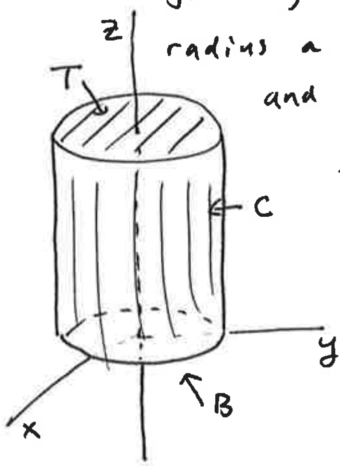
Mathematician spotlight: Katherine Johnson, mathematician at NASA (currently age 99)

- calculated flight trajectories for the first Americans in space
- created the backup system that helped the Apollo 13 crew return safely
- helped NASA transition from human computers to digital computing machines

Plan for last two weeks: scalar surface integrals ← last time, and again today

vector surface integrals ← start today, and study until the end.

Example. Suppose that the amount of mold at the point (x,y,z) on an old can of beans is given by $f(x,y,z) = x^2+y^2+z$. Further suppose that the can is a cylinder of radius a and height h, centered on the z-axis from $z=0$ to $z=h$, with the top and bottom disks attached. How much total mold is on the can?



Plan: Compute the scalar surface integral of f over the three surfaces — the top T, bottom B and cylinder C — and add them.

Bottom:
$$\iint_B \overbrace{(x^2+y^2+z)}^{f(x,y,z)}\,dS = \iint_{D} (x^2+y^2+\underset{\substack{z=0\\ \text{on } B}}{0})\,\underset{xy\text{-plane}}{dA} = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2\cdot r\,dr\,d\theta = 2\pi\cdot\frac{a^4}{4} = \frac{\pi a^4}{2}.$$

shadow is unit disk → D

Top:
$$\iint_T (x^2+y^2+z)\,dS = \iint_D (x^2+y^2+\underset{\substack{z=h\\ \text{on } B}}{h})\,\underset{xy\text{-plane}}{dA} = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}(r^2+h)\,dr\,d\theta = 2\pi\left(\frac{a^4}{4}+\frac{ha^2}{2}\right).$$

shadow is unit disk → D

Cylinder: We need to set up a scalar surface integral. First, we need to parameterize C.

$\vec{X}(\theta, z) = [a\cdot\cos\theta,\ a\cdot\sin\theta,\ z]$ for $0 \le \theta \le 2\pi$, $0 \le z \le h$.

then
$$\left.\begin{aligned}\vec{X}_\theta &= [-a\cdot\sin\theta,\ a\cdot\cos\theta,\ 0]\\ \vec{X}_z &= [0,\ 0,\ 1]\end{aligned}\right\} \Rightarrow \vec{X}_\theta \times \vec{X}_z = [a\cdot\cos\theta,\ a\cdot\sin\theta,\ 0] \Rightarrow \|\vec{X}_\theta\times\vec{X}_z\| = a.$$

So we can compute the scalar surface integral:
$$\iint_C f(x,y,z)\,dS = \iint_R f(\vec{X}(\theta,z))\,\|\vec{X}_\theta\times\vec{X}_z\|\,dz\,d\theta = \int_{\theta=0}^{2\pi}\int_{z=0}^{h} \overbrace{\Big(\underbrace{(a\cdot\cos\theta)^2}_{x^2}+\underbrace{(a\cdot\sin\theta)^2}_{y^2}+z\Big)}^{f(\vec{X}(\theta,z))}\cdot\overbrace{a}^{\|\vec{X}_\theta\times\vec{X}_z\|}\cdot dz\,d\theta$$

region in the θz-plane → R

$$= \int_{\theta=0}^{2\pi}\int_{z=0}^{h}(a^2+z)\cdot a\cdot dz\cdot d\theta = \int_{\theta=0}^{2\pi}\int_{z=0}^{h} a^3+za = 2\pi\left(a^3h+a\frac{h^2}{2}\right).$$

Total:
$$\iint_{\text{can}} f\,dS = \iint_B f\,dS + \iint_T f\,dS + \iint_C f\,dS = \underbrace{\frac{\pi a^4}{2}}_{B} + \underbrace{\frac{\pi a^4}{2}+\pi h a^2}_{T} + \underbrace{2\pi a^3 h + \pi a h^2}_{C}.$$

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All integrals are variations on the same idea: adding up function values over some geometric object.
$1, f(x,y), f(x,y,z), \ldots$ interval, region, solid, curve, surface...

Vector surface integrals!

Suppose there is a rapidly flowing stream full of tiny fishes, ← a vector field $\vec{F}$

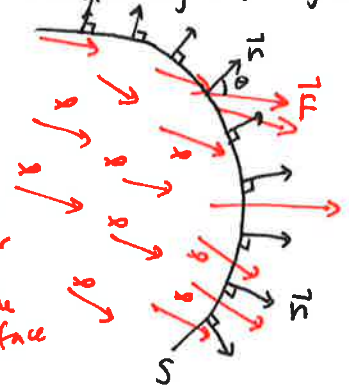
and you are holding a net that has an inside and an outside, ← a surface S with a chosen orientation

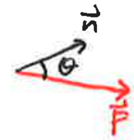
and you wish to measure how many fish you catch. ← how much $\vec{F}$ flows through S.

Use a vector surface integral:

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\; dS$$

vector field — unit normal at each point of surface — add it up over the whole surface





$\vec{F}\cdot\vec{n}$ measures how much $\vec{F}$ points in the same direction as $\vec{n}$.

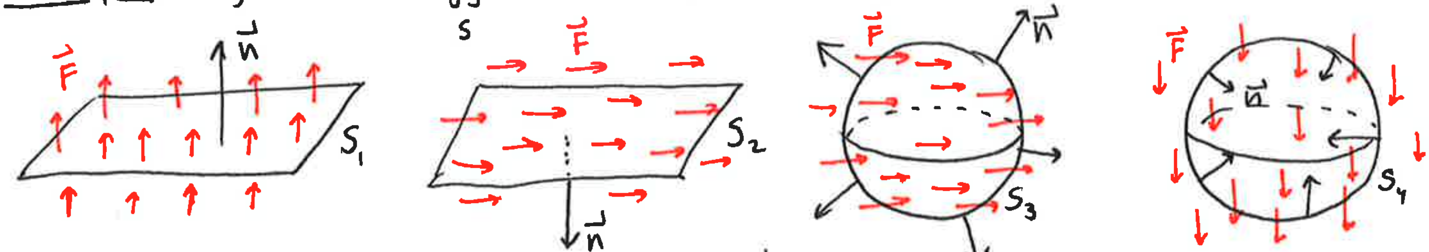
$\vec{F}\cdot\vec{n} > 0$: same direction! catch fish. :)

$\vec{F}\cdot\vec{n} < 0$: opposite direction! lose fish. :(

$\vec{F}\cdot\vec{n} = 0$: perpendicular! fish swim along net.

This quantity is called the "flux" of $\vec{F}$ through S.

Example. Say whether $\iint_S \vec{F}\cdot\vec{n}\, dS$ is positive, negative or 0 for each:



$\iint_{S_1}\vec{F}\cdot\vec{n}\,dS$ ___ 0 because:	$\iint_{S_2}\vec{F}\cdot\vec{n}\,dS$ ___ 0 because:	$\iint_{S_3}\vec{F}\cdot\vec{n}\,dS$ ___ 0 because:	$\iint_{S_4}\vec{F}\cdot\vec{n}\,dS$ ___ 0 because:

Example. Let S be the unit square in the xy-plane with upward unit normal, and $\vec{F} = [\cos(xz), e^y, 5]$.

Compute $\iint_S \vec{F}\cdot\vec{n}\, dS$. Here we can see $\vec{n} = [0,0,1]$, so $\vec{F}\cdot\vec{n} = [\sim, \sim, 5]\cdot[0,0,1] = 5$.

So $\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S 5\, dS = 5\,(\text{area of } S) = 5\cdot 1 = \underline{5}.$ ← makes sense because, as for S_1 above, all the $\vec{F}$ vectors point (generally) "up", so in the same direction as $\vec{n}$.

How to compute vector surface integrals in general:

- For a surface S: $\vec{X}(s,t)$, normal vector is $\vec{X_s}\times\vec{X_t}$, so unit normal vector is $\dfrac{\vec{X_s}\times\vec{X_t}}{\|\vec{X_s}\times\vec{X_t}\|}$.
- dS = area of parallelogram spanned by $\vec{X_s}\,ds$ and $\vec{X_t}\,dt$ $= \|X_s \times X_t\|\,ds\,dt$.

So
$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_R \vec{F}\cdot\underbrace{\left(\frac{\vec{X_s}\times\vec{X_t}}{\cancel{\|\vec{X_s}\times\vec{X_t}\|}}\right)}_{\vec{n}}\underbrace{\cancel{\|X_s\times X_t\|}\,ds\,dt}_{dS} = \boxed{\iint_R \vec{F}(\vec{X}(s,t))\cdot(\vec{X_s}\times\vec{X_t})\,ds\,dt}$$

region in st-plane → R

We'll use this next time!