Some intentional themes of the mathematicians I decided to spotlight:

- they are all alive
- they alternated male and female
- at most two per week were caucasian

Happy national day of silence! Lots of adults, lots of professors, lots of mathematicians are LGBTQ+.

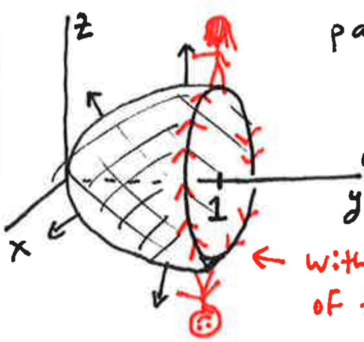
Including:
- me (Diana Davis)
- Moon Duchin
- Emily Riehl
- Autumn Kent
- Dylan Thurston
- Harrison Bray

... and many more!

Last time: Stokes' Theorem: $\int_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S (\text{curl}\,\vec{F}) \cdot d\vec{S}$, when S and ∂S have compatible orientation.

This time:
- Lots of examples
- "surface independence" property of curl vector fields, analogous to path independence of conservative vector fields.

Example. Let $\vec{F} = [(y-1)\sin(e^{xz^y}), xyz\, e^{xyz}, xz+y]$, and let S be the piece of the paraboloid $y = x^2 + z^2$ with $y \le 1$, oriented outwards. Compute $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$.

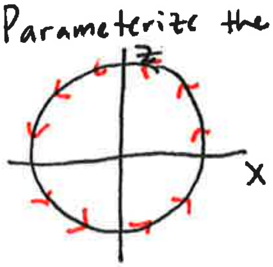


Option 1: Just do it! But this is tough, because the $\vec{i}$ & $\vec{j}$ functions are tough.

Option 2: Stokes' Theorem! $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$, as long as we orient ∂S compatibly.

← with this orientation, if you walk along ∂S with your head in the direction of the chosen normal vectors (orientation) of S, your left arm is over S.

Parameterize the boundary curve:



$x(t) = \cos t$
$z(t) = \sin t$
and
$y(t) = 1$

$\Rightarrow \vec{x}(t) = [\cos t, 1, \sin t]$, $0 \le t \le 2\pi$

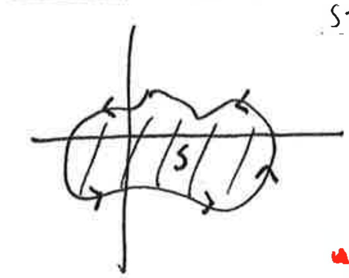
$\Rightarrow \vec{x}'(t) = [-\sin t, 0, \cos t]$.

so $\vec{F}(\vec{x}(t)) = [0, \underbrace{\text{~~mess~~}}_{\text{something messy}}, \cos t \cdot \sin t + 1]$.

Now $\int_{\partial S} \vec{F} \cdot d\vec{s} = \int_{t=0}^{2\pi} F(\vec{x}(t)) \cdot \vec{x}'(t)\, dt = \int_0^{2\pi} [0, \text{~~m~~}, \cos t \cdot \sin t + 1] \cdot [-\sin t, 0, \cos t]\, dt = \int_0^{2\pi} (\cos^2 t \sin t + \cos t)\, dt = 0$

$\Rightarrow \iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \underline{\underline{0}}.$

Check this out! If S and ∂S are in the xy-plane, then Stokes' Theorem is just Green's Theorem:



Stokes' Theorem says: $\int_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$

same

$= \iint_S \text{curl}\,\vec{F} \cdot \vec{n}\, dS$

because with CCW orientation your head is always pointing up out of the page

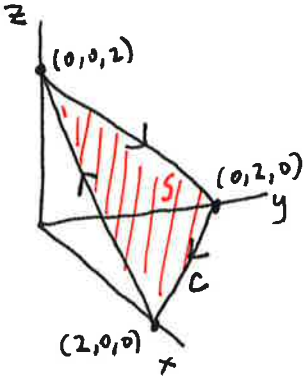
xy plane

$= \iint_S \text{curl}\,\vec{F} \cdot \vec{k}\, dA$

$= \iint_S [-Q_z, P_z, Q_x - P_y] \cdot [0,0,1]\, dA$

$= \iint_S (Q_x - P_y)\, dA$ ← Green's Theorem! ☺

Example of going the other way: Let $\vec{F} = [x \cdot \sin(e^x) - xz, -2xy, z^2+y]$ and let C be the triangular path from $(2,0,0) \to (0,0,2) \to (0,2,0) \to (2,0,0)$.



Compute $\int_C \vec{F} \cdot d\vec{s}$.

<u>Option 1</u>: Parameterize each of the three parts, and deal with $x \cdot \sin(e^x)$. ☹

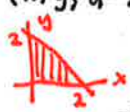
<u>Option 2</u>: Stokes' Theorem! $\int_C \vec{F} \cdot d\vec{s} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$ ← where S is <u>any</u> surface whose boundary is C, with appropriate orientation.

Let's use the simplest surface bounded by C: the flat triangle, which is part of the plane $x+y+z=2$.

Orientation of S: upward normal or downward normal? ______

Parameterize S: $\vec{X}(x,y) = (x, y, 2-x-y) \Rightarrow \vec{X}_x = [1,0,-1]$, $\vec{X}_y = [0,1,-1]$ $\Rightarrow \vec{X}_x \times \vec{X}_y = [1,1,1]$ ← we want ↗ so use the opposite: $\vec{X}_y \times \vec{X}_x = [-1,-1,-1]$.

for x, y in



Compute curl $\vec{F}$: $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x\cdot \sin(e^x) - xz & -2xy & z^2+y \end{vmatrix} = [1, x, -2y]$.

So $\int_C \vec{F} \cdot d\vec{s} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_D \text{curl}\,\vec{F}(\vec{X}(x,y)) \cdot (\vec{X}_y \times \vec{X}_x)\,dx\,dy = \int_{y=0}^{y=2} \int_{x=0}^{x=2-y} [1, x, -2y] \cdot [-1,-1,-1]\,dx\,dy = \int_0^2 \int_0^{2-y} (-1 + x + 2y)\,dx\,dy$

D ← region in the xy-plane parameterizing S

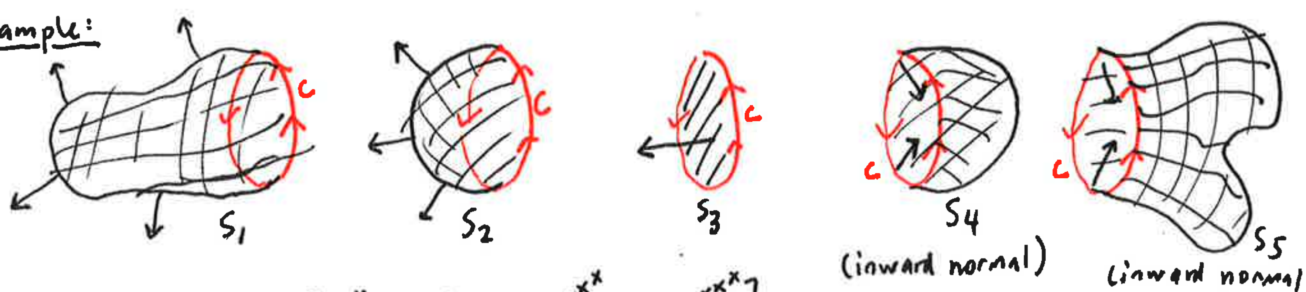
$= \underline{\underline{2.}}$

But actually, let's think about this: Stokes' Theorem says $\int_C \vec{F} \cdot d\vec{s} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$, where S is <u>any</u> surface whose boundary is C (with correct orientation).

So if two surfaces S_1, S_2 have the same boundary, $\iint_{S_1} \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_{S_2} \text{curl}\,\vec{F} \cdot d\vec{S}$. Wow!

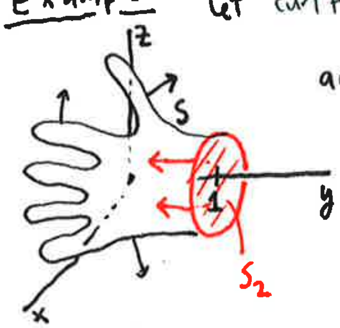
(and compatible/same orientation)

<u>Example:</u>



$\iint_{S_1} \text{curl}\,\vec{F} \cdot d\vec{S} = \vdots = \iint_{S_5} \text{curl}\,\vec{F} \cdot d\vec{S}$

<u>Example.</u> Let $\text{curl}\,\vec{F} = [y^{y^y}\sin(z^2), (y-1)e^{x^{x^x}} + 2, z e^{x^{x^x}}]$ and compute $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$ over the "hand surface" shown. No thank you!



<u>Method 1</u>: do it! No.

<u>Method 2</u>: Compute $\int_{\partial S} \vec{F} \cdot d\vec{s}$ using Stokes' Thm. But we don't have $\vec{F}$. ☹

<u>Method 3</u>: Replace S with a simpler surface with the same boundary!

$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_{S_2} \text{curl}\,\vec{F} \cdot d\vec{S}$ ← unit disk at $y=1$ with "left" normal

$= \iint_{S_2} \text{curl}\,\vec{F} \cdot \vec{n}\,dS = \iint_{S_2} \text{curl}\,\vec{F} \cdot [0,-1,0]\,dS$

$= \iint_{S_2} -((y-1)e^{x^{x^x}} + 2)\,dS = \iint_{S_2} -2\,dS = \underline{\underline{-2\pi.}}$ ☺

↖ $y=1$ on S_2