Mathematician spotlight: Illya V. Hicks, Professor of Computational & Applied Math., Rice Univ.

- math major at Texas State University - aced calculus - also played football
- studies combinatorial optimization, graph theory, integer programming
- applications to cancer treatment, social networks, network design

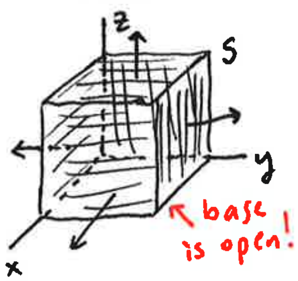
<u>Today</u>: More examples of Gauss's Theorem, including "closing off" a surface.

<u>Friday</u>: An example that combines Stokes' Theorem and Gauss's Theorem, plus a wrap-up overview of the last part of the course.

<u>Example</u>. Compute $\iint_S \overbrace{[P(y,z), Q(x,z), 3z]}^{\vec{F}} \cdot d\vec{S}$, where S is the surface of the unit cube $[0,1]\times[0,1]\times[0,1]$ <u>without</u> the base, oriented <u>out</u>.

horrible function of y and z

horrible function of x and z



<u>Option 1</u>: Just do it! Parameterize each of the 5 faces (not too hard) & compute.
→ but we don't know P and Q, and we're told they're horrible. :(

<u>Option 2</u>: Apply Gauss's Theorem and integrate the divergence over the enclosed solid.
→ but S is not a closed surface. :(

<u>Idea</u>: Close off S, apply Gauss's Theorem, then subtract off the vector surface integral of the surface we added.

Let S_1 denote the bottom square face, oriented ______, so $\vec{n} = [\ \ ,\ \ ,\ \ ]$. By Gauss's Theorem,

$$\iint_{S+S_1} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV \Rightarrow \iint_S \vec{F}\cdot d\vec{S} + \iint_{S_1} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$$

$S+S_1$: the whole closed cube surface; E ← solid unit cube; S ← 5 faces; S_1 ← bottom face

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV - \iint_{S_1} \vec{F}\cdot d\vec{S}$$

what we want = integral of div $\vec{F}$ over solid − vector surface integral over bottom face.

Ok, let's compute these two parts.

$\vec{F} = [P(y,z), Q(x,z), 3z] \Rightarrow \operatorname{div}\vec{F} = 0+0+3 = 3.$

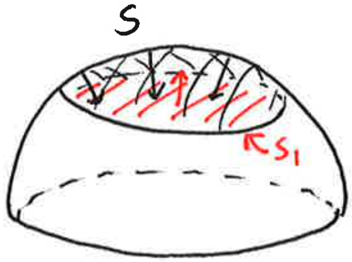
So $\iiint_E \operatorname{div}\vec{F}\, dV = \iiint_E 3\, dV = 3\iiint_E dV = 3(\text{volume of unit cube}) = 3\times 1 = \underline{\underline{3}}.$

And $\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot\vec{n}\, dS = \iint_{S_1} [P,Q,3z]\cdot[0,0,-1]\, dS = \iint_{S_1} -3z\, dS = \iint_{S_1} 0\, dS = \underline{\underline{0}}.$

$z=0$ on the base of the cube, which is the surface S_1

So $\iint_S [P,Q,3z]\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV - \iint_{S_1} [P,Q,3z]\cdot d\vec{S} = 3 - 0 = \underline{\underline{3}}$

Example. Compute $\iint_S \underbrace{[2xy-z^2,\ y-y^2,\ -z]}_{\vec{F}} \cdot d\vec{S}$, where S is the "spherical cap" $x^2+y^2+z^2=2$, $z \geq 1$, oriented down.



↖ this function looks tough to integrate; we'd rather use the derivative, which is easier.

We want to use Gauss's Theorem, so we need to do two things:

1. close off S by attaching any surface with the same boundary as S:
   → let's use the unit disk at height $z=1$, oriented ______.
   This orientation is chosen so the closed surface has consistent orientation.
2. Change the sign, since S has inward orientation.

So now by Gauss's Theorem, $\iint_{\underset{\text{closed surface}}{S+S_1}} \vec{F}\cdot d\vec{S} = -\iiint_{E} \operatorname{div}\vec{F}\,dV$ (E ← enclosed region) $\Rightarrow \underbrace{\iint_S \vec{F}\cdot d\vec{S}}_{\text{(what we want)}} = -\iiint_E \operatorname{div}\vec{F}\,dV - \iint_{S_1}\vec{F}\cdot d\vec{S}$.

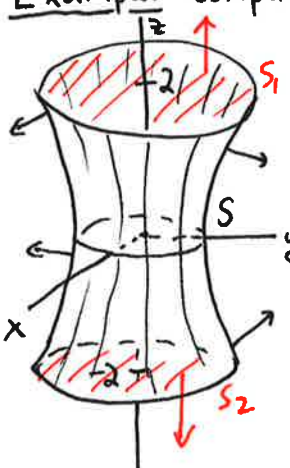
$$= -0-(-\pi) = \pi.$$

Let's compute these:

$$\iiint_E \operatorname{div}\vec{F}\,dV = \iiint_E (2y+1-2y-1)\,dV = \iiint_E 0\,dV = 0.$$

$$\iint_{S_1}\vec{F}\cdot d\vec{S} = \iint_{S_1}\vec{F}\cdot\vec{n}\,dS = \iint_{S_1}[\sim,\sim,-z]\cdot[0,0,1]\,dS = \iint_{S_1} -z\,dS = \iint_{S_1} -1\,dS = -\iint_{S_1} dS = -(\text{area of } S_1) = -\pi.$$

Example. Compute $\iint_S \underbrace{[ye^{\cos(\sin z)},\ x^{100}e^{z^{x^z}},\ x-z^2]}_{F}\cdot d\vec{S}$, where S is part of the hyperboloid $x^2+y^2-z^2=1$ between $z=-2$ and $z=2$, oriented out.



To use Gauss's Theorem, we have to attach two surfaces!

S_1: the disk of radius $\sqrt{5}$* in the plane $z=2$, with $\vec{n}=[0,0,1]$

S_2: " " " " " $z=-2$, " $\vec{n}=[0,0,-1]$.

* $x^2+y^2-z^2=1$
$x^2+y^2-2^2=1$
$x^2+y^2=5$
$r=\sqrt{x^2+y^2}=\sqrt{5}$

Then $\underbrace{\iint_S \vec{F}\cdot d\vec{S}}_{\text{what we want}} = \iiint_E \operatorname{div}\vec{F}\,dV - \underbrace{\iint_{S_1}\vec{F}\cdot d\vec{S}}_{\text{flux over top cap}} - \underbrace{\iint_{S_2}\vec{F}\cdot d\vec{S}}_{\text{flux over bottom cap.}}$, by Gauss's Theorem. (E ← solid enclosed)

Let's compute them:

$\iiint_E \operatorname{div}\vec{F}\,dV = \iiint_E (0+0-2z)\,dV = \iiint_E -2z\,dV = 0$ because the solid region E is symmetric across $z=0$ and $-2z$ is odd with respect to z.

$\iint_{S_1}\vec{F}\cdot d\vec{S} = \iint_{S_1}\vec{F}\cdot\vec{n}\,dS = \iint_{S_1}[\sim,\sim,x-z^2]\cdot[0,0,1]\,dS = \iint_{S_1}(x-z^2)\,dS = \iint_{S_1} x\,dS - \iint_{S_1} z^2\,dS = 0 - \iint_{S_1} 4\,dS = -4(\text{area of } S_1)$

S_1 is symmetric with respect to x

$z=2$ on S_1

similarly,

$\iint_{S_2}\vec{F}\cdot d\vec{S} = 0 - \iint_{S_2}(-4)\,dS = 4(\text{area of } S_2)$.

because $\vec{n}=[0,0,-1]$ for S_2.

So $\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV - \iint_{S_1}\vec{F}\cdot d\vec{S} - \iint_{S_2}\vec{F}\cdot d\vec{S} = 0 - 4(\text{area of } S_1) + 4(\text{area of } S_2) = \underline{0}$. (have same area)