Diana Davis — Math 34 — April 20, 2018 — Class #34 ① §7.1

Mathematician spotlight: Henry Segerman, Assistant Professor, Oklahoma State University
- Studies 3-manifolds, triangulations, hyperbolic geometry
- does mathematical visualization, 3D printing, virtual reality for visualization
- creates 3D printed objects that blend math and art (IMHO)

Last time: We discovered how to describe a surface parametrically, with two parameters: $\vec{x}(s,t)$, $\vec{x}(r,\theta)$ etc.
We determined that the normal vector to the surface is $\vec{n} = \vec{X_s} \times \vec{X_t}$ (etc.)

Today: We are integrating $\left\{\begin{matrix}\text{scalar}\\ \text{vector}\end{matrix}\right.$ functions over $\left\{\begin{matrix}\text{curves}\\ \text{surfaces}\end{matrix}\right.$: Scalar surface integrals!

First, let's find a tangent plane.

Example: Find an equation for the tangent plane to DD's favorite surface,
$\vec{x}(r,\theta) = [r\cdot\cos\theta,\ r\cdot\sin\theta,\ \cos r]$, for $\begin{matrix}0 \le r \le \infty\\ 0 \le \theta \le 2\pi,\end{matrix}$ at $(1, 0, \cos 1)$, i.e. when $r=1, \theta=0$.

r-curves:
- hold θ constant
- vary r

$\left.\begin{matrix}\theta = 0 \Rightarrow \vec{x}(r,0) = [r, 0, \cos r]\\ \theta = \frac{\pi}{2} \Rightarrow \vec{x}(r,\frac{\pi}{2}) = [0, r, \cos r]\end{matrix}\right\}$ cosine curves radiating out from 0

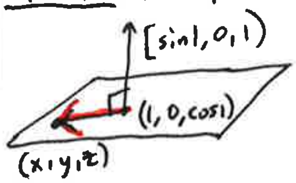
θ-curves:
- hold r constant
- vary θ

$\left.\begin{matrix}r=1 \Rightarrow \vec{x}(1,\theta) = [\cos\theta, \sin\theta, \cos 1]\\ r=10 \Rightarrow \vec{x}(10,\theta) = [10\cos\theta, 10\sin\theta, \cos 10]\end{matrix}\right\}$ circles at undulating heights

looks like circular waves propogating out in the water

$$\vec{X_r} = [\cos\theta, \sin\theta, -\sin r] \Rightarrow \vec{X_r}(1,0) = [1, 0, -\sin 1]$$
$$\vec{X_\theta} = [-r\sin\theta, r\cos\theta, 0] \Rightarrow \vec{X_\theta}(1,0) = [0, 1, 0]$$
$$\Rightarrow \vec{X_r} \times \vec{X_\theta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & -\sin 1\\ 0 & 1 & 0\end{vmatrix} = \underbrace{[\sin 1, 0, 1]}_{\text{normal vector to surface at } (1,0,\cos 1)} = \vec{n}$$

Recall: The plane passing through $(1, 0, \cos 1)$ with normal vector $[\sin 1, 0, 1]$ is the collection of points (x,y,z) such that

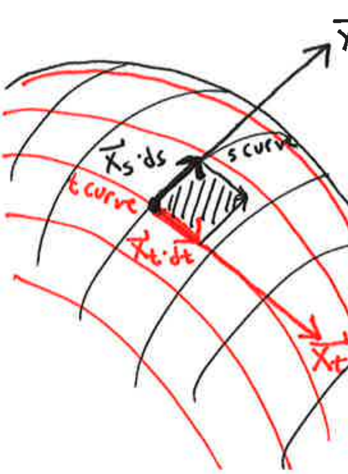


$$\begin{bmatrix}\sin 1\\ 0\\ 1\end{bmatrix} \cdot \begin{bmatrix} x-1\\ y-0\\ z-\cos 1\end{bmatrix} = 0 \Rightarrow \sin 1\,(x-1) + 0\,(y-0) + 1\,(z-\cos 1) = 0$$
$$\Rightarrow x\cdot\sin 1 - \sin 1 + z - \cos 1 = 0 \Rightarrow \underline{\underline{x\cdot\sin 1 + z = \cos 1 - \sin 1.}}$$

tangent plane equation.

What a cool surface! How would we ever calculate its surface area, for instance?

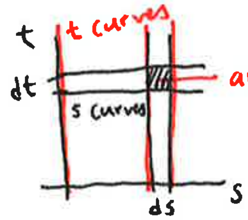
Surface area. Consider a surface S parameterized by $\vec{X}(s,t)$, defined over a region D of the st-plane.
Last time, we showed that $\vec{X_s}$ and $\vec{X_t}$ are tangent to the surface in different directions.
Recall that $\|\vec{u} \times \vec{v}\|$ is the area of the parallelogram spanned by vectors $\vec{u}$ and $\vec{v}$,
so $\|\vec{X_s} \times \vec{X_t}\|$ " " " " " " $\vec{X_s}$ and $\vec{X_t}$.



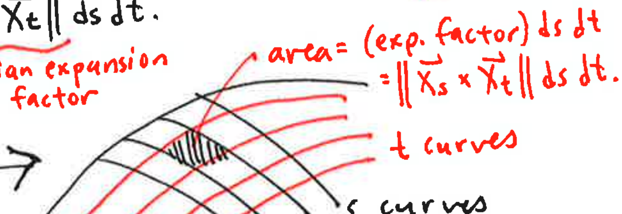
Idea: Make a tiny vector in the s-direction: $\vec{X_s}\cdot ds$ ← tiny distance
and " " " " t-direction: $\vec{X_t}\cdot dt$ ←

Then $\|\vec{X_s}\,ds \times \vec{X_t}\,dt\|$ = area of tiny parallelogram $= \|\vec{X_s} \times \vec{X_t}\| \underbrace{ds\,dt}_{\text{pull out constants}}$

So Surface area of $S = \iint_D \underbrace{\|\vec{X_s} \times \vec{X_t}\|}_{\text{Jacobian expansion factor}} ds\,dt.$

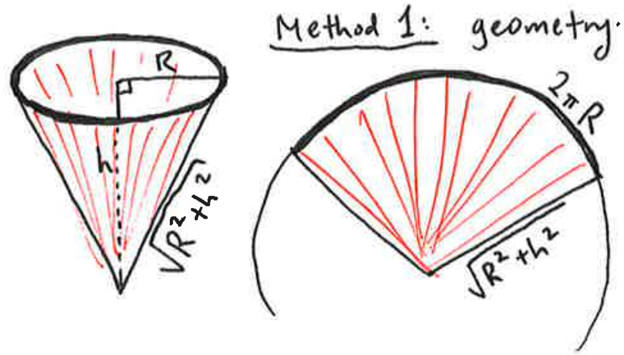


$\xrightarrow{\vec{X}(s,t)}$



Example. Find the surface area of a cone of radius R and height h.

Method 1: geometry. Cut open the cone and lay it flat.



$$\text{area} = \begin{pmatrix}\text{area of disk}\\ \text{of radius}\\ \sqrt{R^2+h^2}\end{pmatrix} \times \begin{pmatrix}\text{proportion}\\ \text{of the disk}\\ \text{that we have}\end{pmatrix}$$

$$= \pi\left(\sqrt{R^2+h^2}\right)^{\cancel{2}} \times \left(\frac{\cancel{2\pi} R}{\cancel{2\pi}\cancel{\sqrt{R^2+h^2}}}\right)$$

$$= \pi R\sqrt{R^2+h^2}.$$

Method 2: Parameterize the surface and integrate $\iint_D \|\vec{X}_r \times \vec{X}_\theta\|\, dr\, d\theta$.

$$\vec{X}(r,\theta) = \left[r\cdot\cos\theta,\ r\cdot\sin\theta,\ \frac{h}{R}\cdot r\right]$$

$$\vec{X}_r = \left[\cos\theta,\ \sin\theta,\ \frac{h}{R}\right] \Rightarrow \vec{X}_r \times \vec{X}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & h/R \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \left[-\frac{h}{R}\cdot r\cos\theta,\ -\frac{h}{R} r\sin\theta,\ r\right]$$

$$\vec{X}_\theta = \left[-r\sin\theta,\ r\cos\theta,\ 0\right] \qquad \Rightarrow \|\vec{X}_r \times \vec{X}_\theta\| = \underbrace{r}_{\text{variable}}\ \underbrace{\frac{\sqrt{R^2+h^2}}{R}}_{\text{constant}}$$

So surface area $= \displaystyle\int_{\theta=0}^{2\pi}\int_{r=0}^{R} \overbrace{r\frac{\sqrt{R^2+h^2}}{R}}^{\|\vec{X}_r\times\vec{X}_\theta\|}\, dr\, d\theta = \frac{\sqrt{R^2+h^2}}{R}\cdot\int_{\theta=0}^{2\pi} d\theta \cdot \int_{r=0}^{R} r\, dr = \frac{\sqrt{R^2+h^2}}{\cancel{R}}\cdot\cancel{2}\pi\cdot\frac{R^{\cancel{2}}}{\cancel{2}} = \pi R\sqrt{R^2+h^2}.$ ☺

Scalar surface integrals! Integrating a scalar function over a surface.

dS = tiny piece of area on our surface S

$$\text{Surface area} = \iint_D 1\, dS = \iint_D 1 \underbrace{\|\vec{X}_s \times \vec{X}_t\|\, ds\, dt}_{dS}$$

we can replace 1 with a function that assigns a value (temperature, density, charge) to each point, and integrate it.

Scalar surface integral of $f(x,y,z)$ over $S = \vec{X}(s,t)$ on D $= \displaystyle\iint_D f(x,y,z)\, dS = \iint_D \underbrace{f(\vec{X}(s,t))}_{\text{function at each point}} \underbrace{\|\vec{X}_s \times \vec{X}_t\|\, ds\, dt}_{dS}$

Example. Compute $\iint_S xy\, dS$, where S is the unit disk in the xy-plane.

Here, the "surface" we are integrating over is a part of our familiar xy-plane.

Method 1: set it up in polar coords: $\displaystyle\iint_S xy\cdot dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \underbrace{(r\cdot\cos\theta)}_{x}\underbrace{(r\cdot\sin\theta)}_{y}\cdot\underbrace{r\cdot dr\cdot d\theta}_{dA} = \ldots$

Method 2: set it up as a surface integral: $\vec{X}(r,\theta) = [r\cdot\cos\theta,\ r\cdot\sin\theta,\ 0]$ for $\left.\begin{matrix}0\le\theta\le 2\pi\\ 0\le r\le 1\end{matrix}\right\} D$

$$\vec{X}_r = [\cos\theta,\ \sin\theta,\ 0] \Rightarrow \|\vec{X}_r \times \vec{X}_\theta\| = \|[0,0,r]\| = r.$$
$$\vec{X}_\theta = [-r\sin\theta,\ r\cos\theta,\ 0]$$

so $\displaystyle\iint_S xy\cdot dS = \iint_D f(r\cdot\cos\theta, r\cdot\sin\theta, 0)\|\vec{X}_r\times\vec{X}_\theta\|\, dr\, d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \underbrace{(r\cos\theta)(r\sin\theta)}_{f(x,y,z)=xy}\cdot\overset{\|\vec{X}_r\times\vec{X}_\theta\|}{r}\cdot dr\, d\theta$ as before!