

Math 15

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Swathmore College
Swarthmore, PA
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To the Student

Contents: As you work through this book, you will discover that the various topics of single-variable calculus have been integrated into a mathematical whole. There is no Chapter 5, nor is there a section on second derivatives. The curriculum is problem-centered, rather than topic-centered. Techniques and theorems will become apparent as you work through the problems, and you will need to keep appropriate notes for your records — there are no boxes containing important theorems. There is no index as such, but the reference section at the end should help you recall the meanings of key words that are defined in the problems (where they usually appear italicized).

Your homework: Each page of this book contains the homework assignment for one night. The first day of class, we will work on the problems on page 1, and your homework is page 2; on the second day of class, we will discuss the problems on page 2, and your homework will be page 3, and so on for each day of the semester. You should plan to spend two to three hours solving problems for each class meeting.

Comments on problem-solving: You should approach each problem as an exploration. Reading each question carefully is essential, especially since definitions, highlighted in italics, are routinely inserted into the problem texts. It is important to make accurate diagrams whenever appropriate. Useful strategies to keep in mind are: create an easier problem, guess and check, work backwards, and recall a similar problem. It is important that you work on each problem when assigned, since the questions you may have about a problem will likely motivate class discussion the next day.

Problem-solving requires persistence as much as it requires ingenuity. When you get stuck, or solve a problem incorrectly, back up and start over. Keep in mind that you're probably not the only one who is stuck, and that may even include your teacher. If you have taken the time to think about a problem, you should bring to class a written record of your efforts, not just a blank space in your notebook. The methods that you use to solve a problem, the corrections that you make in your approach, the means by which you test the validity of your solutions, and your ability to communicate ideas are just as important as getting the correct answer.

The problems in this text

This set of problems is based on the curriculum at Phillips Exeter Academy, a private high school in Exeter, NH. Most of the problems and figures are taken directly from the **Mathematics 3, 4, 5** books, written by Rick Parris and other members of the PEA Mathematics Department. Some problems are based on those from **Calculus, 6th edition**, by Hughes-Hallett, Gleason, McCallum, et al. Some of the problems were written by Diana Davis (who has been both a student and a faculty member at PEA), for a single-variable calculus class at Swarthmore College. Anyone is welcome to use this text, and these problems, so long as you do not sell the result for profit. If you create your own text using these problems, please give appropriate attribution, as I am doing here.

Help me help you!

Please be patient with me, as I try to be patient with you. I have a long time two working on this set of problems, thinking hard about each problem and how they all connect and build the ideas, step by step. I've done my best, so trust me, you have the tools to solve the problems! On the other hand, I may have made some mistakes, and for that, I apologize in advance.

Just remember that we are all in this together. Our goal is for each student to learn the ideas and skills of Math 15, really learn them — and along the way I will learn new things, too. That's the beauty of this teaching and learning method, that it recognizes the humanity in each of us, and allows us to communicate authentically, person to person.

One way of describing this method is “the student bears the laboring oar.” This is a metaphor: You are rowing the boat; you are not merely along for the ride. You do the work, and in this way you do the learning. The next page gives some ideas for ways that you can do this work of moving the “boat,” which is our class and your learning, forward.

You might wonder, what is my job as your teacher? Part of my job is to give you good problems to think about, which are in this book. During class, my job is to help you learn to talk about math with each other, and help you build a set of problem-solving strategies. At the beginning, I will give you lots of pointers, and as you improve your skills I won't need to help as much.

I might say things like

- “Please go up to the board and write down what you're saying.”
- “Get some colored chalk and add that to the picture on the board.”
- “You were confused before, and now it sounds like you understand; could you please explain what happened in your head?”

I am so excited to see what you can do and hear what you have to say.

Discussion Skills

1. Contribute to the class every day
2. Speak to classmates, not to the instructor
3. Put up a difficult problem, even if not correct
4. Use other students' names
5. Ask questions
6. Answer other students' questions
7. Suggest an alternate solution method
8. Draw a picture
9. Connect to a similar problem
10. Summarize the discussion of a problem

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1. After being dropped from the top of a tall building, the height of an object is described by $y = 400 - 16t^2$, where y is measured in feet and t is measured in seconds.

(a) How many seconds did it take for the object to reach the ground, where $y = 0$?

(b) How high is the projectile when $t = 2$, and (approximately) how fast is it falling?

2. Draw a graph that displays plausibly how the temperature changes during a 48-hour period at a desert site. Assume that the air is still, the sky is cloudless, the Sun rises at 7 am and sets at 7 pm. Be prepared to explain the details of your graph.

3. The x -intercepts of $y = f(x)$ are -1 , 3 , and 6 . Find the x -intercepts of

(a) $y = f(2x)$ (b) $y = 2f(x)$ (c) $y = f(x + 2)$ (d) $y = f(mx)$

Compare the appearance of each graph to the appearance of the graph $y = f(x)$.

4. Find a function f for which $f(x)f(a) = f(x + a)$ for all numbers x and a .

5. A potato is taken from the oven, its temperature having reached 350 degrees. After sitting on a plate in a 70-degree room for twelve minutes, its temperature has dropped to 250 degrees. In how many more minutes will the potato's temperature reach 120 degrees? Assume *Newton's Law of Cooling*, which says that the difference between an object's temperature and the ambient temperature is an exponential function of time.

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1. After being thrown from the top of a tall building, a projectile follows a path described parametrically by $(x, y) = (48t, 400 - 16t^2)$, where x and y are in feet and t is in seconds.

(a) How many seconds did it take for the object to reach the ground, where $y = 0$? How far from the building did the projectile land?

(b) How fast was the projectile moving at $t = 0$ when it was thrown?

(c) Where was the projectile when $t = 2$, and (approximately) how fast was it moving?

2. A Butterball® turkey whose core temperature is 70 degrees is placed in an oven that has been preheated to 325 degrees. After one hour, the core temperature has risen to 100 degrees. The turkey will be ready to serve when its core temperature reaches 190 degrees. To the nearest minute, how much more time will this take?

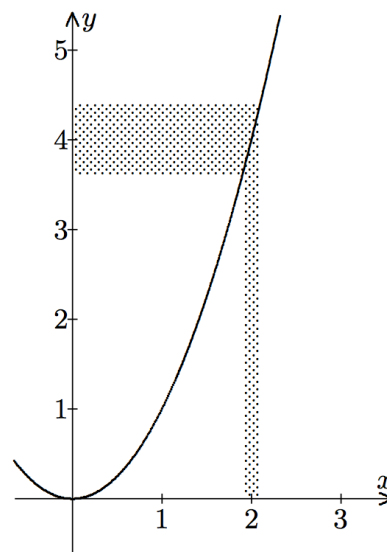
3. Inflation in the country of Eromhtraws has reached alarming levels. Many banks are paying 100 percent annual interest, some banks are paying 100/12 percent monthly interest, a few are paying 100/365 percent daily interest, and so forth. Trying to make sense of all these promotions, Milou decides to graph the function E defined by $E(x) = \left(1 + \frac{1}{x}\right)^x$. What does this graph reveal about the function $E(x)$? Calculate the specific integer values $E(1)$, $E(12)$, $E(365)$, and $E(31536000)$.

4. (Continuation for class discussion) This function has a *limiting value* as $x \rightarrow \infty$. This example is so important that a special letter is reserved for the limiting value (as is done for π). It is now traditional to write $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$, which means that $\left(1 + \frac{1}{n}\right)^n$ approaches e as n approaches infinity. This limit is an example of an *indeterminate form*. For some additional limit practice, use your calculator or computer to evaluate $\lim_{n \rightarrow \infty} \left(1 + \frac{0.09}{n}\right)^n$, which is greater than 1. Make up a story to go with this question.

5. Find a function f for which $f(x/a) = f(x) - f(a)$ for all positive numbers x and a .

6. Verify that $|x^2 - 4| < 0.0001$ is true for all numbers x that are sufficiently close to 2. In other words, show that the inequality is satisfied if the distance from 2 to x is small enough. How small is “small enough”?

7. (Continuation for class discussion) *What “limit” means:* Let p be any small positive number. Show that there is another small positive number d , which depends on p , that has the following property: whenever $|x - 2| < d$ it is true, it is also true that $|x^2 - 4| < p$. This means —intuitively— that x^2 can be brought arbitrarily close to 4 by making x suitably close to 2. It is customary to summarize this situation by writing $\lim_{x \rightarrow 2} x^2 = 4$. In describing how to find the number d , it will be convenient to assume at the outset (there being no loss of generality in doing so) that p -values greater than 4 are not under consideration.



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8. If h is a number that is close to 0, the ratio $\frac{2^h - 1}{h}$ is close to 0.693 ... Express this using limit notation. Interpret the answer by using a *secant line* for the graph of $y = 2^x$. Notice that this limit provides another example of an *indeterminate form*.
9. Working in radian mode, sketch a large, clear graph of $y = \sin x$ for $-2\pi \leq x \leq 2\pi$. Find the slope of this curve at the origin. Would the slope have been different if you had worked in degree mode?
10. (Continuation for class discussion) Working in radian mode, evaluate $\lim_{x \rightarrow 0} \frac{\sin x}{x}$. Interpret your answer.
11. Consider the cubic graph $y = 3x^2 - x^3$.
- (a) Write $3x^2 - x^3$ in factored form.
- (b) Use this form to explain why the graph lies below the x -axis only when $x > 3$, and why the origin is therefore an *extreme point* on the graph.
- (c) Use preceding information to sketch the cubic graph.
12. Graph $f(x) = \frac{x^3 - 1}{x - 1}$. What are the *domain* and *range* of f ?

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1. Differential calculus is all about finding slopes of curves. Okay, that may be an oversimplification, but it's the main idea! In Page 2 # 8, you saw that the slope of $y = 2^x$ at $x = 0$ is about 0.693. This problem extends this idea. (Problems extending ideas... also a theme!)

(a) The *slope of the curve* $y = 2^x$ at its y -intercept is slightly less than 0.7. First, make a large, clear sketch of the curve in your notebook. Then verify this claim by computing the slope of the *secant line* to the graph of $y = 2^x$ between $x = 0$ and $x = 0.01$. This means that you need to compute

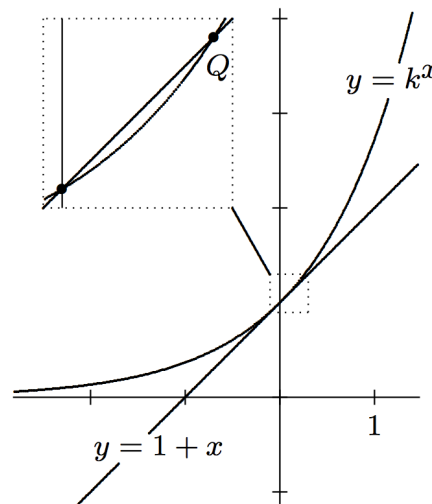
$$\frac{2^{0.01} - 2^0}{0.01 - 0}.$$

Add this secant line to your sketch, with the coordinates of its endpoints clearly labeled. (You will probably want to draw a “zoomed-in” view of the curve near the y -intercept.)

(b) The slope of the curve $y = 3^x$ at its y -intercept is nearly 1.1. Repeat part (a) for this curve: make a sketch of the graph, compute the slope of the secant line between $x = 0$ and $x = 0.01$, and add the line to your sketch. What if you had used $x = -0.01$ and $x = 0$ instead?

(c) The above data suggest that there is a number b for which the slope of the curve $y = b^x$ is exactly 1 at its y -intercept. Choose values of b between 2 and 3, and repeat what you did in parts (a) and (b) to find the slope of the curve $y = b^x$ at its y -intercept, until you find a value for b that gives a slope as close to 1 as you can. Do you recognize this number b ?

2. (Continuation for class discussion) The figure shows the line $y = 1 + x$, along with the graph of $y = k^x$, where k is slightly smaller than the special number b . The curve crosses the line at $(0, 1)$ and (as the magnified view shows) at another point Q nearby in the first quadrant. Given the x -coordinate of Q , it is possible to calculate k by just solving the equation $k^x = 1 + x$ for k . Do so when $x = 0.1$, when $x = 0.01$, and when $x = \frac{1}{n}$. The last answer expresses k in terms of n ; evaluate the limit of this expression as n approaches infinity, and deduce the value of b . What happens to Q as n approaches infinity?



3. A function f is said to be *continuous at* a , provided that $f(a) = \lim_{x \rightarrow a} f(x)$. If f is continuous at every point in its domain, it is called a *continuous function*. Most functions in this book are continuous, but here is an odd example: Let $F(x)$ be the fractional part of x . Thus $F(98.6) = 0.6$, $F(\pi) \approx 0.1416$, $F(2) = 0$, and $F(-2.54) = -0.54$. In your notebook, make a large, clear sketch of the graph of $F(x)$ from $x = -10$ to $x = 10$. Then describe *all* of the x -values at which F is *discontinuous*.

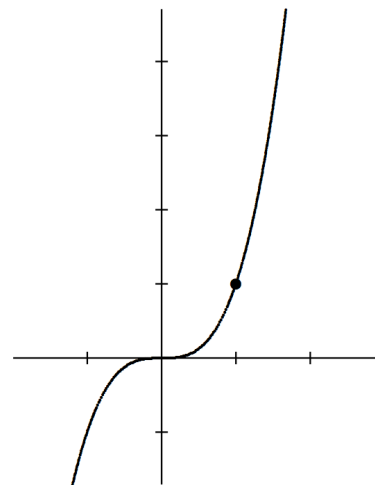
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4. The expression $\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n$ may remind you of the definition $e = \lim_{k \rightarrow \infty} \left(1 + \frac{1}{k}\right)^k$. It is in fact possible to find a simple relationship between the values of the two limits. You could start by replacing n by $2k$.

5. (Continuation for class discussion) Try to generalize by expressing $\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n$ in terms of e and r , for any positive value of r .

6. Tell how the slope of the curve $y = 3^x$ at its y -intercept compares with the slope of the curve $y = 2 \cdot 3^x$ at its y -intercept.

7. The point $(1, 1)$ is on the graph of $y = x^3$. Find coordinates for another point *on the graph* and very close to $(1, 1)$. Find the slope of the line that goes through these points. Explain how this slope is related to the value of $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1}$. This limit is an example of an indeterminate form.



8. (Continuation for class discussion) First show that $x^3 - 1$ is divisible by $x - 1$, then show that $x^3 - 8$ is divisible by $x - 2$. One way to proceed is to use the long-division process for polynomials (see the example in the index at the end of this book). Show that your results could be useful in finding the slope of a certain curve.

9. Recall that the slope of the curve $y = e^x$ at its y -intercept is 1. Use this information to find the slopes of the curves $y = e^{2x}$ and $y = e^{mx}$ at their y -intercepts.

10. *One-sided limits.* Consider the *sign function*, which is defined for all nonzero values of x by $\text{sgn}(x) = \frac{x}{|x|}$.

(a) In your notebook, make a large, clear sketch of the graph of $\text{sgn}(x)$ from $x = -10$ to $x = 10$.

(b) Confirm that both $\lim_{x \rightarrow 0^-} \text{sgn}(x)$ and $\lim_{x \rightarrow 0^+} \text{sgn}(x)$ exist, and notice that they have different values. Here, the new notation $x \rightarrow 0^+$ means that x approaches 0 from the right, and $x \rightarrow 0^-$ means that x approaches 0 from the left.

Because the two one-sided limits do not agree, $\text{sgn}(x)$ does not approach a (two-sided) limit as $x \rightarrow 0$. Notice that $\text{sgn}(0)$ remains undefined.

(c) (Continuation for class discussion) Read again the definition of a *continuous* function and of a *discontinuous* function, given in the index and also earlier in this assignment. Is the function $\text{sgn}(x)$ continuous or discontinuous? Is it possible to “fix” it and make it continuous?

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1. Let $g(x) = 3^x$, and let $h(x) = \frac{g(x + 0.001) - g(x)}{0.001}$.

(a) Make a table of values of $g(x)$ and $h(x)$, for several x -values between -1 and 2 , as shown on the right.

(b) Do you notice a relationship between the two columns? If not, compute the ratio $h(x)/g(x)$ for each x -value in your table. This investigation should suggest a simple method for finding the slope at *any* point on the curve $y = 3^x$. What is the method?

x	$g(x)$	$h(x)$
-1		
-0.5		
0		
0.5		
1		
1.5		
2		

2. (Continuation) First apply your knowledge of exponents to rewrite the expression $\frac{3^{a+h} - 3^a}{h}$ so that 3^a appears as a factor. Then use your calculator or computer to evaluate $\lim_{h \rightarrow 0} \frac{3^h - 1}{h}$. One way to do this is to go to wolframalpha.com and enter in the search bar:

limit as h goes to 0 of (3^h-1)/h

Another way to do this is to enter into a TI-89 or other calculator:

limit((3^h-1)/h,h,0)

Explain the relevance of the limit it gives you, to the pattern observed in #1.

3. (Continuation)

(a) Rewrite the equation $x = \frac{3^h - 1}{h}$ so that 3 appears by itself on one side of the equation.

(b) Then find the limiting value of the other side of the equation as h approaches 0 . Show that $e^x = \lim_{h \rightarrow 0} (1 + xh)^{1/h}$. Use this to find the value of x , and then explain the answers you got in #1 and #2. *Hint:* Use a trick similar to the one used in Page 3 # 4-5.

4. Let f be the quadratic function defined by $f(x) = x^2 - 3x$.

(a) Make a large, clear sketch in your notebook of the graph of the function.

(b) Compute $f(2)$ and $f(2 + h)$.

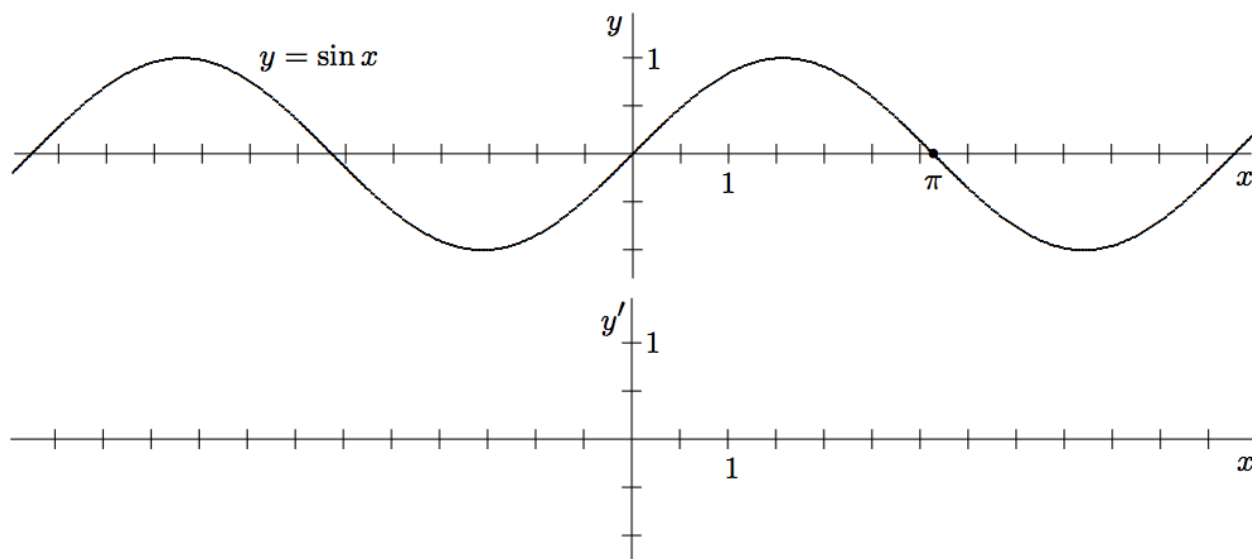
(c) Use algebra to evaluate $\lim_{h \rightarrow 0} \frac{f(2 + h) - f(2)}{h}$. (This means: multiply out everything, cancel like terms in the numerator and denominator if you can, and then plug in $h = 0$ if you can do so without getting something of an indeterminate form.) What is the meaning of the answer?

5. (Continuation) Calculate $\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$. (See #2 above for how to do this using a calculator or computer.) What does this limit represent?

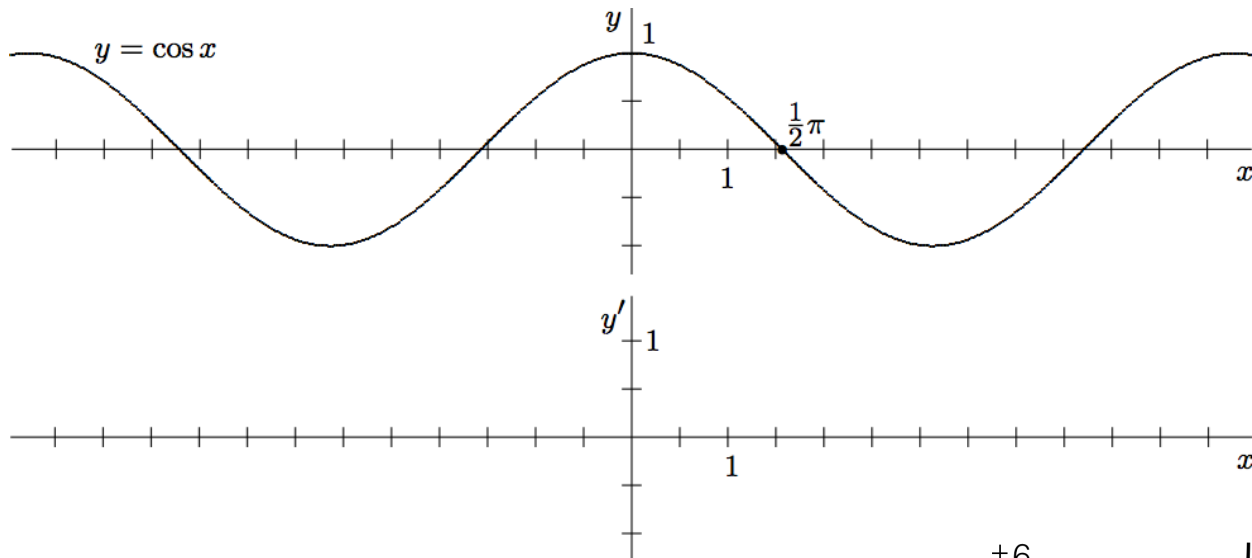
6. After being dropped from the top of a tall building, the height of an object is described by $y = 400 - 16t^2$, where y is measured in feet and t is measured in seconds. Find a formula for the *rate of descent* (in feet per second) for this object. Your answer will depend on t .

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7. Below you will find a radian-mode graph of $y = \sin x$ for $-2\pi \leq x \leq 2\pi$. The scales on the axes are the same. Estimate the slope of the graph at a dozen points of your choosing. On the second (blank) system of axes, plot this data (slope versus x). Connect the dots. Do you recognize the pattern? Why is radian measure for angles essential here?

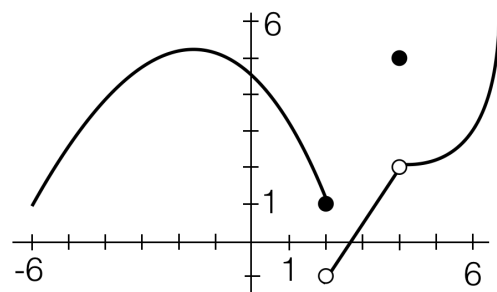


8. (Continuation) Repeat the process for the curve $y = \cos x$, using the axes below.



9. To the right is the graph of $f(x)$, defined on the interval $[-6, 6]$. Use the picture to give approximate values for each of the following (if they exist).

- (a) $\lim_{x \rightarrow -4} f(x)$ (b) $\lim_{x \rightarrow 2} f(x)$ (c) $\lim_{x \rightarrow 2^-} f(x)$
 (d) $\lim_{x \rightarrow 2^+} f(x)$ (e) $\lim_{x \rightarrow 4} f(x)$ (f) $f(4)$



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1. The point $P = (0.5, 0.25)$ lies on the graph of $y = x^2$. Sketch this graph. Use the zoom feature of your calculator or computer to look at very small portions of the graph as it passes through P . When the magnification is high enough, the graph looks like a straight line. Find a value for the slope of this apparent line. (Do this using nearby points, as we have done in previous problems.) It is not surprising that this number is called *the slope of $y = x^2$ at $x = 0.5$* . Now calculate the slope of the same curve at a different x -value. Notice that the slope is twice the x -value you chose.

2. (Continuation) The preceding item stated that $2x$ serves as a *slope formula* for any point on the curve $y = x^2$. Confirm this by using algebra (as in Page 4 # 4c) to evaluate $\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$. Make a diagram that shows clearly what the symbol h represents.

3. (Continuation) You would achieve the same result by evaluating $\lim_{u \rightarrow x} \frac{u^2 - x^2}{u - x}$. Explain. Make a diagram that shows clearly what the symbol u represents.

4. Find an equation for each of the following lines:

(a) The secant line to the graph of $y = x^2$ that intersects it at $x = 1$ and $x = 2$.

(b) The *tangent line* to the graph of $y = x^2$ at the point $P = (0.5, 0.25)$ from #1, using the slope you found there.

Add both of these lines to your sketch.

5. Consider the function $f(x) = |x|$.

(a) Make a large, clear sketch of the graph of $f(x)$ in your notebook.

(b) The function $f(x)$ is *nondifferentiable* at $x = 0$. Justify the description by explaining what the difficulty is (refer to the definition in the index).

(c) Find another example of a function that has the same type of nondifferentiability at a single point in its domain.

6. The core temperature of a potato that has been baking in a 375-degree oven for t minutes is modeled by the equation $C = 375 - 300(0.96)^t$. Find a formula for the *rate of temperature rise* (in degrees per minute) for this potato. Your answer will depend on t .

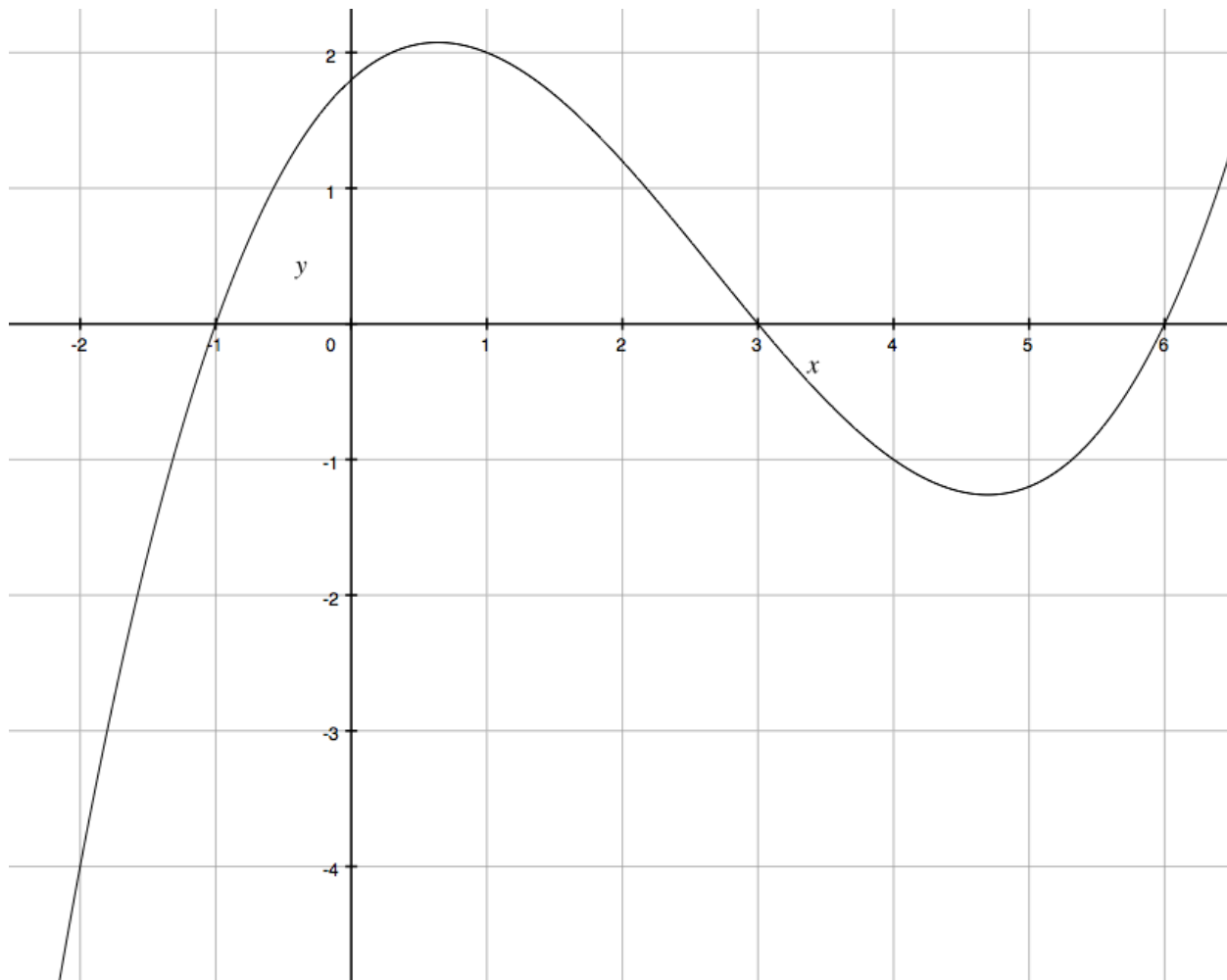
Hint: Apply what you learned in Page 4 # 1-3.

We will do the problems on the other side of this sheet in groups at the board in class. Feel free to look at them and work on them, but we won't write them up on the board at the beginning of class.

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7. Below is the graph of the function $f(x) = \frac{1}{10}(x+1)(x-3)(x-6)$ for the domain $[-2, 6]$.

- (a) Find the x -intercepts of $f(x)$. Use algebra to do this, then check with the picture.
- (b) What is the slope of $f(x)$ at its y -intercept? Use your best estimate.



8. (Continuation)

- (a) Find the x -intercepts of $f(2x)$.
- (b) Add a sketch of $f(2x)$ to the picture.
- (c) Describe the geometric transformation that transforms $f(x)$ into $f(2x)$.
- (d) What is the slope of $f(2x)$ at its y -intercept? Use your best estimate.

9. (Continuation)

- (a) Repeat #8 (a)-(d) for $f(mx)$ for some nonzero number $m \neq 2$ of your choice.
- (b) Hypothesize a rule that describes how the slopes of $f(x)$ and $f(mx)$ are related: *The slope of $f(mx)$ at its y -intercept is* _____
Justify your answer.

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1. *The derivative.* You have recently answered several rate questions that illustrate a fundamental mathematical process. For example, you have seen that:

- Given any point on the graph $y = \sin t$, the slope of the curve at that point is $\cos t$.

- Given any point on the graph $y = x^2$, the slope of the curve at that point is exactly twice the x -coordinate of the point. _____

- If the height of a falling object is $400 - 16t^2$ at time t , then the velocity of the object is $-32t$ at time t . _____

- Given any point on the graph $y = 3^x$, the slope of the curve at the point is exactly $\ln 3$ times the y -coordinate of the point. _____

- If an object is heated so that its temperature is $375 - 300(0.96)^t$ degrees at time t , then its temperature is increasing at a rate of $-300(0.96)^t \ln 0.96$ degrees per minute at time t .

(a) Fill in the blanks above with the problem where we discovered the fact, e.g. Page 2#7.

In each of the five examples, a function is given:

$$y(t) = \sin t ; f(x) = x^2 ; h(t) = 400 - 16t^2 ; F(x) = 3^x ; C(t) = 375 - 300(0.96)^t$$

From these functions, new functions are derived by means of a limiting process:

$$m(t) = \cos t ; m(x) = 2x ; V(t) = -32t ; M(x) = 3^x \ln 3 ; R(t) = -300(0.96)^t \ln 0.96$$

A function derived in this way from a given function f is called the *derived function of f* , or simply the *derivative of f* , and it is often denoted f' to emphasize its relationship to f . The five functions derived above could therefore be named y' , f' , h' , F' , and C' , instead of m , m , V , M , and R , respectively. Each of them provides *rate information* about the given function. By the way, the use of primes ($'$) to indicate derivatives is due to *Lagrange*.

(b) What is the derivative of the function E defined by $E(x) = b^x$? As usual, you should assume that b is a positive constant.

(c) Use algebra to find the derivative of the power function defined by $p(x) = x^3$.

This means: find the slope (the “difference quotient”) of $p(x) = x^3$ between the x -values x and $x + h$, where h is a small number, multiply everything out, cancel what you can, and then take the limit as $h \rightarrow 0$ (making sure to avoid an indeterminate form).

2. The slope of the curve $y = 2^x$ at its y -intercept is $\ln 2$, which is approximately 0.693.

(a) We have seen this before, in problem(s) _____.

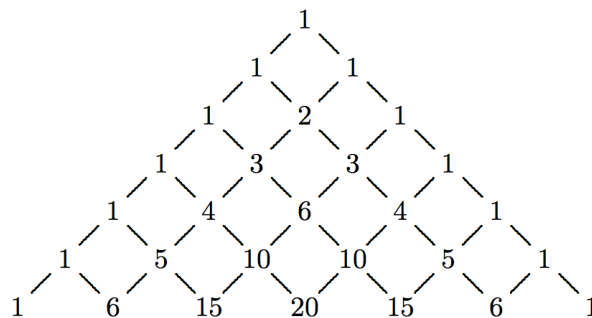
(b) Explain where this number 0.693 comes from, in two ways: (1) write down the associated limit whose value is 0.693, (2) use the formula you found in problem #1(b) above.

(d) Find the slope of the curve $y = 2^x$ at the point $(3, 8)$.

3. (Continuation) What is the slope of the curve $y = 2^x$ at the point on the curve whose y -coordinate is 5?

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4. *Pascal's Triangle* is an infinite array that is partially shown at right.



(a) Can you figure out how each row is obtained from the row above? Add another row to the bottom of the triangle.

(b) Multiply out the following, collect like terms, and then arrange the terms so that the highest power of a is first and the highest power of b is last, for example $a^2 + 2ab + b^2$:

$(a + b)^1, (a + b)^2, (a + b)^3, (a + b)^4, (a + b)^5, \dots$

(c) Explain the relationship between your result to (b) and Pascal's triangle.

(d) Use (c) to write down the expanded form of $(a + b)^6$ without multiplying out the terms.

(e) It is customary to call the top row of this array the 0^{th} row. According to this convention, how many entries appear in the n^{th} row? What is the first entry of the n^{th} row? What is the second entry of the n^{th} row? *We will use this last bit later. Get psyched up!*

(f) (Challenge, optional) What is the third entry of the n^{th} row? What is the sum of all the entries in the n^{th} row?

Do (yes, do!) these problems for homework. We'll discuss them in groups at the board, so please don't write them up at the beginning of class.

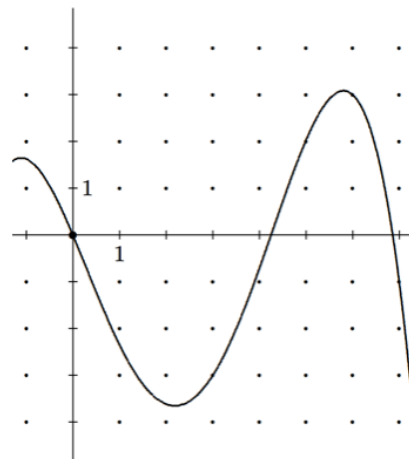
5. On the graph $y = f(x)$ shown at right, draw lines whose slopes are:

(a) $\frac{f(7) - f(3)}{7 - 3}$

(b) $\lim_{h \rightarrow 0} \frac{f(6 + h) - f(6)}{h}$

(c) $\frac{f(7)}{7}$

(d) $\lim_{h \rightarrow 0} \frac{f(h)}{h}$



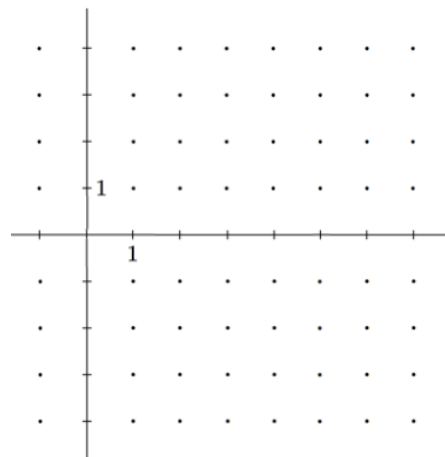
6. (Continuation) On the same graph, mark points where the x -coordinate has the following properties (a different point for each equation):

(a) $\frac{f(x) - f(2)}{x - 2} = \frac{1}{2}$

(b) $\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = -1$

(c) $\frac{f(x)}{x} = \frac{1}{2}$

(d) $\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = 0$



7. (Continuation) Graph the slope function f' on the axes at right.

Math 15

1. Recall that the slope of the curve $y = e^x$ at its y -intercept is 1. Use this information, and the result of Page 5 # 9b, to find the slopes of the curves $y = e^{2x}$ and $y = e^{mx}$ at their y -intercepts.

2. (Continuation) Justify the identity $2^x = e^{x \ln 2}$. Then apply this equality to the problem of finding the slope of the curve $y = 2^x$ at its y -intercept.

3. You have found the derivatives of at least two *power functions*.

(a) To be specific, you have shown that the derivative of $f(x) = x^2$ is $f'(x) = 2x$, in _____, and that the derivative of $p(x) = x^3$ is $p'(x) = 3x^2$, in _____ (fill in blanks with page and problem number). This evidence suggests that there is a general formula for the derivative of any function defined by $g(x) = x^n$, at least when n is a positive integer.

(b) First conjecture what you think the formula for $g'(x)$ is.

(c) To obtain the correct formula, first multiply out the binomial $(x+h)^n$, using your results from Page 6 # 4. Then multiply out and simplify $\lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}$.

4. A *linear function* has the form $L(x) = mx + b$, where m and b are constants (this means that the values of m and b do not depend on the value of x).

(a) Draw a graph of $L(x)$, labeling m and b in your picture.

(b) Write down a formula for the derivative $L'(x)$, and use your graph to justify your answer.

5. A ratio of changes $\frac{f(t+h) - f(t)}{h}$ can be written $\frac{f(t+\Delta t) - f(t)}{\Delta t}$, in which h is replaced by Δt . The symbol Δ (*Greek* “delta”) is chosen to represent the word *difference*. It is customary to call Δt the *change in t* . The corresponding *change in $f(t)$* is $f(t+\Delta t) - f(t)$, which can be abbreviated $\Delta f(t)$.

A ratio of changes $\frac{f(t+\Delta t) - f(t)}{\Delta t}$ is called a *difference quotient*, and the process of evaluating limits of difference quotients is often called *differentiation*.

(a) Employ differentiation (meaning: use difference quotients) to find the velocity of an object that moves along the x -axis according to the equation $x = f(t) = 4t - t^2$.

(b) Use this derivative to find the velocity and speed of the object each time that it passes the point $x = 0$. (You will need to find the values of t for which the object passes $x = 0$.)

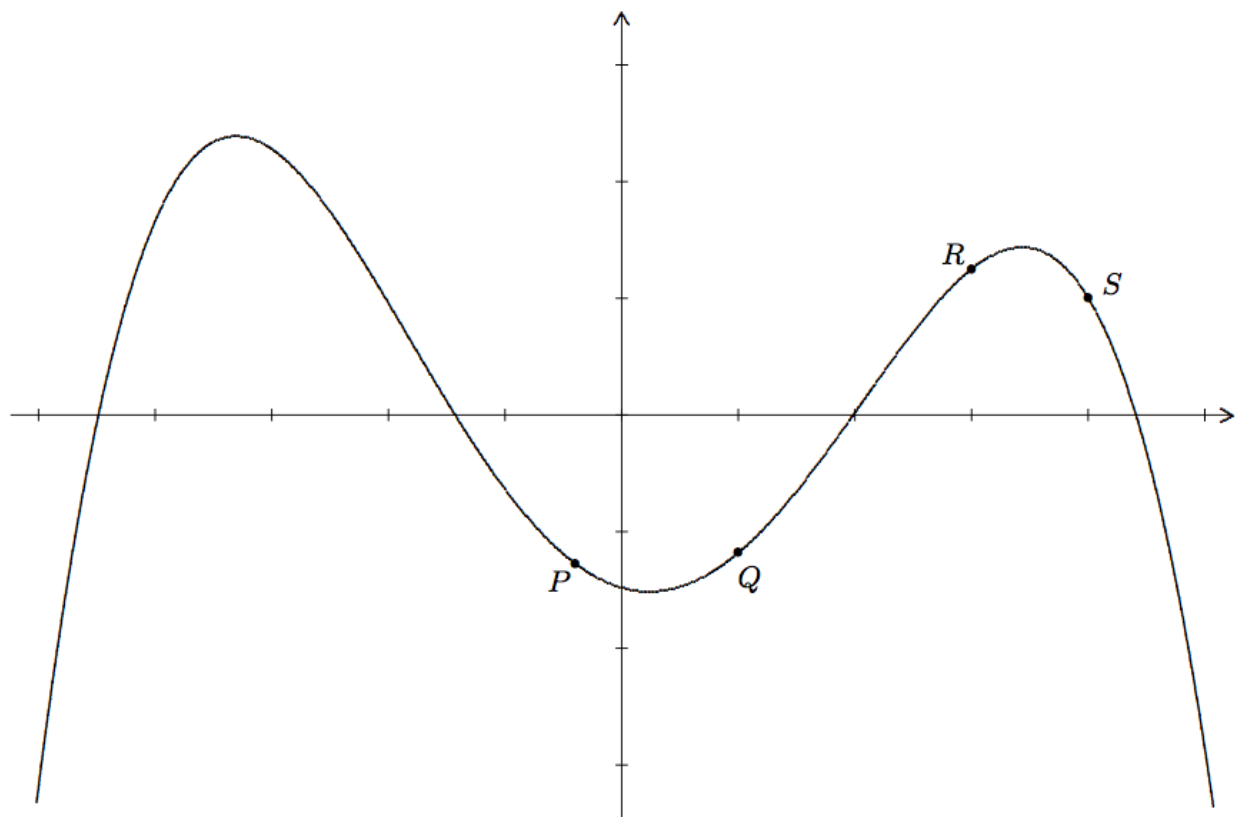
6. (Continuation) The equation $f'(t) = 4 - 2t$ is sometimes expressed $\frac{dx}{dt} = 4 - 2t$. This illustrates the *Leibniz notation* (pronounced libe-nits) for derivatives. After you explain why Δx is a good name for $\Delta f = f(t+\Delta t) - f(t)$, write an equation involving a limit that relates $\frac{dx}{dt}$ and $\frac{\Delta x}{\Delta t}$. This should help you understand what Leibniz had in mind.

Math 15

Do (yes, do!) these problems for homework. We'll discuss them in groups at the board, so please don't write them up at the beginning of class.

7. The figure below shows the graph of $y = f(x)$, where f is a differentiable function. The points P , Q , R , and S are on the graph. At each of these points, determine which of the following statements applies:

- (a) f' is positive (b) f' is negative
- (c) f is increasing (d) f is decreasing
- (e) f' is increasing (f) f' is decreasing



8. (Continuation)

- (a) The graph $y = f(x)$ is called *concave up* at points P and Q , and it is called *concave down* at points R and S . Explain the terminology.
- (b) Give an interval of x -values for which $f(x)$ is concave up.
- (c) Give an interval of x -values for which $f(x)$ is concave down.
- (d) How can you tell where $f(x)$ changes from concave up to concave down, or vice-versa?

Math 15

1. In this problem, we'll find the derivative of $R(x) = \frac{1}{x}$.

(a) Use the algebra of difference quotients to find a formula for the derivative. In other words, please calculate $R'(x) = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$. You'll need to find a common denominator.

(b) You should have found in (a) that all values of $R'(x)$ are negative. What does this tell you about the graph of $y = \frac{1}{x}$?

(c) Is your result consistent with the *Power Rule*? Write $R(x)$ in the form $R(x) = x^k$, and discuss whether $R'(x)$ follows the Power Rule formula you found in Page 7 # 3c.

2. Riding a train that is traveling at 72 mph, Morgan walks at 4 mph toward the front of the train, in search of the snack bar. How fast is Morgan traveling, *relative to the ground*?

3. (Continuation) Given *differentiable* functions f and g , let $k(t) = f(t) + g(t)$. Use the definition of the derivative (i.e. write out a limit of difference quotients) to show that $k'(t) = f'(t) + g'(t)$ must hold. This justifies *term-by-term differentiation*.

Use term-by-term differentiation to do the next two problems.

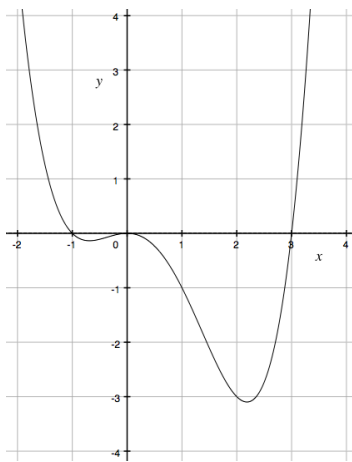
4. Find the derivative of $C(x) = (x^2 + 5)^3$. Please do this by multiplying everything out and then using term-by-term differentiation. Soon, we will learn an easier way!

5. Find the derivative of each of the following functions:

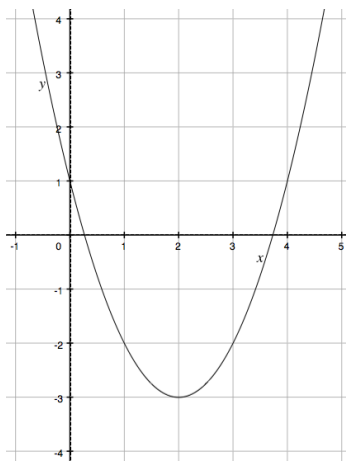
(a) $f(x) = 2^x + x^2$ (b) $g(t) = 3t - 5 \sin t$

(d) $P(t) = 12 + 4 \cos(\pi t)$ (c) $L(x) = (\ln 2)^x$

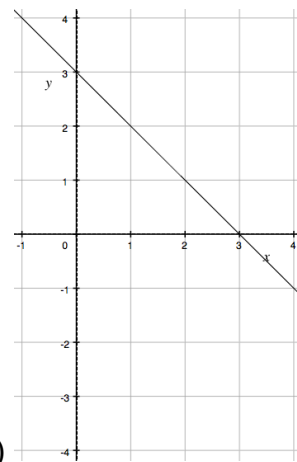
6. For each graph $f(x)$ below, estimate the values $f'(0)$, $f'(2)$ and $f'(3)$.



(a)



(b)



(c)

7. (Continuation) For each graph $f(x)$ above, sketch the graph of $f'(x)$. Make sure to label your axes.

Math 15

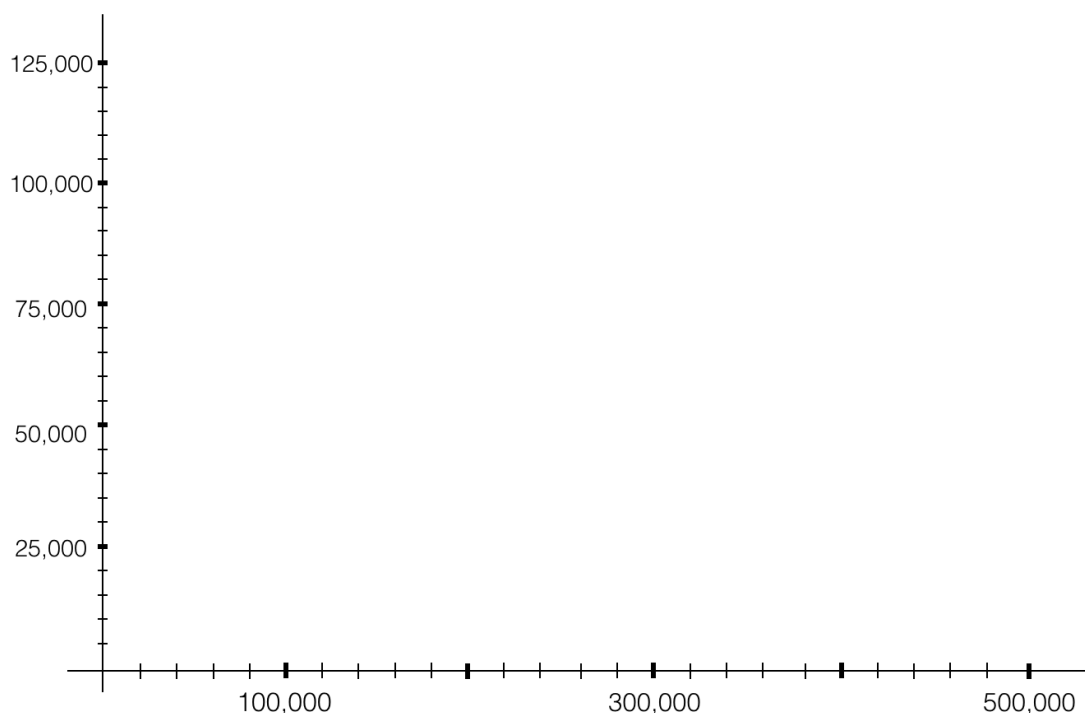
8. The IRS tax formula for married couples is a *piecewise-linear* function. In 2013 it was

$$T(x) = \begin{cases} 0.1x & \text{for } x \leq 17,850 \\ 1785 + 0.15(x - 17850) & \text{for } 17,850 < x \leq 72,500 \\ 9982.5 + 0.25(x - 72500) & \text{for } 72,500 < x \leq 146,400 \\ 28457.5 + 0.28(x - 146400) & \text{for } 146,400 < x \leq 223,050 \\ 49919.5 + 0.33(x - 223050) & \text{for } 223,050 < x \leq 398,350 \\ 107768.5 + 0.35(x - 398350) & \text{for } 398,350 < x \leq 450,000 \\ 125846 + 0.396(x - 450000) & \text{for } 450,000 < x \end{cases}$$

This function prescribed the tax $T(x)$ for each nonnegative taxable income x .

Find the tax paid by a married couple with:

- (a) \$10,000 of taxable income (b) \$40,000 of taxable income (answer: \$ 5107.50)
 (c) \$100,000 of taxable income (d) \$300,000 of taxable income.
 (e) This is known as a *progressive* tax structure: those with higher income pay a *higher percentage* of income in taxes. Find this percentage for each part (a)-(d) above. (b: 12.8 %)
9. (a) Explain why $T(x)$ passes through the points $(0, 0)$, $(17850, 1785)$, $(72500, 9982.5)$, $(146400, 28457.5)$, $(223050, 49919.5)$, $(398350, 107768.5)$, and $(450000, 125846)$.
 (b) Graph the function T below. Is $T(x)$ a continuous function?
 (c) Explain why $T'(x)$ makes sense for all but seven nonnegative values of x . For these seven values, T is said to be *nondifferentiable*.
 (d) Graph the derived function T' . How many different values does T' have?



Math 15

1. The function defined by $C(x) = (x^2 + 5)^3$ is an example of a *composite function*, meaning that it is built by substituting one function into another. One of the functions is $f(x) = x^2 + 5$ and the other is $G(x) = x^3$.

(a) First explain why that $C(x) = G(f(x))$.

(b) Then confirm that $C'(x) = G'(f(x)) \cdot f'(x)$, by calculating each side separately. *Hint:* You already calculated $C'(x)$ in a previous problem.

This is an example of the *Chain Rule* for derivatives.

2. (Continuation) Let $f(x) = e^{(x^2)}$. Use the Chain Rule to find $f'(x)$.

3. Use a limit of difference quotients (as in Page 7 # 1) to find a formula for the derivative of the power function defined by $Q(x) = \frac{1}{x^2}$. Does your result conform to the pattern established by the derivatives of other power functions?

4. (Continuation)

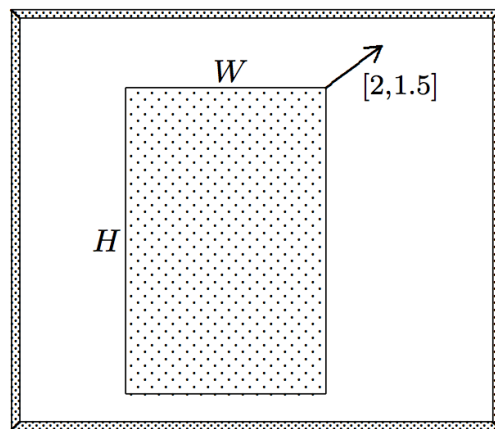
(a) Graph $y = \frac{1}{x^2}$ and $y = \frac{1}{(x-5)^2}$ on the same axes, on your calculator or computer.

(b) By examining the graphs, explain how the derivative of $P(x) = \frac{1}{(x-5)^2}$ can be obtained in a simple way from the derivative of $Q(x)$.

(c) Make up your own example of a similarly constructed function, and find its derivative.

5. (Continuation) Given a function f and a constant c , determine a general relationship between the derivative of f and the derivative of $g(x) = f(x - c)$.

6. Kelly is using a mouse to enlarge a rectangular frame on a computer screen. As shown at right, Kelly is dragging the upper right corner at 2 cm per second horizontally and 1.5 cm per second vertically. Because the width and height of the rectangle are increasing, the enclosed area is also increasing.



(a) At a certain instant, the rectangle is 11 cm wide and 17 cm tall. What are its dimensions 0.1 seconds later? How much area has the rectangle gained during this 0.1 second?

(b) Make calculations (and maybe sketch on the figure above) to show that most of the additional area comes from two sources — a contribution due solely to increased width, and a contribution due solely to increased height. Your calculations should also show that the rest of the increase is insignificant — amounting to less than 1%.

Math 15

7. (Continuation) Repeat the calculations, using a time increment of 0.001 second. As above, part of the increase in area is due solely to increased width, and part is due solely to increased height. What fractional part of the change is not due solely to either effect?

8. (Continuation) Let $A(t) = W(t) \cdot H(t)$, where A , W , and H stand for the area, width, and height of a rectangle, respectively, as functions of time. The previous examples illustrate the validity of the equation

$$\Delta A = W \cdot \Delta H + H \cdot \Delta W + \Delta W \cdot \Delta H.$$

(Refer back to Page 7 # 6 for an explanation of the notation Δ for a tiny difference.)

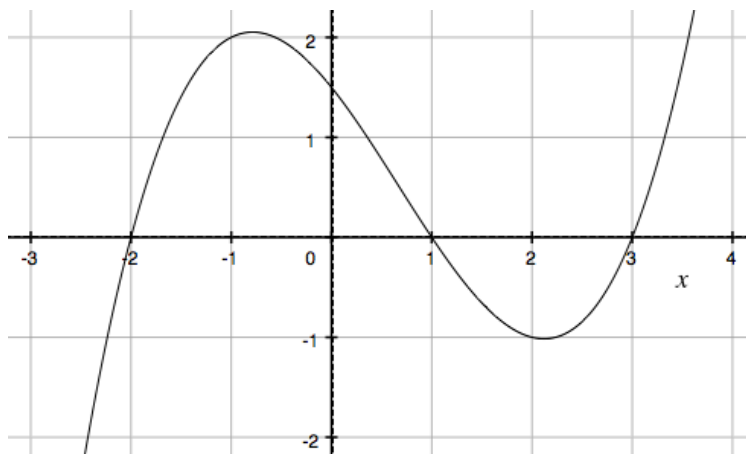
(a) Draw a $(W + \Delta W) \times (H + \Delta H)$ rectangle. Label the lengths W , ΔW , H and ΔH in your picture. Label the area $W \times H$, the area ΔA , and the areas corresponding to each of the parts of the right hand side of the above equation, in your picture.

(b) Explain why the term $\Delta W \cdot \Delta H$ plays an insignificant role as $\Delta t \rightarrow 0$.

(c) Divide both sides of the above equation by Δt and find limits as $\Delta t \rightarrow 0$, thus showing that the functions $\frac{dA}{dt}$, W , H , $\frac{dW}{dt}$, and $\frac{dH}{dt}$ are related in a special way. This relationship illustrates a theorem called the *Product Rule*.

8. Let $g(t) = t \cdot \cos(t)$. Use the Product Rule to find $g'(t)$.

9. The picture to the right shows the graph of $f'(x)$. Answer the following questions, which are not about $f'(x)$, but are about the *original* function $f(x)$:



(a) Where (this means: for which intervals of x -values) is $f(x)$ increasing?

(b) Where is $f(x)$ decreasing?

(c) For which value of x does $f(x)$ have the steepest positive slope?

(d) For which value of x does $f(x)$ have the steepest negative slope?

(e) Where is $f(x)$ concave up?

(f) Where is $f(x)$ concave down?

(g) Sketch $f(x)$. Explain why there is more than one right answer.

Math 15

1. Some notation. Given a function $f(x)$, the related function $f'(x)$, also known as $\frac{df}{dx}$, tells us the slope of $f(x)$ at each point x . If we are interested in the slope of $f(x)$ at a given x -value a , we evaluate $f'(a)$, also known as $\frac{df}{dx}|_{x=a}$.

Given the function $f(x) = (1 + 2x)^2$, find: **(a)** $f'(1)$ **(b)** $\frac{df}{dx}|_{x=-1}$

2. Given that f is a differentiable function and that the value of c does not depend on x , explain (justify why it is true) each of the following differentiation properties:

(a) If $g(x) = f(x - c)$, then $g'(x) = f'(x - c)$.

(b) If $g(x) = c \cdot f(x)$, then $g'(x) = c \cdot f'(x)$.

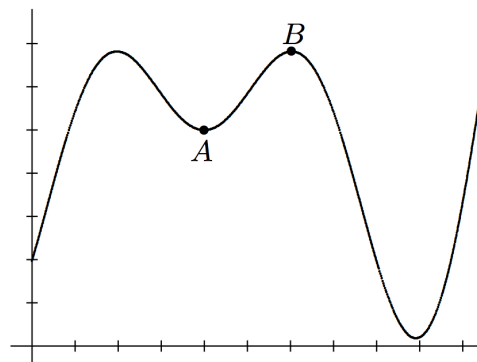
(c) If $g(x) = f(cx)$, then $g'(x) = c \cdot f'(cx)$.

(d) Which of these differentiation properties illustrates the Chain Rule?

3. Let $P(x) = x^{-n}$, where n is a positive whole number. Use the method of difference quotients to find $P'(x)$. *Hint:* Use a common denominator and Pascal's Triangle.

4. The diagram shows the graph of $y = f(x)$, which can be interpreted in the following two ways:

(a) It shows the elevation during a hike along a mountain ridge, as a function of time. During the part of the hike represented by the curve that joins point A to point B , there is a moment when the hiker is working hardest. If you had a formula for the function f , how would you calculate this time?



(b) It represents a bird's-eye view of a winding road. As you drive along the section of road from point A towards point B , there is a point where the car stops turning to the left and starts turning to the right. If you had a formula for the function f , how would you locate this point?

5. It is often useful to calculate the derivative of a derivative f' . The result is called the *second derivative of f* and denoted f'' . For each of the following, calculate f' and f'' :

(a) $f(x) = x^2 - 1$ **(b)** $f(z) = 3 \cdot 5^z$ **(c)** $f(u) = \cos(2u)$ **(d)** $f(t) = e^{-t^2}$

6. (Continuation)

(a) In example (a), notice that $f'(-1) < 0$ and $0 < f''(-1)$. What does this tell you about the graph of $y = f(x)$?

(b) Find a similar point (where f' is negative and f'' is positive) on the graph of (c). *Hint:* use the unit circle

(c) On the graph of example (d), find all points where f' is negative and f'' is positive.

Math 15

7. (From Hughes-Hallett et al, *Calculus*, §2.3 # 23): Sketch a graph of a smooth curve whose *slope* meets the given condition:

- (a) It is everywhere positive, and increasing gradually.
- (b) It is everywhere positive, and decreasing gradually.
- (c) It is everywhere negative, and increasing gradually.
- (d) It is everywhere negative, and decreasing gradually.
- (e) Assuming that the graph is of a function $f(x)$, rewrite the statements of parts (a)-(d) in notation, using only symbols from the list: $f(x)$, $f'(x)$, $f''(x)$, $>$, $<$, $=$, 0 , $\&$.

8. For each of the following, find an example of a function that satisfies the given condition:

- (a) $E'(x) = E(x)$
- (b) $T''(x) = -T(x)$
- (c) $p'(x) = x^2$
- (d) $f(x + a) = f(x) \cdot f(a)$ (Page 1 # 4)

Math 15

1.

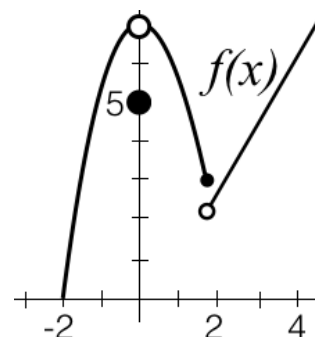
- (a) State the definition of the derivative, using difference quotients.
- (b) Use this definition to compute the derivative of the function $f(x) = \frac{1}{x}$ at $x = 1$.
- (c) Verify you get the same answer by using rules of differentiation.
- (d) Determine the equation of the line tangent to the curve at $x = 1$.
- (e) Draw a picture of the graph and its tangent line.
- (f) Find the equation of the normal line as well at $x = 1$. (Recall that the *normal line* is the line perpendicular to the tangent line.)

2. Sketch the graph of a function $f(x)$ that satisfies all of the properties listed below:

- $f(x)$ is defined for all real numbers x
- $\lim_{x \rightarrow \infty} f(x) = 0$ and $\lim_{x \rightarrow -\infty} f(x) = 7$
- f is continuous for all real numbers but f is not differentiable at $x = 0$
- f' is decreasing for $x < 0$ and f' is increasing for $x > 0$.

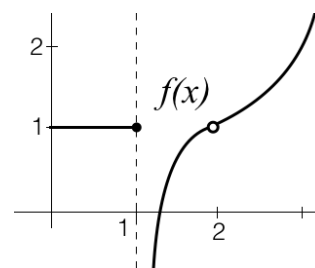
3. Use the figure on the right to give an approximate value for each of the following limits, or explain why it does not exist:

- (a) $\lim_{x \rightarrow -2} f(x)$
- (b) $\lim_{x \rightarrow 0} f(x)$
- (c) $\lim_{x \rightarrow 2} f(x)$
- (d) $\lim_{x \rightarrow 4} f(x)$



4. Use the figure on the right to give an approximate value for each of the following limits, or explain why it does not exist:

- (a) $\lim_{x \rightarrow 1^-} f(x)$
- (b) $\lim_{x \rightarrow 1^+} f(x)$
- (c) $\lim_{x \rightarrow 1} f(x)$
- (d) $\lim_{x \rightarrow 2^-} f(x)$
- (e) $\lim_{x \rightarrow 2^+} f(x)$
- (f) $\lim_{x \rightarrow 2} f(x)$



5. The power that a battery supplies to an apparatus such as a laptop depends on the internal resistance of the battery. For a battery of voltage V and internal resistance r , the total power delivered to an apparatus of resistance R is given by $P = \frac{V^2 R}{(R + r)^2}$.

- (a) Find $\frac{dP}{dR}$, assuming V and r are constants.
- (b) For what value(s) of R is the tangent line to the graph of $P(R)$ horizontal?

6. Consider the function $H(x) = \sqrt[2017]{7x^6 - x^5 + 4x^4 - 9x^3 + 3x - \pi^2}$.

- (a) Determine two functions f and g such that $H(x) = (f \circ g)(x)$.
- (b) Now determine $H'(x)$ using the chain rule.

Math 15

7. Consider the function $N(t) = v$, which describes the volume of a certain pumpkin, which is measured in liters, as a function of time, which is measured in weeks.

- (a) Say what the statement $N(3) = 2$ means, in words, **with units**.
- (b) Say what the statement $N^{-1}(3.7) = 5$ means, in words, **with units**.
- (c) Say what the statement $N'(3) = 3.1$ means, in words, **with units**.

8. For each part, sketch a graph of a function $f(x)$ that satisfies the given property.

- (a) $f(x)$ is increasing and $f'(x)$ is decreasing.
- (b) $f(x)$ is decreasing and $f'(x)$ is increasing.
- (c) $f(x)$ is concave down and $f(x)$ is increasing.
- (e) $f(x)$ is concave up and $f(x)$ is decreasing.
- (f) $f(x)$ is continuous at $x = 1$ and is *not* differentiable at $x = 1$.
- (g) $f(x)$ is differentiable and satisfies $\lim_{x \rightarrow 2} f(x) = f(2)$.

9. Let $F(x) = h(x)e^{r(x)}$, where $h(x)$ and $r(x)$ are functions. Find a formula for $F'(x)$.

10. Find the derivative of the given functions. You do not have to simplify your answer.

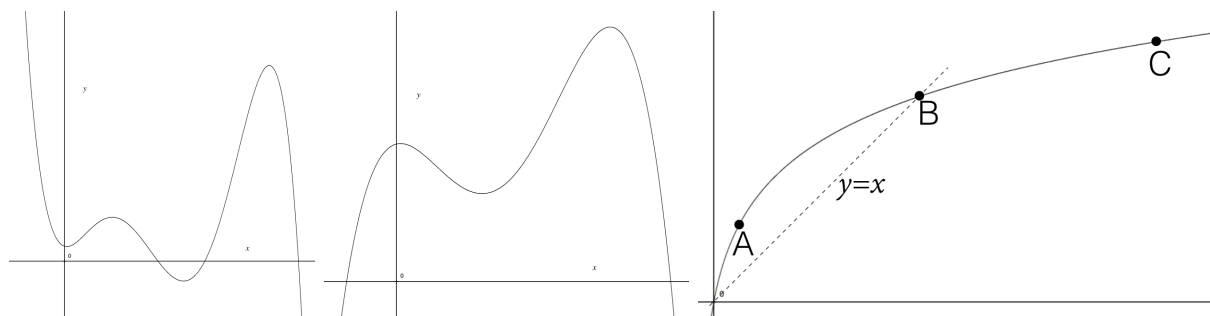
- (a) $P(t) = (\sqrt{t^2 + e^{3t}})3^{t^2-5}$.
- (b) $R(x) = \frac{\sqrt[3]{x} + x^5 - 2x^2 + x^{\pi+e}}{3x^7}$
- (c) $S(x) = \frac{x^e - 1}{x^e + 1} - 3\pi^2$
- (d) $g(x) = e^x(x-2)^{15}3^x$
- (e) $Q(t) = 3^{t^\pi - e^2}$. Find $Q''(t)$

11.

(a) For the function on the left below, mark with a letter a point where the curve has each of the following slopes: $A : -3, B : -1, C : 0, D : 1/2, E : 1, F : 2$.

(b) For the function in the middle below, at what points is the slope of the graph positive? at what points is it negative? At which point does the graph have the greatest slope? The lowest (largest negative) slope?

(c) For the function on the right below, arrange the following numbers from smallest to largest: the slope of the graph at A , the slope of the graph at B , the slope of the graph at C , the slope of the line AB , the number 0, the number 1.



Math 15

1. Pat is deciding how many hours per night to spend studying Math 15. Previous data collected from generations of Swarthmore students suggests that the knowledge k obtained from studying t hours per night varies according to the function:

$$k(t) = (3 - t)(t - 1)^2.$$

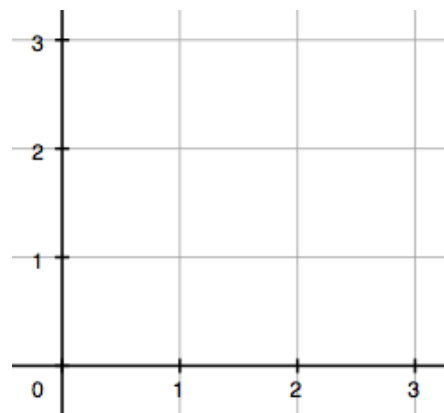
(a) Pat is able to allocate any amount of time t to studying, that satisfies $0 \leq t \leq 3$. The first inequality $0 \leq t$ arises because it's not possible to study for a negative amount of time. Make up an explanation for the second inequality.

Understandably, Pat wants to understand how much knowledge will be gained from different amounts of studying – what amount will result in the most learning? the least? So Pat wishes to graph $k(t)$ for $0 < t < 3$.

(b) Compute $k(t)$ for $t = 0, 1, 2, 3$ and add these data points to the graph.

(c) Suppose that $k(t)$ has a *local maximum* or *local minimum* for some time value t_0 . What would the graph look like at that point? How could you use the above equation for $k(t)$, to find this value?

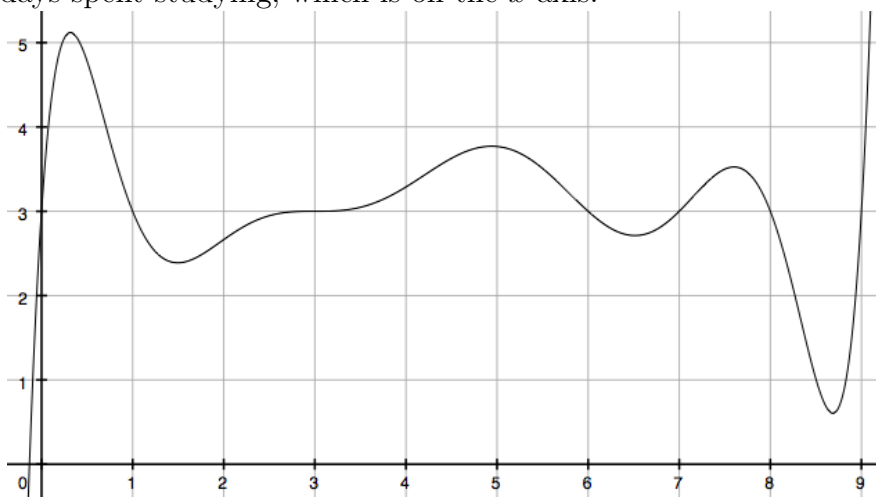
(d) Do it! Use $k(t)$ to find the *critical points* of the function. Then use your critical point data to sketch the graph.



2. Eugene is deciding how many days of the 9-day Fall Break to spend studying. The graph below shows the amount of Knowledge and Understanding (KU) gained on the y -axis, which depends on the number of days spent studying, which is on the x -axis.

(a) Mark each *critical point* on the graph (you may wish to look up the definition). Then label them A, B, C , etc. from left to right so that you can refer to them below.

(b) Color red the parts of the graph that are concave-up. Color blue the parts that are concave-down.



(c) Suppose the you are given the set of x -values that correspond to critical points. Explain how to use concavity to classify which ones are maxima (yay!) and which are minima (sad!).

(d) Write a recipe for finding and then classifying critical points, **using only the symbols:** $x, f(x), f'(x), f''(x), =, >, <, 0$ **and the words:** if then and is a local max min

(e) Some critical points are *neither maxima nor minima*. Explain. What is the concavity of the function, at such a critical point? Explain using words, and also using symbols as in the previous question.

Math 15

3. Kim is eating cookies before soccer practice. The increased performance p depends on the number of cookies c eaten, according to the equation

$$p(c) = (\cos^2(c - \pi/2) + 2) \sin(c - \pi/2) + 2.$$

While still in the dorm, Kim graphs the function, yielding the graph below.

Kim wants to know how much performance increase will result from eating cookies, of course. Back in the dorm, it's a simple matter to plug c into $p(c)$ and find out. But out on the sidelines, with only a phone to do calculations, Kim can only use a linear equation.

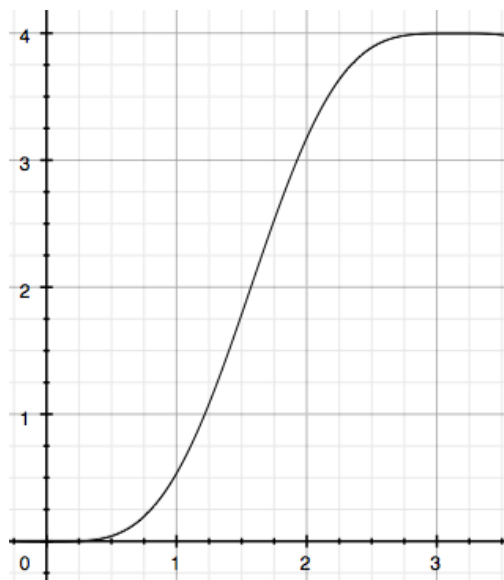
(a) Kim wants to eat around 1-2 cookies, and conveniently $1.5 \approx \pi/2$ lies in this range. Find the tangent line to $p(c)$ at $c = \pi/2$, and sketch it on the graph. Comment on how well or poorly the line fits the curve.

Call your tangent line equation $L(c)$, for *linearization*.

(b) Out on the field, the team wants to know how much performance increase to expect from 2 cookies. Use $L(c)$ to answer their question. Mark the corresponding point on the picture above.

(c) Concerned that the value may not be close enough, the team calls back to an injured teammate who stayed back in the dorm, and asks them to plug 2 into the equation for $p(c)$. What is the teammate's answer?

(d) How reliable was the outdoor estimate? For which values of c would you trust the simple, linearized equation to give you a reasonable estimate of the actual function?



4. Consider the function $f(x) = 3x^4 - 16x^3 + 18x^2$.

(a) Find the critical points of the function.

(b) For each critical point you found in part (a), classify it as a local maximum, local minimum, or neither.

(c) Use this information to sketch the function as best you can.

5. (Continuation) Consider the same function $f(x) = 3x^4 - 16x^3 + 18x^2$ as above.

(a) Find an equation for the tangent line $L(x)$ to the function at $x = 2$.

(b) Add a sketch of the tangent line to your graph.

(c) Compute the values $f(3)$ and $L(3)$, and mark them on your graph. How accurate is the linear approximation function $L(x)$ for this value?

(d) Repeat part (c) for $x = 2.5$ and $x = 2.1$.

6. Consider the function $x^2 + y^2 = 5$.

(a) What does the graph of the function look like? Sketch it.

(b) Can you express this graph as a function $y = f(x)$?

Math 15

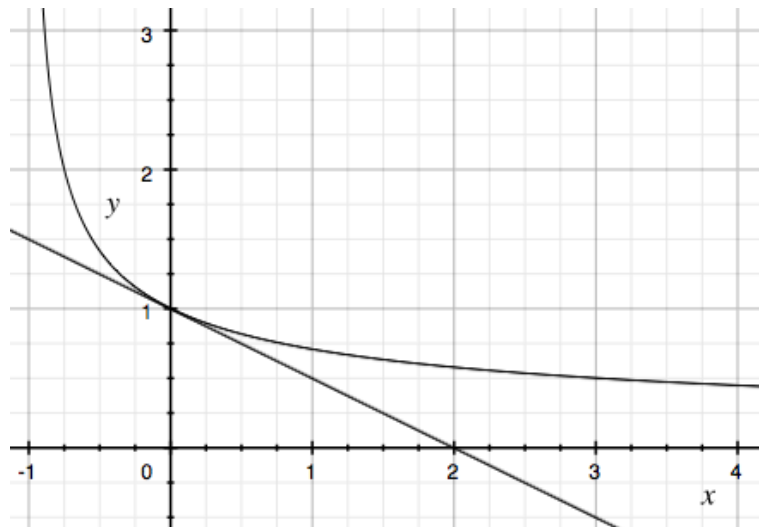
1. Consider the function $f(x) = x^5 + 15x^4 + 25$.

(a) For a function $f(x)$, if $f'(a) = 0$ or if $f'(a)$ is not defined, we say that a is a *critical point* of $f(x)$. Find all the critical points of f .

(b) For a function $f(x)$, if the concavity of f changes from concave-up ($f'' > 0$) to concave-down ($f'' < 0$) at $x = a$, we say that a is an *inflection point* for f . Find all the inflection points of f .

2. Show that $1 - \frac{x}{2}$ is the best linear approximation (tangent line) of $\frac{1}{\sqrt{1+x}}$ near $x = 0$.

These two functions are graphed to the right. Which is which?



3. In this problem, we'll find the derivative of $\ln(x)$.

(a) Explain why, if $f(x)$ is a function of x , it is true that

$$\frac{d}{dx} b^{f(x)} = b^x \cdot \ln(b) \cdot \frac{d}{dx} f(x).$$

(b) Explain why $e^{\ln x} = x$.

(c) Use the Chain Rule to differentiate each side of the equation in (b), and use your result to solve for $\frac{d}{dx} \ln x = \frac{1}{x}$.

4. Suppose that for a certain function $f(x)$, we know that $f(2) = 5$ and that $f'(2) = -0.4$.

(a) Use this information to estimate $f(2.1)$.

(b) Suppose that we also know that $f''(x) < 0$ for all x . Considering this, was your answer to (a) an overestimate or an underestimate?

5. Find the derivatives of the following functions:

(a) $s(\theta) = \sin \theta \cos \theta$

(b) $r(x) = \sqrt{2 + \sin(3x)}$

6. Let's find the derivative of $\tan x$:

(a) Fill in the numerator and denominator of each of the following fractions with the words **adjacent**, **opposite** and **hypotenuse**:

$$\sin \theta = \frac{\text{---}}{\text{---}}$$

$$\cos \theta = \frac{\text{---}}{\text{---}}$$

$$\tan \theta = \frac{\text{---}}{\text{---}}$$

(b) Express $\tan \theta$ in terms of $\sin \theta$ and $\cos \theta$.

(c) Use the Product or Quotient Rule to find the derivative of $f(\theta) = \tan \theta$.

Math 15

7. Consider the circle equation $x^2 + y^2 = r^2$.

(a) Solve for y as a function of x . Explain why there are two possibilities.

(b) Find $\frac{dy}{dx}$, from your equation in part (a).

(c) Here's another method: Consider y to be a function of x , and differentiate both sides of the circle equation: $\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}r^2$. Explain why the result is $2x + 2y\frac{dy}{dx} = 0$.

(d) Solve for $\frac{dy}{dx}$ in the result from (c). Does it agree with your answer to (b)?

(e) Which method was easier, (b) or (c)-(d), for finding $\frac{dy}{dx}$?

The method used in (c)-(d) is called *implicit differentiation*.

8. Use your results from problem 1 to sketch the function $f(x)$ from that problem.

Math 15

1. In the Amphitheater Adventure and in Page 13 # 2c, you explained how to use concavity to tell whether a given critical point is a maximum, minimum or neither. This is known as the *Second Derivative Test*, and it says that, for a function $f(x)$:

- If $f'(a) = 0$ and $f''(a) < 0$, then a is a local maximum for f .
- If $f'(a) = 0$ and $f''(a) > 0$, then a is a local minimum for f .
- If $f'(a) = 0$ and $f''(a) = 0$, the test tells us nothing.

Use this test to classify as a maximum or minimum the critical points of $f(x) = x^5 + 15x^4 + 25$.
Hint: you already found the critical points of this function, in Page 14 # 1.

2. Find the derivatives of the following functions:

(a) $s(t) = \ln(5t^2 + 7)$

(b) $c(x) = 2x(\ln x + \ln 5) - 3x + \pi$

3. We proved the Power Rule, which says that $\frac{d}{dx}x^n = nx^{n-1}$, for all integers n . We have assumed that it also holds for non-integer powers, such as $f(x) = x^{1/2}$. Let's prove it now.

(a) Explain why $(f(x))^2 = x$.

(b) Use the Chain Rule to differentiate each side of the equation in (a), and use your result to show that $f'(x) = \frac{1}{2f(x)}$.

(c) Explain why this proves that $f'(x) = \frac{1}{2}x^{-1/2}$.

4. Consider the curve given by

$xy + x + y = 5$, shown at right.

(a) Find a point (x, y) that lies on the curve, and confirm that it satisfies the equation above.

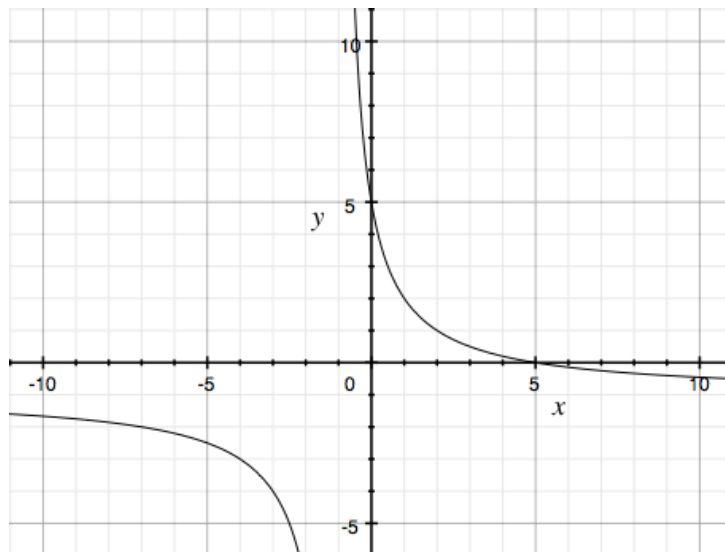
(b) Plot the point you found on the graph at right.

(c) Use implicit differentiation to find $\frac{dy}{dx}$, by differentiating both sides as we did for the circle equation in Page 14 # 7.

(d) Find an equation for the tangent line to the curve at your point from (a).

(e) Sketch in your tangent line on the graph above.

(f) Explain why the tangent line you found is the *best linear approximation* of the curve near your point.



Math 15

5. Consider the function $f(x) = 3x^4 - 4x^3 + 6$.

(a) Find all critical points of the function.

(b) At a critical point a , $f'(a) = 0$. Suppose that a is a local minimum. What is the *sign* of $f(x)$, a little to the left of a ? How about a little to the right? Answer the same questions when a is a local maximum. *Hint*: draw a picture

(c) Determine which of the critical points you found in (a) are local maxima, minima or neither, by testing the *sign* of $f'(x)$ on the left and right of each of your critical points. *Hint*: factor your expression for $f'(x)$

(d) Check your answers to (a) and (b) by graphing the function.

This method, of using the sign of $f'(x)$ near a critical point to classify it, is called the *First Derivative Test*. It sometimes allows us to classify a critical point, even when the Second Derivative Test fails.

6. Consider the curve $e^x + x = 2$.

(a) Explain why there is a solution between $x = 0$ and $x = 1$.

(b) Replace the left side of the equation by its best linear approximation at $x = 0$, and use it to find a (approximate) solution for x .

(c) Explain why it is not possible to find an exact solution for x , and thus we would want a linear approximation.

7. Now that we know the derivative of $\ln x$, we can give a shorter proof of the derivative of $f(x) = b^x$. Let's do it!

(a) Explain why $\ln(b^x) = x \cdot \ln b$.

(b) Differentiate each side of the equation in (a).

(c) Use your answer from (b) to show that $\frac{d}{dx}b^x = b^x \cdot \ln b$.

(d) Why didn't find the derivative this way when we did it weeks ago?

Math 15

1. (a) Differentiate both sides of the equation $\tan(\arctan x) = x$ to show that

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}.$$

(b) Your answer should have been $\cos^2(\arctan x)$. We can simplify this! Draw a right triangle and label the lengths of the legs of the triangle 1 and x . Label one of the angles α , in such a way that $\arctan(x/1) = \alpha$. Now find the length of the hypotenuse, and solve for $\cos^2(\alpha) = \cos^2(\arctan x)$. Then simplify your answer so that it is just a function of numbers and x , with no \arctan or \cos or α .

2. Dale throws a pumpkin straight up in the air, with an initial velocity of 50ft/sec. The pumpkin, released from Dale's hands in this exceptionally powerful toss, is 5 ft above the ground when it is released.

- (a) Explain why the height of the pumpkin after t seconds should be

$$h(t) = -16t^2 + 50t + 5.$$

(b) What is the maximum height of the pumpkin on this throw?

(c) For the time t_0 at which the pumpkin achieves its maximum height, is $h''(t_0)$ positive or negative? Make a guess, and then compute it.

(d) At what time does the pumpkin smash on the ground?

3. Consider the function $x^2 + xy = y^3 + xy^2$.

(a) Find a **nonzero** point (x, y) that lies on the graph of the function. *Hint:* use guess and check.

(b) Graph this function on your calculator or computer. Confirm that the point you found in (a) lies on the curve. Sketch the graph in your notebook.

(c) Use implicit differentiation to find $\frac{dy}{dx}$.

(d) Find an equation for the tangent line to the curve at your point from (a). Add the tangent line to your sketch, and confirm that the numbers you got make sense.

4. Sketch a possible graph of $f(x)$, given that:

- $y' < 0$ for all x
- $y'' = 0$ for $x = 2, 5, 8$
- $y'' > 0$ for $x < 2$ and for $5 < x < 8$
- $y'' < 0$ for $2 < x < 5$ and for $x > 8$

5. Consider the function $f(x) = 5x - \ln(x)$.

(a) Find the critical points and inflection points of the function.

(b) Use the Second Derivative Test (and, if it fails, some other method of your choice) to classify each critical point as a maximum, minimum or neither.

Math 15

6. The four graphs to the right show the temperature in Swarthmore over four different days.

(a) For each day, estimate the maximum and minimum temperatures that occurred, and mark the corresponding points on each graph.

(b) For which days did the maximum temperature occur: (A) at the left or right endpoints of the day (B) at a local maximum or minimum inside the day (C) somewhere else?

7. The task you just did is called *optimizing under a constraint*.

(a) In problem 6, what were we seeking to optimize, and what was the constraint?

(b) Give an example of something in your own life that you wish to maximize or minimize. Then give three examples of constraints in your life that apply to that thing.

8. (From *Calculus*, Hughes-Hallett et al):

An oscillating mass m at the end of a spring is at a distance y from its equilibrium position given by $y = A \cdot \sin\left(\sqrt{\frac{k}{m}} \cdot t\right)$. The constant k measures the stiffness of the spring.

(a) Find a time at which the mass is farthest from its equilibrium position.

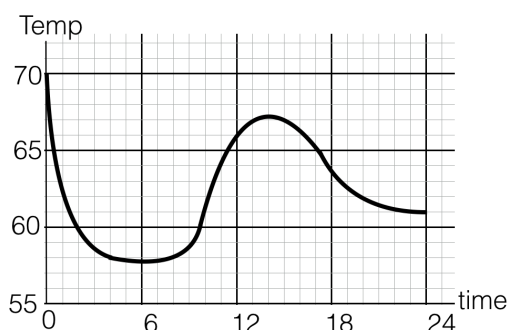
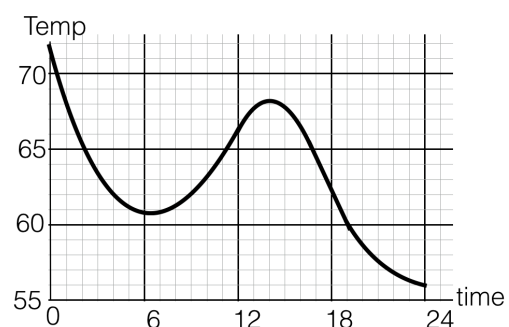
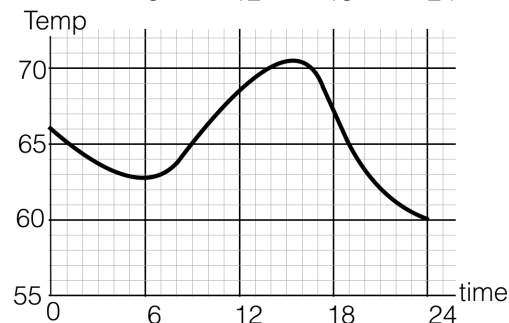
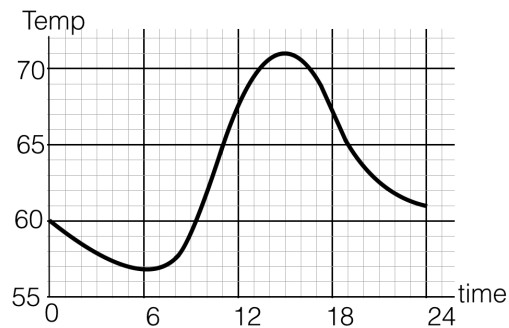
(b) Find a time at which the mass is moving fastest.

(c) Find a time at which the mass is accelerating fastest.

(d) What is the period T of the oscillation?

(e) Find dT/dm . What does the sign of dT/dm tell you?

9. Cat says to Pat, “if 1 is a critical point of $f(x)$, then we know $f'(1) = 0$.” Pat says, “I think there’s some other way to get a critical point, but I don’t remember how.” Who is right – Cat or Pat? Find evidence to support your claim.



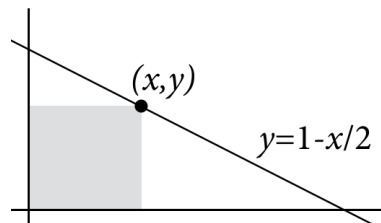
Math 15

1. Find the derivative of $\arcsin x$, using the same method we used for $\arctan x$.
2. Consider the curve $e^t = 0.03t + 0.97$.
 - (a) Explain why there is a solution near $t = 0$.
 - (b) Replace e^t by its linearization, and use this to find a (approximate) solution for t .
3. We have previously shown that the Power Rule applies to functions of the form $f(x) = x^k$ where k is an integer, or the number $1/2$. Let's prove it for all rational powers, so for

$$y = x^{m/n}$$

where m, n are integers.

- (a) Raise each side of the equation above to a power to eliminate fractions in the exponents.
 - (b) Use implicit differentiation to find $\frac{dy}{dx}$, and simplify your answer to a familiar form.
4. Suppose that you wish to maximize the area of a rectangle, which has one corner at the origin and the opposite corner on the line $y = 1 - x/2$, as shown.



- (a) Write an expression for the area of the rectangle, which is a function of x and y .
 - (b) Rewrite the expression for the area, in terms of just x .
 - (c) Use your differentiation skills to find the dimensions of the rectangle that maximize its area!
 - (d) Choose a different point on the line, find the area of the corresponding rectangle, and confirm that it is less than the area of the one you found in (c).
5. Find the value of c so that the function $f(x) = x \cdot e^{cx}$ has a critical point at $x = 5$.
6. Sketch a possible graph of $f(x)$, given that:
- $y' = 0$ for $x = 0$ and $x = 10$
 - $y' > 0$ for $x < 10$
 - $y' < 0$ for $x > 10$
 - $y'' = 0$ for $x = 0$ and $x = 5$
 - $y'' < 0$ for $x < 0$ and for $x = 10$
 - $y'' > 0$ for $0 < x < 5$

Math 15

7. A 400-meter running track is made of two parallel straightaways, connected by semicircular curves, as shown. Suppose that you want to choose the dimensions of the track to maximize the area of the rectangular soccer field at its center. Let's call the length of the straightaway x .



- (a) What range of x -values makes physical sense?
- (b) Find the perimeter of one curved end, in terms of x .
- (c) Use part (b) to find the width of the field, in terms of x .
- (d) Find the area of the field, in terms of x .
- (e) Use your skills to determine how long the straightaways should be!

8. Find the derivatives of the following functions:

- (a) $L(t) = \ln(1 + t^2)$
- (b) $T(x) = \arctan(4x)$

9. Sam says to Cam, “if $f''(0) = 0$, then f has an inflection point at $x = 0$.” Cam says, “I don’t think that’s correct, but I’m having trouble thinking of a counterexample.” Who is right – Sam or Cam? Find evidence to support your claim.

Math 15

1. The *Extreme value theorem* says that:

- A continuous function on a closed, bounded interval $[a, b]$ attains a global maximum value and a global minimum value, and
- to find the maximum and minimum function values, evaluate the function at each critical point that lies in the interval, and at the two endpoints; the maximum is the greatest of these values, and the minimum is the least of the values.

Follow the recipe from the second part above to find the global maximum and minimum value of $f(x) = x^3 - 3x^2 + 20$ on the interval $[-1, 1]$.

2. Two Swarthmore alums started a business making stroopwafel cookies. It cost them \$2000 to set up their business (kitchen, packaging machine, board of health certification). They sell stroopwafels on Parrish Beach for \$1 each. Producing each stroopwafel costs \$0.60.

(a) What are the *fixed costs* of the business – the costs that the company must pay, whether they sell any product or not?

(b) What is the *marginal cost* of producing one additional stroopwafel?

(c) What is the *marginal revenue* of selling one additional stroopwafel?

(d) What is the total *profit* (revenue minus all costs) when selling 3000 stroopwafels?

(e) How many stroopwafels do they need to sell, to break even?

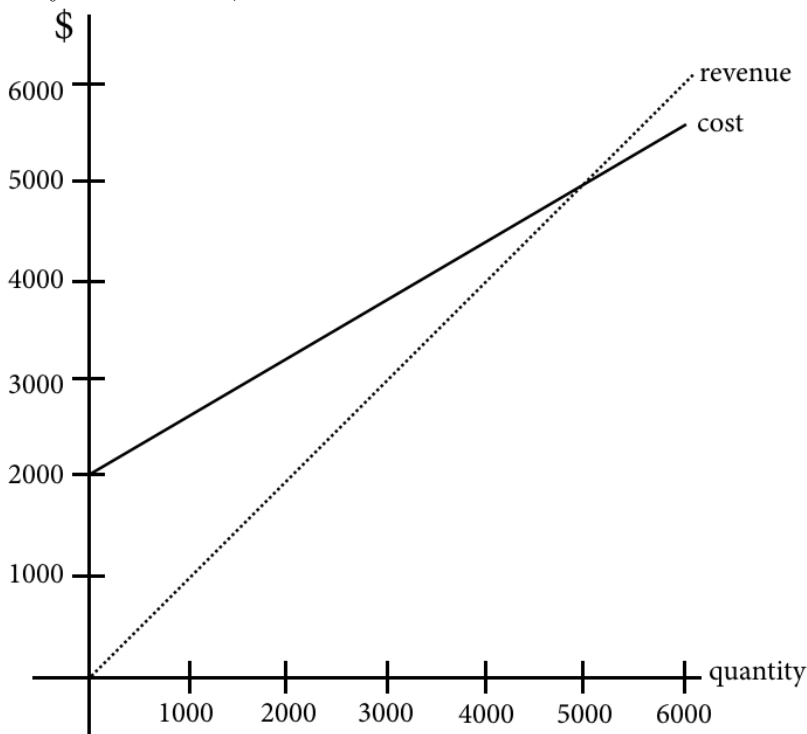
3. (Continuation)

(a) Find an equation for the total cost $C(q)$ of producing q stroopwafels.

(b) Find an equation for the total revenue $R(q)$ from selling q stroopwafels.

(c) Find the value of q where $C(q) = R(q)$. Interpret the meaning of this number in the context of the problem.

(d) The picture to the right illustrates the stroopwafel business. Explain the meaning of each part of the picture, in the context of the stroopwafel business: the y -intercept of each line, the slope of each line, and the intersection point of the two lines.



(e) Calculate the number $R(3000) - C(3000)$. What does this number represent? Find this number as the length of a vertical line segment in the picture.

Math 15

4. You have received a sheet of paper with 19×24 squares. Your mission, should you choose to accept it, is to make the largest possible open-top box that can be created from this piece of paper by removing a square from each corner and folding up the sides.

(a) Convert the first letter of your first name into a number via A=1, B=2, etc. Your n is the remainder when this number is divided by 9. For example, Jan has J=10 so $n = 1$.

(b) Cut off (I mean it! Use scissors!) an $n \times n$ square from each corner, fold up the sides to make an open-top box (I mean it!), and find its volume. **Bring your box to class.**

(c) Find n for your best friend's name, and find the volume of their associated box. Whose is bigger? What n do you think would make a box with the largest possible volume?

(d) Find a formula for the volume of the box, as a function of n .

(e) Use your differentiation skills to find the value of n that gives the maximum volume. How close was your estimate in (c)? Note that this is only a *local* maximum; you also need to justify that your answer is actually a *global* maximum.

5. A spherical snowperson, finding itself in balmy Swarthmore, is melting. The snowperson's radius upon arrival is 90 cm, and its radius is decreasing at 2 cm/min.

(a) Write an equation for the snowperson's radius after t minutes. **include units!**

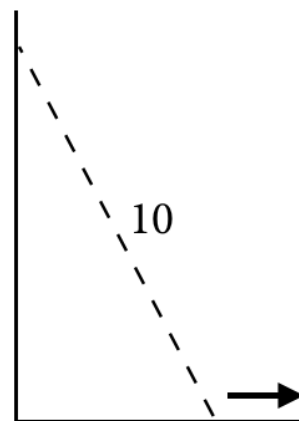
(b) The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. Write an equation for the volume of the snowperson after t minutes. **include units!**

(c) Find the rate at which the snowperson's volume is decreasing, t minutes after it started melting, and also 30 minutes after it started melting. **include units!**

6. *The ladder problem.* A 10-foot ladder leans against a vertical wall, as shown.

(a) Given that the bottom of the ladder is x feet from the wall, find a formula for the height $h(x)$ of the top of the ladder.

(b) In an unfortunate, but sadly very common situation, the base of the ladder is not firmly attached to the ground, but is rather on a Slip-N-Slide, so the distance x is actually a function $x(t)$ of time. Give an expression for $h'(t)$, given that h is a function of x , and x is a function of t .



Suppose that the foot of the ladder is sliding away from the wall at a rate of $1/2$ foot per second. Determine how fast the top of the ladder is sliding down the wall, when the foot of the ladder is: (c) 4 feet from the wall (d) 8 feet from the wall

7. Justify the following statement: *If marginal revenue is greater than marginal cost, then you should produce an additional unit of goods.*

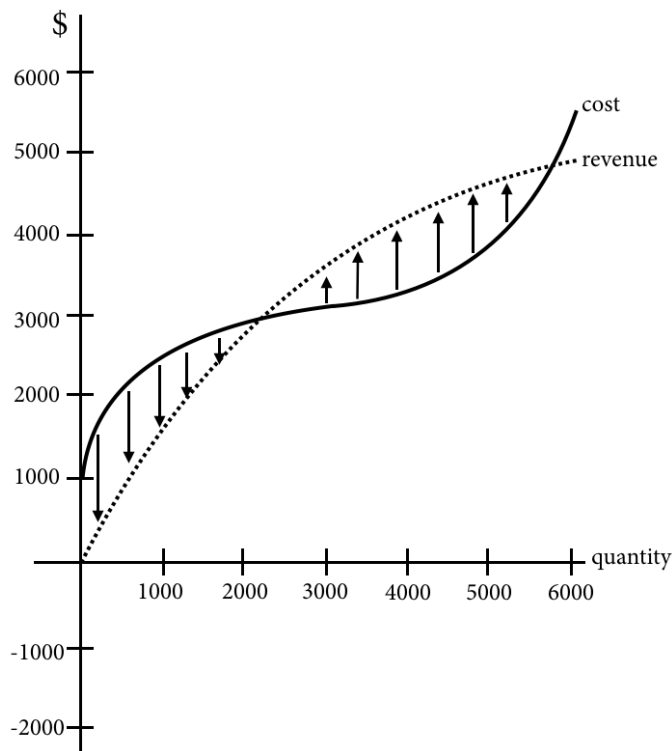
Math 15

1. The graphs to the right show the production cost, and revenue, for Morgan's coffee bus business, which each depend on the quantity of coffees sold.

(a) What is the cost of producing 1000 coffees? What is the revenue from selling 1000 coffees?

(b) Using the graph, estimate the profit (revenue minus costs) of the business at each of the following production levels: 0, 1000, 2000, 3000, 4000, 5000, and 6000 coffees sold.

(c) What quantity of coffees (in the range 0-6000) should Morgan aim to sell, in order to maximize profit? What is this associated maximum profit?



2. (Continuation)

(a) An important quantity in a business is $\text{profit} = \text{revenue} - \text{total costs}$. Sketch in the profit curve $P(q) = R(q) - C(q)$ on the same axes. Check whether the maximum of this function agrees with your answer to (c) in the previous problem.

(b) If you had an equation for the profit function $P(q)$, how would you use it to figure out the quantity q that maximizes profit?

(c) Justify the following statement: *Profit is maximized when $R'(q) = C'(q)$.*

3. (Continuation) The *marginal cost* of production is the cost of producing one additional unit of goods, such as one more cup of coffee.

(a) Explain how to use the *slope* of the cost graph above, to determine the marginal cost of production.

(b) Estimate the marginal cost of production when producing each of the following quantities of coffees: 1000, 3000, 5000.

4. The cost of educating q Swarthmore students is $C(q)$, and the revenue from enrolling q students is $R(q)$. If $C'(1500) = 50000$ and $R'(1500) = 66000$, and their goal is to increase profits, should Swarthmore enroll an additional student, or not?

Math 15

5. *More related rates.* At noon, a red sports car was 15 miles south of an intersection, heading due north along a straight highway at 40 mph. Also at noon, a blue sports car was 20 miles west of the same intersection, heading due east along a straight highway at 80 mph.

(a) Write an equation relating the distance r separating the two cars, the distance x between the blue car and the intersection, and the distance y between the red car and the intersection. *Hint:* avoid using a square root; it will make your life easier in the next step

(b) Use implicit differentiation of your equation in (a), remembering that r , x and y are all functions of time, to obtain the equation

$$2r \frac{dr}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}.$$

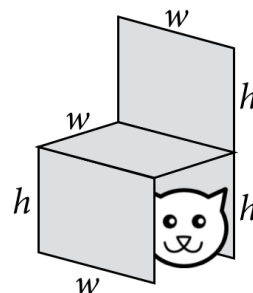
(c) The distance r separating the two cars was 25 miles at noon. At what rate was this separation decreasing?

Parts (d) and (e) to discuss in class

(d) At 1 pm, the cars were 65 miles apart. At what rate was this separation increasing?

(e) At what time of day were the cars closest together, and how far apart were they?

6. Ryan is making a cardboard chair in architecture class. The height of each part is the same (some length h), and the width and depth are the same, some length w . Ryan's cat likes to sit under the chair, so Ryan wishes to maximize the volume of the space under the chair. The instructor has given the students 300 in² of cardboard to make their chairs. What dimensions should Ryan use? (After determining an answer, also remember to justify that it gives a maximum volume, rather than a minimum.)



Math 15

1. Cameron, a student of Calculus, was instructed to find the derivative — with respect to x — of five functions, each expressed in terms of an unknown function y . Below are Cameron's five answers. For each, reconstruct the expression that Cameron differentiated. You will have to write your answers in terms of y (and x), of course. Can you be absolutely sure that your answers agree with the questions on Cameron's assignment?

(a) $2.54 + \frac{dy}{dx}$ (b) $\frac{dy}{dx} \sec^2 y$ (c) $\sqrt{5y} \frac{dy}{dx}$ (d) $\frac{7}{y^2} \frac{dy}{dx}$ (e) $(y - \cos x) \left(\frac{dy}{dx} + \sin x \right)$

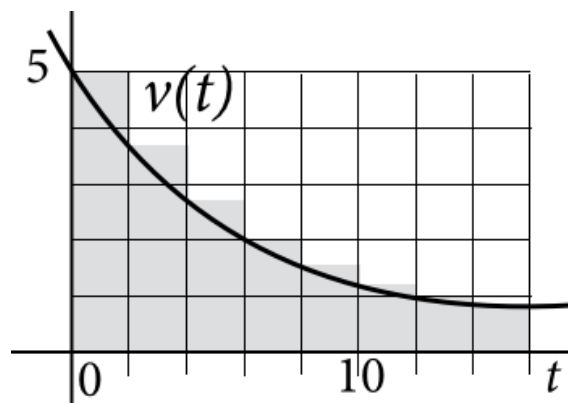
The functions you found are called *antiderivatives* of the functions on Cameron's sheet.

2. (For fun) We're about to learn the Fundamental Theorem of Calculus. Let's explore some other fundamental theorems first.

(a) The *Fundamental Theorem of Arithmetic* says that any number can be uniquely factored into primes. Factor the number 16100. The word *uniquely* means that there's only one way to do it, i.e. that everyone's answer should be the same.

(b) The *Fundamental Theorem of Algebra* says that any polynomial can be factored into terms of the form $(x - a)$, where each a is a complex number (or possibly just a real number). Factor the polynomial $x^3 + 2x^2 - 5x - 6$.

3. The curve in the figure shows the speed $v(t)$, in meters per second, of a bicycle that is decelerating over the course of 16 seconds. We want to figure out how far the bike traveled during this 16 seconds while it was slowing down.



(a) Estimate the speed of the bike at each of the following times: $t = 0$, $t = 2$, $t = 10$, $t = 16$.

(b) The boxes in the figure suggest a way of estimating the distance traveled. Suppose that, rather than decelerating between $t = 0$ and $t = 2$, the bike had gone its $t = 0$ speed for that entire 2 seconds. Estimate how far would it have traveled during those 2 seconds.

(c) Suppose the bike went its $t = 2$ speed for the entire time from $t = 2$ to $t = 4$. Estimate how far would it have traveled during *those* 2 seconds.

(d) Repeat this calculation for each 2-second interval, and use it to find an estimate for the total distance traveled. Explain how this calculation is related to the boxes in the picture.

(e) Is your answer to part (d) an overestimate or underestimate of the actual distance?

4. (Continuation) The estimation method we used in the previous problem is called a *left sum*.

(a) Explain why it is reasonable to call it a *left sum*.

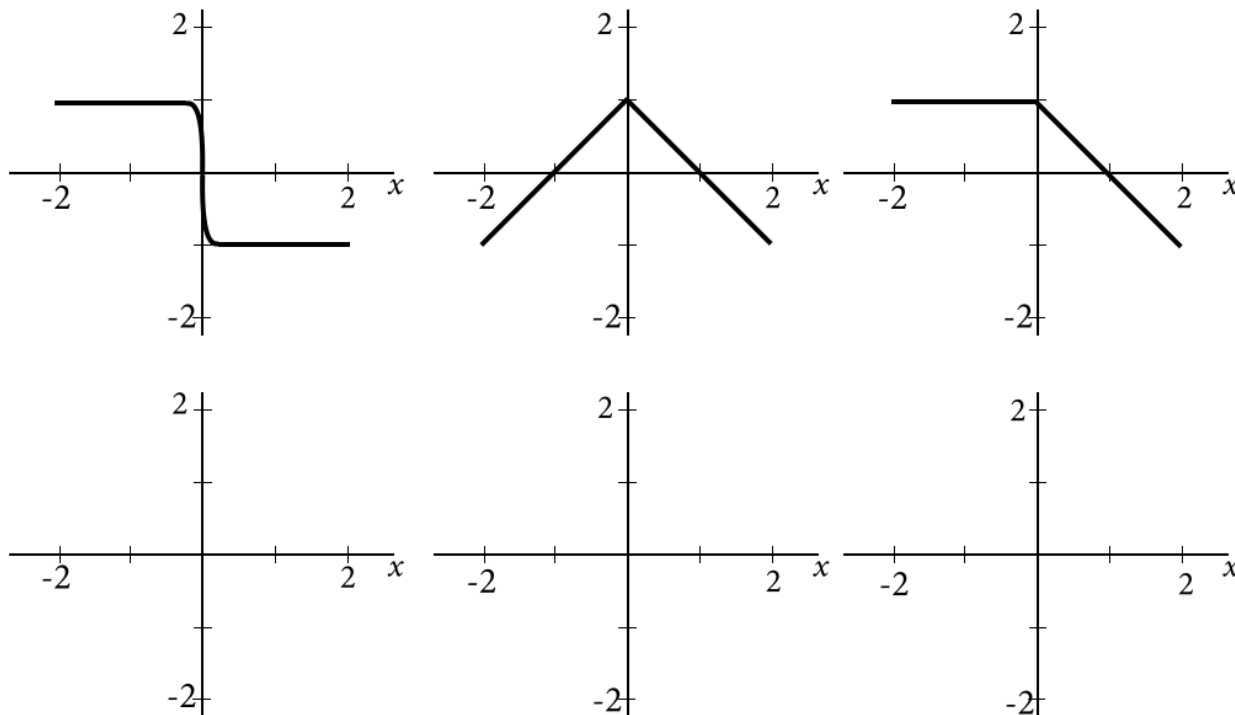
(b) In the picture above, sketch in rectangles whose *upper-right* corner is on the curve. Estimating the distance traveled using these rectangles is called a *right sum*.

(c) Estimate the distance traveled using a right sum, as in part (d) in the previous problem.

(d) Is your estimate an overestimate or underestimate of the actual distance?

Math 15

5. For each graph of $f'(x)$ below, sketch $f(x)$ on the axes provided.



6. (Continuation)

(a) Suppose that you are given the additional information that, for each function, $f(0) = 1$. Add such a function to each of the three axes in blue.

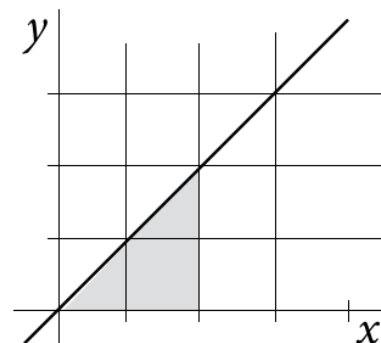
(b) Suppose that instead $f(0) = -1$ for each function. Sketch these functions in red.

7. An *accumulation function* is a vacuum-like creature that scoops up area as it goes along, and counts how much it has so far. For example, an accumulation function (which we'll call $F(x)$) for the function $f(x) = x$ scoops up the area under the graph $y = x$. Let's assume that it starts scooping at $x = 0$.

(a) Use the picture to explain why $F(2) = 2$.

(b) Find $F(3)$.

(c) Find $F(k)$ for any positive value k .



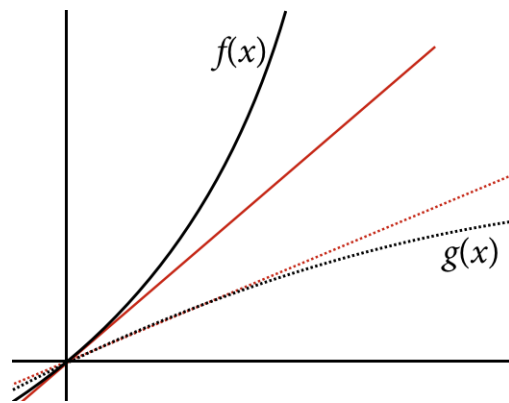
8. Soon, we will learn a technique (l'Hôpital's rule) for dealing with limits of the form $0/0$ or ∞/∞ . Show, however, that it is possible to evaluate each of the following limits using strategies that you already know.

(a) $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4}$ (b) $\lim_{x \rightarrow 1} \frac{\sin(2x)}{x}$ (c) $\lim_{x \rightarrow \infty} \left(\frac{x+1}{x} \right)^x$

Math 15

1. *L'Hôpital's Rule.* Let f and g be differentiable functions, and suppose that $f(a) = 0 = g(a)$. Then $\lim_{t \rightarrow a} \frac{g(t)}{f(t)} = \lim_{t \rightarrow a} \frac{g'(t)}{f'(t)}$, provided that the second limit exists. Use the picture to explain why.

2. (Continuation) The previous problem, and its picture, describe a limit of the form $0/0$. In fact, l'Hôpital's rule applies to any limit $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ that is



of one of the following *indeterminate forms*: $0/0$, ∞/∞ , $0 \times \infty$, $\infty - \infty$, 1^∞ , 0^0 , or ∞^0 . For each of the following, describe its form as one of the above or say that it is not indeterminate. Then evaluate it, using l'Hôpital's rule if it applies.

(a) $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4}$

(b) $\lim_{x \rightarrow 1} \frac{\sin(2x)}{x}$

(c) $\lim_{x \rightarrow \infty} \frac{\ln x}{x}$

(d) $\lim_{x \rightarrow 1} \frac{x}{x-1}$

(e) $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

3. We have been approximating the area under a graph by adding up the area of boxes that have a corner on the graph. This is called a *Riemann sum*, which can be left-handed or right-handed. The number of boxes used is called n , and the width of each box is called Δt (or Δx or whatever your variable is), where Δ is the Greek letter “delta” and stands for “difference.” Draw rectangular boxes for each of the following Riemann sums, and then find an approximate value for the sum (by counting boxes):

- (a) On top picture, in red, a left sum with $\Delta t = 4$.
- (b) On top picture, in blue, a right sum with $\Delta t = 4$.
- (c) On bottom picture, in red, a left sum with $\Delta t = 2$.
- (d) On bottom picture, in blue, a right sum with $\Delta t = 2$.

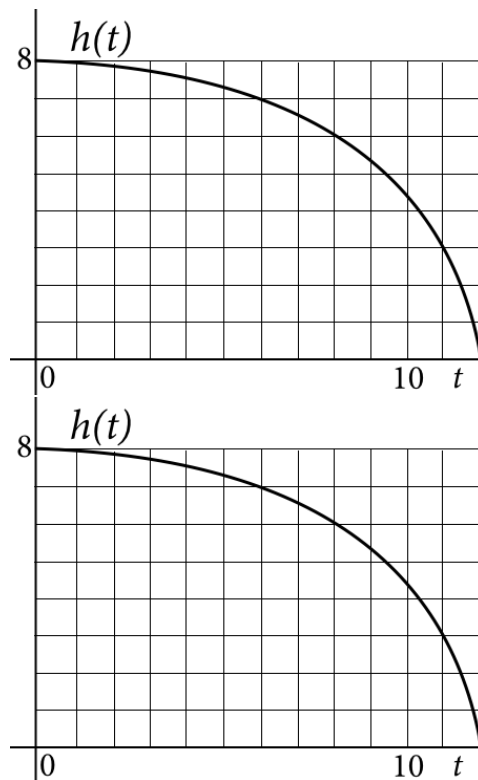
4. (Continuation) As Δx gets smaller, the rectangles get narrower and the estimate gets better.

(a) Explain why, as $\Delta x \rightarrow 0$, the left sum estimate and the right sum estimate get closer together. *Hint:* use your pictures.

(b) When we take the limit of the Riemann sum for $f(x)$ as $\Delta x \rightarrow 0$, we get the *area* under the curve. If we are finding the area from $x = a$ to $x = b$, the notation for this is

$$\int_a^b f(x) dx,$$

where the symbol \int is an elongated “S” that stands for “sum.” Explain why the *actual* area $\int_a^b f(x) dx$ under the curve is between the left sum and the right sum *estimates*.



Math 15

5. Using $\Delta x = 1/2$, fill in a table of values (provided to the right) that you could use to estimate $\int_0^2 x^2 dx$.

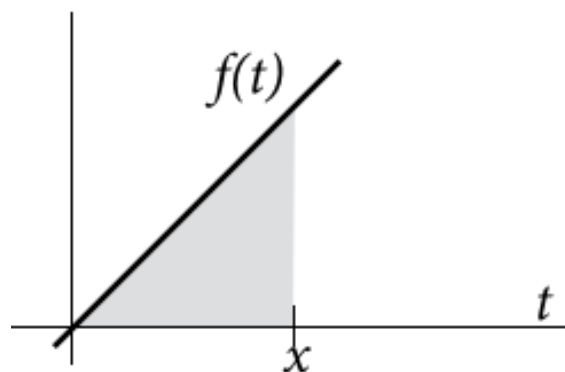
x	0	1/2	1	3/2	2
$f(x)$					

6. For each of the following, find two different functions that satisfy the criterion.

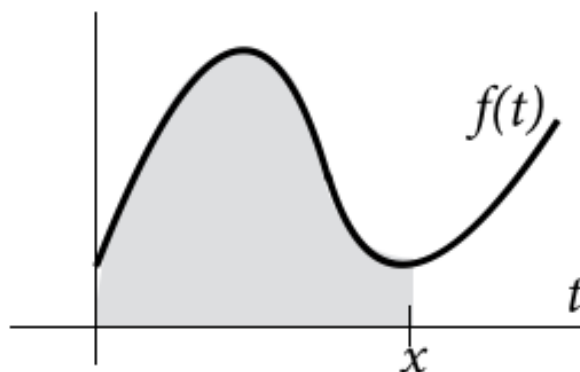
(a) $f'(x) = 2x$ (b) $dg/dx = \cos x$ (c) $dy/dx = 1/x$ (d) $p'(x) = 2p(x)$

These are called *differential equations*.

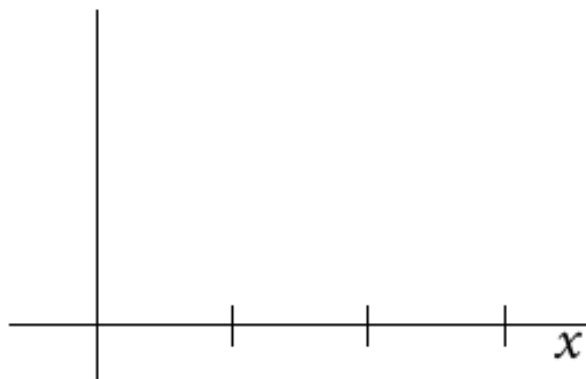
7. For each function $f(t)$ shown below, the related function $F(x) = \int_0^x f(t) dt$ is an accumulation function (see Page 21 # 7), adding up all the area under $f(t)$ from $t = 0$ to $t = x$. An example x is shown for each function. Sketch each $F(x)$ on the axes below.



$F(x)$



$F(x)$



8. A function $f(x)$ is said to *dominate* a function $g(x)$ if $\lim_{x \rightarrow \infty} \frac{g(x)}{f(x)} = 0$. For example, x^2 dominates x , which we write as $x^2 \gg x$.

(a) Explain why x^2 dominates x .

(b) Explain why $2^x \gg x^2$.

(c) Make up your own example, that no one else in the class will think of, of two functions where one dominates the other.

(d) Which function dominates as $x \rightarrow \infty$, $\ln(x+3)$ or $x^{0.2}$?

Math 15

1. Evaluate each of the following limits, using l'Hôpital's rule if it applies.

(a) $\lim_{x \rightarrow 0} \frac{e^x - 1}{\sin x}$ (b) $\lim_{x \rightarrow \infty} \frac{\ln x}{x}$ (c) $\lim_{x \rightarrow \infty} \frac{\sin x}{x}$ (d) $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x}$

2. The space shuttle is taking off from Cape Canaveral in Florida, and a NASA observer on the ground is measuring its speed $v(t)$, in meters per second, at intervals of 3 seconds, for the 12 seconds when it is still close enough to do so. The collected data is shown in the table.

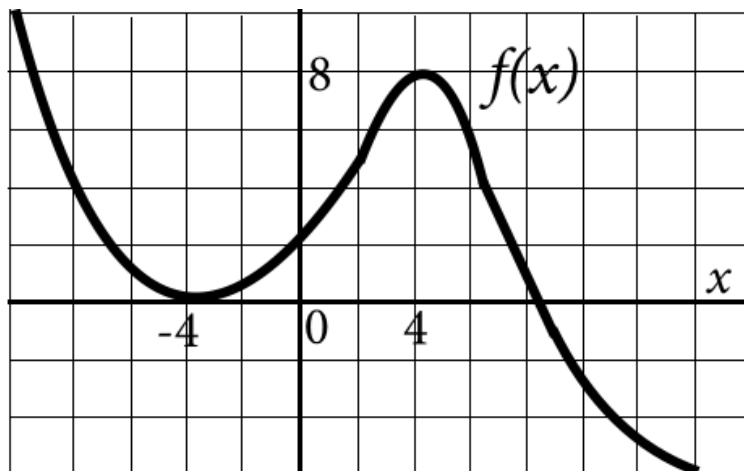
t	0	3	6	9	12
$v(t)$	20	23	24	26	31

- (a) Make a graph and plot the data points.
- (b) Find an upper estimate (you will need to choose between a left sum and a right sum) for the distance traveled by the space shuttle between in the first 12 seconds of flight.
- (c) Find a lower estimate of the distance traveled.
- (d) How could you find a more accurate estimate? What do you think the actual distance is?

3. The picture to the right shows the graph of $f(x)$.

(a) The area of a region below the x -axis is usually taken to be *negative*. Why do you think this is?

(b) Explain the difference between “the area under $f(x)$ ” and “the area between $f(x)$ and the x -axis.” On the graph above, shade in the regions corresponding to each of these descriptions. Then estimate each of these two numbers, for the graph of $f(x)$ shown.



(c) Use the picture to estimate

$$\int_{-8}^{14} f(x) \, dx.$$

Math 15

4. Suppose that the function $f(t)$ gives the velocity of a car at time t .

(a) Explain why the *distance* traveled by the car from time $t = a$ to $t = b$ is given by

$$\int_a^b f(x) \, dx.$$

(b) Suppose that you had a function, let's call it $F(t)$, for the position of the same car at time t . Explain why the distance traveled by the car from time $t = a$ to $t = b$ is $F(b) - F(a)$.

(c) Explain why $F'(t) = f(t)$.

5. (Continuation) The *Fundamental Theorem of Calculus* states that, if the function $f(x)$ is continuous on the interval $[a, b]$, and there is a function $F(x)$ so that $f(x) = F'(x)$, then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

The function F is called an *antiderivative* for f .

(a) For the function $f(x) = x$, find the function $F(x)$.

(b) Use the FTC to compute $\int_0^2 x \, dx$.

(c) Check that your answer agrees with your function from Page 21 # 7 and your sketch from Page 22 # 7.

6. For each part, let $f(t) = F'(t)$, write out the integral $\int_a^b f(x) \, dx$, and evaluate it using the Fundamental Theorem of Calculus.

(a) $F(t) = t^2$, $a = 2$, $b = 5$

(b) $F(t) = 3t^2 + 2t$, $a = 1$, $b = 3$

(c) $F(t) = \sin t$, $a = 0$, $b = \pi/2$

7. An *initial value problem* is a differential equation where a value of the function is given, so that you know where the function is located vertically, or in other words so that you can find the value of the constant C in the “ $+C$.” Solve the following initial value problems.

(a) $dy/dx = 4x^3$, $y(0) = 3$.

(b) $dp/dx = x^3 + x^4$, $p(1) = 4$.

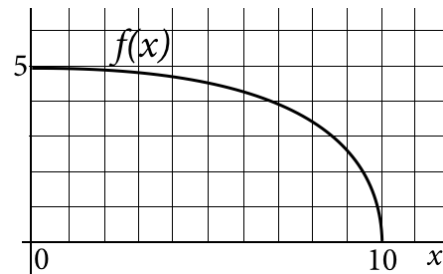
Math 15

1. Use the Fundamental Theorem of Calculus to evaluate $\int_0^2 (4x^3 - 3x^2) dx$.

2. The value of the integral $\int_0^{10} f(x) dx$ is either 30, 40, 50 or 60. Which one is it? Explain.

3. Find an *antiderivative* (a function whose derivative is the given function) for each of the following functions.

- (a) $f(x) = 3$ (b) $g(w) = \frac{1}{w} - w$ (c) $p(z) = \frac{1}{z^3}$
 (d) $h(t) = \sqrt{t}$



4. (Continuation) Suppose that you also know that $F(0) = 2$, $G(1) = 3$, $P(1) = 1/2$, and $H(1) = -1/3$, where F , G , P and H are the antiderivatives of f , g , p and h , respectively. Find $F(x)$, $G(w)$, $P(z)$ and $H(t)$.

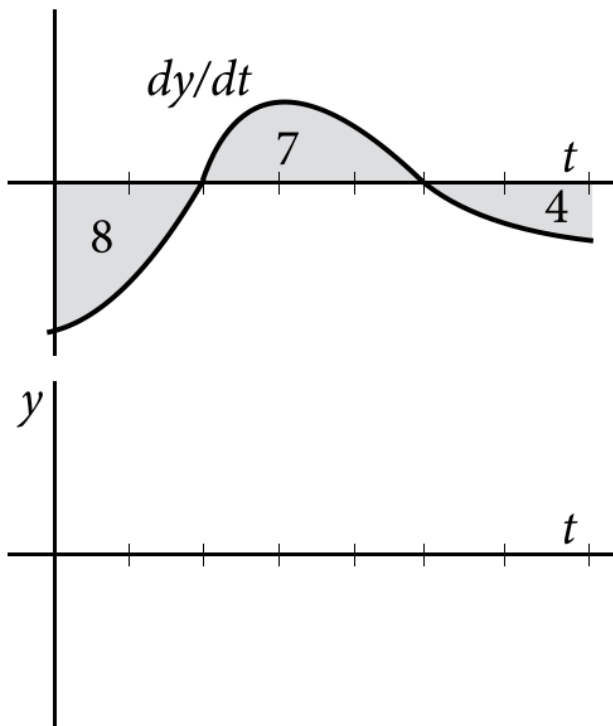
5. The graph of dy/dt is shown to the right. The area of each region is as indicated. Suppose that $y = 3$ when $t = 0$, i.e. $y(0) = 3$.

(a) Which y -values can you determine? Determine them.

(b) Where (which t -values) are the maxima, minima and inflection points of $y(t)$?

(c) dy/dt has a maximum at $t = 3$. What happens at the corresponding point on the graph of $y(t)$?

(d) Use all of this information to sketch an accurate graph of $y(t)$.



6. To *find an indefinite integral* means to give a function that is an antiderivative for the given function, and then indicate that the constant is unknown. For example, to find an indefinite integral for the function x^2 , we would say

$$\int x^2 dx = \frac{x^3}{3} + C.$$

Find an indefinite integral for each of the following:

- (a) $\int (5e^x - 4 \cos x) dx$ (b) $\int \frac{7}{\sqrt[3]{x}} dx$ (c) $\int \left(\sqrt{x^5} - \frac{3}{x} \right) dx$

Math 15

7. The position $h(t)$ of a turnip, which was thrown vertically upward, is given by

$$h(t) = -4.9t^2 + 10t + 2,$$

where t is the number of seconds since the throw, and the position is measured in meters above the ground.

- (a) Find a function $v(t)$ for the velocity of the turnip.
 - (b) At what speed was the turnip thrown?
 - (c) Find a function $a(t)$ for the acceleration of the turnip.
 - (d) You should have found that the acceleration is constant. Use your knowledge of physics to explain why.
8. Suppose that F is an accumulation function for $f(t)$. Previously, we've assumed that F started accumulating area at 0, but now we'll let it start at any value a .
- (a) Draw a picture to help you explain why

$$F(x) = \int_{t=a}^{t=x} f(t) \, dt.$$

- (b) Explain why it would be confusing and bad to write $F(x) = \int_a^x f(x) \, dx$.
- (c) Explain why $F'(x) = f(x)$.
- (d) Look at your sketches for Page 22 # 7, and confirm that for each of the two functions, the graph of $f(t)$ gives the slope of the graph of $F(t)$, i.e. $f(t) = F'(t)$.

Math 15

1. Give an example of a function $f(x)$ and an interval $[a, b]$ so that $\int_a^b f(x) dx$ is negative. Also sketch the graph your function on your interval, and explain why your example works.

2. The area under the graph of $y = 1/x$ from $x = 1$ to $x = a$ is 5. Use the FTC to find a . *Hint*: draw a picture

3. Let F be an accumulation function for $f(t)$, scooping up area starting at some fixed t -value a and ending at some t -value x :

$$F(x) = \int_a^x f(t) dt.$$

(a) Explain the difference between the meaning of the variable t and the meaning of the variable x in the above equation.

(b) Suppose we want to find the rate of change $F'(x)$ of the function F . Explain why

$$F'(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h}.$$

(c) Mark $x+h$, h , $F(x+h)$ and $F(x+h) - F(x)$ on the picture above (assume h is a small positive number).

(d) Using an argument using the picture above and what you marked on it in (c), and involving the statement “height is area divided by width,” explain why $F'(x) = f(x)$.

4. (Continuation) The FTC actually has two parts. The First Fundamental Theorem of Calculus (which we already saw) states that if $f(t)$ is continuous on $[a, b]$ and $F'(t) = f(t)$, then $\int_a^b f(t) dt = F(b) - F(a)$. The Second Fundamental Theorem of Calculus states that, under the same hypotheses (i.e. in the same situation),

$$F(x) = \int_a^x f(t) dt$$

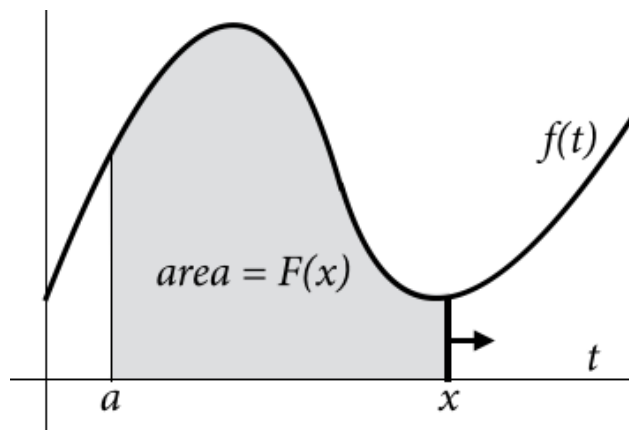
is an antiderivative of f , and that

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

(a) Use problem 3(d) to explain why $\frac{d}{dx} \int_a^x f(t) dt = f(x)$.

(b) Explain why it is not possible to compute the definite integral $\int_0^x \cos(t^2) dt$.

(c) Nonetheless, compute $\frac{d}{dx} \int_0^x \cos(t^2) dt$.



Math 15

5. Determine each limit, or explain why the limit does not exist.

(a) $\lim_{x \rightarrow \infty} \frac{6}{1 + 1/x}$ (b) $\lim_{x \rightarrow 0} \frac{1 - 2^x + \sin x}{x^3}$ (c) $\lim_{x \rightarrow 0} \frac{2^x - 3^x}{x^2}$

(d) $\lim_{x \rightarrow 3} \frac{4x + 1}{x^2 - 9}$ (e) $\lim_{x \rightarrow \infty} \frac{x}{x^2 + 1}$

6. The acceleration due to gravity of (falling) objects near the earth's surface is $a(t) = -9.8 \text{ m/s}^2$.

(a) Explain why the acceleration is a negative number.

(b) Suppose that a potato pellet is shot vertically upwards at a speed of 5 meters per second. Using the fact that it is subject to gravity (and ignoring the effects of air resistance, etc.), find an equation $v(t)$ for its speed t seconds after being shot.

(c) Further suppose that the potato was shot from 8 meters above the ground. Find an equation for its height $h(t)$ above the ground t seconds after being shot.

Hint: Refer to Page 24 # 7 and work backwards.

7. Evaluate the following integrals:

(a) $\int_0^2 (x^2 - 3x + 2) \, dx$ (b) $\int_0^3 \left(\frac{x^2}{2} - 3x \right) \, dx$ (c) $\int_0^{\pi/2} (\cos x + \sec^2 x) \, dx$

8. (Continuation) Find the *average value* of each function over the given interval:

(a) $f(x) = x^2 - 3x + 2$ over the interval $[0, 2]$

(b) $g(x) = \frac{x^2}{2} - 3x$ over the interval $[0, 3]$

(c) $h(x) = \cos x + \sec^2 x$ over the interval $[0, \pi/2]$

Math 15

1. The graph of $f'(x)$ is shown to the right.

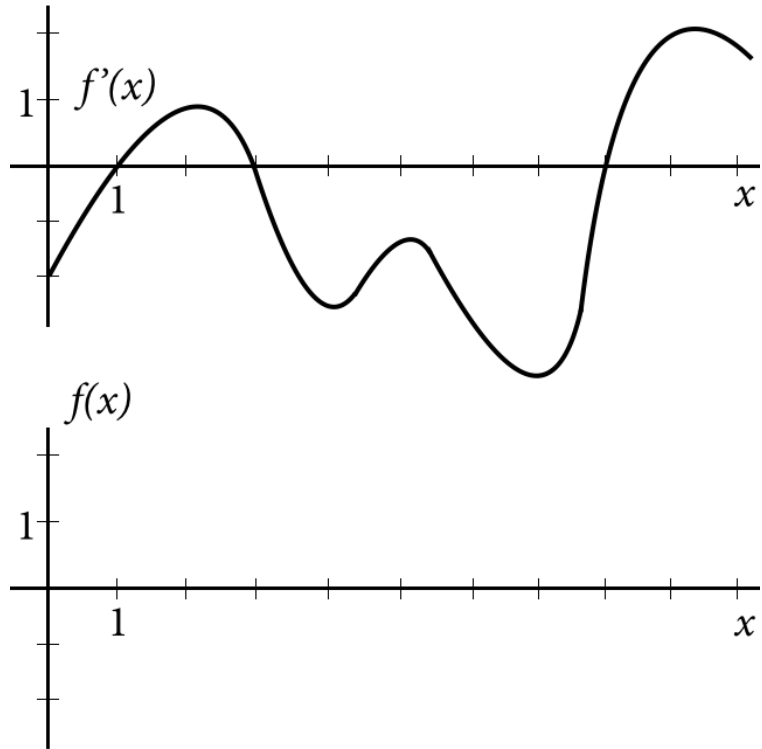
(a) Find the x -coordinates of any local maxima and minima.

(b) Find the x -coordinates of any inflection point(s).

(c) Determine the interval(s) over which the function is decreasing.

(d) Determine the interval(s) over which the function is concave-up.

(e) Suppose $f(0) = 1$. Sketch a graph of $f(x)$ that is as accurate as you can make it.



2. Suppose that you are standing on a cliff, 100 meters above sea level. You throw a small tchotchke down towards the sea at a speed of 3 meters per second. Find an equation $h(t)$ for the height of the tchotchke t seconds after it is released from your hand.

3. Let $F(x)$ be an antiderivative for $f(x)$.

(a) Suppose that $\int_1^4 f(x) dx = 7$ and $F(4) = 9$. Find $F(1)$.

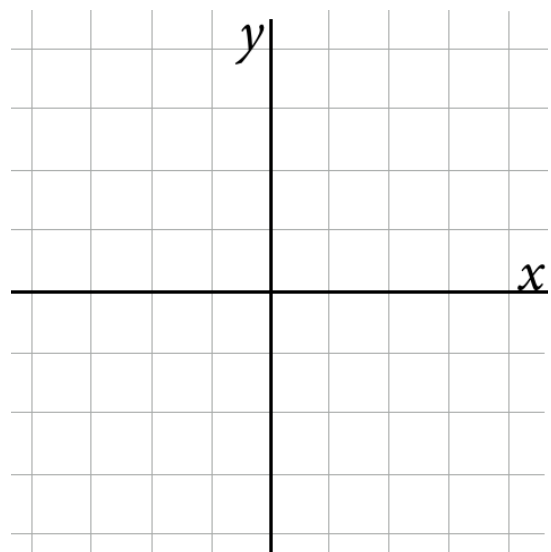
(b) Suppose that $\int_0^{10} f(x) dx = 0$. What is the relationship between $F(0)$ and $F(10)$?

4. In this problem, you will find the area between the curves $y = x^2$ and $y = x + 2$.

(a) Sketch these two curves on the axes at right.

(b) Find the two points where the curves intersect, and mark them on your picture.

(c) Use integrals to find the area between the two curves. *Hint*: you know how to use an integral to find the area under a curve



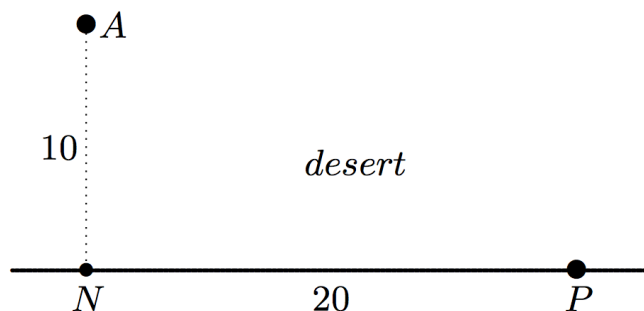
5. If $F(x) = \int_1^x (\cos(2t) + 3^t) dt$, find $F'(x)$.
Hint: see Page 25 # 4

Math 15

6. Pat's speed while driving home for winter vacation was $60 + 4 \sin(\pi t)$, where t is the number of hours since noon. Pat began driving at 1pm and finished the drive at 3:30pm. What was Pat's average speed for the trip?

7. (From *Calculus*, Hughes-Hallett et al) Ice is forming on a pond at a rate given by $dy/dt = k\sqrt{t}$, where y is the thickness of the ice t hours since the ice started forming, and k is a positive constant. Find the thickness y as a function of t . *Hint*: you can do this

8. *A classic problem.* Alex the Geologist is in the desert in a jeep, 10 km from a long, straight road. On the road, the jeep can do 50 kph, but in the desert sands, it can manage only 30 kph. Alex is very thirsty, knows that there is a gas station 20 km down the road (from the nearest point on the road) that has ice-cold Pepsi, and decides to drive there. Alex follows a straight path through the desert, and reaches the road at a point that is between N and P , and x km from N . The total time $T(x)$ for the drive to the gas station is a function of this quantity x .



(a) Find an explicit expression for $T(x)$. *Hint*: time = distance / rate

(b) Calculate $T'(x)$. Use algebra to find the minimum value of $T(x)$ and the value of x that produces it.

Math 15

acceleration: The rate of change of velocity.

accumulate: To integrate a function, which becomes the rate of accumulation.

angle-addition identities: For any angles α and β , $\cos(\alpha + \beta) \equiv \cos \alpha \cos \beta - \sin \alpha \sin \beta$ and $\sin(\alpha + \beta) \equiv \sin \alpha \cos \beta + \cos \alpha \sin \beta$.

antiderivative: If f is the derivative of g , then g is called *an antiderivative of f* . For example, $g(x) = 2x\sqrt{x} + 5$ is an antiderivative of $f(x) = 3\sqrt{x}$, because $g' = f$.

arccos: This is another name for the inverse cosine function, commonly denoted \cos^{-1} .

arcsin: This is another name for the inverse sine function, commonly denoted \sin^{-1} .

arctan: This is another name for the inverse tangent function, commonly denoted \tan^{-1} .

arithmetic mean: The *arithmetic mean* of two numbers p and q is $\frac{1}{2}(p + q)$.

asymptote: Two graphs are *asymptotic* if they become indistinguishable as the plotted points get further from the origin. Either graph is an asymptote for the other.

average value: If $f(x)$ is defined on an interval $a \leq x \leq b$, the average of the values of f on this interval is $\frac{1}{b-a} \int_a^b f(x) dx$.

average velocity is displacement divided by elapsed time; it is calculated during a time interval.

Chain Rule: If a composite function is defined by $C(x) = f(g(x))$, its derivative is a product of derivatives, namely $C'(x) = f'(g(x))g'(x)$.

chord: A segment that joins two points on a curve.

composite: A function that is obtained by applying two functions in succession. For example, $f(x) = (2x-60)^3$ is a composite of $g(x) = 2x-60$ and $h(x) = x^3$, because $f(x) = h(g(x))$. Another composite of g and h is $k(x) = g(h(x)) = 2x^3 - 60$. Notice also that f is a composite of $p(x) = 2x$ and $q(x) = (x - 60)^3$.

compound interest: When interest is left in an account (instead of being withdrawn), the additional money in the account itself earns interest.

concavity: A graph $y = f(x)$ is *concave up* on an interval if f'' is positive on the interval. The graph is *concave down* on an interval if f'' is negative on the interval.

Math 15

conic section: Any graph obtainable by slicing a cone with a cutting plane. This might be an ellipse, a parabola, a hyperbola, or some other special case.

conjugate: Irrational roots to polynomial equations sometimes come in pairs.

constant function: A function that has only one value.

continuity: A function f is *continuous at a* if $f(a) = \lim_{x \rightarrow a} f(x)$. A function is called *continuous* if it is continuous at every point in its domain. For example, $f(x) = 1/x$ (which is undefined at $x = 0$) is continuous. If a function is continuous at every point in an interval, the function is said to be continuous *on* that interval.

cosecant: The reciprocal of the sine.

critical point: A number c in the domain of a function f is called *critical* if $f'(c) = 0$ or if $f'(c)$ is undefined.

curvature: In an absolute sense, the rate at which the direction of a curve is changing, with respect to the distance traveled along it. For a circle, this is just the reciprocal of the radius. [58,61,73] The sign of the curvature indicates on which side of the tangent vector the curve is found.

decreasing: A function f is *decreasing* on an interval $a \leq x \leq b$ if $f(v) < f(u)$ holds whenever $a \leq u < v \leq b$ does.

degree: See *polynomial degree*.

derivative: Given a function f , its derivative is another function f' , whose values are defined by $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$, which is the derivative of f at a .

derivative at a point: Given a function f , its *derivative at a* is the limiting value of the difference quotient $\frac{f(x) - f(a)}{x - a}$ as x approaches a .

difference quotient: The slope of a chord that joins two points $(a, f(a))$ and $(b, f(b))$ on a graph $y = f(x)$ is $\frac{f(b) - f(a)}{b - a}$, a quotient of two differences.

differentiable: A function that has derivatives at all the points in its domain.

differential equation: An equation that is expressed in terms of an unknown function and its derivative. A solution to a differential equation is a function.

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differentials: Things like dx , dt , and dy . Called “ghosts of departed quantities” by George Berkeley (1685-1753), who was skeptical of Newton’s approach to mathematics.

differentiation: The process of finding a derivative, perhaps by evaluating the limit of a difference quotient, perhaps by applying a formula such as the *Power Rule*.

discontinuous: A function f has a *discontinuity at a* if $f(a)$ is defined but does not equal $\lim_{x \rightarrow a} f(x)$; a function is *discontinuous* if it has one or more discontinuities.

domain: The domain of a function consists of all the numbers for which the function returns a value. For example, the domain of a logarithm function consists of positive numbers only.

double-angle identities: Best-known are $\sin 2\theta \equiv 2 \sin \theta \cos \theta$, $\cos 2\theta \equiv 2 \cos^2 \theta - 1$, and $\cos 2\theta \equiv 1 - 2 \sin^2 \theta$; special cases of the *angle-addition identities*.

e is approximately 2.71828. This irrational number frequently appears in scientific investigations. One of the many ways of defining it is $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$.

even function: A function whose graph has reflective symmetry in the y -axis. Such a function satisfies the identity $f(x) = f(-x)$ for all x . The name *even* comes from the fact that $f(x) = x^n$ is an even function whenever the exponent n is an even integer.

exponential functions have the strict form $f(x) = b^x$, with a constant base and a *variable exponent*. It is also common practice to use this terminology to refer to functions of the form $f(x) = k + a \cdot b^x$, although most of them do not satisfy the rules of exponents.

Extended Mean-Value Theorem: If f is a function that is $n + 1$ times differentiable for $0 \leq x \leq b$, then

$$f(b) = f(0) + f'(0)b + \frac{1}{2}f''(0)b^2 + \cdots + \frac{1}{n!}f^{(n)}(0)b^n + \frac{1}{(n+1)!}f^{(n+1)}(c)b^{n+1}$$

for some c between 0 and b . This version of the theorem is due to *Lagrange*.

Extended Power Rule: The derivative of $p(x) = [f(x)]^n$ is $p'(x) = n[f(x)]^{n-1}f'(x)$.

extreme point: either a *local minimum* or a *local maximum*. Also called an *extremum*.

Extreme-value Theorem: If $f(x)$ is continuous for $a \leq x \leq b$, then $f(x)$ attains a maximum and a minimum value. In other words, $m \leq f(x) \leq M$, where $m = f(p)$, $a \leq p \leq b$, $M = f(q)$, and $a \leq q \leq b$. Furthermore, p and q are critical values or endpoints for f .

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Fibonacci sequence: A list of numbers, each of which is the sum of the two preceding. [15] Leonardo of Pisa (1180-1250), who was called *Fibonacci* (literally “Filius Bonaccio”), learned mathematics from his Arab teachers, and introduced algebra to Europe.

frustrum: There is no such word. See *frustum*.

frustum: When a cone is sliced by a cutting plane that is parallel to its base, one of the resulting pieces is another (similar) cone; the other piece is a *frustum*.

functional notation: For identification purposes, functions are given short names (usually just one to three letters long). If f is the name of a function, then $f(a)$ refers to the number that f assigns to the value a .

Fundamental Theorem of Algebra: Every complex polynomial of degree n can be factored (in essentially only one way) into n linear factors.

Fundamental Theorem of Calculus: In a certain sense, differentiation and integration are inverse procedures.

geometric mean: The *geometric mean* of two positive numbers p and q is \sqrt{pq} .

global maximum: Given a function f , this may or may not exist. It is the value $f(c)$ that satisfies $f(x) \leq f(c)$ for all x in the domain of f .

global minimum: Given a function f , this may or may not exist. It is the value $f(c)$ that satisfies $f(c) \leq f(x)$ for all x in the domain of f .

Greek letters: Apparently unavoidable in reading and writing mathematics! Some that are found in this book are α (alpha), β (beta), Δ (delta), π (pi), ψ (psi), Σ (sigma), and θ (theta). Where does the word *alphabet* come from?

half-life: When a quantity is described by a decreasing exponential function of t , this is the time needed for half of the current amount to disappear.

Heaviside operator: The use of a symbol, such as D or D_x , to indicate the differentiation process. The scientist Oliver Heaviside (1850-1925) advocated the use of vector methods, clarified Maxwell’s equations, and introduced operator notation so that solving differential equations would become a workout in algebra.

identity: An equation (sometimes written using \equiv) that is true no matter what values are assigned to the variables that appear in it. One example is $(x + y)^3 \equiv x^3 + 3x^2y + 3xy^2 + y^3$, and another is $\sin x \equiv \sin(180 - x)$.

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implicitly defined function: Equations such as $x^2 + y^2 = 1$ do not express y explicitly in terms of x . As the examples $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$ illustrate, there in fact could be several values of y that correspond to a given value of x . These functions are said to be *implicitly defined* by the equation $x^2 + y^2 = 1$.

improper fraction: Not a *proper fraction*.

improper integral: $\int_a^b f(x) dx$ is *improper* when either a or b is infinite or when the values $f(x)$ of the integrand are undefined, or not bounded, for all $a \leq x \leq b$.

increasing: A function f is *increasing* on an interval $a \leq x \leq b$ if $f(u) < f(v)$ holds whenever $a \leq u < v \leq b$ does.

indeterminate form: This is an ambiguous limit expression, whose actual value can be deduced only by looking at the given example. The five most common types are:

$\frac{0}{0}$, examples of which are $\lim_{t \rightarrow 0} \frac{\sin t}{t}$ and $\lim_{h \rightarrow 0} \frac{2^h - 1}{h}$ [8,17,36,68,70]

1^∞ , examples of which are $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$ and $\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n$ [7,11,105]

$\frac{\infty}{\infty}$, examples of which are $\lim_{x \rightarrow \infty} \frac{2x + 1}{3x + 5}$ and $\lim_{x \rightarrow 0} \frac{\log_2 x}{\log_3 x}$ [13,85]

$0 \cdot \infty$, examples of which are $\lim_{x \rightarrow 0} x \ln x$ and $\lim_{x \rightarrow \pi/2} \left(x - \frac{1}{2}\pi\right) \tan x$ [88]

$\infty - \infty$, examples being $\lim_{x \rightarrow \infty} \sqrt{x^2 + 4x} - x$ and $\lim_{x \rightarrow \pi/2} \sec x \tan x - \sec^2 x$ [111]

The preceding limit examples all have different values.

inflection point: A point on a graph $y = f(x)$ where f'' changes sign.

instantaneous velocity is unmeasurable, and must therefore be calculated as a limiting value of average velocities, as the time interval diminishes to zero.

integral: The precise answer to an accumulation problem. A limit of *Riemann sums*.

integrand: A function that is integrated.

intermediate-value property: A function f has this property if, for any k between $f(a)$ and $f(b)$, there is a number p between a and b , for which $k = f(p)$. For example, f has this property if it is *continuous* on the interval $a \leq x \leq b$.

interval notation: A system of shorthand, in which " $a \leq x \leq b$ " is replaced by the statement " x is in $[a, b]$ ", and " θ is in $(0, 2\pi)$ " means $0 < \theta < 2\pi$.

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inverse function: Any function f processes input values to obtain output values. A function that undoes what f does is said to be *inverse* to f , and often denoted f^{-1} . In other words, $f^{-1}(b) = a$ must hold whenever $f(a) = b$ does. For some functions ($f(x) = x^2$, for example), it is necessary to restrict the domain in order to define an inverse.

isocline: A curve, all of whose points are assigned the same slope by a differential equation.

Lagrange's error formula: Given a function f and one of its Taylor polynomials p_n based at $x = a$, the difference between $f(x)$ and $p_n(x)$ is $\frac{1}{(n+1)!}f^{(n+1)}(c)(x-a)^{n+1}$, for some c that is between a and x . [108,109] Joseph Lagrange (1736-1813) made many contributions to calculus and analytic geometry, including a simple notation for derivatives.

Lagrange notation: The use of primes to indicate derivatives.

Law of Cosines: This theorem can be expressed in the SAS form $c^2 = a^2 + b^2 - 2ab \cos C$ or in the equivalent SSS form $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$.

Leibniz notation: A method of naming a derived function by making reference to the variables used to define the function. For example, the volume V of a sphere is a function of the radius r . The derivative of this function can be denoted $\frac{dV}{dr}$ instead of V' . [29] The philosopher Gottfried Wilhelm Leibniz (1646-1716) is given credit for inventing calculus (along with his contemporary, Isaac Newton).

l'Hôpital's Rule: A method for dealing with indeterminate forms: If f and g are differentiable, and $f(a) = 0 = g(a)$, then $\lim_{t \rightarrow a} \frac{f(t)}{g(t)}$ equals $\lim_{t \rightarrow a} \frac{f'(t)}{g'(t)}$, provided that the latter limit exists. The Marquis de l'Hôpital (1661-1704) wrote the first textbook on calculus.

limit: A number that the values of a function get arbitrarily close to.

linear interpolation: To calculate coordinates for an unknown point that is between two known points, this method makes the assumption that the three points are collinear.

ln: An abbreviation of *natural logarithm*, it means \log_e . It should be read "log" or "natural log".

local maximum: Given a function f and a point c in its domain, $f(c)$ is a local maximum of f if there is a positive number d such that $f(x) \leq f(c)$ for all x in the domain of f that satisfy $|x - c| < d$.

local minimum: Given a function f and a point c in its domain, $f(c)$ is a local minimum of f if there is a positive number d such that $f(c) \leq f(x)$ for all x in the domain of f that satisfy $|x - c| < d$.

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logarithm: The exponent needed to express a given positive number as a power of a given positive base. Using a base of 4, the logarithm of 64 is 3, because $64 = 4^3$.

logarithmic derivative: Dividing the derivative of a function by the function produces a relative rate of change.

logistic equation: A differential equation that describes population growth in situations where limited resources constrain the growth.

long-division algorithm: The process by which an *improper fraction* is converted to a mixed fraction. For example, the polynomial division scheme shown at right was used to convert the improper fraction $\frac{2x^2 + 3}{x - 2}$ into the equivalent mixed form $2x + 4 + \frac{11}{x - 2}$. The process is terminated because the remainder 11 cannot be divided by $x - 2$. (In other words, $\frac{11}{x - 2}$ is a *proper fraction*.)

$$\begin{array}{r} 2x + 4 \\ x - 2 \overline{) 2x^2 + 0x + 3} \\ \underline{2x^2 - 4x} \\ 4x + 3 \\ \underline{4x - 8} \\ 11 \end{array}$$

Mean-Value Theorem: If the curve $y = f(x)$ is continuous for $a \leq x \leq b$, and differentiable for $a < x < b$, then the slope of the line through $(a, f(a))$ and $(b, f(b))$ equals $f'(c)$, where c is strictly between a and b . There is also a version of this statement that applies to integrals.

mho: The basic unit of conductance, which is the reciprocal of resistance, which is measured in ohms. This was probably someone's idea of a joke.

mixed expression: The sum of a polynomial and a proper fraction, e.g. $2x - 3 + \frac{5x}{x^2 + 4}$.

moment: Quantifies the effect of a force that is magnified by applying it to a lever. Multiply the length of the lever by the magnitude of the force.

monotonic: A function that either increases or decreases is called *monotonic*.

natural logarithm: The exponent needed to express a given positive number as a power of e .

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Newton's Law of Cooling is described by exponential equations $D = D_0 b^t$, in which t represents time, D is the difference between the temperature of the cooling object and the surrounding temperature, D_0 is the initial temperature difference, and b is a positive constant that incorporates the rate of cooling. Isaac Newton (1642-1727) contributed deep, original ideas to physics and mathematics.

nondifferentiable: A function is nondifferentiable at a point if its graph does not have a tangent line at that point, or if the tangent line has no slope.

normal line: The line that is perpendicular to a tangent line at the point of tangency.

nth derivative: The standard notation for the result of performing n successive differentiations of a function f is $f^{(n)}$. For example, $f^{(6)}$ means f'''''' . It thus follows that $f^{(1)}$ means f' and $f^{(0)}$ means f .

odd function: A function whose graph has half-turn symmetry at the origin. Such a function satisfies the identity $f(-x) = -f(x)$ for all x . The name *odd* comes from the fact that $f(x) = x^n$ is an odd function whenever the exponent n is an odd integer.

one-sided limit: Just what the name says.

operator notation: A method of naming a derivative by means of a prefix, usually D , as in $D \cos x = -\sin x$, or $\frac{d}{dx} \ln x = \frac{1}{x}$, or $D_x(u^x) = u^x(\ln u)D_x u$.

parabola: This curve consists of all the points that are equidistant from a given point (the *focus*) and a given line (the *directrix*).

partial fractions: Converting a proper fraction with a complicated denominator into a sum of fractions with simpler denominators, as in $\frac{3x+2}{x^2+x} = \frac{2}{x} + \frac{1}{x+1}$.

period: A function f has positive number p as a period if $f(x+p) = f(x)$ holds for all x .

piecewise-defined function: A function can be defined by different rules on different intervals of its domain. For example, $|x|$ equals x when $0 \leq x$, and $|x|$ equals $-x$ when $x < 0$.

polynomial: A sum of terms, each being the product of a numerical coefficient and a nonnegative integer power of a variable, for examples $1 + t + 2t^2 + 3t^3 + 5t^4 + 8t^5$ and $2x^3 - 11x$.

polynomial degree: The degree of a polynomial is its largest exponent. For example, the degree of $p(x) = 2x^5 - 11x^3 + 6x^2 - 9x - 87$ is 5, and the degree of the constant polynomial $q(x) = 7$ is 0.

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polynomial division: See *long division*.

Power Rule: The derivative of $p(x) = x^n$ is $p'(x) = nx^{n-1}$.

Product Rule: The derivative of $p(x) = f(x)g(x)$ is $p'(x) = f(x)g'(x) + g(x)f'(x)$.

proper fraction: The degree of the numerator is less than the degree of the denominator, as in $\frac{5x-1}{x^2+4}$. Improper fractions can be converted by *long division* to *mixed expressions*.

quadratic formula: The solutions to $ax^2 + bx + c = 0$ are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

quadratic function: A polynomial function of the second degree.

quartic function: A polynomial function of the fourth degree.

Quotient Rule: The derivative of $p(x) = \frac{f(x)}{g(x)}$ is $p'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$.

range: The range of a function consists of all possible values the function can return. For example, the range of the sine function is the interval $-1 \leq y \leq 1$.

relative maximum means the same thing as *local maximum*.

relative minimum means the same thing as *local minimum*.

Riemann sum: This has the form $f(x_1)w_1 + f(x_2)w_2 + f(x_3)w_3 + \cdots + f(x_n)w_n$. It is an approximation to the integral $\int_a^b f(x) dx$. The interval of integration $a \leq x \leq b$ is divided into subintervals I_1, I_2, I_3, \dots , and I_n , whose lengths are w_1, w_2, w_3, \dots , and w_n , respectively. For each subinterval I_k , the value $f(x_k)$ is calculated using a value x_k from I_k . [61,65] Bernhard Riemann (1826-1866) applied calculus to geometry in original ways.

Rolle's Theorem: If f is a differentiable function, and $f(a) = 0 = f(b)$, then $f'(c) = 0$ for at least one c between a and b . [79] Michel Rolle (1652-1719) described the emerging calculus as a collection of ingenious fallacies.

root: Another name for *zero*.

secant: The reciprocal of the cosine.

secant line: A line that intersects a (nonlinear) graph in two places.

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separable: A differential equation that can be written in the form $f(y)\frac{dy}{dx} = g(x)$.

sign function: This is defined for all nonzero values of x by $\operatorname{sgn}(x) = \frac{x}{|x|}$.

slope of a curve at a point: The slope of the tangent line at that point.

speed: The magnitude of *velocity*. For a parametric curve $(x, y) = (f(t), g(t))$, speed is expressed by the formula $\sqrt{(x')^2 + (y')^2}$, which is sometimes denoted $\frac{ds}{dt}$. [120] Notice that that speed is *not* the same as dy/dx .

standard position: An angle in the xy -plane is said to be in *standard position* if its initial ray points in the positive x -direction. Angles that open in the counterclockwise direction are *positive*; angles that open in the clockwise direction are *negative*.

step function: A piecewise constant function.

tangent line: A line is tangent to a curve at a point P if the line and the curve become indistinguishable when arbitrarily small neighborhoods of P are examined.

term-by-term differentiation: The derivative of a sum of functions is the sum of the derivatives of the functions.

triangle inequality: The inequality $PQ \leq PR + RQ$ says that any side of any triangle is at most equal to the sum of the other two sides.

velocity vector: The *velocity vector* of a differentiable curve $(x, y) = (f(t), g(t))$ is $\left[\frac{df}{dt}, \frac{dg}{dt}\right]$ or $\left[\frac{dx}{dt}, \frac{dy}{dt}\right]$, which is tangent to the curve. Its magnitude $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is the speed. Its *components* are derivatives of the component functions.

zero: A number that produces 0 as a functional value. For example, $\sqrt{2}$ is one of the zeros of the function $f(x) = x^2 - 2$. Notice that 1 is a zero of any logarithm function, because $\log 1$ is 0, and the sine and tangent functions both have 0 as a zero.