Mathematician spotlight: Gwyn Coogan, math teacher, Phillips Exeter Academy
- math major at Smith College
- ran in 1992 Olympics in Barcelona (10,000m)
- PhD at UC Boulder in number theory and postdoc at Wisconsin
- now teaches, coaches, and teaches other math teachers new ways to teach.

Recently, we've been exploring Stokes' Theorem:

surface integral of curl $\longleftrightarrow$ boundary curve integral of vector field.
2D integral of "derivative" $\longleftrightarrow$ 1D integral of function

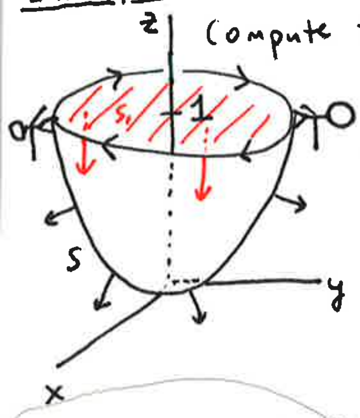
Today, we'll start exploring our last theorem of the semester, Gauss's Theorem:

solid integral of divergence $\longleftrightarrow$ boundary surface integral of vector field.
3D integral of "derivative" $\longleftrightarrow$ 2D integral of function.

But first, another example of Stokes' Theorem, using the "surface independence" from last time.

Example. Let $\vec{F}$ be a field such that $\operatorname{curl}\vec{F} = [y e^{y^{z^2}}, xy\sin(\cos z) - y^2, 2yz]$. ← we'll see later why such an $\vec{F}$ must exist.

Compute the flux of $\operatorname{curl}\vec{F}$ over S: $z = x^2 + y^2$ with $z \le 1$, outward normal.



Option 1: Just do it! Parameterize S, integrate $\operatorname{curl}\vec{F}$. ☹ looks maybe impossible.

Option 2: Stokes' Thm says $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_{\partial S} \vec{F}\cdot d\vec{s}$, so find $\vec{F}$ and integrate over the boundary circle. ☹ Did you say find $\vec{F}$??

Option 3: Replace S by a simpler surface with same boundary & compatible orientation. Let's use the unit disk at $z=1$, oriented down.
So on this surface S_1, $\vec{n} = [0,0,-1]$.

$z=1$ on S_1

$$\text{Now } \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_{S_1} \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_{S_1} \operatorname{curl}\vec{F}\cdot\vec{n}\, dS = \iint_{S_1} [\text{m}, \text{m}, 2yz]\cdot[0,0,-1]\, dS = \iint_{S_1} -2yz\, dS = \iint_{S_1} -2y\, dS.$$

To compute this (wonderfully simple!) integral, either

- convert to polar coordinates: $\int_{\theta=0}^{2\pi}\int_{r=0}^{1} -2\cdot r\sin\theta\cdot r\, dr\, d\theta = \int_0^{2\pi}\sin\theta\, d\theta \cdot \int_0^1 -2r^2\, dr = 0\cdot \text{m} = \underline{\underline{0}}.$ (m: something finite)

- Notice that S_1 is symmetric across $y=0$ and $-2y$ is odd with respect to y, so integral is 0.

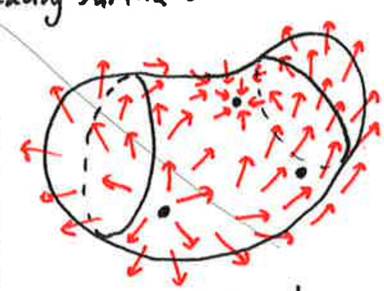
Gauss's Theorem! Suppose that $\vec{F}$ is a vector field with continuous partial derivatives throughout some solid region E in $\mathbb{R}^3$, where the boundary surface ∂E of E is oriented outward.

$$\text{Then } \iiint_E (\operatorname{div}\vec{F})\, dV = \iint_{\partial E} \vec{F}\cdot d\vec{S}.$$

divergence is a scalar function, so this is a scalar triple integral.

$\vec{F}$ is a vector field, so this is a vector surface integral.

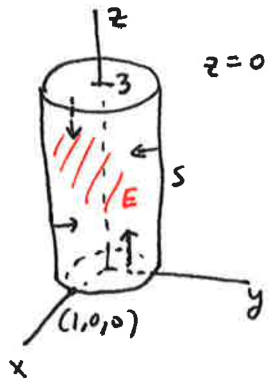
"the amount of stuff that flows into or out of each point enclosed by surface" = "the (net) amount that the vector field $\vec{F}$ flows into or out of the surface"



the double integral adds up $\vec{F}\cdot d\vec{S}$ at each point on the surface of the solid. (Surface is oriented out.)

the triple integral adds up $\operatorname{div}\vec{F}$ at each point of the solid interior

Example. Compute $\iint_S \underbrace{\left[y^{123} e^{\sin(yz)}, y - x^{z^x}, z^2 - z\right]}_{\vec{F}} \cdot d\vec{S}$, where S is the cylinder $r=1$ from $z=0$ to $z=3$, with top and bottom disks attached, and inward normal.



Can we apply Gauss's Theorem? - $\vec{F}$ has continuous partial derivatives everywhere, so ok ✓
- boundary surface is oriented outward ✗
don't worry! just change the sign. ok.

$$\iint_S \vec{F} \cdot d\vec{S} = -\iiint_E \operatorname{div} \vec{F}\, dV = -\iiint_E (0+1+2z-1)\, dV = -\iiint_E 2z\, dV.$$

solid region inside cylinder

Estimate: should $-\iiint_E 2z\, dV$ be positive, negative or zero? ______

Convert to cylindrical coordinates:

$$\iiint_E 2z \cdot dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{3} 2z \cdot \underbrace{r}_{\text{expansion factor!}} \cdot dz \cdot dr \cdot d\theta = \int_0^{2\pi} d\theta \cdot \int_0^1 r\, dr \cdot \int_0^3 2z\, dz = 2\pi \cdot \frac{1}{2} \cdot 9 = \underline{\underline{9\pi}},$$

so $\iint_S \vec{F} \cdot d\vec{S} = -\iiint_E 2z\, dV = \underline{\underline{-9\pi}}$. negative, as expected! :)

Here are some things we can deduce now.

Analogy: curls : surface integrals :: cons. vector fields : line integrals

<u>The flux of a curl vector field through any closed surface is 0.</u>

Let $\vec{F}$ be continuous everywhere and let S be any closed surface.

Then $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \pm \iiint_E \operatorname{div}(\operatorname{curl} \vec{F})\, dV = \pm \iiint_E 0\, dV = \underline{\underline{0}}$.

depending on orientation of S

E ← the solid enclosed by the surface S

we proved that $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$ using Clairaut's Theorem

This is similar to how the line integral of a conservative vector field over any closed path is 0.

You can prove the same result using Stokes' Theorem, if you agree to think about the empty set:

$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$, and S is closed, so S has no boundary, so $\partial S = \emptyset$, ← notation for the empty set
and the integral of <u>any</u> function over the empty set is 0.

<u>A given field is the curl of another if its divergence is 0.</u>

- Is there a field $\vec{F}$ so that $\operatorname{curl} \vec{F} = \left[y e^{y^{z^2}}, xz \sin(\cos(z)) - y^2, 2yz\right]$?
  Well, the divergence of this field is $0 + 0 - 2y + 2y = 0$, so yes!
- Is there a field $\vec{G}$ so that $\operatorname{curl} \vec{G} = \left[y^{199} \sin(\cos(z)), x e^{z^4}, z^3 x\right]$?
  Compute the divergence of this field: ______________ Is there such a G? ______

Amazing! You can know that such a field does or does not exist, without explicitly finding it!

(this resolves the "we'll see later why..." from the first example on the first page.)