Math 150

Multivariable Calculus

Mathematics and Statistics Department
Williams College
Williamstown, MA
Spring 2017

To the Student

Contents: As you work through this book, you will discover that the various topics of multivariable calculus have been integrated into a mathematical whole. There is no Chapter 5, nor is there a section on the gradient. The curriculum is problem-centered, rather than topic-centered. Techniques and theorems will become apparent as you work through the problems, and you will need to keep appropriate notes for your records — there are no boxes containing important theorems.

Your homework: Each page number of this book contains the homework assignment for one night. The first day of class, we will work on the problems on page 1, and your homework is page 2; on the second day of class, we will discuss the problems on page 2, and your homework is page 3, and so on for the 36 class days of the semester. Sometimes the night's homework will require more space than one page, so it will be numbered, e.g., pages 3a-3b for day 3. You should plan to spend 2-3 hours each night solving problems for this class.

Comments on problem-solving: You should approach each problem as an exploration. Reading each question carefully is essential, especially since definitions, highlighted in italics, are routinely inserted into the problem texts. It is important to make accurate diagrams whenever appropriate. Useful strategies to keep in mind are: draw a picture, create an easier problem, guess and check, work backwards, and recall a similar problem. It is important that you work on each problem when assigned, since the questions you may have about a problem will likely motivate class discussion the next day.

Problem-solving requires persistence as much as it requires ingenuity. When you get stuck, or solve a problem incorrectly, back up and start over. Keep in mind that you're probably not the only one who is stuck, and that may even include your teacher. If you have taken the time to think about a problem, you should bring to class a written record of your efforts, not just a blank space in your notebook. The methods that you use to solve a problem, the corrections that you make in your approach, the means by which you test the validity of your solutions, and your ability to communicate ideas are just as important as getting the correct answer.

The problems in this text

This set of problems is based on the curriculum at Phillips Exeter Academy, a private high school in Exeter, NH. Some of the problems and figures are taken directly from the *Mathematics 5* book, written by Rick Parris and other members of the PEA Mathematics Department. Many of the problems are taken directly from *Calculus*, by Jon Rogawski and Colin Adams. The rest of the problems were written by Diana Davis, for a multivariable calculus class at Williams College. Anyone is welcome to use this text, and these problems, so long as you do not sell the result for profit. If you create your own text using these problems, please give appropriate attribution, as I am doing here.

Help me help you!

Please be patient with me, as I am patient with you. I have created this set of problems for you, thinking hard about each problem and how they all connect and build the ideas, step by step. When you don't catch those connections, I will wait for you to make the connection, or maybe help you, or maybe wait for a classmate to make the connection.

Just remember that we are all in this together. Our goal is for each student to learn the ideas and skills of multivariable calculus, really learn them — and along the way I will learn new things, too. That's the beauty of this teaching and learning method, that it recognizes the humanity in each of us, and allows us to communicate authentically, person to person.

One way of describing this method is "the student bears the laboring oar." This is a metaphor: You are rowing the boat; you are not merely along for the ride. You do the work, and in this way you do the learning. The next page gives some ideas for ways that you can do this work of moving the "boat," which is our class and your learning, forward.

You might wonder, what is my job as your teacher? Part of my job is to give you good problems to think about, which are in this book. During class, my job is to help you learn to talk about math with each other, and help you build a set of problem-solving strategies. At the beginning, I will give you lots of pointers, and as you improve your skills I won't need to help as much.

I might say things like

- "Please go up to the board and write down what you're saying."
- "Get some colored chalk and add that to the picture on the board."
- "You were confused before, and now it sounds like you understand; could you please explain what happened in your head?"

I am so excited to see what you can do, and hear what you have to say.

Discussion Skills

1. Contribute to the class every day
2. Speak to classmates, not to the instructor
3. Put up a difficult problem, even if not correct
4. Use other students' names
5. Ask questions
6. Answer other students' questions
7. Suggest an alternate solution method
8. Draw a picture
9. Connect to a similar problem
10. Summarize the discussion of a problem

Math 150

First day - in class

1. Find several examples of nonzero numbers a, b, c, d so that $ac + bd = 0$.

2. A bug moves linearly with constant speed across my graph paper. I first notice the bug when it is at $(3, 4)$. It reaches $(9, 8)$ after two seconds and $(15, 12)$ after four seconds. Draw a clear, accurate diagram of this situation. Then predict the position of the bug after six seconds; after nine seconds; after t seconds.

3. Draw the following segments. What do they have in common?

from $(3, -1)$ to $(10, 3)$; from $(1.3, 0.8)$ to $(8.3, 4.8)$; from $(\pi, \sqrt{2})$ to $(7 + \pi, 4 + \sqrt{2})$.

4. (Continuation) The *directed segments* have the same *length* and the same *direction*. Each represents the *vector* $[7, 4]$. The *components* of the vector are the numbers 7 and 4.

(a) Find another example of a directed segment that represents this vector. The initial point of your segment is called the *tail* of the vector, and the final point is called the *head*.

(b) Which of the following directed segments represents [7,4]? from $(-2, -3)$ to $(5, -1)$; from $(-3, -2)$ to $(11, 6)$; from $(10, 5)$ to $(3, 1)$; from $(-7, -4)$ to $(0, 0)$.

5. For a *line* in $\mathbf{R}^2$, you are familiar with the notion of its *slope*. We would like to define an analogous measure for a *plane* in $\mathbf{R}^3$. What would you want this number to mean geometrically?

Math 150

1. Find a way to show that points $A = (-4, -1), B = (4, 3)$, and $C = (8, 5)$ are collinear.

2. Suppose that the vectors $[a, b]$ and $[c, d]$ are perpendicular. Show that $ac + bd$=0.

3. Suppose that $ac + bd = 0$. Show that the vectors $[a, b]$ and $[c, d]$ are perpendicular.

4. The *dot product* of vectors $\mathbf{u} = [a, b]$ and $\mathbf{v} = [m, n]$ is the number $\mathbf{u} \bullet \mathbf{v} = am + bn$. The *dot product* of vectors $\mathbf{u} = [a, b, c]$ and $\mathbf{v} = [m, n, p]$ is the number $\mathbf{u} \bullet \mathbf{v} = am + bn + cp$. In general, the dot product of two vectors is the sum of all the products of corresponding components. Let $\mathbf{u} = [-2, 3, 1], \mathbf{v} = [0, 1, 2]$, and $\mathbf{w} = [1, 2, -1]$. Calculate

(a) $4\mathbf{u}$ **(b)** $\mathbf{u} + \mathbf{v}$ **(c)** $4\mathbf{u} - v$ **(d)** $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w})$ **(e)** $\mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$.

5. The x- and y-coordinates of a point are given by the equations shown below. The position of the point depends on the value assigned to t. Use graph paper to plot points corresponding to the values $t = -4, -3, -2, -1, 0, 1, 2, 3$, and 4. Do you recognize any patterns? Describe them.

$$\begin{cases} x = -2 + 2t \\ y = 10 - t \end{cases}$$

6. (Continuation) The path of the bug in Page 1 # 2 intersects the line given by the equations above. At what point? First answer this question by making a careful sketch on graph paper, and then find a way to solve it using a system of equations. You will have to think carefully about t.

7. Another way to write an equation for a line is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} t$. Explain. This is called a *parametric equation*, and t is called the *parameter*. Give two other parametric equations that describe this same line.

8. Show that the dot product of vectors $\mathbf{u} = [2, -3, 6]$ and $\mathbf{v} = [-6, 2, 3]$ is zero. Can you find a nonzero vector $\mathbf{w} = [x, y, z]$ so that $\mathbf{w} \bullet \mathbf{u} = 0$ *and* $\mathbf{w} \bullet \mathbf{v} = 0$?

9. (Continuation) An alternative notation for expressing these vectors is $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ and $\mathbf{v} = -6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, where $\mathbf{i} = [1, 0, 0], \mathbf{j} = [0, 1, 0]$, and $\mathbf{k} = [0, 0, 1]$. Compute $\mathbf{i} \bullet \mathbf{i}$ and $\mathbf{j} \bullet \mathbf{k}$, and explain why your answers make sense.

10. Find four different points (x, y, z) that satisfy the equation $2x + y + 3z = 6$. Make a clear, accurate diagram in your notebook of the x-, y- and z-axes, and plot the four points on your sketch. What kind of object do you think this equation represents?

Math 150

1. Find coordinates for two points that belong to the plane $2x + 3y + 5z = 15$, trying to choose points that no one else in the class will think of. Show that the vector $[2, 3, 5]$ is perpendicular to the segment that joins your two points.

2. (Continuation for class discussion) Explain why $[2, 3, 5]$ is perpendicular to the plane.

3. Write a *parametric equation* for each of the following:

(a) A line passing through the points $(1, 7)$ and $(9, 3)$.

(b) A line passing through the points $(1, 7, 4)$ and $(9, 3, 5)$.

4. You know how to compute the dot product of two vectors, $[a, b] \bullet [c, d] = ac + bd$, and similarly for three dimensions – but what does it *mean*? We'll give two answers, one now and one later.

Answer 1. The dot product measures how much two vectors point in the "same direction."

Carefully sketch the vectors $\mathbf{u} = [5, 1], \mathbf{v} = [-1, 5], \mathbf{w} = [-3, 2]$. Compute all three pairwise dot products, and use your observations from this data (and you are welcome to collect more!) to fill in each blank below with one of the following characterizations:

are perpendicular   point in similar directions   point generally in opposite directions

- $\mathbf{u} \bullet \mathbf{v} > 0$ when $\mathbf{u}$ and $\mathbf{v}$ __________________________________.
- $\mathbf{u} \bullet \mathbf{v} = 0$ when $\mathbf{u}$ and $\mathbf{v}$ __________________________________.
- $\mathrm{I}\mathbf{u} \bullet \mathbf{v} < 0$ when $\mathbf{u}$ and $\mathbf{v}$ __________________________________.

5. The point $P = (-5, 8)$ is in the second quadrant. You are used to describing it by using the *rectangular coordinates* -5 and 8. It is also possible to accurately describe the location of P by using a different pair of coordinates: its *distance from the origin* and an *angle in standard position* (measured counter-clockwise from the positive x-axis). These numbers are called *polar coordinates*. Calculate polar coordinates for P, and notice that there is more than one correct answer.

THERE ARE MORE PROBLEMS FOR THIS HOMEWORK ON THE NEXT PAGE

6. The graph of the equation $z = 9 - x^2 - y^2$ is a surface called a *paraboloid.*

(a) For what points (x, y) is the surface defined?

(b) Why do you think the surface was named as it is?

(c) Through any point on the paraboloid passes a circle that lies entirely on the paraboloid. Explain. Could there be more than one circle through a single point?

(d) The plane that is tangent to the paraboloid at $(0, 0, 9)$ is parallel to the xy-plane. This should be evident. (If it isn't, do problem 7 now, and graph the surface.) It should also be evident that the plane that is tangent to the paraboloid at $(1, 2, 4)$ is *not* parallel to the xy-plane. Can you think of a way to describe the "steepness" of this plane numerically?

(e) Consider the related function $f(x, y) = 9 - x^2 - y^2$. The set $f(x, y) = 5$ is known as a *level curve.* Explain this terminology. Also sketch this level curve, which lives in the xy-plane.

7. (Continuation) *A note on making your life even more awesome:* Anytime you are investigating a surface, you should graph it.

- Easiest way: Type the equation into Google. Try it now:

  `z=9-x^2-y^2`

- Next-easiest: Type the same equation into WolframAlpha, a super powerful web site.
- If you are on a Mac, search for "Grapher" − it is a wonderful program for graphing that comes standard on the Mac but is not usually listed in the Applications folder.
- There are many free graphing apps for your mobile device.

Try these, and be prepared to report to the class which one(s) you like best.

8. Let $|\mathbf{u}|$ denote the *length* of vector $\mathbf{u}$. Show that $|\mathbf{u}|^2 = \mathbf{u} \bullet \mathbf{u}$.

9. Verify that the area of the parallelogram defined by two vectors $[a, b]$ and $[c, d]$ is $|ad-bc|$. Explain the significance of the absolute-value signs in this formula. The expression $ad - bc$ is an example of a *determinant.* The sign of a determinant is an indication of *orientation*, meaning that it can be used to distinguish clockwise from counterclockwise. Explain.

10. Show that the dot product *distributes over addition*: $\mathbf{v} \bullet (\mathbf{u} + \mathbf{w}) = \mathbf{v} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{w}$. Is the dot product *commutative* − is it always true that $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$?

Math 150

1. Write an equation for the plane that is perpendicular to the vector $[4, 7, -4]$ and that goes through the point $(2, 3, 5)$.

2. We will prove the *Vector Law of Cosines*, using the SSS version of the Law of Cosines that you may remember from a geometry course:

For a triangle where the sides with lengths a and b come together to form angle C, and the side opposite angle C has length c, we have $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

Draw vectors $\mathbf{u}$ and $\mathbf{v}$ tail-to-tail so that they make a θ-degree angle. Draw the vector $\mathbf{u}-\mathbf{v}$, the third side of the triangle, and check to see that it points in the right direction.

(a) Solve for $\cos\theta$ using the SSS version of the Law of Cosines, expressing all lengths in terms of $\mathbf{u}, \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$.

(b) If you use vector algebra to simplify the numerator as much as possible, you will discover the relationship between $\mathbf{u} \bullet \mathbf{v}$ and $\cos\theta$.

3. Consider the equations $2x + 3y + 5z = 15$ and $x - 2y + 2z = 3$. There are many points (x, y, z) whose coordinates fit *both* equations. Two of them are $(1, 1, 2)$ and $(17, 2, -5)$. Find another point whose coordinates fit both equations. What does the configuration of *all* the common solutions look like?

4. *The cross product.* Given two vectors $\mathbf{u} = [p, q, r]$ and $\mathbf{v} = [d, e, f]$, there are infinitely many vectors $[a, b, c]$ that are perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. It is a routine exercise in algebra to find one, and it requires that you make a choice during the process. It so happens that there is a "natural" way to make this choice, and an interesting formula results.

(a) Confirm that $\mathbf{w} = [\,qf - re, rd - pf, pe - qd\,]$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

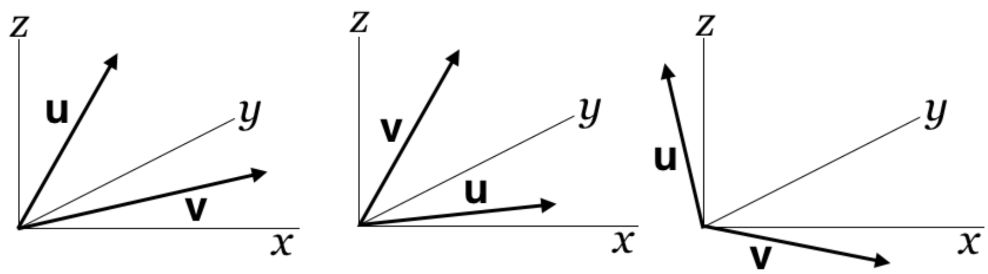
(b) It is customary to call $\mathbf{w}$ the *cross product* of $\mathbf{u}$ and $\mathbf{v}$, and to write $\mathbf{w} = \mathbf{u} \times \mathbf{v}$. Is it true that $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$?

(c) The *direction* of $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. It so happens that the *length* of $\mathbf{u} \times \mathbf{v}$ is the area of the parallelogram spanned by the vectors $\mathbf{u}$ and $\mathbf{v}$. Confirm this fact for vectors $\mathbf{u}$ and $\mathbf{v}$ of your choice, trying to choose vectors that no one else in the class will think of.

(d) Give three explanations of the fact $\mathbf{u} \times \mathbf{u} = \mathbf{0}$. Also explain why the zero is in boldface type.

THERE ARE MORE PROBLEMS FOR THIS HOMEWORK ON THE NEXT PAGE

5. The direction of the cross product $\mathbf{u} \times \mathbf{v}$ is given by the *right hand rule*: Place vectors $\mathbf{u}$ and $\mathbf{v}$ tail-to-tail. Flatten your right hand, and point your fingers in the direction of $\mathbf{u}$. Now curl your fingers in the direction of $\mathbf{v}$ (you may have to flip over your hand to do this). Your thumb points in the direction of $\mathbf{u} \times \mathbf{v}$. For each set of vectors $\mathbf{u}$ and $\mathbf{v}$ below, sketch a vector in the direction of $\mathbf{u} \times \mathbf{v}$. (In these pictures, the x- and z-axes are in the plane of the page, and the y-axis extends away from you. Use your 3D imagination!)



6. If you want to specify a direction, you can use a vector of any length. Give vectors of length 1, 5 and 14 in:

(a) the direction of vector $[3, -4]$,

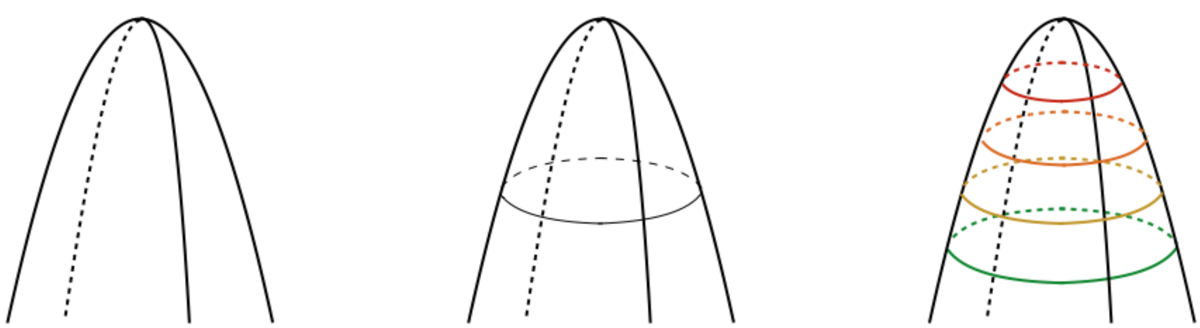
(b) the direction of vector $[-2, 3, 6]$.

A vector of length 1 is called a *unit vector*; it is sometimes convenient to use these.

7. *How to draw a paraboloid in three easy steps.*

1. Draw two parabolas that open in the same direction and share a vertex, and make the back part dashed.
2. Draw an (elliptical) horizontal cross section. Make the back part dashed.
3. Sketch in some more horizontal cross sections in different colors.

Try it: draw a paraboloid in your notebook.



8. The *level curves* of a function $z = f(x, y)$ are the "shadows" of the horizontal cross sections of the surface, in the xy-plane. They are exactly like contour lines on a topographical map (see the 2016 Mountain Day T-shirt – please wear it to class if possible). Make a sketch in the xy-plane, showing the four level curves corresponding to the colored horizontal cross sections in the graph above of the paraboloid surface. Use colors!

1. What does the dot product *mean*?

Answer 2. $\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cos\theta$, where θ is the counter-clockwise angle from $\mathbf{u}$ to $\mathbf{v}$.

Armed with this knowledge, fill in the following statements with right/obtuse/acute:

- $\mathbf{u} \bullet \mathbf{v} > 0$ when the angle between $\mathbf{u}$ and $\mathbf{v}$ is ____________.
- $\mathbf{u} \bullet \mathbf{v} = 0$ when the angle between $\mathbf{u}$ and $\mathbf{v}$ is ____________.
- $\mathbf{u} \bullet \mathbf{v} < 0$ when the angle between $\mathbf{u}$ and $\mathbf{v}$ is ____________.

2. Polar coordinates for a point P in the xy-plane consist of two numbers, r and θ, where r is the distance from P to the origin O, and θ is the size of an angle in standard position that has OP as its terminal ray. Find polar coordinates for each of the following points:

(a) $(0, 1)$ **(b)** $(-1, 1)$ **(c)** $(4, -3)$ **(d)** $(1, 7)$ **(e)** $(-1, -7)$

3. Describe the configuration of all points whose polar coordinate r is 3. Describe the configuration of all points whose polar coordinate θ is 110.

4. Write an equation for a plane that is perpendicular to the plane $2x - y + 3z = 6$ and that passes through the origin.

5. *More on the cross product.* You now have a complicated formula that finds a vector that is perpendicular to both $\mathbf{i}$ and $\mathbf{j}$. Confirm the following special cases:

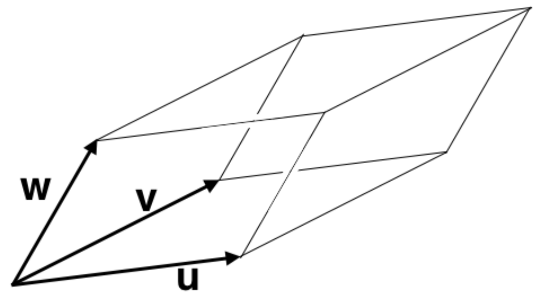
$$\begin{aligned} \mathbf{i} &= \mathbf{j} \times \mathbf{k} \\ \mathbf{j} &= \mathbf{k} \times \mathbf{i} \\ \mathbf{k} &= \mathbf{i} \times \mathbf{j} \\ -\mathbf{i} &= \mathbf{k} \times \mathbf{j} \\ -\mathbf{j} &= \mathbf{i} \times \mathbf{k} \\ -\mathbf{k} &= \mathbf{j} \times \mathbf{i} \end{aligned}$$

Use these results and the usual rules of algebra to derive the rest of the formula for

$$(p\mathbf{i} + q\mathbf{j} + r\mathbf{k}) \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}).$$

6. Write an equation for the plane passing through the points $(3, 1, 1), (-1, 1, 4)$ and $(2, -2, 1)$.

n. Given three arbitrary vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbf{R}^3$, their *triple scalar product* is defined to be $\mathbf{w}\bullet(\mathbf{u}\times\mathbf{v})$.

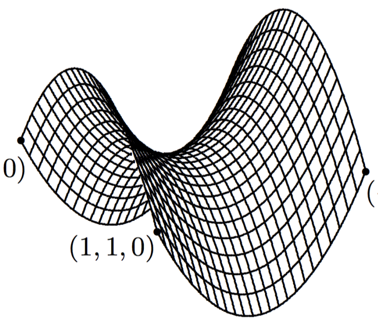


(a) Explain the use of the word "scalar" in the definition.

(b) Show how to interpret the value of this expression as a *volume*.

(c) Show that the same value is obtained from $\mathbf{v}\bullet(\mathbf{w}\times\mathbf{u})$ or $\mathbf{w}\bullet(\mathbf{u}\times\mathbf{v})$.

7. The figure shows the surface $z = f(x, y)$, where $f(x, y) = x^2 - y^2$, and $-1 \le x \le 1$ and $-1 \le y \le 1$. Fifty curves have been traced on the surface, twenty-five in each of the coordinate directions. This saddle-like surfaces is called a *hyperbolic paraboloid*, to distinguish it from the elliptical (or circular) paraboloids you have already encountered. It has some unusual features.



(a) What do all fifty curves have in common?

(b) Confirm that the line through $(1, 1, 0)$ and $(-1, -1, 0)$ lies entirely on the surface.

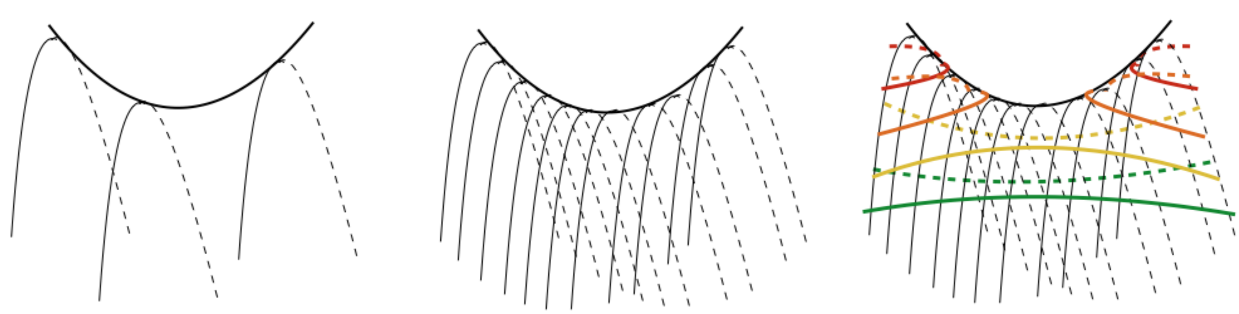
(c) In addition to the line given, there is another line through the origin that lies entirely on the surface. Identify it.

(d) Explain the name "hyperbolic paraboloid."

8. *How to draw a hyperbolic paraboloid in three easy steps.*

1. Draw an upward-facing parabola, and hang some downward-facing parabolas from it.
2. Hang more downward-facing parabolas, with the back part of each dashed or light.
3. Sketch in some horizontal cross sections in different colors (beware – this can make your picture more confusing, so do a good job!).

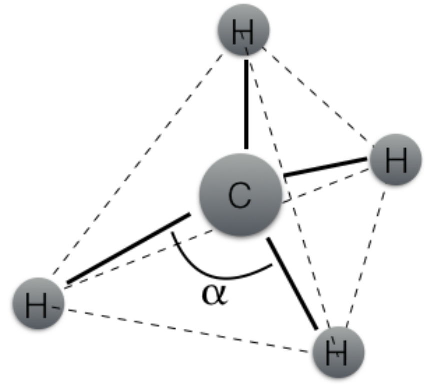
Draw a hyperbolic paraboloid in your notebook.



9. Make a sketch in the xy-plane, showing the level curves corresponding to the colored horizontal cross sections in the graph of the hyperbolic paraboloid surface. Use colors!

Math 150

1. (From Rogawski and Adams, *Calculus*):

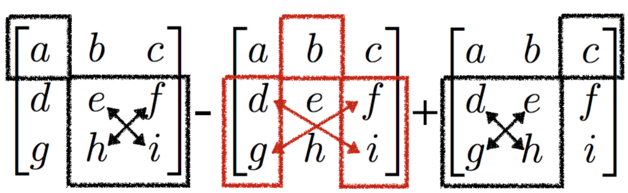


The methane molecule CH_4 consists of a carbon molecule bonded to four hydrogen molecules that are spaced as far apart from each other as possible. The hydrogen atoms then sit at the vertices of a tetrahedron, with the carbon atom at its center, as shown. We can model this with the carbon atom at the point $(1/2, 1/2, 1/2)$ and the hydrogen atoms at $(0,0,0), (1,1,0), (1,0,1)$ and $(0,1,1)$. Find the bond angle α formed between any two of the line segments from the carbon atom to the hydrogen atoms.

2. Given a 3×3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, its *determinant* is, except for sign, the volume of the *parallelepiped* defined by the three column vectors $\mathbf{u} = [a, d, g]$, $\mathbf{v} = [b, e, h]$, and $\mathbf{w} = [c, f, i]$. Show that it can be evaluated as a *triple scalar product* in any of the equivalent forms $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})$, $\mathbf{v} \bullet (\mathbf{w} \times \mathbf{u})$, or $\mathbf{w} \bullet (\mathbf{u} \times \mathbf{v})$, all of which equal $aei + bfg + cdh - afh - bdi - ceg$. The determinant is positive if and only if $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ form a *right-handed coordinate system*, which means that $\mathbf{w}$ is on the same side of the plane formed by $\mathbf{u}$ and $\mathbf{v}$ as $\mathbf{u} \times \mathbf{v}$ is.

3. There is an easier way to remember the formula for the determinant, which in turn gives us an easier formula for the cross product:

$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$. Graphically, this looks like:



(a) Check that this method agrees with the formula given in problem 2 above.

(b) If we allow ourselves to use the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as entries in our matrix (!!!), then the cross product of $\mathbf{u} = [p, q, r]$ with $\mathbf{v} = [d, e, f]$ is given by

$$\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ p & q & r \\ d & e & f \end{bmatrix}.$$

Use this to find a vector that is perpendicular to both $[2, -3, 6]$ and $[-6, 2, 3]$. Does your answer agree with your answer to Page 2 # 8?

4. Let $P_0 = (p, q, r)$, $\mathbf{n} = [a, b, c]$, and $X = (x, y, z)$. Write an equation that says that the vector P_0X is perpendicular to $\mathbf{n}$, and simplify your equation as much as possible. Sketch $P_0, \mathbf{n}$, and an example X. What does the configuration of *all* such points X look like?

5. Convert the polar pair $(r, \theta) = (8, 150)$ to an equivalent Cartesian pair (x, y).

6. Given polar coordinates r and θ for a point, how do you calculate the Cartesian coordinates x and y for the same point?

7. Two intersecting lines define a plane. Confirm this by holding up two pencils in an "X" shape. The lines

$$\ell_1 : \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} t$$

$$\ell_2 : \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} t$$

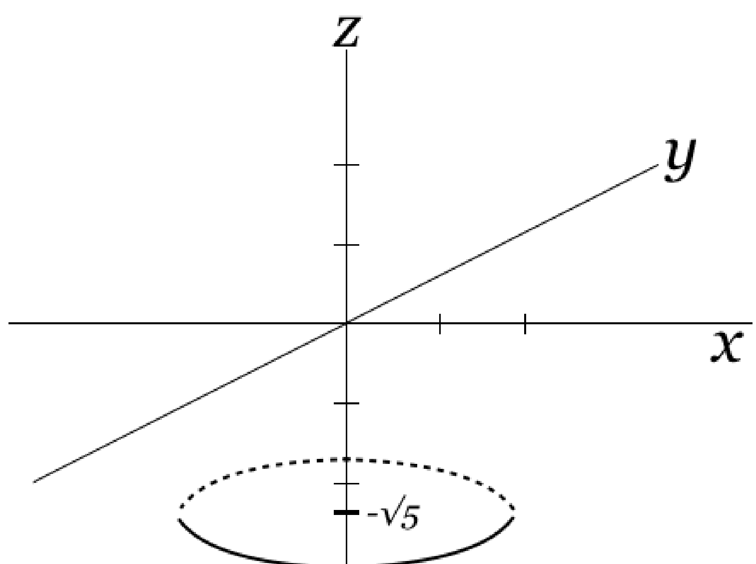
intersect (at what point?), so together they define a plane; find an equation for it.

8. Verify that the point $P = (2, 1, 3)$ is on the *hyperboloid* $x^2 + 4y^2 + 1 = z^2$.

(a) Sketch cross sections of this surface for $z = -\sqrt{5}, -\sqrt{2}, -1, 0, 1, \sqrt{2}, \sqrt{5}$.

For example, we plug in $z = -\sqrt{5}$ and get

$$x^2 + 4y^2 + 1 = 5 \implies x^2 + 4y^2 = 4.$$



This is an ellipse, which contains the points $(2, 0)$, $(-2, 0)$, $(1, 0)$, $(-1, 0)$, so it has a width of 2 in the x-direction and a width of 1 in the y-direction. I have drawn this ellipse in the picture at a height of $z = -\sqrt{5} \approx -2.2$. Now you do the rest.

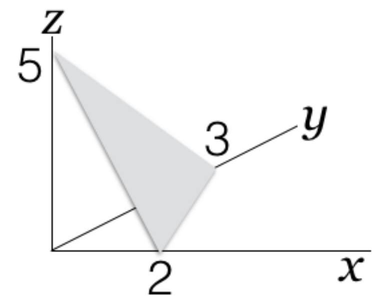
(b) After you have sketched in these seven horizontal cross-sections, describe what you think the entire surface looks like.

9. Verify that the point $Q = (7, 2, 8)$ is on the hyperboloid $x^2 + 4y^2 - 1 = z^2$.

(a) Show that every level curve $z = k$ is an ellipse.

(b) Conclude that this hyperboloid is a connected surface, in contrast to the preceding example, which had two separate parts. (Using the classical terminology, this is a *one-sheeted* hyperboloid, and the preceding example is a *two-sheeted* hyperboloid.) Make a sketch of the surface that is consistent with your findings.

Math 150



1. Find an equation for the plane that contains the triangle shown at right.

2. Sketch the plane given by the equation $6x + 3y - 3z = 12$.

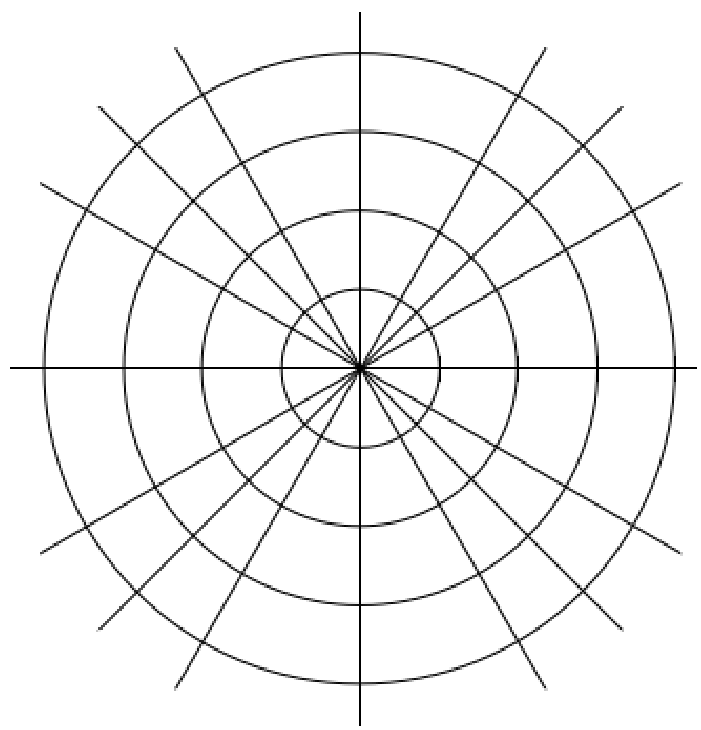
3. Given two planes, how would you choose to define the *angle between* them? Apply your definition to find the angle between two planes that have normal vectors $\mathbf{n}_1 = [1, 0, 1]$ and $\mathbf{n}_2 = [1, 2, -1]$, respectively.

4. Figure out what the curve $r = \sin\theta$ looks like, in two different ways:

(a) Fill in the following table, and then plot the points on the "polar graph paper" below right. It has circles of radius $0.25, 0.5, 0.75$ and 1, and rays at multiples of $\theta = \pi/6$ and $\theta = \pi/4$. (You may want to obtain decimal approximations for $\sin\theta$, to aid you in plotting.) You will have to think about what a "negative radius" means. Then connect the points to create a sketch of what the whole curve looks like.

(b) Multiply both sides of the equation $r = \sin\theta$ by r, convert to rectangular coordinates x and y, and complete the square to put the equation into a familiar form. Does your equation agree with your sketch in part (a)?

θ	$r = \sin\theta$
0	
$\pi/6$	
$\pi/4$	
$\pi/3$	
$\pi/2$	
$2\pi/3$	
$3\pi/4$	
$5\pi/6$	
π	
$7\pi/6$	
$5\pi/4$	
$4\pi/3$	
$3\pi/2$	
$5\pi/3$	
$7\pi/4$	
$11\pi/6$	



5. A mosquito flies at a constant speed according to the equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4+t \\ 1 \\ 1-t \end{bmatrix}$. A spiderweb, with a patient spider, hangs in the position of the plane $2x + 3y + 5z = 15$. Will the mosquito get caught in the web, and if so, when and where?

6. The surface $4x^2 + y^2 + 16z^2 = 16$ is known as an *ellipsoid*. Show that every cross section parallel to a coordinate plane (so, planes of the form $x = c$, or $y = c$, or $z = c$, for any constant c) are ellipses, and then sketch this surface.

7. Explain why the surface $x^2 + 4y^2 = z^2$ is a *double-napped cone*, and sketch it.

8. (From Rogawski and Adams, *Calculus*):

(a) Carefully sketch the vectors $\mathbf{u} = [1, 0, 4], \mathbf{v} = [1, 3, 1], \mathbf{w} = [-4, 2, 6]$.

(b) Find the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.

(c) Find the volume of the parallelepiped spanned by $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$.

9. *Cylindrical coordinates* are a self-explanatory extension of polar coordinates to 3-dimensional space. The coordinate transformation is $(x, y, z) = (r\cos\theta, r\sin\theta, z)$, where $r^2 = x^2 + y^2$. Transform the equation that describes the unit sphere in rectangular coordinates, $x^2 + y^2 + z^2 = 1$, into an equation in cylindrical coordinates.

10. *The cycloid.* A wheel of radius 1 rolls along the x-axis without slipping. A mark on the rim follows a path that starts at $(0, 0)$, as shown in the figure below.

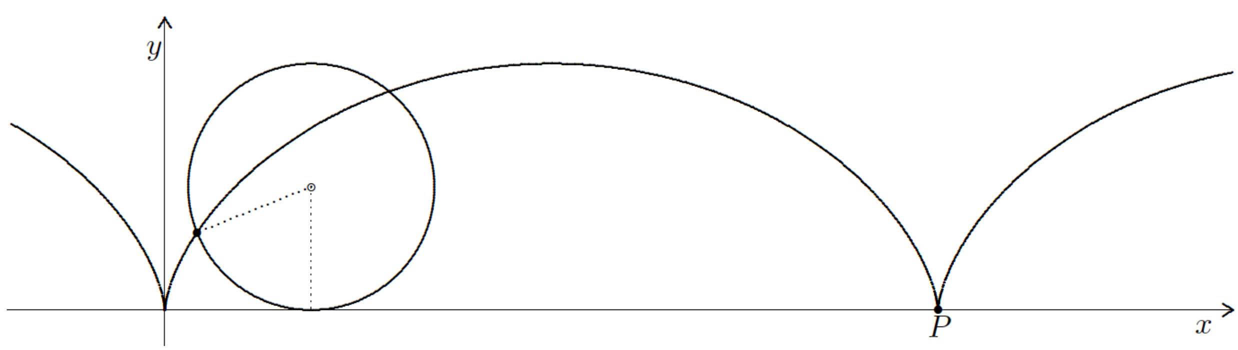
(a) Find the x-coordinate of the point P where the mark first returns to the x-axis.

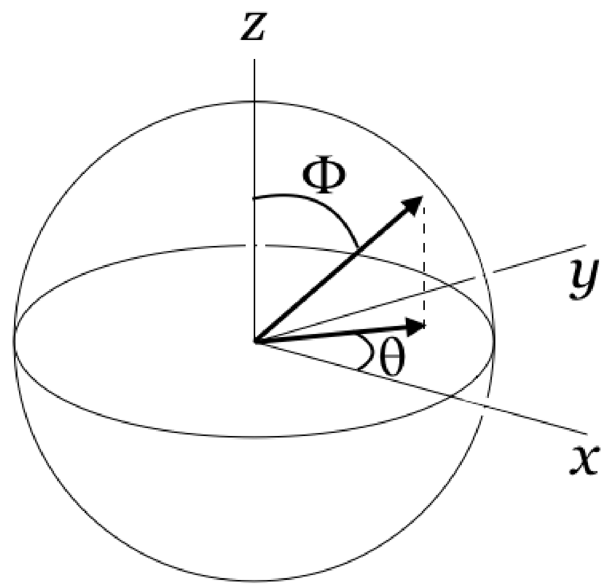
(b) Find both coordinates of the center after the wheel makes a quarter-turn.

(c) Find both coordinates of the mark after the wheel makes a quarter-turn.

(d) Find both coordinates of the mark after the wheel rolls a distance t, where $t < \frac{1}{2}\pi$.

(e) Check your formulas to see whether they are also correct for $\frac{1}{2}\pi \le t$.

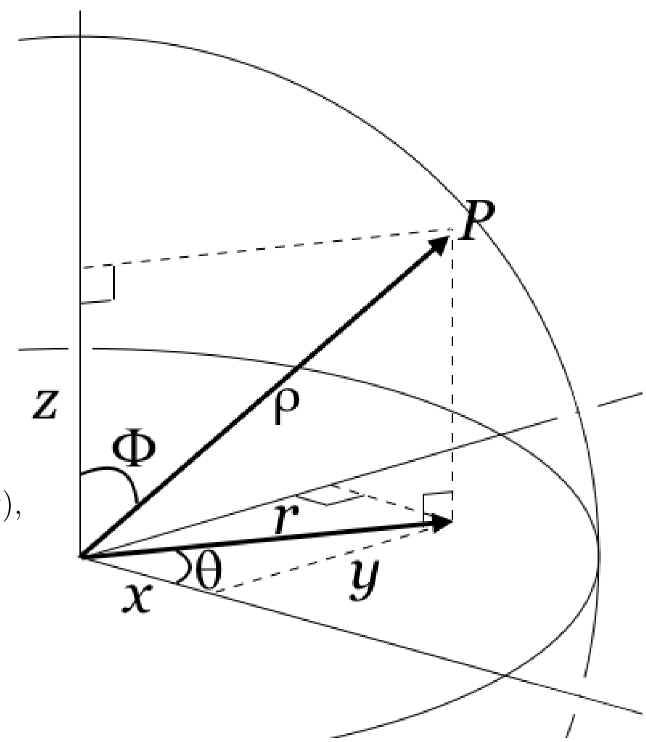




1. We have already seen *Cartesian coordinates* (x, y, z) and *cylindrical coordinates* (r, θ, z). *Spherical coordinates* are yet another way of using 3 numbers to specify a location in 3-space. Points on the unit sphere can be described parametrically by ϕ, the angle measured down from the z-axis, and θ, the angle in standard position measured from the positive x-axis. The third coordinate, ρ, measures distance from the origin.

In navigation on the Earth, θ is the angle usually called *longitude* (assuming that the Prime Meridian intersects the x-axis), and is our familiar θ from polar coordinates. The angle ϕ is the *complement* of the angle usually called *latitude*; it is the angle measured down from the North Pole. The Greek letter ϕ is called "phi," pronounced *fee*. The Greek letter ρ is called "rho" and is pronounced *roe*. We usually take $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$, and $\rho \geq 0$.

Look up the longitude and latitude of Williamstown, and the radius of the earth, and plot our location on the sphere shown. Also explain the (mathematical) difference between the r used in cylindrical coordinates, and the ρ used in spherical coordinates.



2. We would like to be able to translate back and forth between rectangular coordinates (x, y, z) and spherical coordinates (ρ, ϕ, θ). The figure on the right shows a zoomed-in version of the *first octant* in 3-space, with a point P on the surface of a sphere. Use the picture to find the coordinates x, y and z of P in terms of its coordinates ρ, ϕ and θ. *Hint*: first find the distance r in the xy-plane in terms of ρ and ϕ, and then use r to find x and y.

3. (Continuation) Given a point $P = (x, y, z)$, how do you find the spherical coordinate ρ?

4. Describe the configuration of all points whose

- cylindrical coordinate r is 3
- cylindrical coordinate θ is $110°$
- cylindrical coordinate z is -2
- spherical coordinate ρ is 5
- spherical coordinate ϕ is $\pi/4$

5. A minnow swims towards the surface of the water according to the equation

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} t \\ 1+2t \\ -14+4t \end{bmatrix}.$$

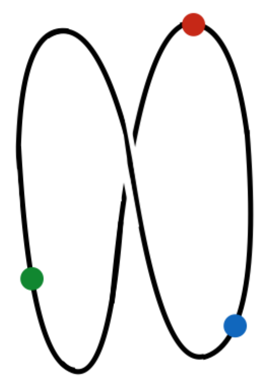
A fishing net hangs in the water in the shape of the surface $z = x^2 + y^2 - 20$. Does the minnow pass through the net?

6. (Continuation) An equivalent expression of the minnow's path is $\mathbf{r}(t) = [t, 1+2t, -14+4t]$. Here t is measured in seconds, and distance is measured in cm.

(a) How fast is the minnow moving (in cm/sec) in the x-direction? How fast is it moving in the y-direction? How fast is it moving in the z-direction? What is the speed of the minnow through the water?

(b) The *velocity vector* of the minnow's path is $\mathbf{r}'(t) = [x'(t), y'(t), z'(t)]$. Find this vector, and explain what it means in the context of the minnow's swim.

7. The figure shows the graph of the curve $\mathbf{r}(t) = [\cos t, \sin t, 2\sin 2t]$.



(a) For which values of t, x and y do the maximum z-values occur?

(b) For which values of t, x and y do the minumum z-values occur?

(c) Use the previous parts to accurately sketch in the x, y, z-axes.

(d) Sketch a *tangent line* to the curve at each of the colored points.

(e) Compute the velocity vector $\mathbf{r}'(t)$, and use it to find an equation for the tangent line to the curve at $t = \pi/4$. Check that your solution agrees with your sketch.

8. You have studied several surfaces described by equations of the form $x^2 + 4y^2 + c = z^2$, for $c = -1, 0, 1$. (Go back and find them now.)

(a) Sketch these three surfaces, and then make a picture that has *all* of them in the same picture, on the same axes.

(b) Now imagine that these are frames from a "movie," at times $t = -1, t = 0, t = 1$. If you smoothly change the value of c, and watch a movie of the surface you get for each value, what would it look like?

9. Consider the parabolic curve $y = x^2$, which goes through the point $P = (1, 1)$.

(a) In terms of c, write an expression for the line that is orthogonal to the curve at (c, c^2).

(b) Assuming that $c \neq 1$, find the intersection R_c of this line with the line $y = \frac{1}{2}(3 - x)$, which is orthogonal (perpendicular) to the curve at P.

(c) The intersection point R_c depends on the value of c. Find the limiting position of R_c as c approaches 1. This is called the *center of curvature* of the parabola at P.

(d) Explain the terminology, and calculate the *radius of curvature* of the parabola at P.

Math 150

1. If $[x(t), y(t)]$ is a parametric curve, then $\left[\frac{dx}{dt}, \frac{dy}{dt}\right]$ is its velocity and $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is its speed. Find a parameterized curve whose speed is $\sqrt{t^4 - 2t^2 + 1 + 4t^2}$.

2. The integral $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ is a template for what type of problem?

3. The graph of an equation $y = f(x)$ is a curve in the xy-plane. The graph of an equation $z = f(x, y)$, on the other hand, is a *surface* in xyz-space. A familiar example is the graph of $z = \sqrt{9 - x^2 - y^2}$, which is a hemisphere of radius 3. For what points (x, y) is this function (and the surface) defined? Find an equation for the *tangent plane* to the hemisphere at $(2, 1, 2)$. *Hint*: Draw a picture.

4. A bumblebee flies lazily through a room according to the equation $\mathbf{r}(t) = [10\cos t, 10\sin t, t]$.

(a) Sketch the path of the bumblebee. (You may want to make a chart of the values of x, y, z at different values of t, and plot the resulting points in 3-space.)

(b) The sun is directly overhead, and is shining through a skylight onto the bumblebee, so that its shadow is on the floor. What does the shadow look like as the bee flies around?

(c) The ceiling is at $z = 10$ feet. Will the bumblebee hit the ceiling? If so, where?

(d) At time $t = \pi$, which direction is the bee flying? You will have to think of a way to express this direction.

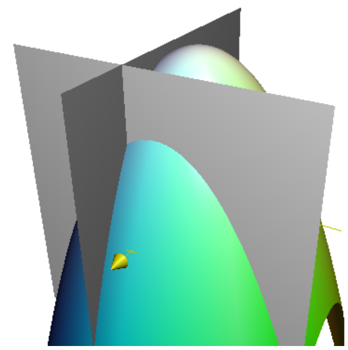
5. The graph of an equation $y = f(x)$ is a curve in the xy-plane. The graph of an equation $z = f(x, y)$, on the other hand, is a *surface* in xyz-space. A familiar example is the graph of $z = \sqrt{9 - x^2 - y^2}$, which is a hemisphere of radius 3. For what points (x, y) is this function (and the surface) defined? Find an equation for the *tangent plane* to the hemisphere at $(2, 1, 2)$. *Hint*: Draw a picture.

6. If you put a ball in a long sock and whirl it around above your head (try it!), the position of the ball at time t will be something like $\mathbf{r}(t) = [\cos t, \sin t, 2]$, where t is measured in seconds and distance is measured in meters.

(a) Velocity is the derivative of position, so it measures how position is changing. Compute the *velocity vector* $\mathbf{r}'(t)$ and explain its meaning in the context of the ball and sock.

(b) Acceleration is the derivative of velocity, so it measures how velocity is changing. Compute the *acceleration vector* $\mathbf{r}''(t)$ and explain its meaning in the context of the ball and sock.

7. A *contour map* is another word for a sketch of the level curves of the surface. Your 2016 Mountain Day T-shirt is a contour map of the elevation function on Mount Greylock and the vicinity. Draw a contour map (in the xy-plane) for the function $f(x, y) = 2x - y$, and label the "elevation" of each line on your contour map. You might start with level curves $f(x, y) = c$ for $c = -1, 0, 1$ and then add some more.



8. A *level curve* is the intersection of a horizontal plane $z = c$ with a surface $z = f(x, y)$. On the other hand, a *vertical trace* is the intersection of a plane of the form $x = c$ or $y = c$ with such a surface. The picture on the right shows the surface $f(x, y) = 5 - x^2/3 - y^2/3$, with the planes $x = 1$ and $y = 2$. The axis that you see pointing out of the paraboloid is the positive y-axis.

(a) Describe what all of the vertical traces, of the form $x = c$ and $y = c$, look like for this surface.

(b) Estimate the slope of the intersection line between the surface and each plane.

9. (Continuation) What do the vertical traces look like for the surface described by the function $f(x, y) = 2x - y$ from problem 7? Draw a picture, like the picture above, with vertical planes slicing through that surface.

10. Let $T(x, y)$ be the temperature at point (x, y) on a rectangular plate (a modern stovetop, perhaps) defined by $a \le x \le b$ and $c \le y \le d$. If T is a non-constant function, then is natural to wonder how to describe rates of temperature change. For example, if the values $T(9.0, 12.0) = 240.0$ and $T(9.03, 12.04) = 239.0$ are measured, then it is possible to calculate an approximate value at $(9.0, 12.0)$ for the *directional derivative* of T in the direction defined by the *unit* vector $\mathbf{u} = [0.6, 0.8]$. Do so. The actual value of the derivative is usually denoted by $D_{\mathbf{u}}T(9.0, 12.0)$. If you wanted to calculate a more accurate value for $D_{\mathbf{u}}T(9.0, 12.0)$, what data would you gather?

Math 150

1. The cycloid $(x, y) = (t - \sin t, 1 - \cos t)$ is the path followed by a point on the edge of a wheel of unit radius that is rolling along the x-axis. The point begins its journey at the origin (when $t = 0$) and returns to the x-axis at $x = 2\pi$ (when $t = 2\pi$), after the wheel has made one complete turn. What is the length of the cycloidal path that joins these x-intercepts?

2. The USA women's soccer team made penalty kicks in their quarterfinal match against Sweden in the 2016 Olympics. Let's assume that the line between the kicker and the goalie is the x-axis, and that the y-coordinate measures height above the ground in feet. The kicker kicks the ball at $(0, 0)$ with an initial velocity vector of $[40, 32]$, measured in feet per second. Make a sketch of this situation.

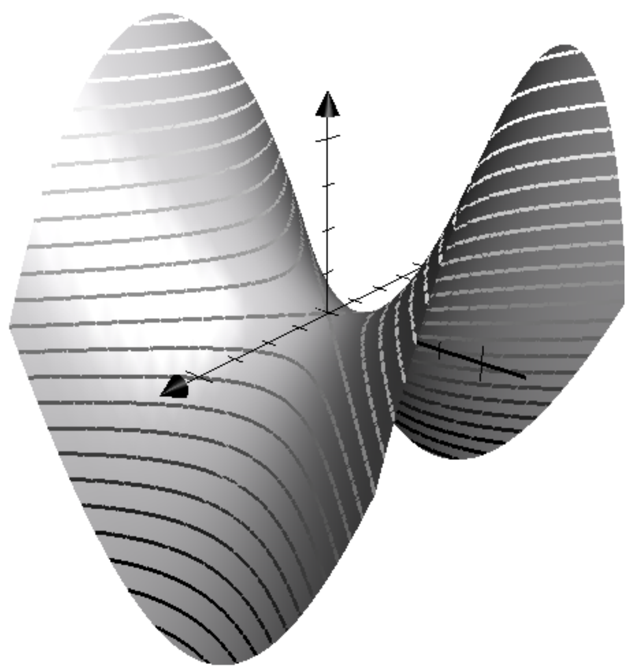
(a) While it is in the air, let's assume there's no wind, so the only force acting on the ball is gravity, which has a force of -32ft/sec^2. So the acceleration vector of the soccer ball is $\mathbf{r}''(t) = [0, -32]$. Integrate this to find $\mathbf{r}'(t)$.

(b) You should have some integration constants in your answer to (**a**). The initial velocity $\mathbf{r}'(0)$ was given in the problem; use this to find the constants and give an expression for $\mathbf{r}'(t)$.

(c) Integrate $\mathbf{r}'(t)$, and use the initial position $\mathbf{r}(0)$ given in the problem to determine what the integration constants should be, and thus give an expression for $\mathbf{r}(t)$.

(d) The goal is 75 feet from the kicker. Will the ball go into the goal?

3. Use the *contour lines* on the graph of $f(x, y) = x^2 - y^2$, shown at right, to sketch a contour plot for this function. Do you know the name for the type of curves you see?



4. Sketch the helix $\mathbf{h}(t) = [a \cos t, a \sin t, bt]$.

(a) Compute the direction vectors $\mathbf{h}'(t)$ and $\mathbf{h}''(t)$. Could you have anticipated their directions?

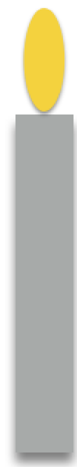
(b) Find the length of the arc from $t = 0$ to $t = 4\pi$.

(c) Find the length of the arc from $t = 0$ to $t = T$, for any value $T > 0$. If $\mathbf{h}(t)$ represents the position of a lazy bumblebee at time T, what does your expression in terms of T represent?

(c) Write an equation for the tangent line to $\mathbf{h}(t)$ at $t = \pi/2$. Add the line to your sketch.

5. We can graph a function $f(x)$ of one variable as a curve in two dimensions, $y = f(x)$. We can graph a function $f(x,y)$ of two variables as a surface in three dimensions, $z = f(x,y)$. It's more difficult to graph a function $f(x,y,z)$ of three variables! One way to think about such a function is that it gives the *temperature* at each point (x,y,z) in space. A good way to visualize the function is to draw its *level surfaces*, the surfaces of the form $f(x,y,z) = c$.

The temperature of a candle flame is about $1500°F$. The temperature of a typical room is $70°F$. Sketch *level surfaces* around the candle flame, which are surfaces for which all points on the surface have the same temperature, and label the temperature of each.



6. For some functions, it's easy to find $f(a,b)$ for any point (a,b) you want. For other functions, it's a little harder. For each of the following, find $f(0,0)$:

(a) $f(x,y) = \dfrac{\cos(\pi + x)}{y^2 - 1}$ **(b)** $f(x,y) = y + \dfrac{\sin x}{x}$ **(c)** $f(x,y) = \dfrac{x+y}{2x+y}$

7. (Continuation) You should have been able to do part **(b)** using a limit, but for part **(c)** it's hard to know quite what to do. Graph the three functions on your computer, and notice that the last surface is *vertical* at the origin! (Some graphing programs work better than others for this sort of thing, so try a few.) We will see that, if you walk towards the origin along different lines, you get *different limits* for $f(0,0)$.

(a) Walk towards the origin along the line $y = 0$, coming from the positive x-axis. This means that we are considering points of the form $(x,0)$, as $x \to 0$:

$\displaystyle\lim_{(x,0)\to(0,0)} \frac{x+y}{2x+y} = \lim_{x\to 0^+} \frac{x}{2x} = \lim_{x\to 0^+} \frac{1}{2} = \frac{1}{2}$. Explain each step of this equation.

(b) Now do the same calculation for walking towards the origin along the y-axis. Does it matter if you are walking from the positive y-axis or from the negative y-axis?

(c) Repeat the calculation one more time, now using a line of the form $y = mx$, so take a limit as $(x, mx) \to (0,0)$. Which numbers can you get as a limit? Does your answer make sense, looking at the graph of the surface?

1. Sketch level surfaces of the function $f(x,y,z) = x^2 + 4y^2 - z^2 = c$ for $c = -1, 0, 1$ and other values of your choice. (*Hint*: You have seen this before.)

2. Consider the function $f(x,y) = \dfrac{x^2}{x^2+y^2}$. Does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist? Explain why or why not, using calculations, graphs, and any other methods of your choice.

3. Once again, a lazy bumblebee flies through a room according to the equation

$$\mathbf{r}(t) = [10\cos t, 10\sin t, t].$$

Now there is also a housefly in the same room, moving according to the path

$$\mathbf{f}(t) = [-70 + 20t, 0, \pi t].$$

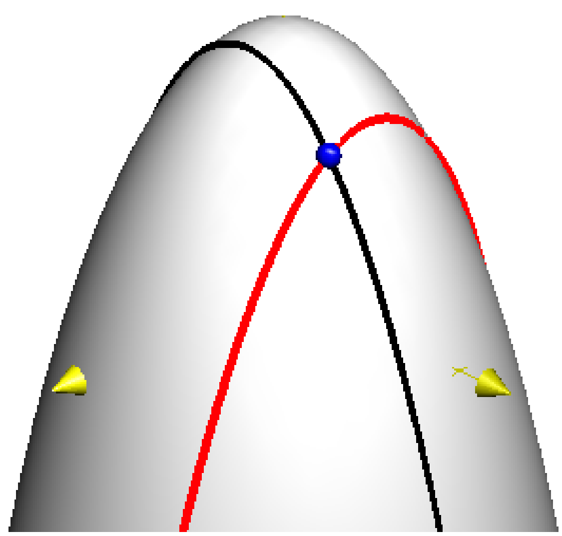
Hint: You can graph parametric equations of curves like these on your computer, if you want some help in sketching them. With *Grapher*, click New Equation From Template, then Curves, and then Cartesian. Make sure your t goes high enough that you can see the important part of the curve, which will be higher than the default $0 \le t \le 10$.

(a) Do the paths of the bumblebee and the fly intersect? Do the two insects collide?

(c) For the bumblebee and the housefly, are they speeding up, slowing down or flying at a constant speed?

(c) A dragonfly moves according to the path $\mathbf{d}(t) = [-70 + 20t^2, 0, \pi t^2]$. How does this path compare to the path of the housefly?

4. The picture shows the surface $z = x^2/3 - y^2/3$, along with the curves cut through this surface by the planes $x = 1$ (black) and $y = 2$ (red). The positive x- and y-axes are pointing towards you out of the surface (use the right hand rule to figure out which one is which!)



(a) What are the (x,y,z) coordinates of the blue point of intersection of the two curves?

(b) Imagine that you are a hiker standing at the blue point. If you walk due north, which is the direction of the positive y-axis, will you be ascending or descending?

(c) If you walk due east, the direction of the positive x-axis, will you be ascending or descending? Will this eastward walk be steeper or less steep than walking north?

(d) Let $\mathbf{e}_1 = [1,0]$ and $\mathbf{e}_2 = [0,1]$. Then $D_{\mathbf{e}_1}(1,2)$ is the *directional derivative at* $(1,2)$ *in the direction* $\mathbf{e}_1$. This is a *number*, which represents the *slope* of the curve that you get when make a vertical slice parallel to vector $\mathbf{e}_1$ that passes through the point $(1,2)$. Is $D_{\mathbf{e}_1}(1,2)$ positive, negative or 0? Answer the same question for $D_{\mathbf{e}_2}(1,2)$, which is defined analogously.

5. Consider the function $f(x, y) = \dfrac{xy^2}{x^2 + y^2}$. Does $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ exist? Explain why or why not. How would you prove your answer correct?

6. (Continuation for class discussion) You can use lines through the origin to show that the limit *doesn't* exist, but simply showing that the limit along any line through the origin is the same isn't enough to prove that the limit *does* exist. Soon, we'll see an example of how it could go wrong. The best way to prove that a limit *does* exist is to convert to polar coordinates, and take a limit as $r \to 0$, which approaches the origin from *every* direction simultaneously. We'll try this together.

Math 150

Review for Midterm Exam 1

For all of the problems on this page, use the following points, lines, vectors and planes:

$$p_1 = (1,-1,1) \qquad p_2 = (0,-2,2) \qquad p_3 = (3,0,-3) \qquad p_4 = (1,2,3)$$

$$\ell_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} t \qquad \ell_2 = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} + \begin{bmatrix} -3 \\ -2 \\ 5 \end{bmatrix} t \qquad \mathbf{v}_1 = [1,1,-1] \qquad \mathbf{v}_2 = [3,-1,4] \qquad \mathbf{v}_3 = [1,2,3]$$

$$P_1 : 3x - 2y + z = 6 \qquad P_2 : 7x - 2y + 3z = 12$$

1. Find an equation for the line ℓ_3 passing through point p_1 in the direction of vector $\mathbf{v}_3$.

2. Find an equation for the line passing through p_1 and p_3.

3. Find an equation for the plane passing through p_1, p_2 and p_3.

4. Confirm that lines ℓ_1 and ℓ_2 intersect, by finding their point of intersection. Then:
(a) Explain why two intersecting lines define a plane.
(b) Find an equation for the plane containing ℓ_1 and ℓ_2.

5. First, confirm that points p_1 and p_3 each lie on plane P_1 *and* on plane P_2. Then:
(a) Explain why two intersecting planes intersect in a line.
(b) Find an equation for the line of intersection of planes P_1 and P_2.

6. Find a vector that is perpendicular to the vector $\mathbf{v}_1$ and to the vector $\mathbf{v}_2$.

7. Find the angle between vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. You may use a calculator.

8. Find the area of the parallelogram spanned by vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

9. Find the volume of the parallelepiped spanned by vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

10. Do the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, in that order, satisfy the right hand rule? Explain.

11. Find the angle between planes P_1 and P_2.

12. Does ℓ_3 (from problem 1) intersect plane P_1? If so, find where; if not, explain why not.

13. Does ℓ_1 intersect plane P_1? You'll have to carefully interpret the result of your algebra.

Math 150

now for some curved stuff

14. Match each equation to a type of surface: paraboloid, hyperbolic paraboloid, one-sheeted hyperboloid, two-sheeted hyperboloid, double-napped cone, or ellipsoid.

(a) $2x^2 + y^2 - 4z^2 = 1$ **(b)** $x = 5 - z^2 - y^2$ **(c)** $z = 4y^2 - x^2$

(d) $3x^2 + 7y^2 + 4z^2 = 10$ **(e)** $2x^2 - y^2 + 4z^2 = -1$ **(e)** $z^2 = 5x^2 + 2y^2$

Hint: Use cross-sections of the form $z = 0$ or $z = c$, or $x = c$, etc. and see how they intersect the surface, whether the intersection curve is an ellipse, a point, or empty. Don't worry, a question like this is too long for an exam, but just one such part would be reasonable.

15. I'll give you an equation in rectangular coordinates (x, y, z) for a certain surface. First, sketch the surface or describe it in words. Then, give me an equation for the same surface, but in cylindrical (C) or spherical (S) coordinates, as indicated.

(a) $z = x^2 + y^2 \to C$ **(b)** $z^2 = x^2 + y^2 \to S$ **(c)** $x^2 + y^2 = 5 \to C$

(d) $x^2 + y^2 + z^2 = 4 \to S$ **(e)** $x^2 + y^2 + z^2 = 4 \to C$

15. Find a tangent line to the curve $\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} t^2-1 \\ 1+5t \\ \sin t \end{bmatrix}$ at the point corresponding to $t = 0$.

16. Consider a particle that moves along a path according to $\mathbf{r}(t) = [t, t^2, t^3]$.

(a) Find the speed of the particle at time $t = 2$.

(b) Write an expression for (but do not compute) the distance traveled by the particle during the three seconds from time $t = 1$ to $t = 4$.

17. Find the length of the helix $\mathbf{h}(t) = [2\cos t, -t, 2\sin t]$ between $t = 0$ and $t = 2\pi$. Is your answer reasonable, thinking about this curve geometrically?

18. For each part, give an example of a path $\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$ that describes a particle:

(a) whose velocity vector is constant;

(b) whose velocity vector is not constant, but that is going at a constant speed;

(c) that is accelerating.

19. A cannon is on a cliff at the point $(0, 100)$. It fires a cannonball with velocity vector $[30, 10]$. The cannonball is affected only by gravity, whose acceleration is $[0, -32]$. How far in the x-direction does the cannonball get before hitting the surface $y = 0$ of the water?

Math 150

1. We've considered several different limits of the form $\lim_{(x,y)\to(0,0)} f(x,y)$. Sometimes (as in Page 10 # 7 and Page 11 # 2), approaching the origin along various lines of the form $y = mx$ gives different values for the limit, and then we know that the limit doesn't exist. Other times (as in Page 11 # 6), approaching the origin along lines of the form $y = mx$ gives the same value for each m, and then we'd like to be able to say that the limit exists. Unfortunately, this can go wrong, when approaching the origin along *curved* paths yields a different value. Let's consider the function $f(x,y) = \dfrac{x^2y}{x^4+y^2}$.

(a) Take a limit of $f(x,y)$ as $(x,y) \to (0,0)$, approaching the origin along lines of the form $y = mx$, and show that the limit is 0 for every value of m.

(b) Now approach the origin along the parabola $y = x^2$, and show that the limit is 1/2.

(c) What a trick, right? You approach along *all* possible lines and get a limit of 0, only to discover that approaching along a parabola gives a different limit. So maybe now we have to try all lines, *and* all parabolas – but even if we get the same limit for all of those, maybe $y = x^3$, or a crazy function where the direction varies wildly, will yield a different limit! How will we ever know if we're done? Well, you can convert to polar coordinates and take the limit as $r \to 0$, which does the job of approaching the origin from all possible directions at once. Do so, and thereby show all at once that $\lim_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$ does not exist.

2. We will figure out how to explicitly compute the two *directional derivatives* of the function $f(x,y) = 5 - x^2/3 - y^2/3$ at the point $(1,2)$ that we estimated in Page 11 # 4.

(a) Along the red curve (refer to the picture in that problem), $y = 2$. So along that curve, the function is just a function of x: $f(x,2) = 5 - x^2/3 + 4/3$. Take the derivative of this function with respect to x. Then plug in $x = 2$, and thereby find the actual slope that the hiker would experience when walking east from the blue point $(1,2,10/3)$.

(b) The notation for this is:

$$f_x = \frac{\partial}{\partial x} f(x,y)\Big|_{y=2} = \frac{d}{dx} f(x,2) = \frac{d}{dx}(5 - x^2/3 + 4/3) = -2x/3.$$

$$f_x(1,2) = -2x/3\Big|_{x=1} = -2/3.$$

The vertical line indicates that we are evaluating the expression at a certain point or value. The symbol ∂ is for a *partial derivative*, which we use for a function of more than one variable, while the symbol d is for a *total derivative* of a function of only one variable. Explain each part of the algebra above, in words, and write your explanation in your notebook.

(c) Find $f_y(1,2)$, which is the slope that the hiker would experience when walking north from the blue point, along the black curve where $x = 1$.

3. (Continuation) Explain why the vector $[1, 0, f_x(1,2)] = [1, 0, -2/3]$ is parallel to the red curve at the blue point. Then give a vector that is parallel to the black curve at the blue point. Finally, see if you can figure out a way to use these to find a *normal vector* to the paraboloid surface at the blue point.

4. Explain why, at a given point on a curve $y = f(x)$, there is only *one* line through that point that is tangent to the curve. Lay a cord on the floor, maybe the charging cord for your phone, and hold your pencil along it to show the tangent line at various points.

5. (Continuation) Now explain why, at a given point on a surface $z = f(x, y)$, there are *many* lines through that point that are tangent to the surface. For the surface of your knee, use your pencil to show several examples of directions that are tangent to the surface.

6. For each part, find the limit, or show that it does not exist:

(a) $\lim_{(x,y)\to(3,4)} \dfrac{1}{\sqrt{x^2+y^2}}$ **(b)** $\lim_{(x,y)\to(0,0)} \dfrac{1}{\sqrt{x^2+y^2}}$

Do you *need* to convert to polar coordinates to find these limits?

Math 150

1. Find the limit, or show that it does not exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2 - y^2}{\sqrt{x^2+y^2}}$.

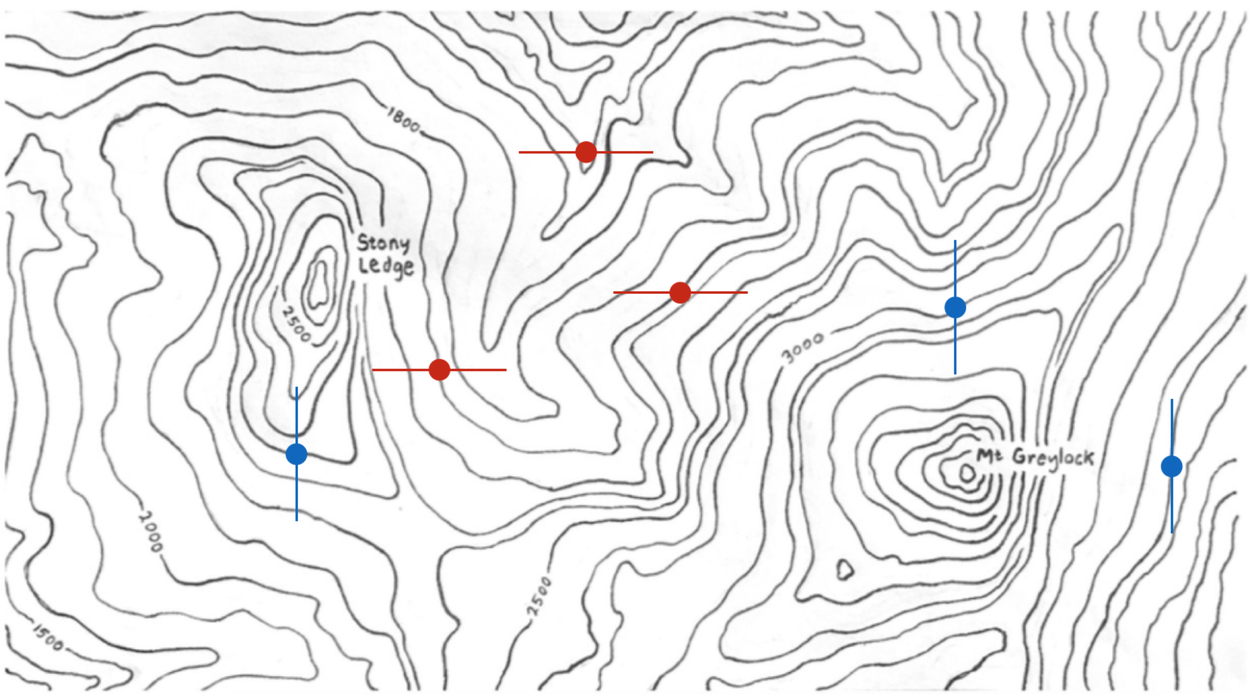
2. People often say: "when you take a partial derivative with respect to x, you just treat y as a constant," and similarly, "when you take a partial derivative with respect to y, you just treat x as a constant."

(a) With this in mind, find the partial derivatives $f_x(x,y)$ and $f_y(x,y)$ of $f(x,y) = x^4y + y^3$.

(b) Notice that each partial derivative is actually a *function* of x and y, which takes different values at different points. Find $f_x(3,2)$ and $f_y(-1,4)$ and explain what these numbers mean geometrically.

3. The picture below is a topographic map for the region of Stony Ledge and Mount Greylock. In fact, it is the picture from the back of the 2016 Mountain Day shirt!

We could say that this map shows the level curves of the function $f(x,y)$, which gives the elevation of a point (x,y). For each red point on the map, say whether f_x is positive, negative or 0 at that point. For each blue point, do the same for f_y.



Thanks to Scott Lewis for allowing us to use this picture.

4. Just as a tangent *line* to a curve gives a good linear approximation to the curve near the point of tangency, a tangent *plane* to a surface gives a good linear approximation near the point of tangency.

(a) For a surface S given by $z = f(x, y)$, and a point $P = (a, b, f(a, b))$ on the surface, explain why the vectors $[1, 0, f_x(a, b)]$ and $[0, 1, f_y(a, b)]$ are both tangent to S at P.

(b) Use these two tangent vectors to find a *normal vector* to S at P.

(c) Find a tangent plane at the point $P = (1, 2, 10/3)$ to the surface $z = 5 - x^2/3 - y^2/3$ shown in Page 11 # 4. (See if you can find a way to check your answer.)

5. Find the limit, when approaching the origin along each of the three coordinate axes, and then decide whether the limit exists: $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 - y^2 + 3z^2}{x^2 + y^2 + z^2}$

1. Find the limit, or show that it does not exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2+5y^2}$.

2. For the function $g(x) = x^2 + \sin x - e^x$, find $g'(x), g''(x)$, and $g'''(x)$.

For problems 3, 4, 5, use the function $\qquad f(x,y) = x^2y + 2x + x\sin y$.

3. Find the partial derivatives of f with respect to x and y.

4. Just as you can take multiple *derivatives* $g'(x), g''(x), g'''(x)$ of a function $g(x)$ of one variable, you can take multiple *partial derivatives* of a function of several variables.

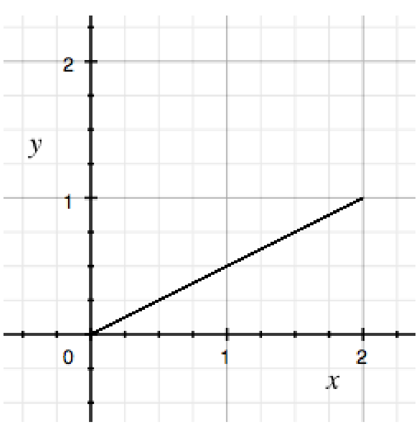
(a) For example, the *second partial derivative* f_{xx} means "the partial derivative of f_x with respect to x." For the function $f(x,y)$, compute f_{xx} and also f_{yy}.

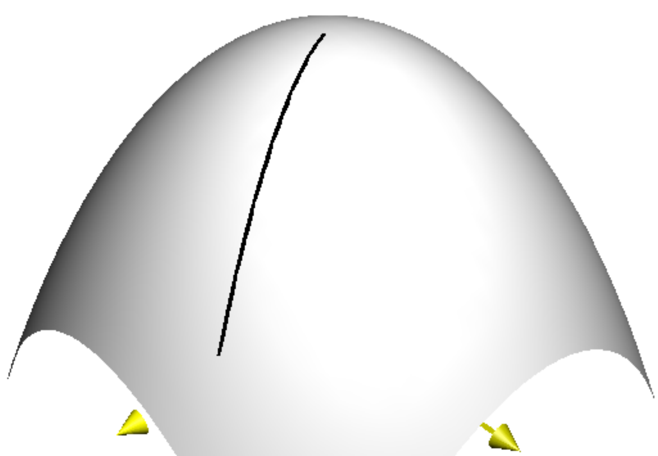
(b) You can also compute a *mixed partial derivative* f_{xy}, which means "the partial derivative of f_x with respect to y." For the function $f(x,y)$, compute f_{xy} and also f_{yx}.

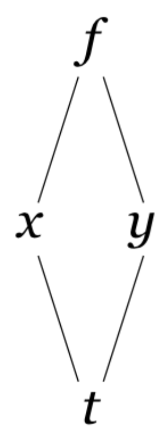
5. Find the tangent plane to the surface $z = x^2y + 2x + x\sin y$ at the point $(1,0,2)$.

6. Suppose that you have just hiked Mount Greylock, and on your map (which is in the xy-plane), the path you took can be parameterized by $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2-2t \\ 1-t \end{bmatrix}$, where t is measured in hours from $t=0$ to $t=1$, and distance is measured in miles east and north from the summit. Further suppose that Mount Greylock's surface can be modeled by the function $f(x,y) = 5 - x^2/3 - y^2/3$.

(a) Explain the line segment and surface graphs below in this context.







(b) Explain why the elevation f is a function of x and y, while x and y are each functions of t. Using the hiking story and the figures, explain why elevation f is a function of t.

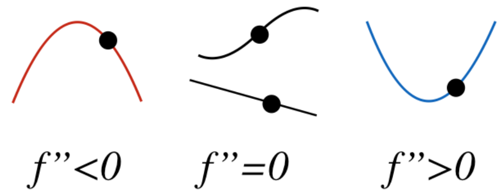
7. Use the *Chain Rule* that you learned in single-variable calculus to find the derivative (with respect to x) of each of the following functions:

(a) $f(x) = \cos(x^2)$ $\qquad$ **(b)** $f(x) = e^{\sin(2x)}$

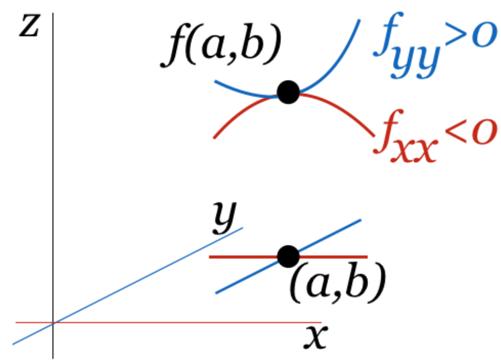
Math 150

1. *Clairaut's Theorem* says that, when a function $f(x,y)$ is defined and continuous, and all of its partial derivatives exist and are continuous, then $f_{xy} = f_{yx}$. Make up an example of a function $f(x,y)$ **that no one else will think of**, and confirm that Clairaut's Theorem works for your example. By the way, "Clairaut" is pronounced "clare-oh."

2. You can *compute* a second partial derivative, but what does it *mean*? Recall from single-variable calculus that when $f''(x) > 0$, the function is *concave up*, when $f''(x) < 0$, the function is *concave down*, and when $f''(x) = 0$, the function is (at least instantaneously) flat. Second partial derivatives measure the same thing, but *in the directions* of the x- and y-axes. f_{yy} measures whether f_y is increasing, decreasing or 0 as you go in the positive y-direction.



In the picture, the red curve is a cross-section of a surface $f(x,y)$ in the x-direction through the point (a,b), and the blue curve is a cross-section in the y-direction through (a,b). This information tells us that the surface has a saddle / pringle shape at $f(a,b)$.



For the function $f(x,y) = 5 - x^2/3 - y^2/3$ shown in Page 11 # 4, look at the picture and say whether f_{xx} and f_{yy} should be positive, negative or 0. Then compute them.

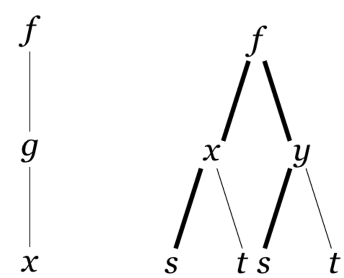
For problems 3 and 4, let $f(x,y) = x^2y$, and let $x(s,t) = st$ and $y(s,t) = e^{st}$.

3. Solve for f as a function of s and t, and then find $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$.

4. The *Chain Rule* from single-variable calculus says:

$$\frac{d}{dx} f\Big(g(x)\Big) = f'\Big(g(x)\Big) \cdot g'(x).$$

We can write this as $\dfrac{df}{dx} = \dfrac{df}{dg}\dfrac{dg}{dx}$, and draw a "dependence tree" that shows that f depends on g, which depends on x.



In multivariable calculus, we can have a function f that depends on several variables (say, x and y), which themselves each depend on several variables (say, s and t). The dependence tree for this example is also shown. In this case, the multivariable chain rule says:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \text{ (shown in thick lines)} \qquad \text{and} \qquad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

The idea is, to find $\dfrac{\partial f}{\partial s}$, you go down every "branch" of the "tree" that connects f to s.

Solve for $\dfrac{\partial f}{\partial x}, \dfrac{\partial x}{\partial s}, \ldots$ – all of the ingredients on the right sides of each equation above – and then plug in for x and y until your expression is entirely in terms of s and t. Compare your answer with that of #3, and discuss which approach you prefer.

5. One reason to find the tangent plane to a surface at a point is that it gives a good *linear approximation* to the function near that point. Consider $f(x,y) = x^2y + 2x + x\sin y$.

(a) Without a calculator, compute $f(1,0)$.

(b) Explain why it would be hard to compute $f(1.1,-0.1)$ without a calculator.

(c) In Page 15 # 5, we found a tangent plane to $z = x^2y + 2x + x\sin y$ at the point $(1,0,2)$, which is $z = 2x + 2y$. Use this linear approximation to find a good estimate for $f(1.1,-0.1)$.

(d) Use your calculator to find $f(1.1,-0.1)$ exactly. How good is our approximation?

Math 150

1. Show that, given a function $f(x,y)$ and a point (a,b), the tangent plane to the surface $z = f(x,y)$ at the point $(a,b,f(a,b))$ is

$$z = f(a,b) \ + \ f_x(a,b)\ (x-a) \ + \ f_y(a,b)\ (y-b).$$

This is also known as the *best linear approximation of f at (a,b)*. Explain the terminology.

2. (Continuation) The following symbols appear in the equation above. Say which ones are variables, and which ones are numbers.

$$z \qquad f(a,b) \qquad f_x(a,b) \qquad x \qquad a \qquad f_y(a,b) \qquad y \qquad b$$

The gradient. We have thought about $f_x(a,b)$ as "the rate of change of the function $f(x,y)$ in the positive x-direction at (a,b)," and similarly for $f_y(a,b)$. It turns out that the *vector* whose entries are these two numbers has some meaning. We call this vector the *gradient*:

$$\text{gradient of } f = \nabla f = \begin{bmatrix} f_x(a,b) \\ f_y(a,b) \end{bmatrix}.$$

3. Show that the best linear approximation L of $f(x,y)$ at (a,b) is given by

$$L(x,y) = f(a,b) + \nabla f(a,b) \bullet \begin{bmatrix} x-a \\ y-b \end{bmatrix}.$$

4. The numbers $f_x(a,b)$ and $f_y(a,b)$ give us rates of change of f at (a,b) in the positive x- and y-directions. What if we want to know the rate of change of f in some *other* direction?

(a) Suppose that you are at the point (a,b), headed in the direction of the vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. Explain why your position can be described by the equation $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} a + u_1 t \\ b + u_2 t \end{bmatrix}$.

(b) Explain why each of the following equalities is true:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(a,b)\ u_1 + f_y(a,b)\ u_2 = \nabla f(a,b) \bullet \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

(c) The symbol ∂ is for a *partial* derivative (when a function depends on more than one variable), while the symbol d is for a *total* derivative (for a function of one variable.) Explain why some of the derivatives in part (b) are ∂ and some are d.

(d) Explain why, if we want to use the equation in part (b) to answer the question "what is the rate of change of f in the direction of the vector $\mathbf{u}$?" we must use a *unit vector* for $\mathbf{u}$.

5. Find the rate of change of the function $f(x,y) = 5 - x^2/3 - y^2/3$, at the point $(1,2)$, in the direction of the vector $[-3,-4]$. Check geometrically whether your answer is reasonable.

Math 150

In fact, Clairaut's Theorem says that, when a function $f(x, y)$ is defined and continuous, and all of its partial derivatives exist and are continuous, *all* of the mixed partial derivatives are equal in *any* order, for example $f_{xyxyxyx} = f_{yyyxxxx}$.

6. Compute f_{xyy} for

$$f(x, y) = ye^{\sin(1/x)} + \cos(\ln(2x^5 - 3\sin x)) + xy^2.$$

Hint: This problem is fun! If you're doing some messy derivatives, find an easier way.

1. Consider the limit $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^3+y^3+z^3}{x^2+y^2+z^2}$.

(a) Show that the limit, when approaching the origin along each of the three coordinate axes, is the same in each case.

(b) Convert to spherical coordinates and determine whether the limit exists.

We now have several tools in our metaphorical mathematical toolbox for dealing with multivariable limits:

- Just plug in the point
- Approach the origin along the axes, or other special lines
- Approach the origin along all lines of the form $y = mx$
- Convert to polar (or spherical) coordinates and take the limit as $r \to 0$ (or $\rho \to 0$)

2. Of the above tools (methods):

(a) Which one(s) do you use when you think the limit doesn't exist?

(b) Which one(s) can you use to prove that the limit does exist?

3. For each of the following, say which method you would use, and why. Then use that method to determine whether the limit exists, and if so, what it is.

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \qquad \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} \qquad \lim_{(x,y)\to(0,0)} \frac{x^2+5}{x-y+3} \qquad \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{xy^2}$$

You are now ready to do the problems in section 14.2 of Rogawski and Adams, about limits.

4. Suppose that you are on a landscape whose elevation can be modeled by the function $f(x,y) = e^{xy} - xy^2$, and you are standing at the point $(1,2)$.

(a) Find the rate of change of your elevation if you were to walk north (positive y-direction), or east (positive x-direction).

(b) Find the rate of change of your elevation if you were to walk south, or west.

5. Consider a surface $z = f(x,y)$.

(a) Explain why, if there is a local maximum or local minimum of the surface over the point (a,b), then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

(b) Graph the surface $z = -x^2 - y^4 + y^2$ on your graphing program. How many local maxima and local minima does it have?

(c) For the function $f(x,y) = -x^2 - y^4 + y^2$, solve the system of equations $\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases}$ and find the three points (x,y) that satisfy both simultaneously. Check that your answer makes sense geometrically, using your graph from (b).

(d) Is it always true that if $f_x(a,b) = 0$ and $f_y(a,b) = 0$, then (a,b) is either a local maximum or a local minimum of $f(x,y)$?

Math 150

1. Find the rate of change of the function $f(x, y) = e^{xy} - xy^2$, at the point $(1, 2)$, in the direction of the vector $[5, 12]$.

2. (Continuation) The notation for the directional derivative of the function f, at the point (a, b), in the direction of the unit vector $\mathbf{u}$, is $D_{\mathbf{u}}f(a, b)$.

(a) Rewrite the question in problem 1, using this new notation.

(b) Explain why $D_{\mathbf{u}}f(a, b)$ is a *number*, whose meaning is a *rate of change.*

(c) Justify each of the following equalities (recall Page 17 # 4b):

$$D_{\mathbf{u}}f(a, b) = \nabla f(a, b) \bullet \mathbf{u} = |\nabla f(a, b)| \cdot |\mathbf{u}| \cdot \cos\theta = |\nabla f(a, b)| \cos\theta.$$

Here $\bullet$ denotes the dot product, $\cdot$ denotes scalar multiplication, and θ is the angle between the vectors $\nabla f(a, b)$ and $\mathbf{u}$ in the xy-plane.

(d) Suppose that you want to go in the direction of the *maximum* rate of change − because $f(x, y)$ describes your elevation on a mountain, say, and you want to ascend as quickly as possible. Which direction should your unit vector $\mathbf{u}$ point, in order to maximize the directional derivative of f at (a, b)?

(e) Explain the geometric meaning of the direction vector $\nabla f(a, b)$.

3. Suppose that the height h of a child, in cm, is a function of her age t, and of her quality of life q, according to the equation $h(t, q) = 60 + 8t + q$. Also suppose that the activity level of the child is a function of her age t, and of her quality of life q, according to the function $a(t, q) = t/2 - q$. Now suppose that her weight w in kg is a function of her height and of her activity level, according to the equation $w(h, a) = 0.3h - a - 10$.

(a) Find the partial derivatives w_h, w_a, w_t, w_q.

(b) Suppose that the pediatrician believes the child is too thin and would like her to gain weight. Which is a more effective strategy, to decrease her activity level by one unit or to increase her quality of life by one unit? (Also explain why the strategies of increasing her height or age by one unit are not good advice.)

4. Find the global maximum of the function $f(x, y) = 4 - x^2 + 2x - y^2 - 4y$ in two ways:

(a) By completing the square, identifying what kind of surface this is, and figuring out geometrically where the maximum point occurs.

(b) By solving for the point where both partial derivatives are 0.

5. The map on the other side shows the high points (red) and low points (green) of each state. Choose your favorite 10 states, and for each one, say whether the high point occurs:

in the interior   on the boundary   on the corner   somewhere else

For each of these categories, choose a state whose high point occurs in that way, and sketch in plausible level curves to show how the highest point ended up there.

Math 150

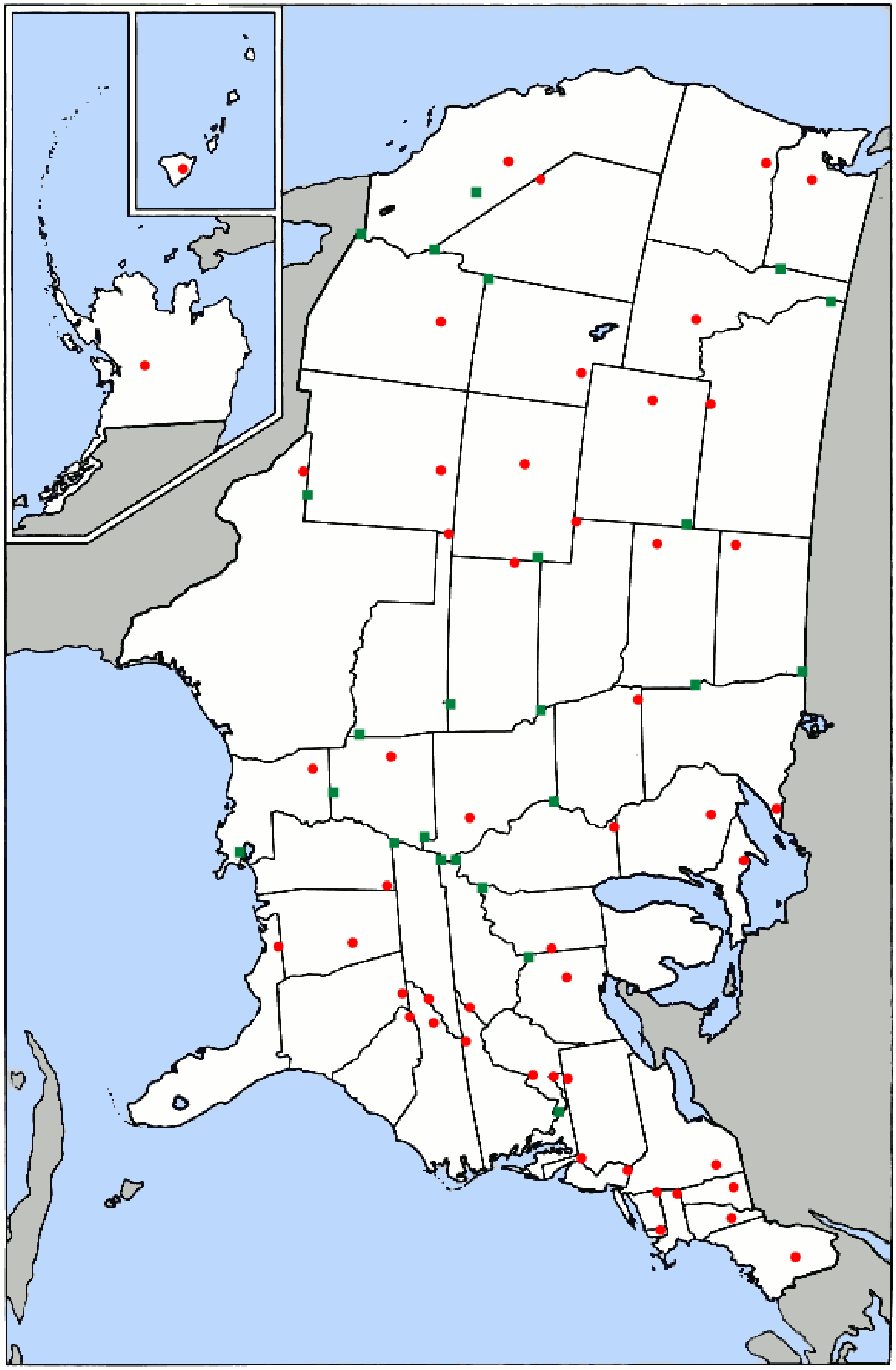


Image from Wikipedia

Math 150

1. *The second derivative test, single-variable calculus.* Faced with a function like

$$f(x) = \frac{1}{4}x^3(x-2)(x+2)$$

and asked to find and classify its critical points, you have learned to do the following:

(a) Find all the *critical points* of $f(x)$, i.e. the values x for which $f'(x) = 0$.

(b) Apply the *second derivative test*: Find $f''(x)$, and for each critical point, determine if $f''(x)$ is positive, negative or 0. Explain how to use this information to classify each critical point as a local maximum, a local minimum, or neither.

(c) Graph $f(x)$ on your graphing program, and check that your answers make sense.

(d) Repeat parts (a)-(c) for the function $g(x) = x^4$, and use this to explain why the second derivative test is sometimes inconclusive, and more information is needed.

2. *The second derivative test, multivariable calculus.* Faced with a function like

$$f(x,y) = x^3 + 2xy - 2y^2 - 10x$$

and asked to find and classify its critical points, do the following:

(a) Find all the *critical points* of $f(x,y)$, i.e. the points (a,b) where $f_x = 0$ and $f_y = 0$.

(b) Apply the *second derivative test*: First, compute $f_{xx}(x,y), f_{xy}(x,y) = f_{yx}(x,y)$, and $f_{yy}(x,y)$. Then, for each critical point (a,b), compute the number

$$D(a,b) = \det \begin{bmatrix} f_{xx}(a,b) & f_{yx}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix} = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b).$$

Then use this information to classify each critical point:

$$\begin{array}{ll} D(a,b) > 0 \text{ and } f_{xx}(a,b) > 0 & \implies f(a,b) \text{ is a local minimum} \\ D(a,b) > 0 \text{ and } f_{xx}(a,b) < 0 & \implies f(a,b) \text{ is a local maximum} \\ D(a,b) < 0 & \implies f(a,b) \text{ is a saddle point} \\ D(a,b) = 0 & \implies \text{ the test is inconclusive} \end{array}$$

(c) Graph $f(x,y)$ on your graphing program, and check that your answers make sense.

3. Find the points on the surface $z = 3x^2 - 4y^2$ where the tangent plane is parallel to $3x + 2y + 2z = 10$.

4. Suppose that you wish to compute $\sqrt{3.01^2 + 3.98^2}$ without using a calculator.

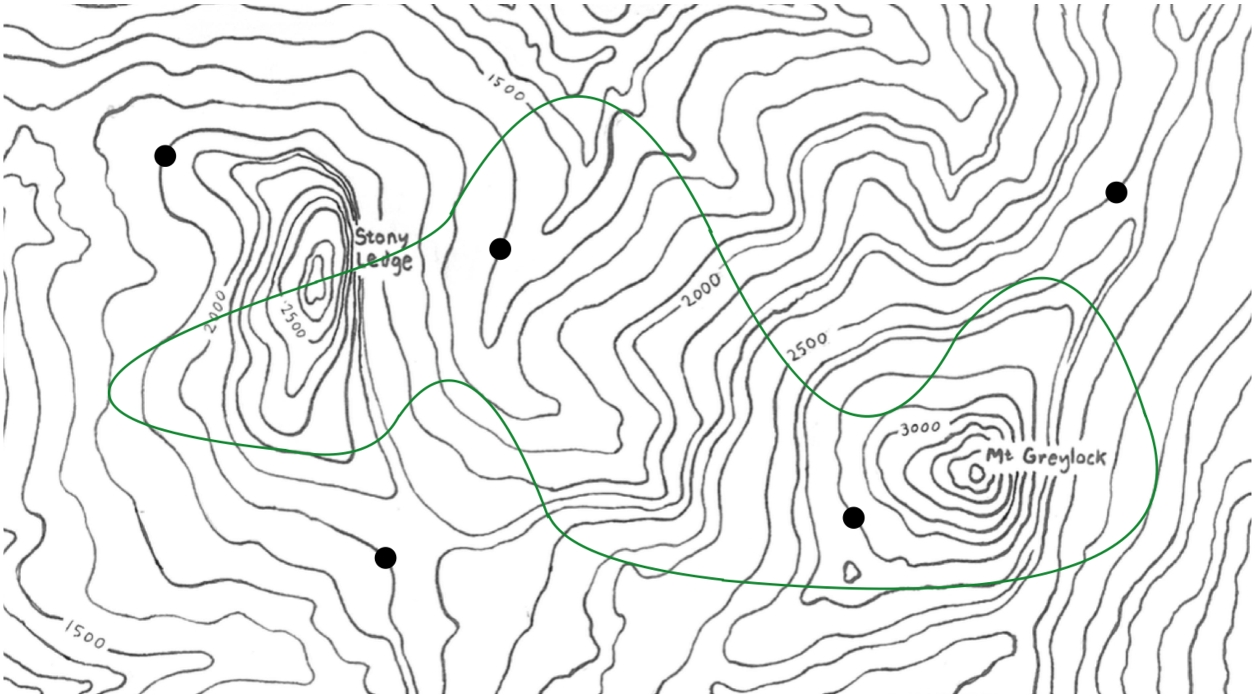
(a) Estimate the answer in your head.

(b) Find a linear approximation (think tangent plane) of the function $f(x,y) = \sqrt{x^2+y^2}$ at a convenient nearby point, and then use it to estimate the answer.

(c) Check your answer with a calculator or computer. How good was the approximation?

You are now ready to do all the practice problems in sections 14.3-14.4 of Rogawski and Adams, about partials and tangents.

5. The topographical map below shows level curves of the elevation function $f(x, y)$. For each of the black points in the map, sketch the gradient vector ∇f at that point.



Thanks again to Scott Lewis for letting us use this image.

6. The Outing Club takes a hike over Stony Ledge and around Mount Greylock, and their path is the green curve shown on the map above. For this hike, determine:

(a) Parts of the hike that were flat,

(b) The steepest part of the hike,

(c) The highest elevation achieved,

(d) The lowest elevation achieved.

For parts (c) and (d), in addition to answering the questions, mark all of the points whose elevation you would need to check, in order to make sure you found the maximum and minimum elevation. What do all of these points have in common?

EXTRA CHALLENGE PROBLEM (optional)

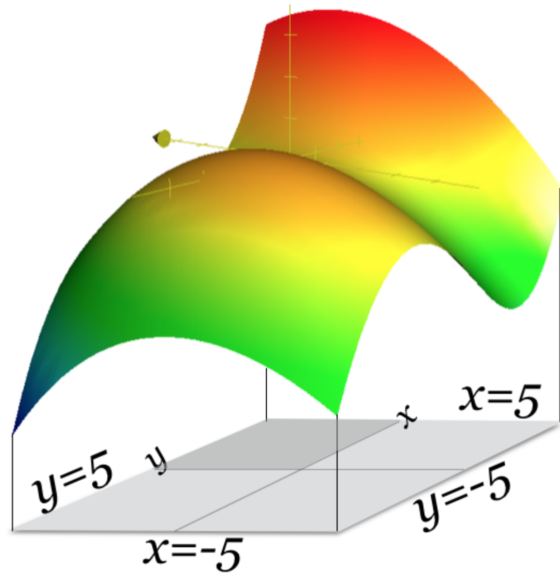
W. For each black point on the map above, imagine that you drop a ball at that point, and it rolls down with gravity, always going in the direction of steepest descent. Sketch the path of the ball. Also sketch in the path that you would take if you started at each black point and wanted to ascent most steeply, which, considered in the opposite direction, is the path that the ball took from the summit to get to the black point. *Hint*: when the contour lines are far apart, it helps to draw in extra contour lines to help you with the direction.

1. The state high point problem, and the Outing Club hike problem, are examples of *optimizing under a constraint*. For the state high points, you are trying to *maximize* elevation, under the *constraint* that you must be in the region of the plane called "Massachusetts."

(a) Give an example of something you are trying to maximize or minimize in your own life, and the associated constraints.

(b) Explain why, if you are trying to find the maximum and minimum values of a function on a (closed, bounded) region of the plane, you need to check the value of the function on all of the following points:

1. The critical points of the function that are inside the region.
2. The critical points of the boundary "cross-section" functions, which are the surface function restricted to each boundary.
3. The corners of the region.



(c) For the surface shown above, which is part of the graph of $f(x, y) = x^3 + 2xy - 2y^2 - 10x$, mark interior critical points in black, critical points of the boundary functions in blue, and the corner points in red. Based on the picture, where do you think the maximum and minimum values of the function occur, over the square region $-5 \leq x, y \leq 5$ shown?

2. (Continuation) Okay, now we're ready to actually do it. For $f(x, y) = x^3 + 2xy - 2y^2 - 10x$:

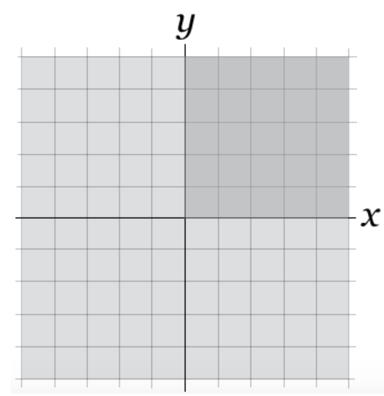
(a) Find the critical points of f that lie inside the region $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$, and write them down on a list. *Hint*: You have already found the critical points; you just have to check which of them are in the region.

(b) We can take a cross section of f along the boundary $x = 5$, which is a function of y:

$$f(5, y) = 5^3 + 2 \cdot 5 \cdot y - 2y^2 - 10 \cdot 5 = 125 + 10y - 2y^2 - 50.$$

Find its critical points, and keep those satisfying $-5 \leq y \leq 5$:

$$\begin{aligned} f(y) &= 75 + 10y - 2y^2 \\ \implies f'(y) &= 10 - 4y = 0 \implies y = 2.5. \end{aligned}$$



so we add the point $(5, 2.5)$ to our list. Now find the critical points along the other three boundaries $x = -5$, $y = 5$, $y = -5$, and add them to your list.

(c) Add the four corner points to your list. (It should now have a total of 11 points listed.) Also plot each point on your list on the square region, which is shown above.

(d) Find the value of f at each of the points on your list, and determine the maximum and minimum values of $f(x, y)$ on the square region! Check that your answer agrees with 1(c).

3. In the first two problems, we considered the region of the xy-plane described by the inequalities $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$, which is a 10×10 square. Sketch (draw a LARGE, CLEAR diagram) the regions described by each of the following sets of inequalities:

(a) $2 \leq r \leq 5$ and $\pi/2 \leq \theta \leq 3\pi/2$

(b) $x > 0$ and $y > x$ and $y < 1$

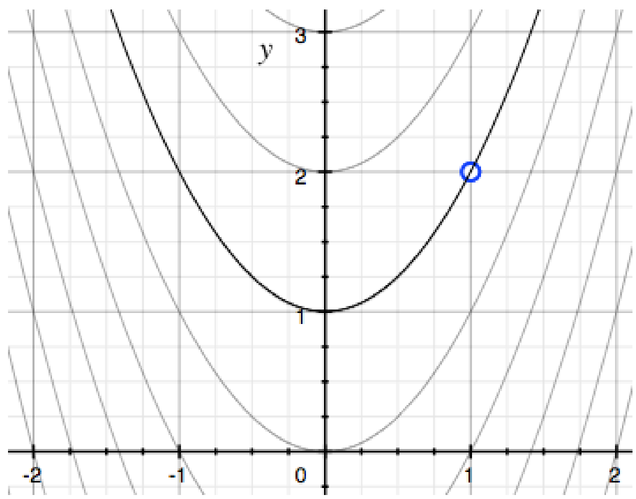
(c) $y > x^2$ and $y < x + 2$

Let's also do some in three dimensions!

(d) $r \leq z$ and $0 \leq z \leq 5$

(e) $0 \leq \phi \leq \pi/3$ and $\rho \leq 1$

4. Suppose that you are asked to find the equation of the line that is perpendicular to the parabola $y = x^2 + 1$ at the point $(1, 2)$, as shown to the right.



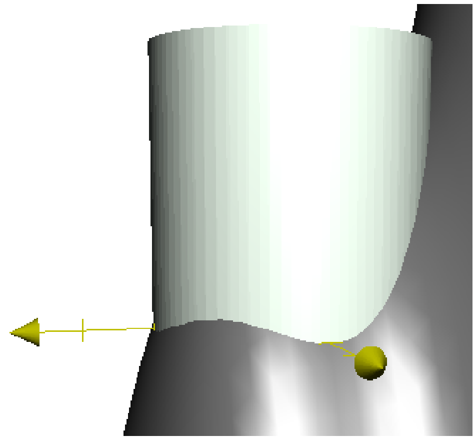
(a) One way to do this is to find the derivative of the function $f(x) = x^2 + 1$ at $x = 1$, and use it to find the line equation. Do so.

(b) Another way to think about the curve $y = x^2 + 1$ is that it is a *level curve* of the function $f(x, y) = y - x^2$. At what level? Label the level of each of the curves shown in the picture.

(c) You know that the gradient vector $\nabla f(1, 2)$ is perpendicular to the level curve of $f(x, y)$ that passes through $(1, 2)$. Use this to find the line equation (probably in parametric form).

5. Find, and classify using the second derivative test, the critical points of the function $f(x, y) = 4x - 3x^3 - 2xy^2$. Also graph this surface on your favorite graphing program, to check your answers, and sketch it in your notebook.

6. Explain why, if you are trying to find the maximum (or minimum) value of a function that occurs on a given constraint curve, you should check all the points where the constraint curve is tangent to a level curve of the function. (This insight will lead us to the idea of *Lagrange multipliers*.)



1. Find the maximum and minimum values of the function $f(x,y) = 1 - x^3 - y^2$, shown as the dark surface, on the unit disk $x^2 + y^2 \leq 1$, by making and checking a list as in Page 21 # 2. *Hint*: for the boundary, write $x = \cos\theta$, $y = \sin\theta$ to find f as a function of θ, and solve $f'(\theta) = 0$. The boundary circle slices the surface like a vertical cylinder, as shown.

2. *Lagrange multipliers.* Suppose we wish to *maximize and minimize* the function $f(x,y) = 1 - x^2 - y^2$ under the *constraint* $x^2 + 4y^2 = 1$.

(a) Sketch and label accurate level curves in the xy-plane for $f(x,y)$ at levels $-2, -1, 0, 1$.

(b) In a different color, add the curve $x^2 + 4y^2 = 1$ to your sketch. Circle the points on it where you think the maximum and minimum values of f occur.

(c) Express the constraint curve as a level curve of the surface $g(x,y) = x^2 + 4y^2$ at level 1, and explain why you can do this.

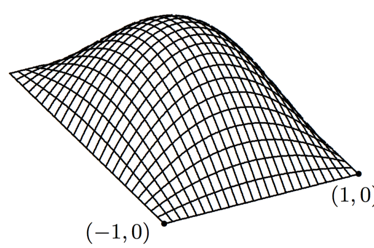
(d) The *Lagrange multipliers equation* says that, at a maximum or minimum point (x,y) of the function $f(x,y)$ on the constraint curve $g(x,y) = c$,

$$\nabla f(x,y) = \lambda \cdot \nabla g(x,y),$$

for some number λ. Explain geometrically what the equation is saying, and why it is true.

(e) The Lagrange multipliers equation above has three variables: x, y and λ, and in fact consists of three equations: one each from the x- and y-components of the gradient, plus the constraint equation. Write down and solve the Lagrange multipliers equation for the given function and constraint, and check that your answers agree with your guess in part (b).

3. The diagram shows $z = (1 - x^2)\sin y$ for the rectangular domain defined by $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$. This surface and the plane $z = 0$ enclose a region $\mathcal{R}$. It is possible to find the volume of $\mathcal{R}$ by integration:



(a) Notice first that $\mathcal{R}$ can be sliced neatly into sections by cutting planes that are perpendicular to the y-axis — one for each value of y between 0 and π, inclusive. Explain why the area $A(y)$ of the slice determined by a specific value of y is given by $A(y) = \displaystyle\int_{x=-1}^{x=1} (1-x^2)\sin y \, dx$. Then evaluate this integral, treating y as a constant.

(b) Explain why the integral $\displaystyle\int_{y=0}^{y=\pi} A(y)\, dy$ gives the volume of $\mathcal{R}$. Then evaluate it.

(c) Notice also that $\mathcal{R}$ can be sliced into sections by cutting planes that are perpendicular to the x-axis — one for each value of x between -1 and 1. As in (a), use ordinary integration to find the area $B(x)$ of the slice determined by a specific value of x.

(d) Integrate $B(x)$ to find the volume of $\mathcal{R}$.

4. The preceding problem illustrates how a problem can be solved using *double integration*. Justify the terminology (it does not mean that the problem was actually solved twice). Notice that the example was made especially simple because the limits on the integrals were constant — the limits on the integral used to find $A(y)$ did not depend on y, nor did the limits on the integral used to find $B(x)$ depend on x. The method of using cross-sections to find volumes can be adapted to other situations, however.

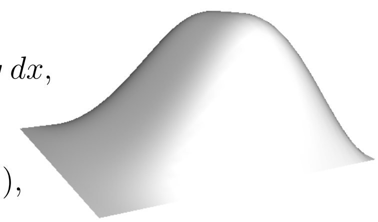
(a) Sketch the region of the plane defined by $0 \le x$, $0 \le y$, and $x + y \le 6$.

(b) Now consider the 3D region $\mathcal{R}$ enclosed by the surface $z = xy(6 - x - y)$ and the plane $z = 0$ for $0 \le x$, $0 \le y$, and $x + y \le 6$. Find the volume of $\mathcal{R}$.

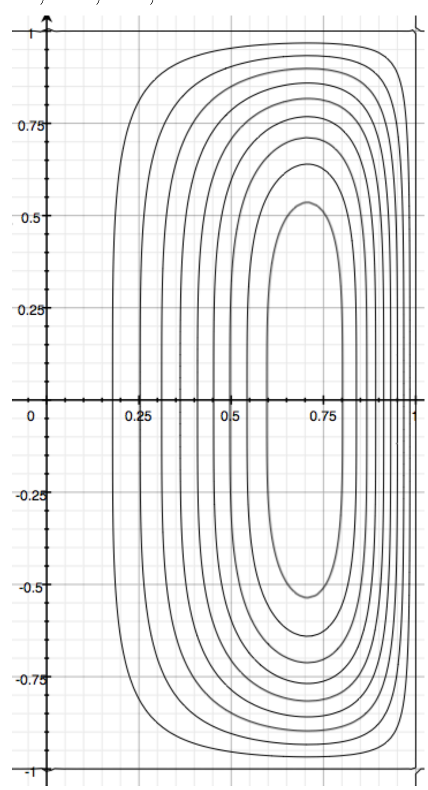
EXTRA CHALLENGE PROBLEM (optional)

5. Consider the integral $V = \displaystyle\int_0^1 \int_{-1}^1 \sin((\sqrt{\pi}x)^2)\cos((\sqrt{\pi}y)^2)\,dy\,dx$, which is the volume between the surface

$$z = f(x, y) = \sin((\sqrt{\pi}x)^2)\cos((\sqrt{\pi}y)^2),$$



shown at right, and the plane $z = 0$. Since $\sin(x^2)$ and $\cos(y^2)$ do not have antiderivatives, it is not possible to find an exact value for this integral. One solution is to use level curves of the function to estimate the value of its integral over a given region, using a *Riemann sum*. The plot below shows level curves of f at levels $0, 0.1, 0.2, \ldots, 0.9$, ordered from outer (rectangular) to inner (oval).



(a) The function has a level "curve" at level 1 in this region, consisting of a single point. Where is it?

(b) Divide the region $[0, 1] \times [-1, 1]$ into eight $1/2 \times 1/2$ squares, and choose a representative value for $f(x, y)$ within each square. Use these to estimate V.

(c) Divide the region into 32 $1/4 \times 1/4$ squares, choose a representative value for $f(x, y)$ within each square, and use these to estimate V.

(d) Is it possible to choose rectangles and representative points so that the Riemann sum estimate for V is 1.5?

(e) Explain the significance of the expression

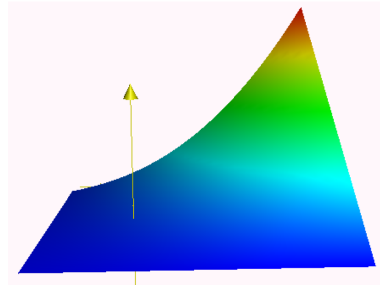
$$\sum_{i=-3}^{4} \sum_{j=1}^{4} \frac{1}{16} f\left(\frac{i}{4}, \frac{j}{4}\right).$$

(f) Explain the significance of the expression

$$\lim_{n\to\infty} \sum_{i=-n+1}^{n} \sum_{j=1}^{n} \frac{1}{n^2} f\left(\frac{i}{n}, \frac{j}{n}\right).$$

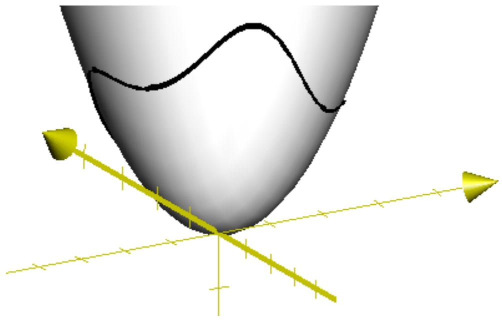
1. Evaluate the double integrals: **(a)** $\displaystyle\int_{x=1}^{x=3}\int_{y=0}^{y=2} x^3 y\; dy\, dx$ **(b)** $\displaystyle\int_{x=-1}^{x=2}\int_{y=x^2}^{y=x+2} 1\; dy\, dx$

2. (Continuation) In part (a), you integrated to find the volume under the *surface* $z = x^3y$ over a rectangular region of the plane, as shown to the right. In part (b), you integrated to find the volume under the surface $z = 1$ over a non-rectangular region of integration. Sketch this region. Then explain why, if you want to find the area of a region of the plane, you can integrate the function $f(x, y) = 1$ over that region.



3. Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x^2 + y^2$, under the constraint $x^4 + y^4 = 1$.

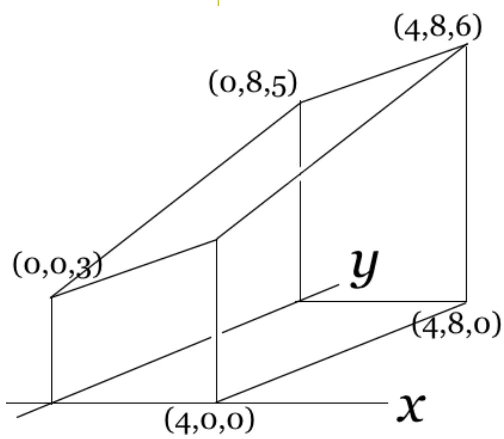
A view of this surface and the constraint curve is shown to the left, from "underneath" the paraboloid.



4. Suppose that the base of your storage shed is the rectangle $0 \le x \le 4$, $0 \le y \le 8$, and its slanted roof is formed by the plane $z = x/4 + y/4 + 3$, as shown. Explain why the following integral gives the *volume* of the shed (a useful number to know, if you wish to store things inside), and calculate the integral.

$$\int_{y=0}^{y=8}\int_{x=0}^{x=4} (x/4 + y/4 + 3)\; dx\, dy$$

Can you express it as a *triple* integral?



5. Suppose that $f(x, y) = 5 - x^2 - y^2/2$ gives the elevation at the point (x, y) of a mountain upon which you are snowshoeing, and you are at the point $(1, 2, 3)$.

(a) Which direction should you hike, if you want to climb most steeply? *Express this direction as a unit vector, and also as an angle θ from the positive x-axis.*

(b) What is the directional derivative of elevation in that direction?

(c) Suppose you only want to climb *half* as steeply as the slope in part (b) that is given by hiking in the direction from part (a). Such a route is shown in the picture. Which direction should you go? *Hint*: Use Page 19 # 2c. *You may find it easiest to express your answer as an angle θ.*

6. Consider the following problem: *Find the points on the surface* $z = 3x^2 - 4y^2$ *where the tangent plane is parallel to* $3x + 2y + 2z = 10$. Last week in Page 20 # 3, you thought about the surface $z = 3x^2 - 4y^2$ as the graph $z = f(x, y)$ of the function $f(x, y) = 3x^2 - 4y^2$, and solved this problem either by writing a tangent plane equation, or by setting partial derivatives equal to each other. Both are very good methods.

Another way to think about this surface is as a *level surface* of the "temperature" function $g(x, y, z) = 3x^2 - 4y^2 - z$, at level 0.

(a) Explain why, at any point (a, b, c) on the surface $g(x, y, z) = 3x^2 - 4y^2 - z = 0$, the gradient vector $\nabla g(a, b, c)$ is perpendicular to the surface.

(b) Use this insight to solve the problem.

You are now ready to do the problems in section 14.6 of Rogawski and Adams, about gradient.

Math 150

1. Let $V(x,y) = 1 - x^2 - y^2$ be interpreted as the speed (cm/sec) of fluid that is flowing through point (x,y) in a pipe whose cross section is the unit disk $x^2 + y^2 \leq 1$. Assume that the flow is the same through every cross-section of the pipe. Notice that the flow is most rapid at the center of the pipe, and is rather sluggish near the boundary.

The volume of fluid that passes each second through any *small* cross-sectional box whose area is $\Delta A = \Delta x \cdot \Delta y$ is approximately $V(x,y) \cdot \Delta x \cdot \Delta y$, where (x,y) is a representative point in the small box. (Here the symbol Δ stands for a tiny distance.)

(a) Using an integral with respect to y, combine these approximations to obtain an approximate value for the volume of fluid that flows each second through a strip of width Δx that is parallel to the y-axis. The result will depend on the value of x that represents the position of this strip.

(b) Use integration with respect to x to show that the volume of fluid that leaves the pipe (through the cross-section at the end) each second is $\pi/2 \approx 1.57$ cc.

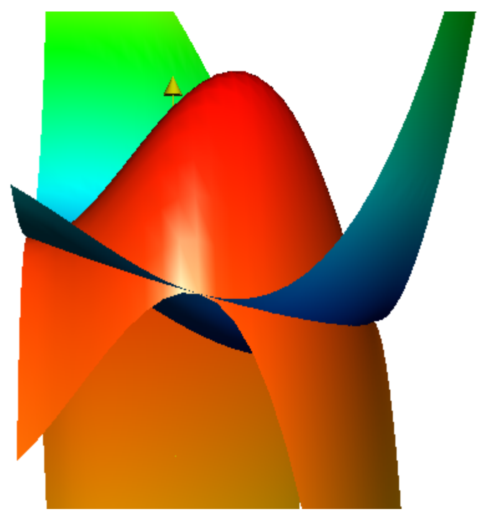
Hint: trig substitution, $x = \sin\theta$. This requires some clever single-variable calculus, so if you get stuck at some point, it's ok, we'll later discover a better way to work out this one.

2. A metal plate consists of the region bounded by the curves $y = x$ and $y = x^2$.

(a) Sketch this region, in a LARGE, CLEAR diagram, and set up a double integral to integrate a function $f(x,y)$ over this region.

(b) The amount of electric charge at a point (x,y) of the plate is $f(x,y) = 2xy$ coulombs per square cm. Find the total amount of charge on the plate.

3. You have found the volume under a given surface (such as x^3y or $x/4+y/4+3$) over a given region. But what about the volume *between two surfaces*? For this, you have to find the region of integration in the xy-plane, and then set up the limits of integration.



(a) Consider the surfaces $z = -(x^2 - y)(y - x - 2) - x$ and $z = 4(x^2 - y)(y - x - 2) - x$, pieces of which are shown to the right. Find the curves in the xy-plane that are the shadows of the intersection curves of these surfaces. Sketch the curves in the xy-plane and shade the area inside, which is our region of integration.

(b) Write a double integral to find the volume of the region between the surfaces, and then compute its value. *Note: This will require some tedious algebra. Consider it to be integrating practice.*

4. In setting up a double integral, it is customary to tile the domain of integration using little rectangles whose areas are $\Delta x \cdot \Delta y$. In some situations, however, it is better to use small tiles whose areas can be described as $r \cdot \Delta\theta \cdot \Delta r$. Sketch such a tile, and explain the formula for its area. In what situations would such tiles be useful?

5. Many foods are sold in cylindrical aluminum cans. It is plausible that the dimensions of the cans are chosen to *minimize* the amount of aluminum used in their production.

(a) Find the volume and surface area of a cylinder of radius r and height h.

(b) One of these is the function we wish to optimize (minimize or maximize), and the other is the constraint. Write and solve the Lagrange multipliers equation for this problem.

(c) You weren't given the volume or the surface area, so your solution to part (c) only tells you the most efficient *ratio* of r to h, i.e. the most efficient *proportions* of such a can. To determine the exact dimensions, you need to know an actual constraint. Given that the volume of a soda can is 355 cm^3, find the height and radius that minimizes surface area.

(d) Find and measure a soda can. Is it made in these optimal proportions? If not, why do you think the manufacturer made it in a non-optimal shape?

(e) In part (b), you got an answer, but maybe you are worried that this value gives you a *maximum* surface area for a given volume, rather than a minimum! Explain, using any method you like, why you are confident that the value you found really does give a minimum.

EXTRA CHALLENGE PROBLEM (optional)

6. Suppose that we had chosen to *maximize* the *volume* of the can, under a constraint of a fixed *surface area*. How would this change the Lagrange multipliers equation in part (b)? Which parts of our solution would change, and which parts would stay the same? If you are not sure, repeat the problem, with a fixed surface area of 300 cm^2.

Math 150

1. In Page 24 # 1, you integrated $f(x,y) = 1 - x^2 - y^2$ over the unit disk $x^2 + y^2 \leq 1$. This is much easier in *polar coordinates*, replacing the Cartesian area form $dA = dx\, dy$ with the polar area form $dA = r\, dr\, d\theta$. Explain why the following two integrals are equal, and then compute the latter.

$$\int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} (1 - x^2 - y^2)\; dy\; dx = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (1 - r^2)\; r\; dr\; d\theta$$

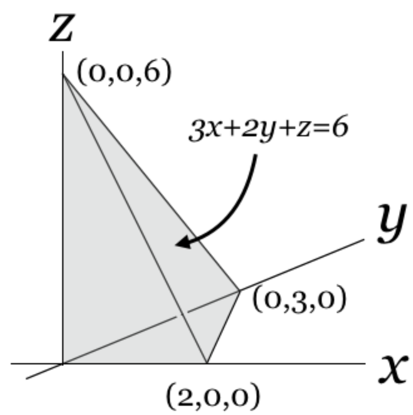
Hint: Compute the integral with r and θ just as you have been computing integrals with x and y.

2. Given a point inside the unit circle, the distance to the origin is some number between 0 and 1. What is the *average* of all these distances? It should also be a number between 0 and 1. Justify your approach. *Hint*: To evaluate your integral, sketch the integrand (the function you are integrating) as a surface and compute a volume using basic geometry.

3. A 400-meter running track is made of two parallel straightaways, connected by semicircular curves, as shown. Suppose that you want to choose the dimensions of the track to maximize the area of the rectangular field at its center. How long should the straightaways be? *Note:* It is possible to solve this using single-variable calculus. To practice our new skills, please try using multivariable calculus to solve it.

4. Evaluate the double integral $\displaystyle\int_{x=0}^{x=1} \int_{y=x}^{y=1} \cos\left(y^2\right)\, dy\, dx$ without using a calculator. You need to describe the domain of the integration in a way that is different from the given description. This is called *reversing the order of integration.*

5. The purpose of this problem is to find the volume in the first octant (the part of 3-space where x, y and z are all positive) bounded by the coordinate planes and the plane $3x + 2y + z = 6$.



(a) Find the volume of the region using basic geometry.

(b) Find the volume of the region using a double integral in the order $dy\ dx$.

When we use a double integral over a region $\mathcal{R}$ in the xy-plane to find a volume sitting over that plane, we can think of $\mathcal{R}$ as the *shadow* of the volume we want to compute. We don't have to use the shadow in the xy-plane – we can use the shadow in *any* of the three coordinate planes!

(c) Use the yz-plane as the "shadow plane," and write a double integral that finds the volume of the region using a double integral in the order $dz\ dy$.

You are now ready to do the problems in §15.1 of Rogawski and Adams, about double integrals.

Math 150

Review problems for Midterm 2

Limits. We have four tools for evaluating limits (top of p. 18), and which tool you use depends on the situation (see p. 18 # 2). If a scalpel will work, don't use a chainsaw!

1. For each of the following, find the limit or show that it does not exist:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^4+x^2y^2+y^4}$ **(b)** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{1+y}$ **(c)** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2y+xy^2}$ **(d)** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{x+y}$

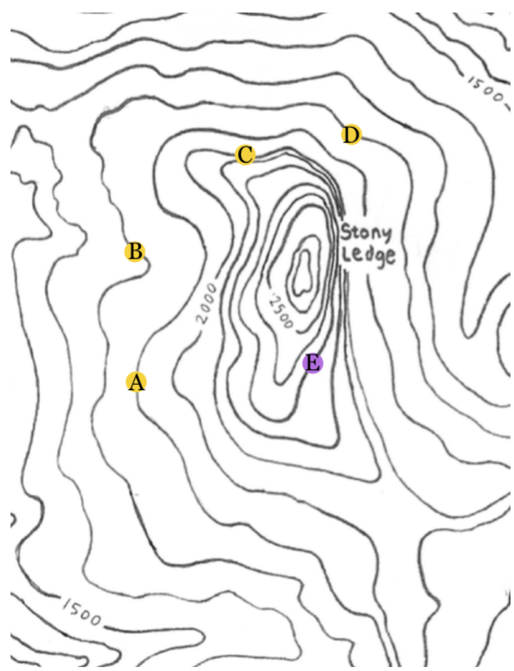
Partial derivatives, a.k.a. *multivariable slopes.*

2. Use a linearization to estimate $\frac{4.1}{7.9}$, similar to what we did in Page 20 # 4.

3. Compute f_{xyxzy} for $f(x,y,z) = y\sin(xz)\sin(x+z) + (x+z^2)\cos(x)y\tan\left(\frac{z+1/z}{x-1/x}\right)$.

Hint: Work smarter, not harder

4. Let $f(x,y)$ be the elevation at point (x,y) near Stony Ledge. The map to the right shows level curves of $f(x,y)$ at each multiple of 100 feet.



(a) For each dot (a,b), say whether $f_x(a,b)$ and $f_y(a,b)$ are positive, negative or 0. For the purple dot, also answer the same question for $f_{xy}(a,b)$.

Gradient, a.k.a. *figuring out which direction is best.*

(b) For each dot (a,b), sketch $\nabla f(a,b)$.

(c) Dale says, "the gradient vector points towards the maximum of a function, or the highest point of the mountain." Is this correct? If not, correct the statement by explaining the meaning of the direction of the gradient vector.

(d) Since the gradient is a *vector*, it has a direction and also a *magnitude*. Some of the gradient vectors in the picture should be longer than others. Explain, and modify your arrows to reflect this.

5. The temperature at (x,y) is given by $T(x,y) = 25y^2 - xy$. A bug is located at $(1,2)$.

(a) What is the bug's current temperature?

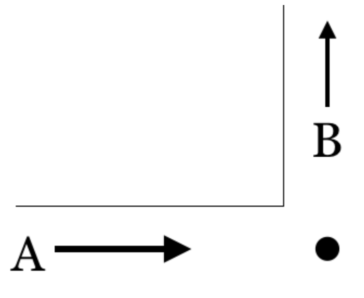
(b) Which direction should the bug walk to get cooler most quickly?

(c) It walks at unit speed in direction $[4,-3]$. Does it get warmer or colder, and how fast?

Chain rule, a.k.a. *functions of functions.*

6. Suppose that f is a function of u, v and w, u is a function of s, v is a function of s and t, and w is a function of s, t and r. First, draw the tree of dependencies, and then write out the chain rule equation that you would use to find $\partial f/\partial s$.

7. Alex realizes that Ben has left his hat in the classroom and runs after him. Ben is walking along the perpendicular hallway, as shown. First, explain why the distance between them is $f(x, y) = \sqrt{x^2 + y^2}$, where x and y are Alex's and Ben's respective distances to the dot. Then, if Alex is running at speed 2 and Ben is walking at speed 1, find the rate at which the distance between them is decreasing when $x = 4$ and $y = 1$.



Optimization, a.k.a. *making life as awesome as possible.*

8. Find and classify the critical points of $f(x, y) = 4x - 3x^3 - 2xy^2$.

9. Find the maximum and minimum values of the function $f(x, y) = x^2 + y^2 - 2x - 4y$ in the region defined by $x \geq 0$, $0 \leq y \leq 3$, $y \geq x$. (Recall the procedure from page 21.)

10. You have \$50.00 to spend on ice cream at Lickety Split for you and your whole entry (a historical event if you are not a first-year). Each scoop s costs \$1.50, and each waffle cone c costs \$1. People's happiness (utility) from eating s scoops and s cones is measured by $U(s, c) = \sqrt{sc}$. How much of each should you buy, to maximize total happiness?

11. Find the point on the line $5 - 2x$ that is closest to the origin. *Hint*: Instead of minimizing the function $\sqrt{x^2 + y^2}$, minimize its *square* $x^2 + y^2$, which is much easier to work with.

Double integrals, a.k.a. *volumes of curvy things.*

12. To make room for the new and improved Bronfman building, part of a hillside needs to be blasted out so that the bottom is flat, down to level $z = 0$. The surface of the hillside is described by $f(x, y) = 1/(x + y)$, and the building will cover the part with $1 \leq x \leq 2$ and $0 \leq y \leq 4$. What is the total volume of material that needs to be be removed?

13. Compute the integral in your head: $\displaystyle\int_{-4}^{-1} \int_{4}^{8} (-5)\, dx\, dy$.

14. Sketch the region of integration in the xy-plane, and then express the integral as a double integral in the opposite order: **(a)** $\displaystyle\int_0^4 \int_x^4 f(x, y)\, dy\, dx$ **(b)** $\displaystyle\int_4^9 \int_{\sqrt{y}}^3 f(x, y)\, dx\, dy$

15. Let $\mathcal{W}$ be the region above the sphere $x^2 + y^2 + z^2 = 6$ and below the paraboloid $z = 4 - x^2 - y^2$.

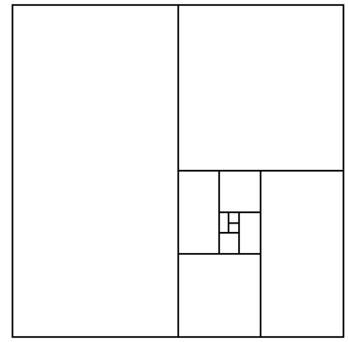
(a) Sketch the two surfaces. Shade the solid $\mathcal{W}$ that lies between them.

(b) Show that the shadow of $\mathcal{W}$ in the xy-plane is the disk $x^2 + y^2 \leq 2$.

(c) Compute the volume of $\mathcal{W}$ using polar coordinates.

16. With whatever tools you like: Find the tangent plane to the surface $z = x^2 + y^{-2}$ at the point $(4, 1, 17)$.

Math 150



1. Consider the *series*

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots$$

The notation "$\cdots$" means that the pattern continues forever. If we added up *all* of the (infinitely many) terms of the sequence, what would the sum be?

2. (Continuation) The *series* above is the sum of the *sequence* $1/2, 1/4, 1/8, 1/16, 1/32, \ldots$. Often it is helpful to give each term a name, so let $a_1 = 1/2, a_2 = 1/4, a_3 = 1/8$, and so on.

(a) Find a_4, a_5 and a_6. Then write a formula for a general term a_n of this sequence.

(b) Consider a sequence where each term $b_n = \dfrac{2^n - 1}{2^n}$. Write out the first five terms of this sequence. What is its significance with respect to problem 1?

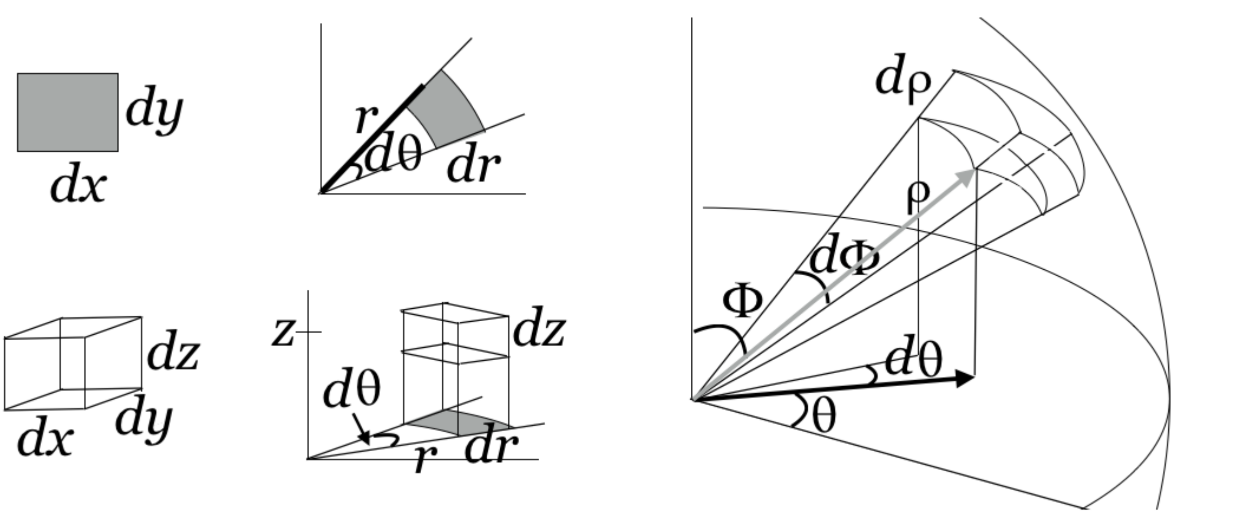
3. The area differential $dx\ dy$, or $dy\ dx$, in a double integral represents the area of a tiny rectangle whose dimensions are δx, or dx, in the x-direction, and δy, or dy, in the y-direction, as shown in the top left of the figure below. The idea here is that you are at some point (x, y), and you "wiggle" x by some infinitesimal amount dx, and you "wiggle" y by some infinitesimal amount dy, and the resulting region shaded in by the "wiggle" is as shown.

Similarly, a tiny "polar rectangle" has area $r\ dr\ d\theta$. In other words, if you are at a point (r, θ), and you wiggle r and θ by infinitesimal amounts dr and $d\theta$, respectively, we have shown that the resulting region, in the top middle of the figure below, has area $(r\ d\theta) \cdot dr$.

(a) Show that the volume of a tiny "brick" obtained by "wiggling" x, y and z, is $dx\ dy\ dz$. This means that the *volume differential* in a Cartesian triple integral is $dV = dx\ dy\ dz$ (or any rearrangement thereof).

(b) Show that a tiny "cylindrical brick" at the point (r, θ, z) in cylindrical coordinates has volume $dV = r\ dr\ d\theta\ dz$.

(c) (OPTIONAL, but read, and note result) Show that a tiny "spherical brick" at the point (ρ, ϕ, θ) in spherical coordinates, as shown on the right side of the figure, has volume $dV = \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$. *Hint*: This product is better expressed as $dV = (d\rho) \cdot (\rho\ d\phi) \cdot (\rho\ \sin\phi\ d\theta)$.



4. You have shown that integrating the function $f(x, y) = 1$ over a region $\mathcal{R}$ of the xy-plane gives the area of $\mathcal{R}$. Similarly, integrating the function 1 over a three-dimensional region gives its volume.

(a) Set up a *cylindrical* integral to find the volume of a cylinder of radius R and height h.

(b) Set up a *spherical* integral to find the volume of a sphere of radius R.

For both of these, check your answer against your previous knowledge of geometry, and remember to use the correct volume differential from problem 3.

5. Fun with sequences. For each sequence below, write out the first six terms of the sequence, starting with $n = 1$. Then say whether you think the sequence *converges*, *diverges to* $\pm\infty$, or *diverges by oscillation.*

(a) $\dfrac{2^n}{n!}$ (recall that $n! = 1 \cdot 2 \cdot 3 \cdots n$.)

(b) $a_n = 3 + \cos(\pi n)$.

(c) $c_1 = 1, c_2 = 1, c_n = c_{n-2} + c_{n-1}$

(d) the n^{th} digit of π

6. More fun with sequences. For each sequence below, write out the general formula for a term a_n.

(a) $-1, 1, -1, 1, \ldots$

(b) $\frac{2}{7}, \frac{3}{8}, \frac{4}{9}, \ldots$

(c) $\frac{1}{1}, \frac{-1}{8}, \frac{1}{27}, \ldots$

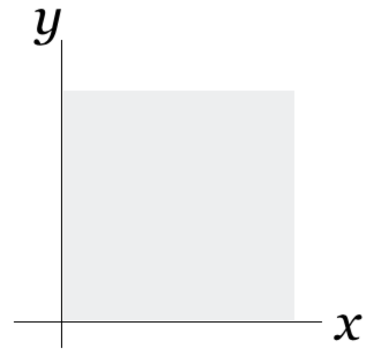
EXTRA CHALLENGE PROBLEM (optional)

S. Perhaps you have an intuitive idea of what it means for a sequence to *converge*, to *diverge to* $\pm\infty$, or to *diverge by oscillation.* Write down a precise definition, or explanation, of each of these notions. Then use your definition to justify your answer to problem 1.

Math 150

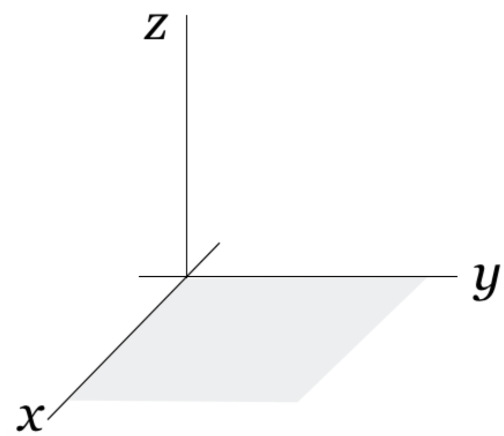
1. For the following double integral, first sketch the region of integration, and then change the order of integration to $dx\ dy$. You will have to use *two* integrals!

$$\int_{x=0}^{x=1} \int_{y=-x}^{y=x} f(x,y)\ dy\ dx$$



2. Sketch the *solid* region of integration for the following integral:

$$\int_{x=0}^{x=1} \int_{y=\sqrt{x}}^{y=1} \int_{z=0}^{z=1-y} f(x,y,z)\ dz\ dy\ dx$$



(a) First, sketch the *shadow* of the solid in the xy-plane, using the limits of integration on the outer two integrals, in the top picture on the right.

(b) Now, draw that shadow again, on the xy-plane in the bottom picture. Then imagine the surfaces $z = 0$ and $z = 1 - y$ in the 3D picture, and draw their intersections with the yz-plane. You can think about the shadow region as an infinitely tall "cookie cutter" slicing vertically through both those surfaces, and the solid region of integration is the part cut out between the surfaces. Sketch the solid in your 3D picture (or maybe just its edges).

3. A *geometric sequence* is a list in which each term is obtained by multiplying its predecessor by a constant. For example, $81, 54, 36, 24, 26, \ldots$ is geometric, with constant multiplier $2/3$. The first term of this sequence is $a_1 = 81$; what is the 40^{th} term? the millionth term? the n^{th} term? Check your formula for $n = 1$, $n = 2$ and $n = 3$.

Definition. If the terms of a sequence come arbitrarily close to a fixed value, the sequence is said to *converge* to that value. For example, we say that the sequence $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$ converges to a limit of 1.

4. What does "arbitrarily close" mean? Here is how it works: I give you a tiny number ϵ, which is the amount of error that I'm allowing you to have, between a term of the sequence and the limit that we claim the sequence converges to. Then you find me an n such that the term a_n and all terms thereafter are within ϵ of the limit L. Let's play, with the example above where $a_1 = 1/2$, $a_2 = 3/4$, $a_3 = 7/8$, etc. and the limit of the sequence is $L = 1$.

Example. I give you $\epsilon = 0.3$. You give me $n = 2$ because $|1 - a_2| = 1 - 3/4 = 1/4 < 0.3$.

(a) Find an n that works for $\epsilon = 0.1 = 1/10$.

(b) Find an n that works for $\epsilon = 0.001 = 1/1000$.

Arbitrarily close means that, no matter what crazy small arbitrary number I give you as ϵ, you can always find me an n that works, to make $|L - a_n| < \epsilon$.

5. A *sequence* is an infinite list of numbers. On the other hand, a *series* is an infinite addition problem, like the following:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$$

(a) Using a calculator or computer, add up the first 20 or 30 terms of this series. What do you think the sum of *all* the terms is? (We'll figure it out later; for now, just guess.)

(b) *Sigma notation* is a compact way of expressing a series. We can write:

$$\sum_{n=1}^{\infty}\frac{1}{n}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$$

Write out the first 6 terms of the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$.

(c) Express the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ from Page 27 # 1 in sigma notation.

One way of understanding what it could possibly mean to do an infinite addition problem is to consider it as a limit of finite addition problems, by considering *partial sums*. The first partial sum S_1 is just the first term, the second partial sum S_2 is the sum of the first two terms, S_3 is the sum of the first three terms, and so on.

(d) Find S_1, S_2, S_3, and a general expression for S_n, for the series in part (c).

(e) For a series $a_1+a_2+a_3+\cdots$, write an expression for S_N in Sigma notation.

Math 150

1. A *geometric series* is a series where each term is obtained by multiplying its predecessor by a constant. It is my favorite kind of series, and it might end up being your favorite, too.

(a) Explain why the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series, and find the constant multiplier.

(b) Explain why the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ from Page 28 # 5b is *not* a geometric series.

(c) A geometric series can be expressed in the form $\sum_{n=0}^{\infty} a \cdot r^n$. Note that n starts at 0 here. Write out the first five terms of this series. Then find a and r for the series in part (a).

2. (Continuation) In this problem, we'll find the (partial) sum of a geometric series that has a finite number of terms. Here's how to do it. First, let's call the sum we want S_n, and write out the terms of the series.

$$\begin{aligned} S_n &= a + ar + ar^2 + \cdots + ar^n \\ S_n \cdot r &= \phantom{a + {}} ar + ar^2 + \cdots + ar^n + ar^{n+1} \end{aligned}$$

The great idea is to multiply both sides of the equation by r, as shown above.

(a) Subtract the second equation from the first, solve the resulting equation for S_n, and thereby obtain a compact formula for the sum we want!

(b) Use your formula to explain why a geometric series with constant multiplier r converges when $|r| < 1$ and diverges when $|r| \geq 1$. Give a simple formula for the sum when $|r| < 1$.

(c) Use your formula to find the sum of the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

For the following problems, first sketch the solid region $\mathcal{W}$ of integration, and then change coordinates as indicated to integrate the given function over the region $\mathcal{W}$.

3. *The weight of a wedge of cheese that gets denser as you move north.*

(a) Sketch the solid region $\mathcal{W}$ described by $x^2 + y^2 \leq 1, x \geq 0, y \geq 0, 0 \leq z \leq 2$.

(b) Calculate $\iiint_{\mathcal{W}} y\, dV$ by converting to cylindrical coordinates.

4. *The weight of a different wedge of cheese, which gets denser as you go up.*

(a) Sketch the solid region $\mathcal{W}$ described by $x^2 + y^2 + z^2 \leq 1$ and $x, y, z \geq 0$.

(b) Calculate $\iiint_{\mathcal{W}} z\, dV$ by converting to spherical coordinates.

Theorem (Divergence Test). If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Equivalently, if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

5. Explain what each part of the Divergence Test is saying, in words. Then use it to show that $\sum_{n=1}^{\infty} \cos n$ does not converge.

6. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is famous enough to have a name, the *harmonic series*. It looks like it should converge, but amazingly, it *diverges*!

(a) Check out this clever trick:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \overbrace{\frac{1}{3} + \frac{1}{4}}^{\geq 1/4} + \overbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}^{\geq 1/8} + + \cdots$$

Each term in the first group is at least $1/4$, and there are two of them. Each term in the second group is at least $1/8$, and there are four of them.... Finish the argument, to show that the sum is infinite.

(b) Explain why this result does not contradict the Divergence Test.

The divergence of the harmonic series is *very* slow:

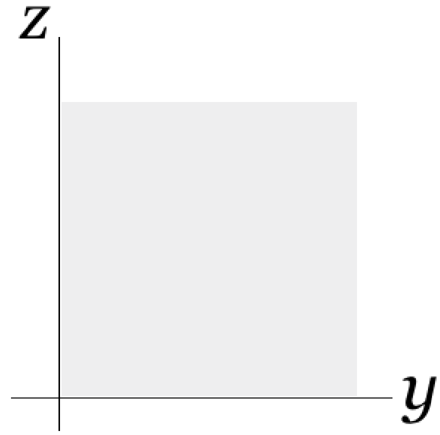
- The sum of the first 10 terms is about 2.9.
- The sum of the first 100 terms is about 5.2.
- The sum of the first 1,000 terms is about 7.5.
- The sum of the first 10,000 terms is still less than 10!

Who would have guessed that the sum of *all* the terms is *infinite*?! Even so, the fact that the harmonic series diverges was first proved in the 1300s.

Math 150

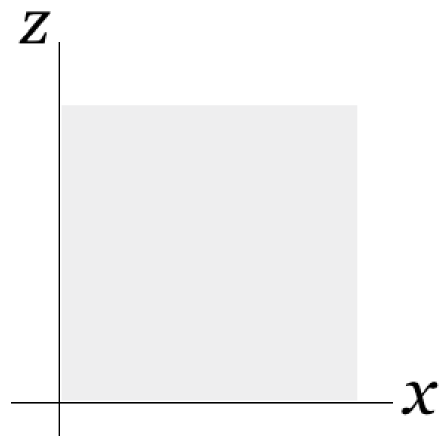
1. In Page 28 # 2, you sketched the *solid* region of integration for the integral:

$$\int_{x=0}^{x=1} \int_{y=\sqrt{x}}^{y=1} \int_{z=0}^{z=1-y} f(x,y,z)\ dz\ dy\ dx.$$

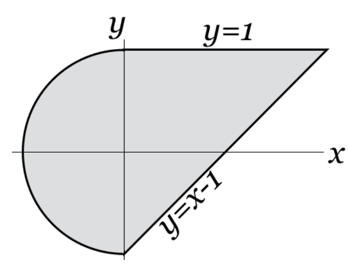


In this problem, we'll write the integral in two other orders of integration, using the other two coordinate planes as the "shadow plane." Refer to your sketch from that problem when you do this one.

(a) Order $dx\ dy\ dz$: First, sketch the *shadow* of the solid in the yz-plane to the right, and use it to write your limits of integration for y and z. Then, for each point (y,z) in the shadow region, determine the surface through which a line, parallel to the x-axis and through the point (y,z), would enter the solid, and where it would exit the solid. Use this to write your limits of integration for x, which will be functions of y and z.



(b) Write the integral in the order $dy\ dz\ dx$: sketch the shadow of the solid in the xz-plane and use this to determine your x and z limits of integration. (You will need to find the curve of intersection of the surfaces, in terms of x and z.) Then determine the surfaces where a line parallel to the y-axis will enter and exit the solid, and use this to find your y limits of integration.



2. Set up a double integral, in both orders of integration, to integrate the function $f(x,y)$ over the region $\mathcal{R}$ shown to the right, which is made from the unit circle and the two lines $y=1$ and $y=x-1$. Which order do you prefer?

3. Tuition at Williams College is about \$51,500 for the 2016-2017 academic year. Tuition at Williams increases, on average, about 3.6% per year.

(a) Predict the tuition at Williams for the 2017-2018 academic year (1 year from now).

(b) Predict the tuition at Williams for the 2026-2027 academic year (10 years from now).

(c) If you are currently a first-year, and you have children when you are 30, and they attend Williams, they will be here from 2046-2050. Predict the total tuition for those four years at Williams.

(d) (Challenge, optional) Suppose that tuition has been increasing by 3.6% every year since Williams was founded in 1793. What was the last year when tuition was under \$1000?

Math 150

Convergence tests for series whose terms are all positive.

Integral test. Suppose that $a_n = f(n)$, where f is a positive, decreasing, and continuous function of x for $x \geq 1$. (This means that the terms of a_n can be given by a function f.) Then:

1. If $\displaystyle\int_1^\infty f(x)\, dx$ converges, then $\sum_{n=1}^\infty a_n$ converges.
2. If $\displaystyle\int_1^\infty f(x)\, dx$ diverges, then $\sum_{n=1}^\infty a_n$ diverges.

Convergence of p-series. The infinite series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise.

Direct comparison test. Assume that there exists $M > 0$ such that $0 \leq a_n \leq b_n$ for all $n > M$. (This means, after a certain point M, each term of b_n is larger than the corresponding term of a_n.) Then

1. If $\sum_{n=1}^\infty b_n$ converges, then $\sum_{n=1}^\infty a_n$ also converges. (bigger one converges $\Longrightarrow$ smaller one converges)
2. $\sum_{n=1}^\infty a_n$ diverges, then $\sum_{n=1}^\infty b_n$ also diverges. (smaller one diverges $\Longrightarrow$ bigger one diverges.)

4. In Page 30 # 6, we used a clever trick to show that the harmonic series $\displaystyle\sum_{n=1}^\infty \frac{1}{n}$ diverges. Prove this in two more ways:

(a) using the integral test, and

(b) using the p-series test.

5. Prove that the series $\displaystyle\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ diverges, in three ways:

(a) using the integral test,

(b) using the p-series test, and

(c) using the direct comparison test, comparing with one of your favorite series.

6. In Page 28 # 5, we conjectured (guessed) that the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n(n+1)}$ converges.

(a) Use the tests above to prove that the series converges.

(b) The tests show that the series converges, but to *what*? Here is a clever trick to figure it out: notice that $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$. Write out the first six terms of the series in this so-called "telescoping" form, and use it to determine the sum of the entire series.

Math 150

1. Given a polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, verify that $p(0) = a_0$, $p'(0) = a_1$, and $p''(0) = 2a_2$. What about $p'''(0)$ and $p^{(4)}(0)$? In general, what is the k^{th} derivative of p at 0?

2. Find an equation for the fifth-degree polynomial $p(x)$ that has the following properties: $p(0) = 0$, $p'(0) = 1$, $p''(0) = 0$, $p'''(0) = -1$, $p^{(4)}(0) = 0$, and $p^{(5)}(0) = 1$.

3. (Continuation) Graph both $y = p(x)$ and $y = \sin(x)$ on the same coordinate-axis system for $-\pi \le x \le \pi$. Can you find an explanation for what you see?

So far, we have considered the convergence of series with all positive terms. How about series that positive terms and also some negative terms? Here are two very useful theorems about such sequences.

Absolute convergence implies convergence. If $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Alternating series test. If $\{b_n\}$ is a positive sequence that is decreasing and converges to 0, i.e.

$$b_1 > b_2 > b_3 > b_4 > \cdots > 0 \qquad \text{and} \qquad \lim_{n\to\infty} b_n = 0,$$

then the *alternating* series with terms b_n converges:

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b + 4 + \cdots \qquad \text{converges.}$$

4. Show that each of the following series converges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

It is one thing to say that a series converges; it is another thing entirely to determine *what* the sum of a convergent series is. Do you have a guess of the sum for part (a) or (b)?

5. Evaluate the integral $\displaystyle\int_{x=0}^{x=1} \int_{y=x}^{y=1} e^{y^2}\, dy\, dx$. *Hint*: Change the order of integration.

6. *Save early, save often.*

(a) On 1 January 2017, you deposit 5500 dollars into an account that pays 6 percent interest annually. How much is this investment worth on 1 January 2063?

(b) On 1 January 2018, you deposit 5500 dollars into an account that pays 6 percent interest annually. How much is *this* investment worth on 1 January 2063? Answer the same question for deposits made on 1 January 2019, 1 January 2020, and so forth, until you see a pattern developing.

(c) Suppose that you deposit 5500 dollars into the same account on 1 January *every* year. Calculate the combined value of *all* of those deposits on 1 January 2063, including the deposit made on that final day.

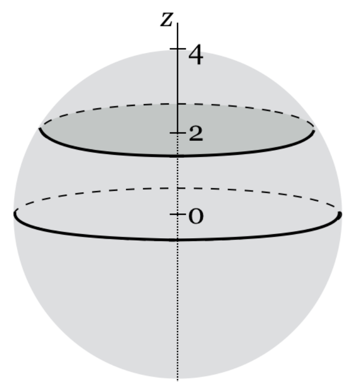
(d) Calculate the total value of *contributions* you made to the account. Compare this to the total value of the account from part (c). Discuss.

Note: If you are 19 years old now and you retire at age 65, you will retire in 2063. The maximum yearly contribution to a Roth IRA account is \$5500. The total stock market index earns approximately 6% appreciation per year. Hence the numbers in the problem above.

7. (Continuation) \$5500 a year is a lot. Suppose instead that you contribute \$1000 a year for the first 10 years, and then $\$n$ a year for the last 37 years. How much do you have to contribute each year (what is n) to end up with \$1,326,000 in 2063 to match the value from problem 6? What is the combined value of contributions under this savings plan? Discuss.

1. Maclaurin polynomials. Given a differentiable function f, you have seen how to use the values $f(0)$, $f'(0)$, $f''(0)$, ... to create polynomials that approximate f near $x = 0$. The coefficients of these polynomials are calculated using values of the derivatives of f. Namely, the coefficient of x^n is $a_n = \dfrac{1}{n!} f^{(n)}(0)$. Use this recipe to calculate the sixth Maclaurin polynomial $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6$ for the cosine function, and graph both the cosine function and the polynomial on the same picture for $-4 \le x \le 4$.

2. Alternating Series Theorem. Choose a sequence of positive numbers $x_0, x_1, x_2, \ldots$ that decreases monotonically to zero. The alternating series $\sum_{n=0}^{\infty} (-1)^n x_n$ must converge, to a sum that is smaller than x_0, larger than $x_0 - x_1$, smaller than $x_0 - x_1 + x_2$, and so forth. Explain. How close the sum of your series is the partial sum $\sum_{n=0}^{\infty} (-1)^n x_n$?



3. The repeating decimal 0.12121212... can be thought of as an infinite geometric series. Write it in the form $a + ar + ar^2 + ar^3 + \cdots$, and also express it using sigma notation. By summing the series, find the rational number equivalent to this repeating decimal.

4. Integrate the function $f(x, y, z) = z$ over the part of the sphere $\rho = 4$ that lies above the plane $z = 2$. *Hint*: Your limits of integration for ρ will depend on ϕ.

5. (Optional, but so useful.) Here are two types of individual retirement accounts (IRAs).

- A *Traditional IRA* is taxed when you take it out: You contribute "pre-tax" money. When you take the money out, you have to pay tax on it before you can use it.
- A *Roth IRA* is taxed when when you put it in: You pay tax on the money, and then put it in the account. When you take it out, you don't have to pay any tax.

Suppose that you take \$10,000 from your paycheck, and you put it in an account paying 6% per year for 50 years. Calculate how much money you have at the end if:

(a) It is a Traditional IRA, so you earn 6% on the full 10,000 for 50 years, and then at the end of 50 years you pay 20% tax on the account.

(b) It is a Roth IRA, so you have to pay 20% tax on the money before you put it in. Then what is left earns 6% interest for 50 years, and then you just take it out.

(c) Explain *why* the answers to parts (a) and (b) are the same.

(d) The difference between the accounts comes from the fact that you will likely be in a *higher tax bracket* later in your life. Suppose that, when you invest the \$10,000, you are in the 15% tax bracket (if your income is \$30,000 per year), and when you take out the money, you are in the 28% tax bracket (income of \$100,000 per year). Calculate how much money you will have if you put the money in a Traditional IRA and how much in a Roth IRA.

Math 150

1. Consider the triple integral $\displaystyle\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x^2} f(x,y,z)\ dz\ dy\ dx$.

(a) Sketch the solid region of integration $\mathcal{W}$.

(b) Rewrite the integral in the order $dx\ dz\ dy$.

(c) (Optional for extra practice later) Rewrite the integral in the order $dy\ dx\ dz$.

2. The *Maclaurin series* for a function f is $\displaystyle\sum_{n=0}^{\infty} a_n x^n$, whose coefficients a_n are defined by the formula $a_n = \dfrac{1}{n!}f^{(n)}(0)$. Notice that the *partial sums* of this series are the Maclaurin polynomials for f. Calculate the Maclaurin series for five important examples:

(a) $\sin x$ **(b)** $\cos x$ **(c)** e^x **(d)** $\ln(1+x)$ **(e)** $\dfrac{1}{1+x}$

In other words, write out a few terms, then express the whole series in sigma notation.

3. (Continuation) A Maclaurin series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ *converges* if its partial sums converge to a limit, called the *sum* of the series. The sum is actually a function of x – it depends on the value of x. In particular, the series might not have a sum for some values of x. This is the case for two of the five examples in the preceding. Which two? For one of the five examples, you have already learned how to find the sum of the series. Which example?

4. The Ratio Test. Suppose that $\sum_{n=1}^{\infty} a_n$ is a series of positive terms, whose ratios $\dfrac{a_{n+1}}{a_n}$ approach a limiting value L as n approaches infinity.

(a) If $L < 1$, you can be sure that the series is convergent. Explain why.

(b) If $L > 1$, you can be sure that the series diverges. Why?

(c) If $L = 1$, then nothing can be said with certainty about the convergence of $\sum_{n=1}^{\infty} a_n$. Show this by finding two specific examples – one that converges and one that diverges – both of which make $L = 1$. This is the ambiguous case.

5. The Limit Comparison Test. Suppose that $\{a_n\}$ and $\{b_n\}$ are two positive sequences, and that $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}$ is a positive number L. Thus $a_n \approx L \cdot b_n$ when n is large.

Explain why the infinite series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

6. Determine convergence or divergence of each series:

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{4n+9}$ **(b)** $\displaystyle\sum_{n=4}^{\infty}\frac{1}{5^n-3^n}$ **(c)** $\displaystyle\sum_{n=1}^{\infty}\frac{3^n+(-2)^n}{5^n}$

7. Approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ to three decimal places. *Hint*: Use Page 32 # 2.

8. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, and that its sum is less than 2.

9. Given two positive numbers p and q whose sum is 1, find the sum of the infinite series $q + pq + p^2q + p^3q + \cdots$. Interpret your answer in the context of probability.

Math 150

1. Working term by term, find the derivative of the following Maclaurin series, using your answers from Page 33 # 2. Check that they agree with what you know to be the derivative.

(a) $\sin x$ **(b)** $\cos x$ **(c)** e^x **(d)** $\ln(1+x)$

2. Given a function f, the terms of its Maclaurin series depend on x. Thus *there is a different series of numbers to consider for each value of* x. Whether such a series converges depends on what x is. The sum of such a series – if there is one – also depends on what x is (not surprising, if you expect the sum to be $f(x)$.) Consider the Maclaurin series of e^x, which is $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$, or $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$. For each of the following, say what the sum should be, and then use your calculator to find a few partial sums to support your answer: **(a)** $\sum_{n=0}^{\infty} \frac{1}{n!}$ **(b)** $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!}$ **(c)** $\sum_{n=0}^{\infty} \frac{1}{n!}(\ln 2)^n$

3. (Continuation) Prove, using our convergence theorems, that each of the three series in the previous problem converges.

4. Change the following integral to cylindrical coordinates, and compute it:

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{3} (x^2 + y^2)\ dz\ dy\ dx$$

5. Because $\ln x$ is defined only for positive values of x (not at 0), there can be no Maclaurin series for $\ln x$. However, the derivative-matching concept can be applied to *any* point on the graph of a differentiable function! Choosing the point $x = 1$ leads to a *Taylor series*

$$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}(x-1)^n$$

for $\ln x$. Explain where the coefficients of this series come from. For which values of x do you think this series is convergent? Graph $y = \ln x$ and its *third-degree Taylor polynomial* $y = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$ on the same system of coordinate axes. What attributes do these curves have in common?

6. (Continuation)

(a) Replace x by 0 in the Taylor series for $\ln x$. Because $\ln x$ approaches $-\infty$ as x approaches 0, it seems likely that the Taylor series at $x = 0$ also diverges. Prove that this is so.

(b) Use the Taylor series for $\ln x$ given above to answer the question in Page 31 # 4a: what is the sum of the convergent series $1 - 1/2 + 1/3 - 1/4 + 1/5 - \cdots$?

7. Determine convergence or divergence of each series:

(a) $\sum_{n=1}^{\infty} \frac{2^n + 4^n}{7^n}$ **(b)** $\sum_{n=1}^{\infty} \frac{n^3}{5^n}$ **(c)** $\sum_{n=1}^{\infty} n^{-0.8}$ **(d)** $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}$

8. (For fun) Find the error in the following argument:

Let's find the sum S of $1 + 2 + 4 + 8 + 16 + \cdots$, using the trick from Page 29 # 2:

$$\begin{aligned} S &= 1 + 2 + 4 + 8 + 16 + \cdots \\ 2S &= \phantom{1 +{}} 2 + 4 + 8 + 16 + \cdots = S - 1. \end{aligned}$$

Therefore $2S = S - 1$, so $S = -1$.

Math 150

Here are topics we studied this semester, and the associated skills you have mastered.

- **Vectors:** Sketching, adding, scaling, dot product and cross product, area of 3D parallelogram, volume of parallelepiped, parametric equations, intersecting lines
- **Planes:** Find the equation of a plane given 3 given points, or 2 intersecting lines, or a point and a line, or a point and a normal vector (perpendicular direction)
- **Quadric surfaces:** Classify what kind it is from an equation, and sketch it
- **Polar, cylindrical, spherical coordinates:** Graph curves and surfaces given an equation in these coordinates; convert Cartesian equations to these coordinates
- **Arclength:** Find the length of a parametric curve, find the speed of a particle
- **Multivariable functions:** $z = f(x, y)$ is a surface; $f(x, y, z)$ gives "temperature," use level curves (topo maps) to understand $f(x, y)$ and level surfaces for $f(x, y, z)$
- **Limits:** When and how to use techniques to show that a limit does or does not exist
- **Partial derivatives:** know what they mean, compute multiple partial derivatives, use Clairaut's Theorem, estimate the sign of f_x and f_y from level curves
- **Tangent planes:** Find the tangent plane to a surface at a point, find the best linear approximation of a given function at a point and use it to estimate the value
- **Gradient:** understand what it means, use it to calculate directional derivative, use it to find the direction of steepest ascent/descent or other desired rate of change
- **Chain rule:** use this to find derivatives of functions of functions
- **Maximizing and minimizing:** Find and classify critical points using the 2^{nd} Derivative Test, find max/min on a constrained region: make a list of interior critical points and boundary critical points and corner points and evaluate the function at each
- **Lagrange multipliers:** Understand why they work, optimize under a constraint
- **Double and triple integrals:** Compute them, sketch the region of integration, change the order of integration
- **Polar, cylindrical, spherical coordinates:** Convert Cartesian coordinates to these, compute integrals in these coordinates using the appropriate dA or dV
- **Sequences and series:** Be able to convert to and from Sigma notation
- **Convergence:** What it means for a sequence to converge, sum of a geometric series
- **Convergence tests:** Divergence test, integral test, p-series test, direct comparison test, alternating series test, absolute convergence implies convergence, alternating series theorem for error estimation, ratio test, limit comparison test
- **Maclaurin and Taylor series:** find power series approximating a function around $x = 0$ or around another given point

Math 150

1. Determine whether the following lines intersect, and if so, find the point of intersection: $\mathbf{r}_1(t) = (2,1,1) + t[-4,0,1]$ and $\mathbf{r}_2(t) = (-4,1,5) + t[2,1,-2]$.

2. Find the angle between the vectors $[0,1,1]$ and $[1,-1,0]$.

3. Find the equation of the plane containing the point $(1,2,3)$ and the line $(x(t), y(t)) = (3-t, 2+2t)$.

4. Sketch and find the volume of the parallelepiped spanned by $[1,0,0]$, $[0,2,0]$ and $[1,1,2]$.

5. Sketch the surface $z^2 = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{8}\right)^2$.

6. Compute the length of the curve $\mathbf{r}(t) = [2t, \ln t, t^2]$ over the interval $1 \le t \le 4$.

7. Find the speed of a particle following the curve $\mathbf{r}(t) = [\sin 3t, \cos 4t, \cos 5t]$ at $t = \pi/2$.

8. Draw level curves at levels $-2, -1, -, 1, 2$ for the function $f(x,y) = x - y^2$.

8. Redo Page 20 # 6.

10. Find each of the following limits, or show that it does not exist:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$ **(b)** $\displaystyle\lim_{(x,y)\to(1,0)} e^{xy}\ln(x-y)$ **(c)** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

11. For $f(x,y) = \cos(x+y^2)$, find f_{yxx}. *Hint*: Choose a convenient order.

12. Redo Page 14 # 3.

13. Use a linear approximation to estimate the value of $\sqrt{1.9 \cdot 1.02}$.

14. Find the tangent plane to the function $f(x,y) = \sqrt{xy}$ at the point $(2,1)$.

15. Suppose that you are on a mountain modeled by $f(x,y) = xy + y^3 - x^2$, at the point $(2,1,1)$. Determine the directional derivative if you walk west. Then determine the directional derivative if you walk southeast.

16. Let $g(x,y) = x^2 - y^2$, let $x = e^u \cos v$, and let $y = e^u \sin v$. Find $\partial g/\partial u$ at $(u,v) = (0,1)$.

17. Find and classify the critical points of $f(x,y) = xye^{-x^2-y^2}$.

18. Determine the global extreme values of the function $f(x,y) = x^3 + x^2y + 2y^2$ on the domain $x, y \ge 0$, $x + y \le 1$.

19. A jewelry box with a volume of 8cm^3 is to be constructed with a gold top, silver bottom, and copper sides. If gold costs \$120 per square cm, silver costs \$40 per square cm, and copper costs \$10 per square cm, find the dimensions to minimize the cost of the box.

Math 150

20. Find the point (a, b) on the graph of $y = e^x$ where the value ab is as small as possible.

21. Compute the double integral of $f(x, y) = x^2$ over the domain $1 \le x \le 3$, $x \le y \le 2x+1$.

22. Compute the integral $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\, dx\, dy$.

23. Evaluate the integral $\displaystyle\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} xy\, dz\, dy\, dx$.

24. Sketch the solid bounded by the planes $x = 0$, $z = 0$, $2x = y$ and $y + z = 2$. Then set up and evaluate the triple integral of the function $f(x, y, z) = z$ in the order $dy\, dx\, dz$.

25. Rewrite the integral $\displaystyle\int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x^2} f(x, y, z)\, dz\, dy\, dx$ in the order $dy\, dx\, dz$.

26. Suppose that your starting salary at age 22 is \$40,000, and that you experience a raise of 3% each year, and that the last year you work is the year you are age 65.

(a) Find your salary the year you are 65.

(b) Find the total amount of money you make in your lifetime. Also express this sum in Sigma notation.

(c) Suppose that your parents held you out of kindergarten when you had just turned 5, and had you start a year later. So now your starting salary of \$40,000 is at age 23. If you still retire at age 65 as above, calculate the total amount of money you make in your lifetime, and compare to parts (a) and (b). Explain.

27. Determine whether each of the following series converges or diverges, and carefully justify your answer using convergence and divergence tests.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!}$ **(c)** $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^4 + 12n}$

28. Approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ to three decimal places. *Hint*: Use Page 32 # 2.

29. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, and that its sum is less than 2.

30. Find the Maclaurin series for **(a)** $f(x) = \sin(x^2)$ and for **(b)** $f(x) = e^{-t^2}$ and for **(c)** $f(x) = \sqrt{1+x}$. **(d)** Use the third Maclaurin polynomial from (c) to approximate $\sqrt{1.8}$. How could you obtain a more accurate approximation?

31. Find the Taylor series for $f(x) = 1/x$ centered at $x = 1$.

Reference

absolutely convergent: A series Σa_n for which $\Sigma |a_n|$ converges. In other words, Σa_n converges, regardless of the pattern of its signs.

acceleration: The derivative of velocity with respect to time.

alternating series: A *series* of real numbers in which every other term is positive.

angle-addition identities: For any angles α and β, $\cos(\alpha+\beta) \equiv \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and $\sin(\alpha+\beta) \equiv \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

angle between vectors: When two vectors $\mathbf{u}$ and $\mathbf{v}$ are placed tail-to-tail, the angle θ they form can be calculated by the dot-product formula $\cos\theta = \dfrac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$. If $\mathbf{u}\bullet\mathbf{v} = 0$ then $\mathbf{u}$ is perpendicular to $\mathbf{v}$. If $\mathbf{u}\bullet\mathbf{v} < 0$ then $\mathbf{u}$ and $\mathbf{v}$ form an obtuse angle.

antiderivative: If f is the derivative of g, then g is called *an antiderivative of f.* For example, $g(x) = 2x\sqrt{x} + 5$ is an antiderivative of $f(x) = 3\sqrt{x}$, because $g' = f$.

average velocity is displacement divided by elapsed time.

bounded: Any subset of $\mathbf{R}^n$ that is contained in a suitably large *disk*.

center of curvature: Given points $\mathbf{p}$ and $\mathbf{q}$ on a differentiable plane curve, let $\mathbf{c}$ be the intersection of the lines *normal* to the curve at $\mathbf{p}$ and $\mathbf{q}$. The limiting position of $\mathbf{c}$ as $\mathbf{q}$ approaches $\mathbf{p}$ is the center of curvature of the curve at $\mathbf{p}$. For non-planar curves, there are many normal lines from which to choose, so an "instantaneous" plane must be specified. One way to select the principal normal direction is to define it as the derivative of the unit tangent vector.

Chain Rule: The derivative of a composite function $C(x) = f(g(x))$ is a product of derivatives, namely $C'(x) = f'(g(x))g'(x)$. The actual appearance of this rule changes from one example to another, because of the variety of function types that can be composed. For example, a curve can be traced in $\mathbf{R}^3$, on which a real-valued temperature distribution is given; the composite $\mathbf{R}^1 \longrightarrow \mathbf{R}^3 \longrightarrow \mathbf{R}^1$ simply expresses temperature as a function of time, and the derivative of this function is the dot product of two vectors.

chord: A segment that joins two points on a curve.

closed: Suppose that $\mathcal{D}$ is a set of points in $\mathbf{R}^n$, and that every convergent sequence of points in $\mathcal{D}$ actually converges to a point in $\mathcal{D}$. Then $\mathcal{D}$ is called "closed."

comparison of series: Given two infinite series Σa_n and Σb_n, about which $0 < a_n \leq b_n$ is known to be true for all n, the convergence of Σb_n implies the convergence of Σa_n, and the divergence of Σa_n implies the divergence of Σb_n.

concavity: A graph $y = f(x)$ is *concave up* on an interval if f'' is positive throughout the interval. The graph is *concave down* on an interval if f'' is negative throughout the interval.

conditionally convergent: A convergent series Σa_n for which $\Sigma |a_n|$ diverges.

content: A technical term that is intended to generalize the special cases length, area, and volume, so that the word can be applied in any dimension.

continuity: A function f is *continuous at a* if $f(a) = \lim_{p \to a} f(p)$. A *continuous function* is continuous at all the points in its domain.

converge (sequence): If the terms of a *sequence* come arbitrarily close to a fixed value, the sequence is said to *converge* to that value.

converge (series): If the *partial sums* of an infinite *series* come arbitrarily close to a fixed value, the series is said to *converge* to that value.

converge (integral): An *improper integral* that has a finite value is said to *converge* to that value, which is defined using a limit of proper integrals.

critical point: A point in the domain of a function f at which f' is either zero or undefined.

cross product: Given $\mathbf{u} = [p, q, r]$ and $\mathbf{v} = [d, e, f]$, a vector that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ is $[\,qf - re, rd - pf, pe - qd\,] = \mathbf{u} \times \mathbf{v}$.

curl: A three-dimensional vector field that describes the rotational tendencies of the three-dimensional field from which it is derived.

curvature: This positive quantity is the rate at which the direction of a curve is changing, with respect to the distance traveled along it. For a circle, this is just the reciprocal of the radius. The principal *normal vector* points towards the center of curvature.

cycloid: A curve traced by a point on a wheel that rolls without slipping. Galileo named the curve, and Torricelli was the first to find its area.

cylindrical coordinates: A three-dimensional system of coordinates obtained by appending z to the usual polar-coordinate pair (r, θ).

decreasing: A function f is *decreasing* on an interval $a \le x \le b$ if $f(v) < f(u)$ holds whenever $a \le u < v \le b$ does.

derivative: Let f be a function that is defined for points $\mathbf{p}$ in $\mathbf{R}^n$, and whose values $f(\mathbf{p})$ are in $\mathbf{R}^m$. If it exists, the derivative $f'(\mathbf{a})$ is the $m \times n$ matrix that represents the best possible linear approximation to f at $\mathbf{a}$. In the case $n = 1$ (a parametrized curve in $\mathbf{R}^m$), $f'(a)$ is the $m \times 1$ matrix that is visualized as the tangent vector at $f(a)$. In the case $m = 1$, the $1 \times n$ matrix $f'(\mathbf{a})$ is visualized as the gradient vector at $\mathbf{a}$.

derivative at a point: Let f be a real-valued function that is defined for points in $\mathbf{R}^n$. Differentiability at a point $\mathbf{a}$ in the domain of f means that there is a linear function L with the property that the difference between $L(\mathbf{p})$ and $f(\mathbf{p})$ approaches 0 faster than $\mathbf{p}$ approaches $\mathbf{a}$, meaning that $0 = \lim\limits_{\mathbf{p}\to\mathbf{a}} \dfrac{f(\mathbf{p}) - L(\mathbf{p})}{|\mathbf{p} - \mathbf{a}|}$. If such an L exists, then $f'(\mathbf{a})$ is the matrix that defines $L(\mathbf{p} - \mathbf{a})$.

determinant: A ratio that is associated with any square matrix, as follows: Except for a possible sign, the determinant of a 2×2 matrix $\mathbf{M}$ is the area of any region $\mathcal{R}$ in 2-dimensional space, divided into the area of the region that results when $\mathbf{M}$ is applied to $\mathcal{R}$. Except for a possible sign, the determinant of a 3×3 matrix $\mathbf{M}$ is the volume of any region $\mathcal{R}$ in 3-dimensional space, divided into the volume of the region that results when $\mathbf{M}$ is applied to $\mathcal{R}$.

differentiable: A function that has derivatives at all the points in its domain.

directional derivative: Given a function f defined at a point $\mathbf{p}$ in $\mathbf{R}^n$, and given a direction $\mathbf{u}$ (a unit vector) in $\mathbf{R}^n$, the derivative $D_{\mathbf{u}}f(\mathbf{p})$ is the instantaneous rate at which the values of f change when the input varies only in the direction specified by $\mathbf{u}$.

discontinuous: A function f has a *discontinuity at a* if $f(a)$ is defined but does not equal $\lim\limits_{p\to a} f(p)$; a function is *discontinuous* if it has one or more discontinuities.

disk: Given a point $\mathbf{c}$ in $\mathbf{R}^n$, the set of all points $\mathbf{p}$ for which the distance $|\mathbf{p} - \mathbf{c}|$ is at most r is called the disk (or "ball") of radius r, centered at $\mathbf{c}$.

diverge means *does not converge*.

divergence: If $\mathbf{v}$ is a vector field, its divergence is the scalar function $\nabla \bullet \mathbf{v}$.

domain: The domain of a function consists of all the numbers for which the function returns a value. For example, the domain of a logarithm function consists of positive numbers only.

double-angle identities: Best-known are $\sin 2\theta \equiv 2\sin\theta\cos\theta$, $\cos 2\theta \equiv 2\cos^2\theta - 1$, and $\cos 2\theta \equiv 1 - 2\sin^2\theta$; special cases of the *angle-addition identities*.

double integral: A descriptive name for an integral whose domain of integration is two-dimensional. When possible, evaluation is an iterative process, whereby two single-variable integrals are evaluated instead.

e is approximately 2.71828182845904523536. This irrational number frequently appears in scientific investigations. One of the many ways of defining it is $e = \lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$.

ellipsoid: A *quadric surface*, all of whose planar sections are ellipses.

extreme point: either a *local minimum* or a *local maximum*. Also called an *extremum*.

Extreme-value Theorem: Suppose that f is a continuous real-valued function that is defined throughout a *closed* and *bounded* set $\mathcal{D}$ of points. Then f attains a maximal value and a minimal value on $\mathcal{D}$. This means that there are points $\mathbf{a}$ and $\mathbf{b}$ in $\mathcal{D}$, such that $f(\mathbf{a}) \le f(\mathbf{p}) \le f(\mathbf{b})$ holds for all $\mathbf{p}$ in $\mathcal{D}$. If f is also differentiable, then $\mathbf{a}$ is either a critical point for f, or it belongs to the boundary of $\mathcal{D}$; the same is true of $\mathbf{b}$.

Fubini's Theorem: Provides conditions under which the value of an integral is independent of the iterative approach applied to it.

Fundamental Theorem of Calculus: In its narrowest sense, differentiation and integration are inverse procedures — integrating a derivative $f'(x)$ along an interval $a \le x \le b$ leads to the same value as forming the difference $f(b) - f(a)$. In multivariable calculus, this concept evolves.

geometric sequence: A list in which each term is obtained by applying a constant multiplier to the preceding term.

geometric series: An infinite example takes the form $a + ar + ar^2 + ar^3 + \cdots = \sum_{n=0}^{\infty} ar^n$. Such a series converges if, and only if, $|r| < 1$, in which case its sum is $\dfrac{a}{1-r}$.

gradient: This is the customary name for the *derivative* of a real-valued function, especially when the domain is multidimensional.

Greek letters: Apparently essential for doing serious math! There are 24 letters. The upper-case characters are

$$\mathrm{A}\ \mathrm{B}\ \Gamma\ \Delta\ \mathrm{E}\ \mathrm{Z}\ \mathrm{H}\ \Theta\ \mathrm{I}\ \mathrm{K}\ \Lambda\ \mathrm{M}\ \mathrm{N}\ \Xi\ \mathrm{O}\ \Pi\ \mathrm{P}\ \Sigma\ \mathrm{T}\ \Upsilon\ \Phi\ \mathrm{X}\ \Psi\ \Omega$$

and the corresponding lower-case characters are

$$\alpha\ \beta\ \gamma\ \delta\ \epsilon\ \zeta\ \eta\ \theta\ \iota\ \kappa\ \lambda\ \mu\ \nu\ \xi\ o\ \pi\ \rho\ \sigma\ \tau\ \upsilon\ \phi\ \chi\ \psi\ \omega$$

Hessian: See *second derivative*.

Math 150

hyperbola I: A hyperbola has two focal points, and the difference between the *focal radii* drawn to any point on the hyperbola is constant.

hyperbola II: A hyperbola is determined by a focal point, a directing line, and an eccentricity greater than 1. Measured from any point on the curve, the distance to the focus divided by the distance to the directrix is always equal to the eccentricity.

hyperboloid: One of the *quadric surfaces.* Its principal plane of reflective symmetry has a special property — every section obtained by slicing the surface perpendicular to this plane is a hyperbola.

improper integral: This is an integral $\int_{\mathcal{D}} f$ for which the domain $\mathcal{D}$ of integration is unbounded, or for which the values of the integrand f are undefined or unbounded.

increasing: A function f is *increasing* on an interval $a \le x \le b$ if $f(u) < f(v)$ holds whenever $a \le u < v \le b$ does.

infinite series: To find the sum of one of these, you must look at the limit of its partial sums. If the limit exists, the series *converges*; otherwise, it *diverges.*

integrable: Given a region $\mathcal{R}$ and a function $f(x, y)$ defined on $\mathcal{R}$, f is said to be *integrable over* $\mathcal{R}$ if the limit of Riemann sums used to define the integral of f over $\mathcal{R}$ exists.

integral test: A method of establishing convergence for positive, decreasing series of terms, by comparing them with improper integrals.

integrand: A function whose integral is requested.

interval of convergence: Given a power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$, the x-values for which the series (absolutely) converges form an interval, by the *Ratio Test.* For example, the *geometric series* $\displaystyle\sum_{n=0}^{\infty} x^n$ converges for $-1 < x < 1$. Also see *radius of convergence.*

Jacobian: A traditional name for the derivative of a function f from $\mathbf{R}^n$ to $\mathbf{R}^m$. For each point $\mathbf{p}$ in the domain space, $f'(\mathbf{p})$ is an $m \times n$ matrix. When $m = n$, the matrix is square, and its determinant is also called "the Jacobian" of f. Carl Gustav Jacobi (1804-1851) was a prolific mathematician; one of his lesser accomplishments was to establish the symbol ∂ for partial differentiation.

l'Hôpital's Rule: A method for dealing with indeterminate forms: If f and g are differentiable, and $f(a) = 0 = g(a)$, then $\displaystyle\lim_{t \to a} \frac{f(t)}{g(t)}$ equals $\displaystyle\lim_{t \to a} \frac{f'(t)}{g'(t)}$, provided that the latter limit exists. The Marquis de l'Hôpital (1661-1704) wrote the first textbook on calculus.

Lagrange multipliers: A method for solving constrained extreme-value problems.

Lagrange notation: The use of primes to indicate derivatives.

level curve: The configuration of points $\mathbf{p}$ that satisfy an equation $f(\mathbf{p}) = k$, where f is a real-valued function defined for points in $\mathbf{R}^2$ and k is a constant.

level surface: The configuration of points $\mathbf{p}$ that satisfy an equation $f(\mathbf{p}) = k$, where f is a real-valued function defined for points in $\mathbf{R}^3$ and k is a constant.

line integral: Given a *vector field* F and a *path* $\mathcal{C}$ (which does not have to be linear) in the domain space, a real number results from "integrating F along $\mathcal{C}$".

Mean-Value Theorem: If the curve $y = f(x)$ is continuous for $a \le x \le b$, and differentiable for $a < x < b$, then the slope of the line through $(a, f(a))$ and $(b, f(b))$ equals $f'(c)$, where c is strictly between a and b. There is also a version of this statement that applies to integrals.

normal vector: In general, this is a vector that is perpendicular to something (a line or a plane). In the analysis of parametrically defined curves, the principal normal vector (which points in the direction of the center of curvature) is the derivative of the unit tangent vector.

odd function: A function whose graph has half-turn symmetry at the origin. Such a function satisfies the identity $f(-x) = -f(x)$ for all x. The name *odd* comes from the fact that $f(x) = x^n$ is an odd function whenever the exponent n is an odd integer.

operator notation: A method of naming a derivative by means of a prefix, usually D, as in $D \cos x = -\sin x$, or $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$, or $D_x(u^x) = u^x(\ln u)D_x u$.

orthonormal: Describes a set of mutually perpendicular vectors of unit length.

parabola: This curve consists of all the points that are equidistant from a given point (the *focus*) and a given line (the *directrix*).

paraboloid: One of the *quadric surfaces*. Sections obtained by slicing this surface with a plane that contains the principal axis are parabolas.

partial derivative: A *directional derivative* that is obtained by allowing only one of the variables to change.

partial sum: Given an infinite series $x_0 + x_1 + x_2 + \cdots$, the finite series $x_0 + x_1 + x_2 + \cdots + x_n$ is called the n^{th} *partial sum*.

path: A parametrization for a curve.

polar coordinates: Polar coordinates for a point P in the xy-plane consist of two numbers r and θ, where r is the distance from P to the origin O, and θ is the size of an angle in standard position that has OP as its terminal ray.

polar equation: An equation written using the polar variables r and θ.

power series: A series of the form $\sum c_n(x-a)^n$. See also *Taylor series.*

Product Rule: The derivative of $p(x) = f(x)g(x)$ is $p'(x) = f(x)g'(x) + g(x)f'(x)$. The actual appearance of this rule depends on what x, f, g, and "product" mean, however. One can multiply numbers times numbers, numbers times vectors, and vectors times vectors — in two different ways.

quadric surface: The graph of a quadratic polynomial in three variables.

Quotient Rule: The derivative of $p(x) = \dfrac{f(x)}{g(x)}$ is $p'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$. This is unchanged in multivariable calculus, because vectors cannot be used as divisors.

radius of convergence: A *power series* $\Sigma c_n(x-a)^n$ converges for all x-values in an interval $a-r < x < a+r$ centered at a. The largest such r is the radius of convergence. It can be 0 or ∞, or anything in between.

radius of curvature: Given a point $\mathbf{p}$ on a differentiable curve, this is the distance from $\mathbf{p}$ to the *center of curvature* for that point.

Ratio Test: Provides a sufficient condition for the convergence of a positive series.

second derivative: The derivative of a derivative. If f is a real-valued function of $\mathbf{p}$, then $f'(\mathbf{p})$ is a vector that is usually called the *gradient* of f, and $f''(\mathbf{p})$ is a square matrix that is often called the *Hessian* of f. The entries in these arrays are *partial derivatives*.

Second-Derivative Test: When it succeeds, this theorem classifies a critical point for a differentiable function as a local maximum, a local minimum, or a saddle point (which in the one-variable case is called an inflection point). The theorem is inconclusive if the determinant of the second-derivative matrix is 0.

speed: The magnitude of *velocity*. For a parametric curve $(x,y) = (f(t), g(t))$, it is given by the formula $\sqrt{(x')^2 + (y')^2}$. Notice that that this is *not* the same as dy/dx.

spherical coordinates: Points in three-dimensional space can be described as (ρ, θ, ϕ), where ρ is the distance to the origin, θ is longitude, and ϕ is co-latitude.

Taylor polynomial: Given a differentiable function f, a Taylor polynomial $\sum c_n(x-a)^n$ matches all derivatives at $x=a$ through a given order. The coefficient of $(x-a)^n$ is given by Taylor's formula $c_n = \frac{1}{n!}f^{(n)}(a)$. Brook Taylor (1685-1731) wrote books on perspective, and re-invented Taylor series.

Taylor series: A *power series* $\sum c_n(x-a)^n$ in which the coefficients are calculated using Taylor's formula $c_n = \frac{1}{n!}f^{(n)}(a)$. The series is said to be "based at a."

Taylor's Theorem: The difference $f(b) - p_n(b)$ between a function f and its n^{th} Taylor polynomial is $\int_a^b f^{(n+1)}(x)\frac{1}{n!}(b-x)^n\,dx$.

triple scalar product: A formula for finding the volume of parallelepiped, in terms of its defining vectors. It is the *determinant* of a 3×3 matrix.

velocity: This n-dimensional vector is the derivative of a differentiable path in $\mathbf{R}^n$. When $n = 2$, whereby a curve $(x, y) = (f(t), g(t))$ is described parametrically, the velocity is $\left[\frac{df}{dt}, \frac{dg}{dt}\right]$ or $\left[\frac{dx}{dt}, \frac{dy}{dt}\right]$, which is tangent to the curve. Its magnitude $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is the speed. The *components* of velocity are themselves derivatives.