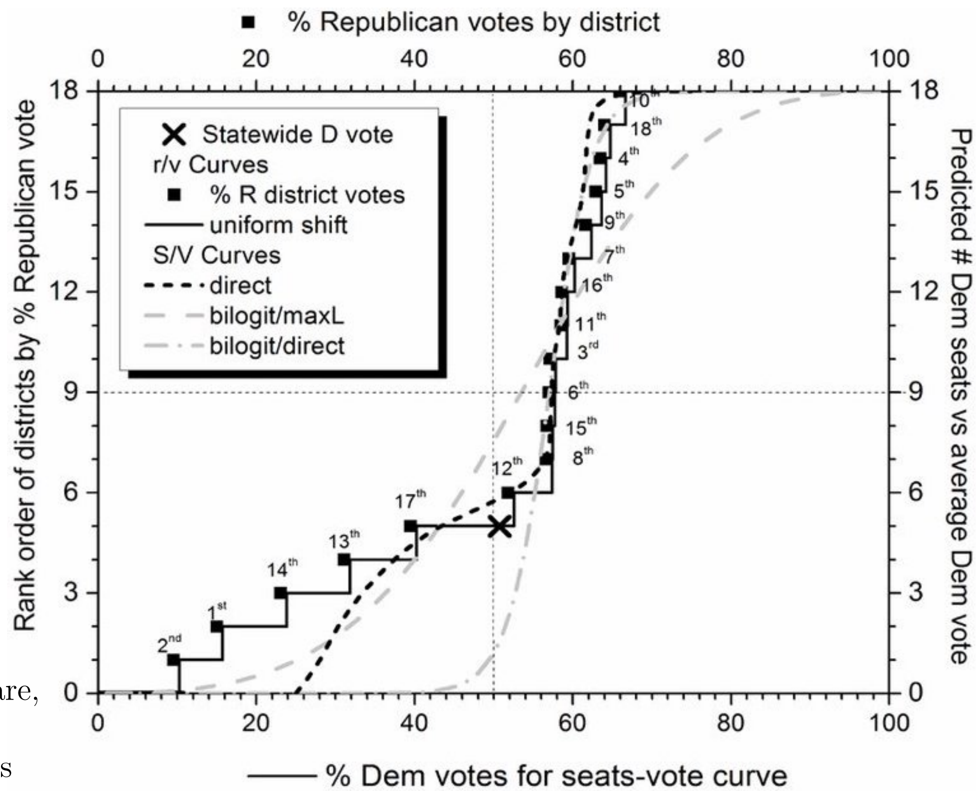
[5]Pictures from Moon Duchin's talk "Random Walks and Gerrymandering," available at `https://vimeo.com/293465324`

Discrete Mathematics

DD **6**. Prove that if a graph contains K_5 or $K_{3,3}$ as a subgraph, then it is not planar.

DD 7. Arrow's Impossibility Theorem (page 17) essentially says that a perfect voting system *does not exist*. For the voting system you devised in Page 15 # 9, which criteria A-E is/are violated?

DD 8. The chart to the right is from the paper *Measures of Partisan Bias for Legislating Fair Elections* by John F. Nagle. You can ignore everything except the bold stair-step curve, which is the seats-votes curve for Pennsylvania's 2012 Congressional election. Here, the x-axis is the Democratic vote share, and the y-axis is the Democratic seats (here *seats*, rather than seat share).



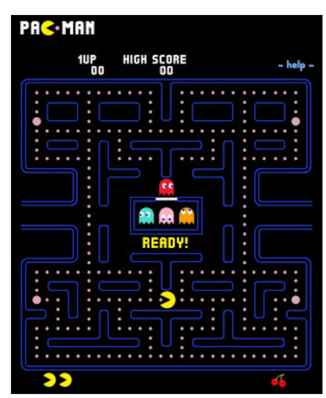
(a) The district that tips over 50% Democratic votes is marked for each stair step. Which districts are packed, and which are cracked? In favor of which party?

(b) The seats-votes curve does *not* pass through the central point $(0.5, 0.5)$ – or in this case, $(0.5, 9)$ – indicating there is partisan bias. In favor of Republicans or Democrats? Explain.

Two measurements in a seats-votes curve have particular meaning.

(c) Consider the horizontal distance from the central point to the curve. This is called the *mean-median* score. What is its meaning?

(d) Consider the vertical distance from the central point to the curve. This is called the *partisan bias* score. What is its meaning?



DD 9. The classic arcade game *Pac-Man* is played on the board shown to the right. Pac-Man, the yellow character below the word "READY!", travels around the tunnels eating dots. The object of the game is to eat all of the dots without running into a ghost.

(a) Play Pac-Man online for at least five minutes. Many web sites allow you to do this; use one that uses the classic board, as shown.

(b) Explain how to use graph theory to find an optimal path for Pac-Man, and then do so.

Discrete Mathematics

Review for midterm 2, which is in the evening after this class.

Here are some topics we have studied, which could appear on the exam:

- graphs, graph coloring, planar graphs, Euler characteristic, Six-Color Theorem, Graph Fact, Planar Graph Fact
- Eulerian paths and circuits, Hamiltonian paths and circuits, applications of these
- maps, districting, gerrymandering, measures of compactness, measures of fairness
- voting methods (lots!), Arrow's impossibility theorem
- apportionment methods (lots!), Huntington-Hill method in particular, geometric mean
- proving the contrapositive, proof by contradiction, proof by induction

The following review problems are provided for your convenience; you are welcome to spend your time working on other problems if you prefer.

0. Make a list of problems, from any page 1-20 in this book, that you would like a classmate or the professor to explain, and any other questions you would like to ask.

DD 1. Give an example of a graph, that no one else will think of, that is not planar. Prove that it is not planar.

PEA 2. For some applications of Eulerian circuits, it is necessary to traverse every edge twice, once in each direction. Give an example of such a situation, and describe how to model it with a graph. Then think about whether it is likely that such a problem has a solution.

PEA 3. The table at right shows adjusted quotas for apportioning delegates to ten states.

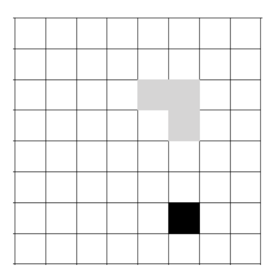
(a) Determine how many delegates each state gets, assuming these quotas are for the Webster method.

(b) Repeat the apportionment, now using the Huntington-Hill method. How many delegates does the state of Euphoria receive? Explain.

State	**Q**
Activity	5.63
Bliss	4.10
Confusion	7.58
Denial	10.52
Euphoria	3.48
Grace	2.85
Hypnosis	8.55
Indecision	9.61
Matrimony	6.17
Panic	6.51

DD 4. Explain the techniques you can use to divide a state into districts in such a way that your own party wins as many districts as possible. Give examples.

DD 5. Which of Arrow's voting methods criteria does the *Point count* method violate? Give an example to justify your answer.



RS 6. (Challenge) Consider a grid of size $2^n \times 2^n$ for $n \geq 1$ (e.g. a chess board is a $2^3 \times 2^3$ grid) with one square missing. Prove that every such grid can be tiled (completely covered with no overlaps, where you are allowed to rotate the shape like in Tetris) by L-shaped pieces made of 3 squares glued together.

n. Revisit problem 0 and make sure you have a list of everything you'd like to discuss in the review.

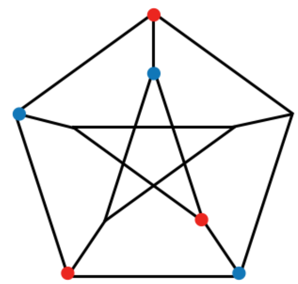
Discrete Mathematics

DD 1. *Kuratowski's Theorem.* So far, we have the following methods to show that a graph is or is not planar:

- To show that it *is* planar, draw the graph with no edge crossings.
- To show that it is *not* planar, use a geometric argument, or the Euler characteristic.

The theorem gives a more precise criterion for when a graph is not planar:

Kuratowski's Theorem: *A graph is nonplanar if and only if it has K_5 or $K_{3,3}$ as a subgraph.*



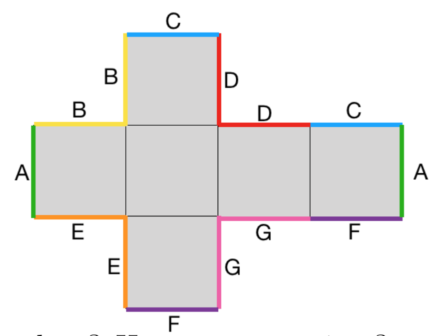
(a) Write statements A and B so that the theorem is in the form $A \iff B$ (A if and only if B).

(b) Which direction is harder to prove, $A \implies B$ or $B \implies A$?

(c) Use Kuratowski's Theorem to show that the *Petersen graph*, shown to the right (which was independently discovered by students in our class!) is not planar, by showing that it contains $K_{3,3}$ as a subgraph. The colored vertices are a hint.

PEA 2. Three investors control all the stock of Amalgamated Consolidated, Inc. One investor owns 46% of the stock, another owns 37%, and the third owns the remaining 17%. When making decisions about the company, each investor gets a proportion of votes based on how much stock they own. Which stockholder is the most powerful?

DD 3. The picture to the right shows a polyhedron that has been cut along some of its edges so that it can be laid flat on the page. The colors (and letters) show which edges in the picture are actually the same edge in the polyhedron.



(a) What polyhedron is this? If you are having trouble figuring it out, cut out the picture and tape the edges of the same color together.

(b) How many faces does the polyhedron have? How many edges? How many vertices?

(c) Counting its faces is pretty easy; counting its edges requires a little more care; counting its vertices is tricky indeed. Can you come up with a method for counting the vertices of this shape, without cutting it out and assembling it?

DD 4. Multiply out the expressions $(L+R)^2$, $(L+R)^3$, $(L+R)^4$, etc. and collect like terms. What does this tell you about a random walk on the number line? Can you design a similar approach for a random walk in the plane?

DD 5. *True story.* My friend Gwyn did a pushup ladder: 1 pushup, then 2, then 3, all the way up to 10, then 9, then 8, back down to 1. I did a smaller ladder: 1, 2, 3, 4, 5, 4, 3, 2, 1.

(a) How many pushups did each of us do? Make a conjecture about the total number of pushups in a ladder up to n and back down, and write the statement down precisely.

Prove your conjecture in three different ways:

(b) By induction; **(c)** using a previous result; **(d)** using geometry.

Here are three measures of gerrymandering.

- The *efficiency gap*. Let's say that a vote is *wasted* if it is a winning vote over 50% in a district, or if it is a losing vote. The *efficiency gap* is

$$\frac{\text{wasted Republican votes} - \text{wasted Democratic votes}}{\text{total votes}},$$

  which is the difference in wasted votes, as a proportion of the total votes in the election. The scholars who popularized the efficiency gap proposed that any election whose efficiency gap is greater than 0.08 in magnitude is probably gerrymandered.

DD 6. In New Mexico's 2012 Congressional election (Page 18 # 10), the Republican vote shares in the three districts were 0.349, 0.627, 0.376, 0.450. Suppose each district had 1000 voters.

(a) Calculate the number of wasted votes for Republicans and Democrats in each district.

(b) Calculate the efficiency gap for New Mexico for this election. (Answer: 0.068.)

(c) Is New Mexico gerrymandered? In favor of which party?

Here are two more measures, which come from the partisan symmetry seats-votes curve:

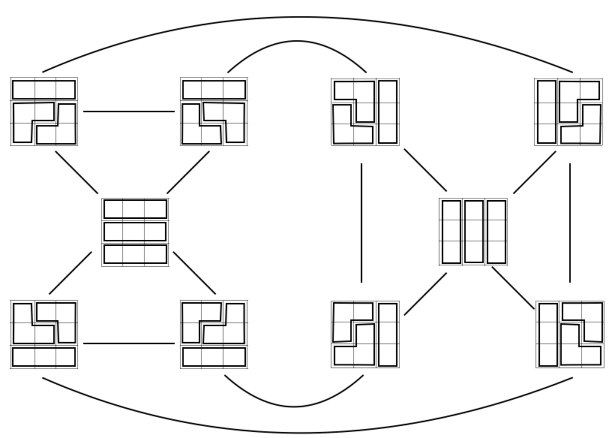
- The *mean-median* score, which is the horizontal distance from the point $(0.5, 0.5)$ to the seats-votes curve (Page 19 # 8b), and
- The *partisan bias* score, which is the vertical distance from $(0.5, 0.5)$ to the same curve.

DD 7. Compute the mean-median and partisan bias scores for New Mexico and Nevada, using the data on page 18c. Also compute the efficiency gap for Nevada using the same data.

DD 8. Which of these measures – efficiency gap, mean-median, and partisan bias – do you think does the best job of detecting gerrymandering? Explain your reasoning.

A new and promising approach to redistricting and gerrymandering analysis is *sampling*, using a random walk on the graph of districting plans.

DD 9. *A random walk on a graph of districting plans.* To the right is a graph of all possible districting plans for the very simple 9-town, 3-district state from Page 18 # 9, where two plans are connected if you can go from one to the other by flipping two towns' district assignments.



(a) Take a 100-step random walk on this graph: Choose a starting point, find some method of assigning numbers 1,2,3 or 1,2,3,4 to its edges, randomly choose one of the numbers, make a step, and repeat.

(b) There are two types of districting plans (2 of one and 8 of the other); does the proportion of how often you land on one or the other depend on what type you start on?

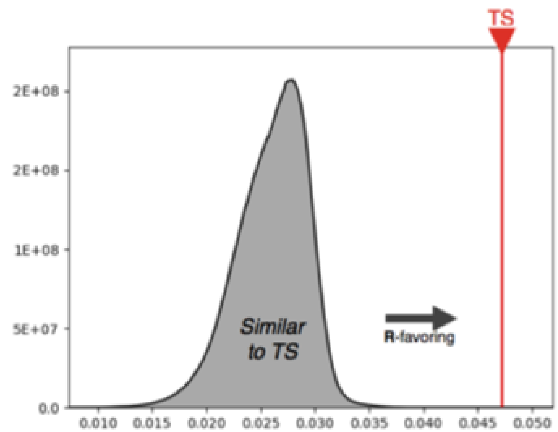
DD 1. *Word graphs*: the vertices are words, and an edge connects two vertices if you can transform one word into the other by changing one letter (see Page 17 # 6).

(a) Make a graph for the word list *LAP, LIP, TAP, TIP, TIN, TOP, TON.*

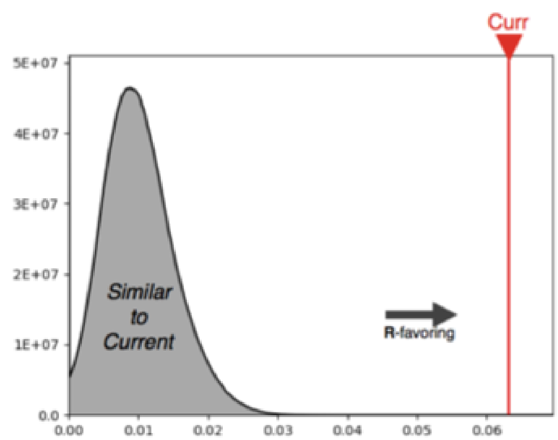
(b) Can you find a word list whose graph is the complete graph on 3 vertices? 4? 5?

(c) Can you find a word list whose associated graph has valence list 2,2,2,2,4?

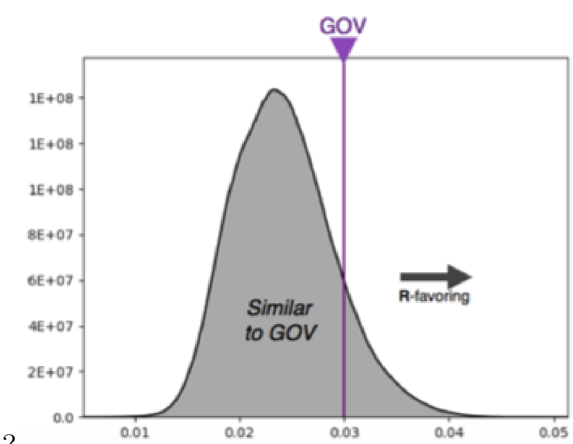
DD *Identifying gerrymandering in Pennsylvania.* As we have discussed, Pennsylvania's "Goofy kicking Donald Duck" congressional district map (labeled here as "Current," and shown in the upper left on Page 18b) was considered to be gerrymandered in favor of Republicans. Two (Republican) senators drew a candidate replacement map ("TS," upper right on 18b), featuring districts that are more compact than in GKDD. The (Democratic) governor also drew a candidate replacement map ("GOV," lower left on 18b). The Governor brought in a mathematician, Professor Moon Duchin, as an expert consultant to assess these plans.



DD 2. Starting with the "Current" (GKDD) districting plan, Duchin had her computer do a random walk on the graph of districting plans for Pennsylvania that were *at least as compact*, and that split *no more counties*, than GKDD. For each of those billion plans, the computer calculated the Mean-Median score (Page 21 # 7), and made a histogram of how many plans had each score. This is the middle picture to the right.[6] The "Current" plan's Mean-Median score is as indicated in red. What can you conclude about whether the districting plan was gerrymandered in favor of Republicans?



DD 3. Duchin repeated the same experiment, starting with the "TS" plan drawn by the senators, and the "GOV" plan drawn by the governor. The results are to the right. What can you conclude about whether each of these districting plans is gerrymandered in favor of one party or the other?



DD 4. The shapes of the histograms in each plot are similar but not the same. Why are they different?

[6] These pictures are from Moon Duchin's expert report, *Outlier analysis for Pennsylvania congressional redistricting*, February 2018, available at `https://sites.tufts.edu/vrdi/files/2018/06/md-report.pdf`.

DD 5. (Proposed by J.C. Charles) For each of the following, prove it or find a counterexample (proofs are eligible for handing in; counterexamples are not):

(a) If a graph has an Eulerian circuit, then it is planar. (b) (the converse)

(c) If a graph has a Hamiltonian circuit, then it is planar. (d) (the converse)

PEA 6. *The game of Sprouts*: Mark a few dots (start with two or three) on a piece of paper. A move consists of joining two dots by an arc that does not touch the rest of the drawing, then marking a new dot on the arc (near the middle). No dot is allowed to have valence greater than three, and new dots introduced during the game begin their lives with valence 2. Play alternates between two players until someone is unable to make a move — this player loses.

(a) Play at least five games of 2-dot and five games of 3-dot Sprouts with a friend.

(b) At most how many moves can the two-dot game last? Who has the advantage?

(c) Do a similar analysis of the three-dot game.

(d) In general, at most how many moves can an n-dot game last?

On Saturday, I (DD) had dinner with other non-tenured faculty and six members of Swarthmore's Board of Managers. I mentioned to one of the Board members (Koof Kalkstein) that we did a problem in class (Page 11 # 1) about scheduling the Board's committee meetings into the least possible number of time slots, and he told me that indeed, this is a big problem they are *actually having*, and that they would love our help! True story. The Board's chairperson, Ed Rowe, later sent me the data (shown on the next page), and asked me to have my students help them out. Here are their requests (also see email).

DD 7. Assume that two committees cannot meet at the same time if they have a member in common. Currently the following meetings take place at the same time: NG and P, AA and D&C, ARM and SR, SA and A&FA – 4 total time slots. Does the current schedule satisfy the requirement that no simultaneous meetings have a member in common? Can you do better – schedule the meetings into 3 time slots? Either do so, or prove that it is impossible.

DD 8. The Board would really like to schedule all of the meetings in 3 time slots. To achieve this, they are willing to have some people miss their meeting – up to one Board member, *maybe two* missing each meeting – in order to achieve this goal. However, a Board member who is chair or vice chair of the committee *cannot* miss that committee's meeting. Can you make this happen? Either do so, or prove that it is impossible.

DD 9. The Board wants to be able to meet in 3 time slots every time in the future. To achieve this, they want you (yes, you) to give them guidelines when choosing which Board members are on each committee, to avoid the problem of needing 4 time slots. For example, you could forbid people from being on more than two committees, or you could have a certain check that you do on a potential table of data. Make at least three suggestions that would help them accomplish this goal, and explain the purpose of each one.

Plan for the future: We will discuss everyone's ideas in class on Tuesday 11/27. Then with a partner, you will write up a 1-2 page report with your ideas and explanations, and you'll present your ideas in class on Tuesday 12/4. We'll combine everyone's ideas into one report from our class. If they like what we send them, we will win a pizza party.

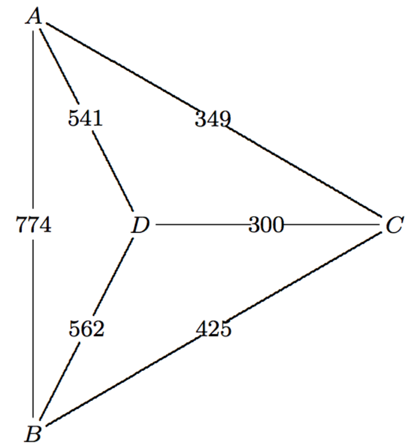
Discrete Mathematics

DD

The table shows Swarthmore's Board of Managers, and their committees: Nominating & Governance, Property, Alumni Affairs, Development & Communication, Audit & Risk Management, Social Responsibility, Student Affairs, and Admissions & Financial Aid. An "X" indicates that they are on the committee; with a VC for vice chair and a C for chair.

Board member	N&G	P	AA	D&C	ARM	SR	SA	A&FA
Sohail Bengali '79		X						
Bill Boulding '77	VC		X			X		
David Bradley '75				X				X
John Chen '76		VC			X			
Rhonda Resnick Cohen '76					X			X
Thom Collins '88		X		X				VC
Elizabeth Economy '84		X					VC	
Thomas Hartnett '94		X						
Marilyn Holifield '69			X		X			C
Emily Anne Jacobstein	X			X		X		
Leslie Jewett '77					C			X
Eleanor Joseph '07						X	X	
Jaky Joseph '06		C		X	X			
Koof Kalkstein '78								
Giles Kemp '72				C	X			
Jane Lang '67		X	X					X
Lucy Lang '03			X			X		X
Cindi Leive '88		X		X			X	
Bennett Lorber '64		X	X				X	
James Lovelace '79			X				X	
Sabrina Martinez '82	X		X					X
David McElhinny '75		X		X				
Cathryn Polinsky '99				VC	X		X	
Vincent Poor		X	X				X	
Lourdes Rosado '85			X			X	X	
Antoinette Sayeh '79							X	
Gus Schwed '84	X							
June Scott '61		X	X					X
Robin Marc Shapiro '78	C			X				
Salem Shuchman '84								
David Singleton '68		X				X		
Thomas Edgar Spock '78		X						
Sujatha Srinivasan '01	X			X	X		C	
Davia Temin '74	X						X	
Joseph Leon Turner '73			X		VC			
Bryan Wolf '84				X			X	
Sam Hayes '57		X						X
Jim Hormel '55								X
Barbara Mathier '65								X

Discrete Mathematics



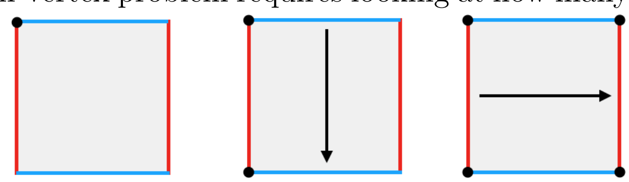
The *Traveling Salesman Problem* (or TSP, for short) is a question about Hamiltonian circuits for complete labeled graphs. Each edge of a complete graph has been labeled with a numerical value, and the objective is to find the Hamiltonian circuit that has the smallest possible sum of edge labels. In the standard application, the vertices represent cities and the labeled edges represent intercity transportation information — mileage, time, or cost, for instance. Other applications: the telephone company wishes to pick up coins from its pay phones; the gas company needs to send out a meter reader; a lobsterman must visit all his traps; an assembly line needs a mechanical device to drill several holes (in succession) in steel plates.

PEA 1. The graph above displays a four-city TSP, with the distances between cities marked on each edge: $AB = 774$, $AC = 349$, $AD = 541$, $BC = 425$, $BD = 562$, and $CD = 300$.

(a) Find the 4-cycle that minimizes the total mileage that our salesman must travel.

(b) When a TSP involves only four vertices, examining every conceivable circuit is a manageable approach, for there are essentially only *three* of them. Explain why.

(c) To apply this *brute-force approach* to a ten-vertex problem requires looking at how many circuits?



DD 2. *Vertex chasing.* As we saw in Page 21 # 3, counting faces is easy, edges are a little harder, and vertices are quite tough indeed. Let's compute the Euler characteristic of the *square torus*, which is a square where parallel edges are glued to each other. To count the vertices, mark any vertex (say, the top left). We want to see which other vertices are the same as this one. The marked vertex is at the left end of the top horizontal edge, so we also mark the left end of the bottom horizontal edge. We can see that the top and bottom ends of the vertical edge on the left are now both marked, so we mark the top and bottom ends of the vertical edge on the right, as well. Now all of the vertices are marked, so the square torus has just one vertex.

(a) Compute the Euler characteristic for the square torus.

(b) What object do you get, if you actually glue up the edges of this square?

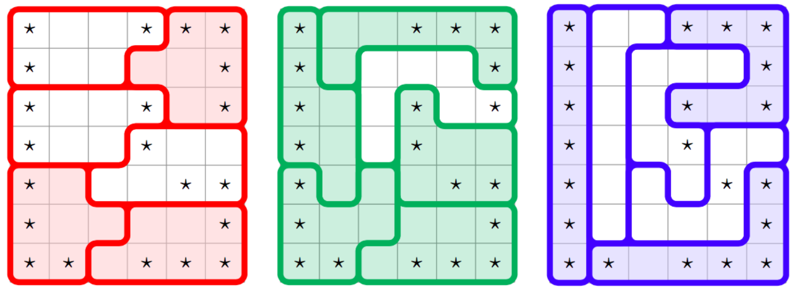
(c) Find the Euler characteristic of a the surface made from a hexagon with its parallel edges glued to each other. What do you think the Euler characteristic measures?

PEA 3. Given a graph, it can be converted into a digraph (**di**rected **graph**) by assigning arrows to each of its edges. This is called *directing a graph.* If the edges of the graph represented streets, the process would correspond to making each street one-way. An interesting question: For what graphs is it possible to assign directions so that the resulting digraph is connected?

(a) Decide what the question means. In particular, what does it mean for a digraph to be *connected*? Why might this matter (if edges were one-way streets, for instance)?

(b) Find a connected graph that *cannot* be made into a connected digraph.

DD 4. A 42-town state has two parties, the *blank* party and the *star* party. The pictures to the right show the locations of the towns voting for each party, and the shading indicates who wins each district with each of the various plans.[7]



(a) For each districting plan shown above, compute the efficiency gap.

(b) For each plan, say which districts are *packed* and which are *cracked* in favor of each party, and discuss the relationship with the efficiency gap you computed for each district.

DD 5. The efficiency gap seems to do a good job in measuring whether districts have been packed or cracked, but here are some undesirable properties that it has.[8] If you are having trouble calculating an efficiency gap, you may assume that the district has 100 voters.

(a) *Penalizes proportionality.* Suppose that the Democratic party wins 60% of the votes, and 60% of the seats. Compute the efficiency gap. It is nonzero and in fact greater than 0.08, so it suggests (see Page 21b) that this is an unacceptable gerrymander. In favor of which party?

(b) *Rewards certain landslides.* For the efficiency gap to be 0 in a district, what must the vote shares be for each party?

(c) *Volatile in competitive races.* For the first (red) districting plan in the previous problem, all of the districts are competitive. Suppose that at the last moment, some last-minute trend (such as the release of damaging information) pushes the voting slightly towards the "star" candidate, so that now there is an additional "star" town in each of the "blank"-winning districts in the picture. What does the efficiency gap say about the districting plan now?

RS **6**. Prove that for each positive integer n, $1^3 + 2^3 + \cdots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$.

PEA 7. A *round-robin* tournament, where each team plays every other team, is to be held among six teams. The results will be recorded in the form of a simple digraph.

(a) How many different digraph summaries are possible?

(b) How many games are to be played in all?

(c) If no team can play more than one game per day, how many days will it take to complete the tournament? Give a day-by-day schedule as justification.

PEA 8. Show by example that it is possible for a six-team round-robin tournament to produce no (undefeated) winner, no (winless) loser, yet no Condorcet 6-cycle, either.

DD 9. With your partner, write up a 1-2-page report for the Board of Managers, solving the scheduling problem as best you can and suggesting solutions (due Tuesday 11/23).

[7] The pictures are from *A formula goes to court: Partisan gerrymandering and the efficiency gap*, by Mira Bernstein and Moon Duchin, May 2017, available at `https://arxiv.org/abs/1705.10812`.

[8] This list of properties is from the same source as above.

Discrete Mathematics

Homework: With your partner, come up with strategies that the Board of Managers can use to assign, or to check the assignments of, Board members to committees.

- Write a 1-2 page document (preferably in LaTeX) that clearly explains each strategy, the reason for it, and how to implement it using a list like the one on Page 22c (or the associated spreadsheet).
- In your explanations, do not assume that your reader knows about graph theory or graph coloring. If you need ideas from this course, you must either explain them completely from the beginning, or find a different way of explaining the same thing that does not use knowledge of graph theory.
- You are welcome to write some code to accomplish whatever you are suggesting; if you do this, please make sure that it is in a form where we could email it to Ed and Koof and they could easily run it in the future on new data.

Plan for class: Each group will have up to 5 minutes (time limit strictly enforced) to explain their ideas. Groups will go in a randomly-chosen order. Each group will have 1/8 of the chalkboard, where you should record the key points of your presentation. At the end, we will discuss as a full class and decide which ideas to send to the Board, and we will write a single document from the class to send to them.

- In your presentation, you do not need to explain the setup of the problem.
- If another group has essentially already explained one of your ideas earlier in class, please *do not* include that idea in your presentation.
- You will hand in your document. Make sure all group members' names are on it. Please also email me (in the text of the email, not as an attachment) the text of your document, either in plaintext or in LaTeX code; this will make it quicker to combine ideas into one document in class.

Discrete Mathematics

DD **1**. Re-prove Page 17 # 1 (about an n-step random walk) using a proof by induction. Make sure to label and correctly use the *base case*, the *inductive hypothesis* and the *inductive step*.

PEA 2. A tennis coach divides her nine players into three teams. Team A has players 1, 6, and 8; team B has 3, 5, and 7; team C has 2, 4, and 9. The strongest player is number 1. In general, player i will defeat player j whenever $i < j$. Which of the three teams is the strongest?

DD

We have studied voting systems (plurality, point count, Condorcet, ...) to elect a *single* candidate. What if we are trying to elect *multiple* candidates to represent a group of people? Here are some election methods that are currently in use.

Geographic districting. The region where the voters live is cut up into geographic *districts*, each of which elects one representative.

At-large voting: Each of the seats (for example, 1 – 18 for PA's Congressional districts) is numbered. Candidates choose a seat number (I'll run for seat #5 since it is my favorite number), and run for just that one seat. Everyone votes for each of the seats.

Pool of candidates: All candidates run in a single election, and everyone votes for their preferred candidate. The candidates with the most votes (the top 18 vote-getters, in the case of PA Congressional representatives) are elected.

DD 3. Geographic districting is used in each state, to elect its allotted number of U.S. Congressional representatives. Describe two strengths and two weaknesses of this system.

DD 4. The city of Santa Clara, CA is 40% Asian, and has six city council members. Despite their large numbers, the Asian community had never been able to elect even *one* city council member using the geographic districting method, so Santa Clara changed to citywide at-large voting.[9] Do you think that the Asian community will be able to elect a candidate of choice under the new method? Explain your reasoning, and justify your answer with an example.

DD 5. The Swarthmore faculty uses the *pool of candidates* method to elect 2 representatives each year to the Committee on Promotion and Tenure (CPT). Each faculty member votes for the person they want on the committee. Here are (made up) election results: Smith 40, Davis 10, Rablen 9, Mathieson 9, Fontes 8.

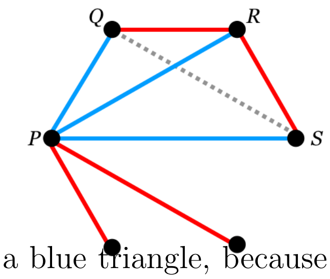
(a) Does this method definitely elect the top two preferred candidates?

(b) Come up with an improvement to this system.

DD 6. Different methods for electing multiple candidates each have strengths and weaknesses. Perhaps your home state or country uses a different system, or perhaps you can think of one that works better. Describe at least one method that we have not discussed yet in this course, and describe the problem(s) it solves.

[9] This example is from Moon Duchin's talk "Random walks and gerrymandering" at 1:02:40, available at `https://vimeo.com/293465324`.

AMS **7**. *Ramsey's theorem.* **Claim.** Take the complete graph on 6 vertices, and color each of its edges red or blue. Then there is always either a red triangle or a blue triangle (or both).



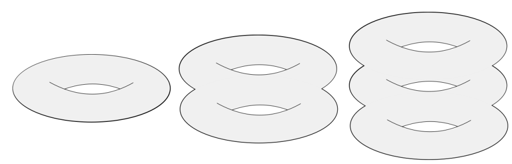
Proof. Choose one vertex and call it P. From P there are five edges going to the other points, so at least three of them must be the same color (say, blue), because (a) ________________. Let's name the vertices on the other ends of the blue edges Q, R, S. Now, if one of the edges QR, RS, or QS is blue, then we can find a blue triangle, because (b) ________________. So let's assume they are all red. But then we can find a red triangle, because (c) ________________.

DD 8. The above claim is *false* for the complete graph on 5 vertices.

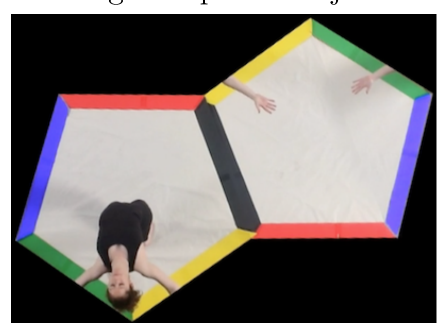
(a) Provide a counterexample.

(b) Explain where the above proof breaks down.

DD 9. One of the main goals of the field of *topology* is to classify surfaces by their *genus*, which, informally speaking, is the number of "holes" it has. The surfaces to the right have genus $1, 2$ and 3. To a topologist, all surfaces of a given genus g are "the same." Here is a joke expressing this fact: "A topologist can't tell the difference between a donut and a coffee mug!" Explain the joke.



2. (Continuation) We can use the Euler characteristic to determine the genus of a surface: A surface with genus g has Euler characteristic $2 - 2g$. Use this to find the genus of the *double pentagon* surface shown to the right. Edges with the same color are glued together. I (DD) studied this surface in my Ph.D. thesis; the award-winning dance video is at `http://tinyurl.com/doublepentagon`.



DD 10. *What the efficiency gap actually measures.* Suppose that a state has n districts, each with P voters. Suppose that Republicans win r of the n districts. Let the overall Republican vote share in the state be V, and let the Republican seat share be $S = r/n$.

(a) Show that the total number of Republican (R) votes in the state is VPn, and find the total number of Democratic (D) votes.

(b) Show that there are $\frac{1}{2}Pr$ non-wasted R votes, and thus $P\left(Vn - \frac{1}{2}r\right)$ wasted R votes.

(c) Calculate the number of wasted D votes. Check your answer by adding it to the wasted R votes and making sure that half of the total votes in the election are wasted (!).

(d) Show that the efficiency gap is $2V - r/n - 1/2$, or in other words $2V - S - 1/2$.

(e) One way of understanding this is that efficiency gap "expects" a party to get a *winner's bonus* of twice as much seat share over 50% as they have vote share over 50%. Explain.

MORE PROBLEMS ON THE THIRD PAGE! #WINNING!

Discrete Mathematics

DD 11. In Page 15 # 6, you considered the case of Massachusetts, which elects 9 representatives, and has 65% Democratic and 35% Republican voters. We made the simplifying assumption that every town has exactly these proportions. The smallest geographic unit for districting purposes is actually not a town, but a *census block*, each of which has about 4,000 people.

(a) Suppose every census block is 65D-35R. Can you draw the districts so that at least one Republican is elected?

(b) Suppose every census block is within 15 percentage points of the average, so that each one has between 50-80% Democratic voters. Can you draw the districts so that at least one Republican is elected?

(c) What has to be true about the proportions of voters in various census blocks, so that it is possible to draw the districts so that at least one Republican is elected?

DD 12. (Continuation) The Voting Rights Data Institute 2018 summer program just (on October 22, 2018) posted a paper, "Locating the representational baseline: Republicans in Massachusetts." Find this paper online and read (at least) the first two pages, and also look Figure 4 and read its caption. Read the whole thing if you have time.

(a) Write down two (or more) sentences explaining why it is *impossible* (not merely difficult) to draw the lines in Massachusetts to create a majority-Republican district.

(b) Can you create a majority-Republican district if it can be made of many small disconnected pieces scattered all over the state? Use evidence from the paper to justify your answer.

For fun, probably in class:

PEA 13. *A map-coloring game*: Player A draws a region. Player B colors it and adds a new region to the diagram. Player A colors B's region and adds a third region. Play alternates in this way until one player is forced to use a fifth color; this player loses the game.

(a) Play this game with a partner at least three times.

(b) Find a winning strategy, or explain why it is not possible.

PEA 14. A high school has 1000 students. Each student is enrolled in five courses, and there are 400 courses being taught.

(a) Explain how to model this situation with a 1400-vertex *bipartite* graph (Page 19 # 4).

(b) How many edges does it have?

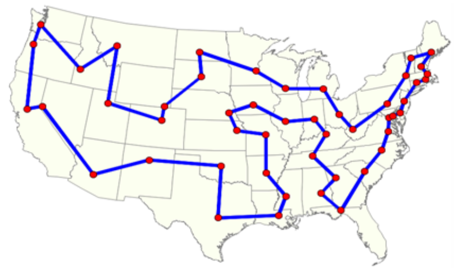
(c) What is the average enrollment per course?

(d) The table below shows the distribution of all the valences for the course vertices. Find x, y, and z, assuming that they are positive integers.

valence	14	13	12	11	10	9	8	7	6
courses	65	155	128	38	6	x	1	y	z

(e) Explain how to model this situation with a different kind of graph.

Discrete Mathematics



DD *More on the Traveling Salesman problem.* The picture shows the minimal-distance tour of the 48 state capitals in the continental U.S. In October 26, after applying some clever math and using 10 months of computer time, a British team published the minimal-distance tour of all 49,687 pubs in the U.K.; you can find a picture online. As you work on the following problems, think about simple examples, and also consider how they would apply to these more complicated TSP applications.

PEA 1. A manageable TSP algorithm: Select any one of the vertices to start at. Travel to that unvisited vertex that is closest to the current vertex. Repeat the preceding step until every vertex has been reached. Finally, close the circuit by traveling back to the starting vertex. This is called the *nearest-neighbor* algorithm, for obvious reasons. Apply it to the example in Page 23 # 1. Does the choice of starting vertex affect the outcome?

PEA 2. Another TSP algorithm: List the edges by cost, in non-decreasing order, breaking ties randomly. The first edge in the list starts the tour. Work through the edge list, rejecting any edge that would make three edges meet at a vertex, or that would close a circuit that does not include every vertex. A complete circuit will eventually be formed. This is called the *sorted-edges* algorithm, or — less clearly — the *cheapest-link* algorithm. Apply it to the same example from Page 23 # 1.

PEA 3. A five-city TSP: The distances are $AB = 60$, $AC = 65$, $AD = 55$, $AE = 70$, $BC = 50$, $BD = 85$, $BE = 95$, $CD = 80$, $CE = 75$, and $DE = 40$. Find the optimal tour.

PEA 4. The nearest-neighbor and sorted-edges algorithms are both examples of *greedy* algorithms, because they only look one step ahead. Give other examples of greedy algorithms.

PEA 5. Neither the sorted-edges algorithm nor the nearest-neighbor algorithm for solving a TSP can be applied to an incomplete graph (a graph that is not the complete graph on the given number of vertices). Explain why.

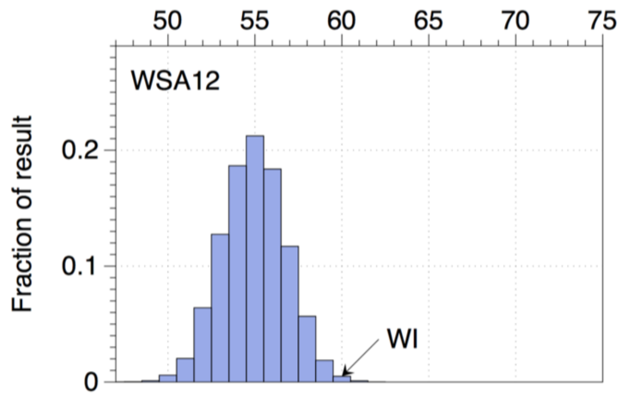
voter	Smith	Davis	Rablen	Mathieson	Fontes
A	1	5	4	3	2
B	1	4	5	3	2
C	1	3	5	4	2
D	1	5	3	2	4
E	1	2	3	4	5
F	5	1	2	3	4
G	4	5	1	2	3
H	3	4	5	1	2

DD 6. The *pool of candidates* method described in Page 25 # 4 is not exactly what the Swarthmore faculty uses, because popular candidates take first-place votes away from other candidates. In fact, everyone submits a ranked preference list of every faculty member they might want on the committee. The method of election is as follows:

- The faculty member with the most first-place votes is elected.
- If the winner got n first-place votes and the second-place person got m first-place votes, the "excess" $n - m$ votes are distributed to the second choices of the n people who voted for the winner, with $(n - m)/n$ from each voter.

The table shows the preference lists for eight faculty members $A, B, \ldots, H$. For example, Professor A's preference list is Smith, Fontes, Mathieson, Rablen, Davis. Which two faculty members are elected under this system? How would you extend the system to elect three or more representatives?

DD 7. For her outlier analysis of Pennsylvania districting plans (Page 22 # 3-5), Moon Duchin used a histogram of the Mean-Median scores for each of the one billion plans visited in the random walk. Other measures are possible, of course. The figure to the right, shows the result of a random walk on districting plans for Wisconsin's state assembly members.[10] The horizontal axis is the *number of seats out of 99 in the Wisconsin state assembly* that would have been won by Republicans, based on the geographic voting patterns in the 2012 election. The vertical axis shows how often that outcome occurred in the 19,184 districting plans visited in a random walk, which range from 49 to 61, with 55 seats being the most frequent outcome. The legislature's plan produced 60 seats for Republicans, making it more extreme than 99.5% of alternative plans.



(a) Write (at least) two sentences explaining why the figure shows that Wisconsin's districting plan was most likely designed to favor Republicans.

(b) Explain why the *number of seats* measure doesn't work as well when the number of seats is small, for example for New Mexico's Congressional representatives (Page 17 & 18 # 10).

DD 8. We have studied various methods to measure whether a districting plan is gerrymandered:

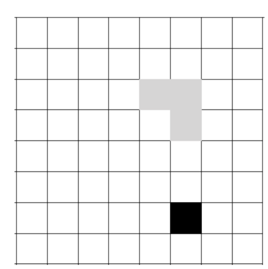
- geometric measures – total perimeter, skew, isoperimetric, . . . (Page 13 # 2)
- the mean-median and partisan bias scores (Page 21 # 7)
- the efficiency gap (Page 21 # 6)
- sampling (random walks on the graph of districting plans) using mean-median score (Page 22 # 3-5) or number of seats (previous problem)

(a) For each method listed above, write down a strength and a weakness.

(b) Make up a gerrymandering measure of your own, and explain the problem(s) it solves.

DD **9**. Prove that in any group of six people, there are three people who all know each other, or three people who all don't know each other.

RS 10. **Claim.** Consider a grid of size $2^n \times 2^n$ for $n \geq 1$ with one square missing. Every such grid can be tiled (completely covered with no overlaps, where you are allowed to rotate the shape like in Tetris) by L-shaped pieces made of 3 squares glued together.



(a) Show by example that the claim holds for 2×2, 4×4 and 8×8 grids. Does it hold for a 6×6 grid?

(b) Prove the claim by induction. Make sure to label and correctly use the *base case*, the *inductive hypothesis* and the *inductive step*.

[10]This figure, and the explanation of it, are from *Gerrymandering metrics: How to measure? What's the baseline?* by Moon Duchin, January 2018, available at `https://arxiv.org/abs/1801.02064`

PEA

Reference

accidental crossing: when two edges of a graph intersect at a non-vertex — an accident of the drawing.

Adams method: a *divisor method* for solving the apportionment problem; favors small states.

adjacent: vertices of a graph that are joined by an edge, or edges of a graph that intersect at a vertex.

adjusted quota: the number you get when you divide a state's population by an arbitrary divisor.

Alabama paradox: an anomaly produced by the *Hamilton method* when the House size is changed.

algorithm: a prescription for carrying out some task mechanically; it need not produce an optimal result.

alkane: in chemistry, an acyclic, saturated hydrocarbon molecule, which can be described by a *tree* in which the valence of each vertex is either 1 (hydrogen) or 4 (carbon).

apportionment: a discrete fair-division problem; the best-known example being to fairly assign each state some of the 435 seats in the House of Representatives.

around-the-world: a puzzle marketed by Hamilton in 1859, this was the first appearance of the Hamiltonian-circuit concept.

Banzhaf index: measures the power of a voter (or voting bloc), by counting pivotal ballots.

bipartite: describes a graph whose vertices can be separated into two groups so that adjacent vertices never belong to the same group. This is is equivalent to saying that the chromatic number of the graph is 2.

Borda count: see the *point-count method*.

Brooks' Theorem: the chromatic number of a graph is at most equal to its largest valence, unless the graph is complete or an odd circuit.

bridge: an edge that — if it were removed — would disconnect an otherwise connected graph.

census: a decennial event, mandated by the Constitution for the purpose of apportionment.

chromatic number: how many colors are needed to color the vertices of a graph.

chromatic index: how many colors are needed to color the edges of a graph.

circuit: a path that starts and finishes at the same vertex.

color a graph: assign colors (or numbers) to the vertices of a graph, so that adjacent vertices receive different colors.

complete graph: a graph in which every pair of vertices is joined by an edge.

Condorcet method: a *round-robin* method of deciding elections that involve more than two candidates.

Condorcet paradox: when all the candidates can be arranged in a cyclic fashion, each one preferred to the next one in the list. [14,20]

Condorcet cycle: see *Condorcet paradox*.

Condorcet winner: a candidate who is preferred to all other candidates; a source in the tournament digraph.

connected graph: a graph in which each pair of vertices can be joined by a chain of adjacent edges.

connected digraph: a directed graph in which, given any starting vertex and any destination vertex, a (directed) path can be found that leads from the starting vertex to the destination vertex.

cost: a numerical label applied to an edge of a graph.

cubic: when referring to graphs, this is synonymous with *trivalent*.

cycle: synonymous with *circuit*.

Dean method: a *divisor method* for solving the apportionment problem.

dictator: in a voting-block situation, a voter who has all the power.

digraph: an abbreviation of "directed graph"; this is a graph in which each edge is assigned a direction.

Dijkstra's algorithm: given a starting vertex in a weighted, connected graph, this method produces a spanning tree that includes minimum-weight paths from the starting vertex to every other vertex in the graph.

direct a graph: apply a direction to each of the edges of a graph, converting it into a digraph.

directed graph: see *digraph*.

disconnected: a graph that is not connected.

district size: the size of the Congressional districts within a given state; obtained by dividing a state's population by its assigned number of representatives.

divisor method: any method of apportionment that applies a rounding rule to *adjusted quotas*, obtained by dividing a fixed but arbitrary number into state populations.

dummy: in a voting-block situation, a voter who has no power.

edge-skeleton: given a polyhedron, this is the graph that results when the faces are removed.

Electoral College: where the President of the United States is chosen; a state's electors typically vote as a bloc.

Eulerian circuit: a circuit in which each edge of a graph appears exactly once.

Eulerian graph: a graph that has an Eulerian circuit.

Eulerian path: a path in which each edge of a graph appears exactly once.

Eulerize: to make a graph Eulerian by adding extra edges.

forest: a disconnected graph that has edges but no circuits.

Four-Color Theorem: the *chromatic number* of a planar graph is at most four.

geometric mean: the geometric mean of two positive numbers a and b is $\sqrt{ab}$.

graph: a network of dots and lines

greedy: an algorithm that looks only one move ahead.

grid: a graph whose vertices represent the intersections of a rectangular system of streets and avenues.

Hamilton method: an algorithm for solving the apportionment problem; rounds up those ideal quotas with the largest fractional parts.

Hamiltonian circuit: a circuit in which each vertex of a graph appears exactly once.

Hamiltonian graph: a graph that has a Hamiltonian circuit.

Hamiltonian path: a path in which each vertex of a graph appears exactly once.

Handshake Theorem: the total of all the valences of a graph must be even; for a digraph, the sum of the invalences must equal the sum of the outvalences.

harmonic mean: the harmonic mean of two positive numbers a and b is $2ab/(a+b)$.

Havel-Hakimi algorithm: a method that determines whether a given list of valences corresponds to an actual graph.

Huntington-Hill method: a *divisor method* for solving the apportionment problem; in current use.

ideal district size: the total population divided by the House size.

ideal quota: a state's fair share of the size of the House.

invalence: counts the number of edges attached to and directed toward a vertex of a digraph.

isomorphism: an equivalence between two graphs, which consists of matching their vertices in such a way that adjacent vertices in one graph correspond to adjacent vertices in the other graph.

Jefferson method: a *divisor method* for solving the apportionment problem; favors large states.

knight's tour: a chessboard recreation that is one of the best-known of all Hamiltonian problems.

Königsberg bridges: there were seven of them, and they became the source of a famous Eulerian puzzle.

Kruskal's algorithm: a sorted-edges method that produces a minimal-cost spanning tree in any (complete) labeled graph.

lower adjusted quota: the largest integer that does not exceed the adjusted quota.

lower quota: the largest integer that does not exceed the ideal quota.

nearest-neighbor algorithm: a strategy for solving the *Traveling-Salesman Problem*; it occasionally produces optimal results.

outvalence: counts the number of edges attached to and directed away from a vertex of a digraph.

path: in a graph, a sequence of edges, with the property that successive edges have exactly one vertex in common; in a digraph, the directions of successive edges also have to agree.

pivotal: a situation where a voter's choice determines the outcome of an election.

planar: a graph that is *isomorphic* to a graph that can be drawn in the plane without accidental crossings.

plurality method: awards an election to the candidate who has the most first-place votes.

point-count method: each candidate receives points from each voter based on that voter's preferences; also called the Borda method.

Prim's algorithm: a multiple-pass method that produces a minimal-cost spanning tree in any (complete) labeled graph; the tree is kept in one connected piece throughout the process.

quota: depending on the context, this can refer to the *ideal quota* or an *adjusted quota*; in the former case, it is the fair share.

regular graph: a graph in which each vertex has the same valence d is called d-regular; for example, a *trivalent* graph is 3-regular.

representational deficiency: the number $r_1(p_2/p_1)-r_2$, where p_i and r_i are the population and the apportionment for State i, respectively, assuming that $r_2/p_2 < r_1/p_1$ (in other words, State 2 has a deficit because it is disadvantaged relative to State 1).

representational surplus: the number $r_2 - (p_2/p_1)r_1$, where p_i and r_i are the population and the apportionment for State i, respectively, assuming that $r_1/p_1 < r_2/p_2$ (in other words, State 2 has a surplus because it is advantaged relative to State 1).

respect quota: an apportionment method does this if it always awards each state either its lower quota or its upper quota, regardless of the census.

round-robin: a tournament in which each competitor faces every other competitor.

sink: in a digraph, a vertex whose outvalence is 0.

sorted-edges algorithm: a strategy for solving the *Traveling-Salesman Problem*; it occasionally produces optimal results.

source: in a digraph, a vertex whose invalence is 0.

spanning tree: given a graph, this is a tree that includes every vertex of the graph.

Sprouts: a pencil-and-paper game that creates networks.

Steiner tree: given a set of vertices in the plane, this is a tree that spans the points and that has minimal total length.

tournament: a complete directed graph.

Traveling Salesman Problem: to find the minimal-cost Hamiltonian circuit in a complete labeled graph.

tree: a connected graph that has no circuits.

trivalent: a graph in which every the valence of every vertex is 3.

upper adjusted quota: the smallest integer that is not less than the *adjusted quota*.

upper quota: the smallest integer that is not less than the ideal quota.

valence: the number of edges attached to a vertex of a graph; see also *invalence* and *outvalence*.

violate quota: this happens when an apportionment awards a state more than its upper quota or fewer than its lower quota.

Vizing's Theorem: the *chromatic index* of a graph is either d or $d + 1$, where d is the largest valence that occurs in the graph.

Webster method: a *divisor method* for solving the apportionment problem.

weight: like a *cost*, a numerical label applied to an edge of a graph; the size of a voting bloc; number used to evaluate a candidate's position on a preferential ballot.

Welsh-Powell algorithm: a method for coloring graphs.

word chain: a puzzle (invented by Lewis Carroll) that is to looking for paths in a large graph.