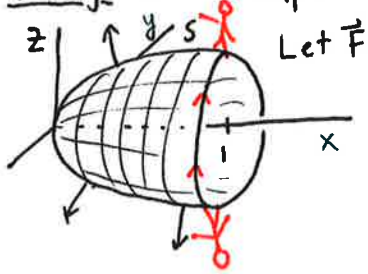
Diana Davis — Math 34 — May 4, 2018 — Class #40 ①

§7.3

Mathematician spotlight: Lila Fontes, Assistant Professor of Computer Science, Swarthmore

- math major at Harvard, PhD at Toronto, postdoc in Paris
- studies privacy and its relationship to communication cost, accuracy & optimality.
- amazingly, it is possible to communicate much while revealing little.

Today: An example bringing together Gauss's and Stokes' Theorems, and then an integral summary!



Let $\vec{F} = [(1-x)e^{\sin(\cos e^y)} - z^2 - y^2,\ y,\ z e^{\sin(\cos e^y)}]$, and let S be $x = y^2+z^2$, $x \le 1$, oriented outward.

Compute $\iint_S (\vec{F} + \operatorname{curl}(\operatorname{curl}\vec{F}))\cdot d\vec{S}$. break into two parts

$= \underbrace{\iint_S \vec{F}\cdot d\vec{S}}_{①} + \underbrace{\iint_S \operatorname{curl}(\operatorname{curl}\vec{F})\cdot d\vec{S}}_{②}$ ← Let's compute these separately.

① Let's apply Gauss's Theorem. We need to close the surface, so we add the unit disk at $x=1$, S_1, oriented in positive x-direction: $\vec{n} = [1,0,0]$ on S_1.

Then $\iint_{S+S_1} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV \Rightarrow \iint_S \vec{F}\cdot d\vec{S} + \iint_{S_1} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$

$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \underbrace{\iiint_E \operatorname{div}\vec{F}\, dV}_{Ⓐ} - \underbrace{\iint_{S_1} \vec{F}\cdot d\vec{S}}_{Ⓑ}$. $= \frac{\pi}{2} - 0 = \frac{\pi}{2}.$

these cancel out since both are $e^{\sin(\cos e^y)}$

Ⓐ $\iiint_E \operatorname{div}\vec{F}\, dV = \iiint_E (-\cancel{\ast}+1+\cancel{\ast})\, dV = \iiint_E 1\, dV = $ volume of E

volume of E: $x = x$, $y = r\cos\theta$, $z = r\sin\theta$; $r^2 \le x \le 1$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$.

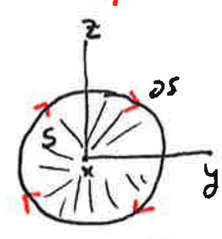
$\Rightarrow dV = r\cdot dx\cdot dr\cdot d\theta$,

$\Rightarrow$ volume $= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{x=r^2}^{1} r\, dx\, dr\, d\theta = \int_0^{2\pi} d\theta \cdot \int_{r=0}^{1}(r-r^3)\, dr = 2\pi \times \frac{1}{4} = \pi/2.$

Ⓑ $\iint_{S_1} (y^2-z^2)\, dS = \iint_{S_1} y^2\, dS - \iint_{S_1} z^2\, dS = 0.$

← we could set these up and compute them, but they would come out the same due to the symmetry of the shape. So they cancel out!

② By Stokes' Theorem, $\iint_S \operatorname{curl}(\operatorname{curl}\vec{F})\cdot d\vec{S} = \int_{\partial S} \operatorname{curl}\vec{F}\cdot d\vec{s}$, if ∂S is oriented correctly: ∂S needs to be clockwise.



So parameterize S by $\vec{x}(t) = [1, \cos(-t), \sin(-t)] = [1, \cos t, -\sin t]$

$\Rightarrow \vec{x}'(t) = [0, -\sin t, -\cos t]$.

Now $\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (1-x)e^{\sin(\cos e^y)} - z^2 - y^2 & y & z e^{\sin(\cos e^y)} \end{vmatrix} = [\cdots, -(0-2z), 0-(-2y)]$

$= [\cdots, -2z, 2y] = [\cdots, 2\sin t, 2\cos t]$.

we don't care because we're going to take the dot product with $\vec{x}'(t)$, and its first component is 0.

So $\iint_S \text{curl}(\text{curl}\vec{F})\cdot d\vec{S} = \int_{\partial S} \text{curl}\,\vec{F}\cdot d\vec{s} = \int_{t=0}^{2\pi} \text{curl}\,\vec{F}(\vec{x}(t))\cdot \vec{x}'(t)\,dt = \int_{t=0}^{2\pi} [-,\ 2\sin t,\ 2\cos t]\cdot[0, -\sin t, -\cos t]\,dt$

$$= \int_{t=0}^{2\pi} (-2\sin^2 t - 2\cos^2 t)\,dt = \int_{t=0}^{2\pi} -2(\cos^2 t + \sin^2 t)\,dt = \int_{t=0}^{2\pi} -2\,dt = \underline{\underline{-4\pi}}.$$

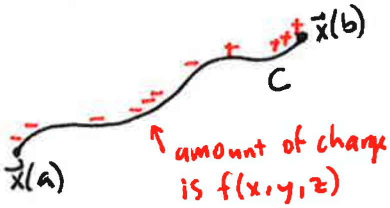
So now, $\iint_S (\vec{F} + \text{curl}(\text{curl}\vec{F}))\cdot d\vec{S} = \iint_S \vec{F}\cdot d\vec{S} + \iint_S \text{curl}(\text{curl}\vec{F})\cdot d\vec{S} = \frac{\pi}{2} - 4\pi = -\frac{7}{2}\pi.$

To do this, we needed:
- two big theorems (Gauss's Theorem and Stokes' Theorem)
- a triple integral
- a vector line integral

all things that we can do! ← strong math.

We can now integrate scalar / vector functions over curves / surfaces. All four!

1. Scalar line integrals. <u>Example:</u> adding up total amount of charge on a wire.



Parameterize the curve $C: \vec{x}(t) = [x(t), y(t), z(t)]$ for $a \le t \le b$.

Then $\int_C f\,ds = \int_{t=a}^{t=b} \underbrace{f(\vec{x}(t))}_{\text{value of function}} \underbrace{\|\vec{x}'(t)\|\,dt}_{\text{distance along the curve for which your function takes that value.}}$

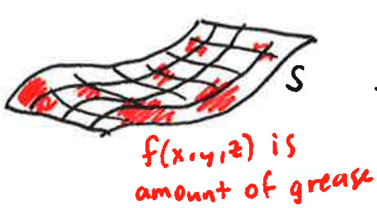
2. Vector line integrals. <u>Example:</u> adding up how much the wind helps/hurts you on your walk.

Parameterize the curve as above. Now direction matters!

Then $\int_C \vec{F}\cdot d\vec{s} = \int_{t=a}^{t=b} \underbrace{\vec{F}(\vec{x}(t))}_{\text{direction of vector field}} \cdot \underbrace{\vec{x}'(t)}_{\text{direction curve is going}}\,dt.$

The dot product measures how much $\vec{F}$ and $\vec{x}'$ point in the same direction.

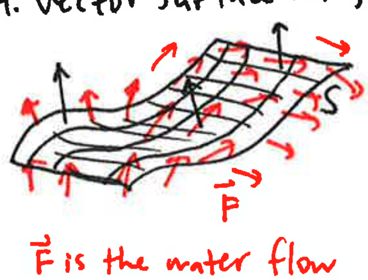
3. Scalar surface integrals. <u>Example:</u> How much total grease is on your greasy pizza napkin.



Parameterize the surface $S: \vec{X}(s,t) = [x(s,t), y(s,t), z(s,t)]$ for s,t in the region R in the st-plane.

Then $\iint_S f\,dS = \iint_R \underbrace{f(\vec{X}(s,t))}_{\text{value of function}} \underbrace{\|\vec{X}_s \times \vec{X}_t\|\,ds\,dt}_{\text{area of the surface for which your function takes that value.}}$

4. Vector surface integrals. <u>Example:</u> How much (net) water flows into your fishing net.



Parameterize the surface as above.

Then $\iint_S \vec{F}\cdot d\vec{S} = \iint_R \underbrace{\vec{F}(\vec{X}(s,t))}_{\text{direction of vector field}} \cdot \underbrace{(\vec{X}_s \times \vec{X}_t)}_{\text{or maybe } \vec{X}_t \times \vec{X}_s \text{, depending on orientation of } S.}\,ds\,dt$ → normal vector to the surface.

Thank you for a great semester! Please let me know about your future endeavors!