Mathematician spotlight: Siddhi Krishna, graduate student, Boston College

- studies topology: the shape of stretchy surfaces
- also knot theory, and 3- and 4-manifolds (surfaces one or two dimensions up)
- teaches at BEAM in the summers, math for smart inner-city kids.

We are learning to integrate $\left\{\begin{matrix}\text{functions}\\ \text{vector fields}\end{matrix}\right.$ over $\left\{\begin{matrix}\text{curves} & \leftarrow \text{we can do this now! yay!}\\ \text{surfaces.} & \leftarrow \text{we'll study this from now to the end.}\end{matrix}\right.$

From last time: If $\vec{F}$ is a gradient vector field, i.e. $\vec{F} = \nabla f$ for some function f, then f is a conservative vector field: $\int_C \vec{F}\cdot\vec{T}\,ds = 0$ for a closed curve C, and $\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2}\vec{F}\cdot\vec{T}\,ds$ if

Check this out: Even if $\vec{F}$ is not conservative, part of it might be; break it into cons. & non-cons.:

Also: To see if $\vec{F}$ is conservative, check if $P_y - Q_x = 0$.
Because if $\vec{F} = \nabla f = [f_x, f_y]$, then (by Clairaut's Theorem) $f_{xy} = f_{yx} \Rightarrow f_{xy} - f_{yx} = 0$.

Example: Is $\vec{F} = [1 - 2y + 2xe^{x^2}y^2,\ 2ye^{x^2} + 2x]$ conservative?

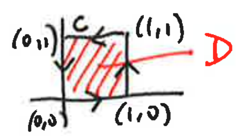
$$\frac{d}{dy}\left(1-2y+2xe^{x^2}y^2\right) - \frac{d}{dx}\left(2ye^{x^2}+2x\right) \stackrel{?}{=} 0$$

$$-2 + \cancel{4xye^{x^2}} - \cancel{4xye^{x^2}} - 2 = -4 \neq 0 \ \therefore\ \text{not conservative.}$$

But part of it is!

$$\vec{F} = \underbrace{\left[1 + 2xe^{x^2}y^2,\ 2ye^{x^2}\right]}_{\text{conservative! } = \nabla f \text{ for } f(x,y) = x + e^{x^2}\cdot y^2} + \underbrace{[-2y, 2x]}_{\text{not conservative.}}$$

Let's do this: Integrate $\vec{F}$ over the ccw unit square, C:



$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \left(\nabla f + [-2y, 2x]\right)\cdot\vec{T}\,ds = \int_C \nabla f\cdot\vec{T}\,ds + \int_C [\underbrace{-2y}_{P}, \underbrace{2x}_{Q}]\cdot\vec{T}\,ds$$

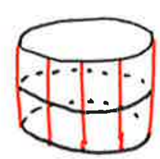
$$= 0 + \iint_D (2 - -2)\,dA = 0 + \iint_D 4\,dA = 4\,(\text{area of } D) = \underline{\underline{4.}}$$

Parameterized surfaces

We can describe any <u>curve</u> in space by $\vec{x}(t) = [x(t), y(t), z(t)]$ ← one variable: a 1-dim'l thing

We can describe any surface in space by $\vec{X}(s,t) = [x(s,t), y(s,t), z(s,t)]$ ← two variables: a 2-dim'l thing

Example: Sketch the surface described by $\vec{X}(s,t) = [\cos t, \sin t, s]$



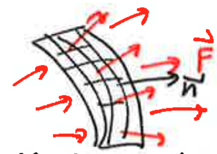
- hold s constant, vary t: get a unit circle at height s
- hold t constant, vary s: get a vertical line

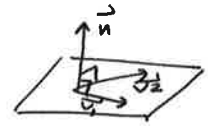
→ (infinite, vertical) cylinder!

Example: Find parametric equations for the sphere of radius R centered at the origin.

- First idea: $\vec{X}(x,y) = (x, y, \pm\sqrt{R^2 - x^2 - y^2})$ for x, y in the unit disk. ← Yuck!
- Better idea: $\vec{X}(r,\theta) = (r\cdot\cos\theta, r\cdot\sin\theta, \pm\sqrt{R^2 - r^2})$, $0 \le r \le R$, $0 \le \theta < 2\pi$. ← Better, but...
- Best idea: $\vec{X}(\phi,\theta) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. ☺

We will frequently want the <u>normal vector</u> to a surface, so that we can use a dot product to measure how much a vector field points through it:





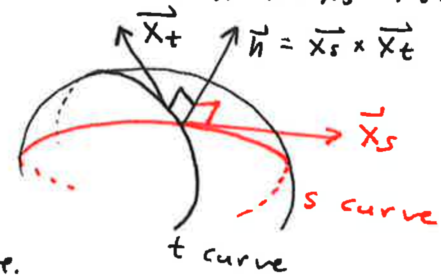
<u>Recall:</u> To find a vector that is perpendicular to a plane, find two vectors $\vec{v}_1, \vec{v}_2$ in the plane and take their cross product: $\vec{n} = \vec{v}_1 \times \vec{v}_2$.

We need to find two vectors that are <u>tangent</u> to the surface and take their cross product.

- We'll get one from s $(\vec{X}_s)$ and one from t $(\vec{X}_t)$ and get a normal vector $\vec{n} = \vec{X}_s \times \vec{X}_t$.

→ Hold t fixed and vary s: get an "s curve." Differentiate $\vec{X}$ with respect to s and get a vector tangent to the s curve, $\vec{X}_s$.

→ Hold s fixed and vary t: get a "t curve." Differentiate $\vec{X}$ with respect to t and get a vector tangent to the t curve, $\vec{X}_t$.



<u>Definition:</u> A surface is <u>smooth</u> at a given point if $\vec{X}_s \times \vec{X}_t \neq \vec{0}$ there.

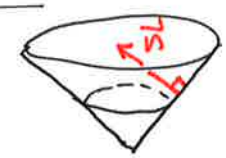
<u>Example:</u> Find a normal vector to the sphere $(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)$.

Expect: vector in radial direction, a multiple of $\vec{X}(\phi,\theta)$.

$$\vec{X}_\phi = [R\cos\phi\cos\theta, R\cos\phi\sin\theta, -R\sin\phi]$$
$$\vec{X}_\theta = [-R\cos\phi\sin\theta, R\cos\phi\cos\theta, 0]$$

$$\Rightarrow \vec{X}_\phi \times \vec{X}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\theta \\ -R\sin\phi\sin\theta & R\sin\phi\cos\theta & 0 \end{vmatrix}$$
$$= [R^2\sin^2\phi\cos\theta, R^2\sin^2\phi\sin\theta, R^2\sin\phi\cos\theta]$$
$$= (R\sin\phi)[R\sin\phi\cos\phi, R\sin\phi\sin\theta, R\cos\theta]$$
$$= (R\sin\phi)\,\vec{X}(\phi,\theta) \leftarrow \text{a multiple of } \vec{X}(\phi,\theta) \text{ as expected.}$$

<u>Example:</u> Parameterize the cone $z=r$ and find the normal vector at $(0,1,1)$.



Let's use polar: $\vec{X}(r,\theta) = [r\cos\theta, r\sin\theta, r]$.

Which r,θ get us to $[\ 0\ ,\ 1\ ,\ 1\ ]$? $r =$ ____, $\theta =$ ____.

- The r-curve through $(0,1,1)$ is $\vec{X}(r,\frac{\pi}{2}) = [r\cos\frac{\pi}{2}, r\sin\frac{\pi}{2}, r] = [0,r,r]$.

  $\Rightarrow \vec{X}_r = [0,1,1]$ (everywhere)

- The θ-curve through $(0,1,1)$ is $\vec{X}(1,\theta) = [\cos\theta, \sin\theta, 1]$

  $\Rightarrow \vec{X}_\theta = [-\sin\theta, \cos\theta, 1]$, so X_θ at $\theta = \frac{\pi}{2}$ is $[-1,0,0]$.

- Now take the cross product: $\vec{X}_r \times \vec{X}_\theta = [0,1,1] \times [-1,0,0] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{vmatrix} = [0,-1,1]$ ← this points "inward"

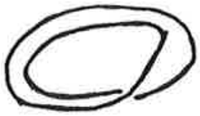
For a general point (r,θ), $\vec{X}_r = [\cos\theta, \sin\theta, 1]$, $\vec{X}_\theta = [-r\sin\theta, r\cos\theta, 0]$

$$\Rightarrow \vec{n} = \vec{X}_r \times \vec{X}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \underbrace{[-r\cos\theta, -r\sin\theta, r]}_{\text{inward, towards the z-axis.}}$$

$\vec{n} = \vec{0}$ at $r=0$, so not smooth at origin.

<u>Orientation:</u> For the cone and sphere, there is a notion of "inward" and "outward."

For a plane, there is no "inside" and "outside," but there is a well-defined notion of sides - you could paint one side red and the other side blue.



For some surfaces, such as a Möbius strip or Klein bottle, they have only one side! They are "nonorientable." We won't study them. :-)