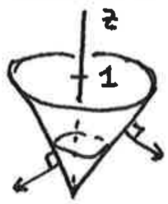
Mathematician spotlight: Edgar Duéñez, senior software engineer, Google
- math contests in high school; math major; PhD in C.S.
- Studied search algorithms & evolutionary bases of social behavior
- at Google, developed machine learning algorithms to identify images.

Today: • more on vector surface integrals
• Stokes' Theorem: relates a vector surface integral to the vector line integral on the boundary of the surface. Is just Green's Theorem in 3D space instead of the plane.

Example. The electric force field $\vec{E}$ from the backup power generator is $\vec{E} = [xz, yz, y]$.



You are standing upside down wearing a tin hat shaped like the cone $z = \sqrt{x^2+y^2}$ from $z=0$ to $z=1$.

Compute the electric flux of $\vec{E}$ flowing out of the hat.

• First, parameterize the hat: $\vec{X}(r,\theta) = [r\cos\theta, r\sin\theta, r]$ for $0 \le \theta \le 2\pi$, $0 \le r \le 1$.

• Now compute tangent vectors: $\vec{X_r} = [\cos\theta, \sin\theta, 1]$

$\vec{X_\theta} = [-r\sin\theta, r\cos\theta, 0]$, now use them to find normal.

$$\Rightarrow \vec{X_r} \times \vec{X_\theta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = [-r\cos\theta, -r\sin\theta, r]$$

← z is positive, so this is the upward (inward) normal vector. So take the opposite. $\vec{X_\theta} \times \vec{X_r}$

we'll use $-\vec{X_r} \times \vec{X_\theta} = \vec{X_\theta} \times \vec{X_r} = [r\cos\theta, r\sin\theta, -r]$.

$$\text{Now } \iint_{\text{hat}} \vec{E} \cdot \vec{n}\, dS = \iint_R \vec{E}(\vec{X}(r,\theta)) \cdot (\vec{X_\theta} \times \vec{X_r})\, dr\, d\theta = \int_0^{2\pi}\int_0^1 [\overbrace{r^2\cos\theta}^{xz}, \overbrace{r^2\sin\theta}^{yz}, \overbrace{r\sin\theta}^{y}] \cdot \overbrace{[r\cos\theta, r\sin\theta, -r]}^{\vec{X_\theta} \times \vec{X_r}}\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(r^3(\cos^2\theta + \sin^2\theta) - r^2\sin\theta\right) dr\, d\theta = \int_0^{2\pi}\int_0^1 (r^3 - r^2\sin\theta)\, dr\, d\theta = \ldots = \frac{\pi}{2}.$$

positive, so the flux is outward!

Example. Let S be the flying pancake formed by slicing the plane $x+z=5$ ($z = 5-x$) with the cylinder $x^2+y^2=9$, like a cookie cutter. Give the pancake upward orientation, and compute $\iint_S (-4\vec{i} - x\vec{k}) \cdot d\vec{S}$.

• First, parameterize the surface: $\vec{X}(x,y) = [x, y, 5-x]$ for x,y in the disk of radius 5 centered at the origin

• Now, compute tangent vectors: $\vec{X_x} = [1, 0, -1]$

$\vec{X_y} = [0, 1, 0]$; now find a normal vector:

$$\vec{X_x} \times \vec{X_y} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = [1, 0, 1]$$

← check: is upward! yay
is the normal vector that you can pull out of the plane coefficients: $1x + 0y + 1z = 5$.

$$\text{Now } \iint_S (-4\vec{i} - x\vec{k}) \cdot d\vec{S} = \iint_{\substack{\text{disk of}\\ \text{radius } 3}} [-4, 0, -x] \cdot [1, 0, 1]\, dA = \iint_{\substack{\text{disk of}\\ \text{radius } 3}} (-4-x)\, dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{3} (-4 - r\cos\theta) \cdot r \cdot dr \cdot d\theta$$

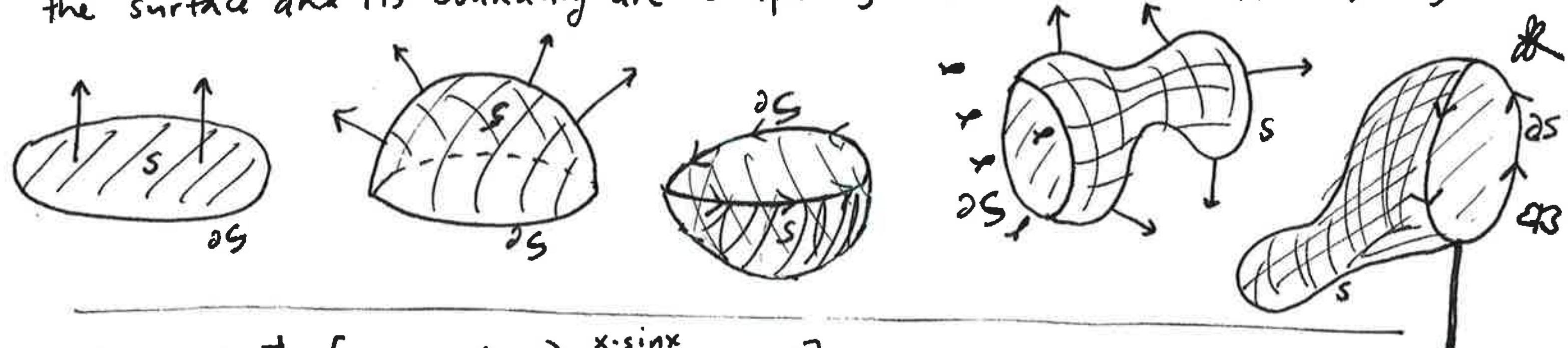
$$= \ldots = -36\pi.$$

negative, so the vector field tends to flow down through the pancake.

Stokes' Theorem. Let $\vec{F}$ be a vector field, and let S be an oriented surface, whose boundary (if any) is ∂S, oriented so that if you walk along ∂S with your head in the direction of the normal vectors for the chosen orientation of S, your left arm is over S.

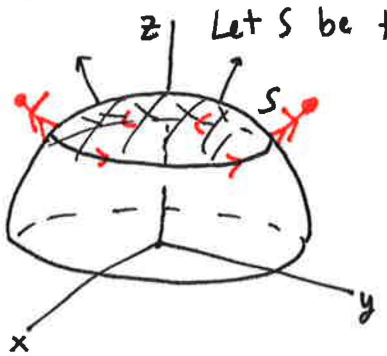
Then $\displaystyle\int_{\partial S} \vec{F}\cdot d\vec{s} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}.$ "The flux of the curl of $\vec{F}$ across a surface S is equal to the line integral of $\vec{F}$ along its boundary."

Draw in normal vectors to the surface, or arrows along the boundary curve, so that the surface and its boundary are "compatibly oriented" (head, left arm, etc.)



Example. Let $\vec{F} = [-y,\ x+(z-1)x^{x\cdot\sin x},\ x^2+y^2]$.

Let S be the piece of the sphere $x^2+y^2+z^2=2$ above $z=1$, with outward orientation.



Compute $\displaystyle\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$.

Method 1: Just do it! Compute $\operatorname{curl}\vec{F}$, and integrate.

ok, $\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x+(z-1)x^{x\cdot\sin x} & x^2+y^2 \end{vmatrix} = \left[\_\_\_,\ \_\_\_,\ \frac{\partial}{\partial x}\left(x+(z-1)x^{x\cdot\sin x}\right) - \ldots\right]$

what a mess! let's not.

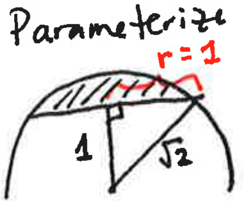
Method 2: Apply Stokes' Theorem.

So $\displaystyle\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_{\partial S}\vec{F}\cdot d\vec{s}$, as long as ∂S is oriented in the correct direction.

Drawing in a left-armed stick figure above, it seems that we should orient the boundary circle as shown: its shadow is CCW in the xy-plane.

Parameterize the boundary curve ∂S. What is its radius? 1.

← a ccw circle of radius 1 at height $z=1$.



side view

So $\vec{x}(\theta) = [\cos\theta,\ \sin\theta,\ 1]$ for $0 \le \theta \le 2\pi$

$\Rightarrow \vec{x}'(\theta) = [-\sin\theta,\ \cos\theta,\ 0]$

Note that on the boundary, $z=1$, so $x+(z-1)x^{x\sin x} = x+(1-1)\cdots = x$.

So now $\displaystyle\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_{\partial S}\vec{F}\cdot d\vec{s} = \int_{\theta=0}^{2\pi} [\underbrace{-\sin\theta}_{-y},\ \underbrace{\cos\theta}_{x},\ \underbrace{1}_{x^2+y^2}]\cdot \underbrace{[-\sin\theta,\ \cos\theta,\ 0]}_{\vec{x}'(\theta)}\,d\theta = \int_0^{2\pi}(\sin^2\theta+\cos^2\theta)\,d\theta = \int_0^{2\pi} d\theta = 2\pi.$

Positive, so the vector field $\vec{F}$ goes (net) CCW, and $\operatorname{curl}\vec{F}$ is (net) outward.