Diana Davis — Math 34 — April 16, 2018 — Class #32 ① §6.3

Marathon Monday

Mathematician spotlight: Harrison Bray, postdoc, University of Michigan

Hamilton College undergrad, Tufts Univ. PhD (Boston!)

- studies geometric structures, geodesic flow
- studying the Thurston set in the complex plane – beautiful pictures

Today: More on Green's Theorem, and another instance where vector line integrals become easier – conservative vector fields.

Example: (danger zone) Compute $\displaystyle\int_C \frac{-y\,dx + x\,dy}{x^2+y^2}$, where C is the CCW unit circle.

Expect a positive result.

error: these vectors should get smaller as you go further out (not unit vectors as shown)

Green's Theorem says: $\displaystyle\int_{\partial D} P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA.$ Let D = unit disk, so $\partial D = C$.

Here, $P = \dfrac{-y}{x^2+y^2} \underset{\text{quotient rule}}{\Rightarrow \dots \Rightarrow} P_y = \dfrac{y^2-x^2}{(x^2+y^2)^2}$, $Q = \dfrac{x}{x^2+y^2} \underset{\text{quotient rule.}}{\Rightarrow \dots \Rightarrow} Q_x = \dfrac{y^2-x^2}{(x^2+y^2)^2}$.

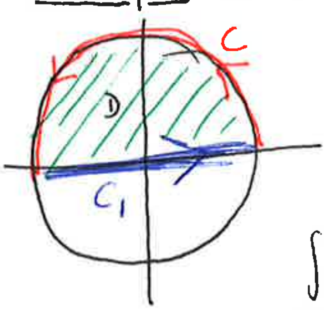
So $\displaystyle\int_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \iint_D \left(\left(\frac{y^2-x^2}{x^2+y^2}\right) - \left(\frac{y^2-x^2}{x^2+y^2}\right)\right) dA = \iint_D 0\,dA = 0.$?! what went wrong??

The problem: Green's Theorem only holds if the vector field $F = [P, Q]$ is continuous everywhere in the region enclosed by D. This is not the case – here we are dividing by 0 at the origin, and $[P, Q]$ is not continuous there. (which direction would it point??)

Note: It would be totally ok to integrate this vector field on a curve not enclosing $\vec{0}$, using Green's Theorem.

Closing off a curve to apply Green's Theorem.

Example. Let $\vec{F} = [xy^2 + x^2,\ x^2y + x - y^{\sin(e^y)}]$ and let C be the top part of the CCW unit circle:



Find $\displaystyle\int_C \vec{F}\cdot\vec{T}\,ds$. Hmm, we want to use Green's Theorem, but C is not closed?!

Idea: ① close off C with another curve C_1, to enclose a region (D).

② use Green's Thm over D, and subtract off $\displaystyle\int_{C_1} \vec{F}\cdot\vec{T}\,ds$ to get $\displaystyle\int_C \vec{F}\cdot\vec{T}\,ds$.

$$\int_{C+C_1} \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{T}\,ds + \int_{C_1} \vec{F}\cdot\vec{T}\,ds = \iint_D (Q_x - P_y)\,dA \Rightarrow \underbrace{\int_C \vec{F}\cdot\vec{T}\,ds}_{\text{want}} = \underbrace{\iint_D (Q_x-P_y)\,dA}_{\text{Green's Thm}} - \underbrace{\int_{C_1} \vec{F}\cdot\vec{T}\,ds}_{\text{compute directly}}.$$

Let's compute each part:

$Q(x,y) = x^2y + x - y^{\sin(e^y)} \Rightarrow Q_x = 2xy + 1$

$P(x,y) = xy^2 + x^2 \Rightarrow P_y = 2xy$

$\displaystyle\Rightarrow \iint_D (Q_x - P_y)\,dA = \iint_D (2xy + 1 - 2xy)\,dA = \iint_D 1\,dA = \text{area of } D = \frac{\pi}{2}.$

On C_1, $y = 0 \Rightarrow \vec{F} = [x^2, x]$ and $\vec{T} = [1, 0]$ and $ds = dx$, so

$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{x=-1}^{x=1} [x^2, x]\cdot[1,0]\,dx = \int_{x=-1}^{x=1} x^2\,dx = \frac{2}{3}.$$

So $\displaystyle\int_C = \iint_D - \int_{C_1} = \frac{\pi}{2} - \frac{2}{3}.$

Some strategies for computing vector line integrals:

(0) Just do it — parameterize the curve, and compute the integral: $\int_{t=a}^{t=b} \vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)\,dt$.

(1) If the curve is horizontal or vertical, simplify $\vec{F}$ and $\vec{T}$ and compute it directly.

(2) If the curve is closed, or if you can close it off, check that $\vec{F}$, C satisfy Green's Thm and use it.

NEW (3) If $\vec{F}$ is a conservative vector field, apply the Fundamental Theorem of Line Integrals.

Fundamental Theorem of Line Integrals: If $\vec{F} = \nabla f$ for some function $f(x,y)$ or $f(x,y,z)$ that is defined everywhere on C and the region it encloses, then

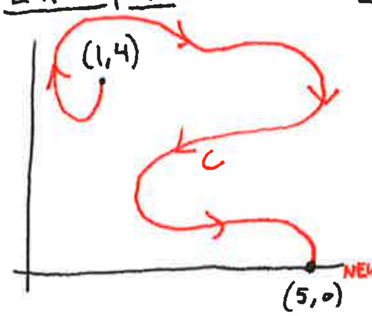
$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \nabla f \cdot \vec{T}\,ds = f(\text{end point of } C) - f(\text{start point of } C).$$

$f(x,y)$ is called the potential function.

rewrite

just like Fundamental Theorem of Calculus, $\int_a^b f(x)\,dx = F(b) - F(a)$, if $F'(x) = f(x)$.

Example. Let $\vec{F} = [e^y + y^2 + 1,\ xe^y + 2xy + \cos y]$ be the wind vector field in Swarthmore.



The classroom is located at (1,4) and the dining hall at (5,0). Let C be the path shown, taken to avoid puddles. Compute $\int_C \vec{F}\cdot\vec{T}\,ds$, which is the amount the wind helped/hurt us get to lunch.

(0) Can't use any previous method, since we don't know enough about C.

NEW (3) See if we can find f so that $\vec{F} = \nabla f$, i.e. hope that $\vec{F}$ is conservative.

Ok, we want $f_x = e^y + y^2 + 1$ and $f_y = xe^y + 2xy + \cos y$

$\Rightarrow f(x,y) = xe^y + xy^2 + x + g(y)$ ← any function of y would have disappeared

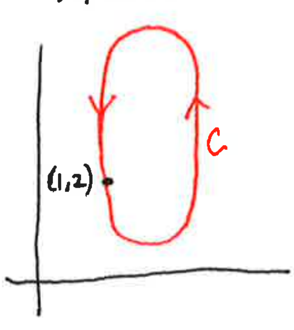
$\Rightarrow f(x,y) = xe^y + xy^2 + \sin(y) + h(x)$ ← any function of x would have disappeared

Combining these, the function that works for both is
$f(x,y) = xe^y + xy^2 + x + \sin(y)$. → check: $f_x = e^y + y^2 + 1$ ✓ $f_y = xe^y + 2xy + \cos y$ ✓

Ok, so $\vec{F}$ is conservative, and its potential function is this $f(x,y)$ we just found.

So by the Fund. Thm. of Line Integrals,
$$\int_C \vec{F}\cdot\vec{T}\,ds = f(5,0) - f(1,4) = \left(5\cdot e^0 + 5\cdot 0^2 + 5 + \sin(0)\right) - \left(1\cdot e^4 + 1\cdot 4^2 + 1 + \sin(4)\right)$$
$$= -7 - e^4 - \sin(4).$$
A tough walk to lunch!

Example. Suppose that, under the same weather conditions as above, you decide to run/swim 10,000m around the track. Thus C consists of 25 ccw laps of the track, as shown.



Compute $\int_C \vec{F}\cdot\vec{T}\,ds$.

We already showed that $\vec{F}$ is a conservative vector field, with $\vec{F} = \nabla f$ for the potential function $f(x,y) = xe^y + xy^2 + x + \sin(y)$ above.

So $\int_C \vec{F}\cdot\vec{T}\,ds = f(\text{end point}) - f(\text{start point})$.

Since the race starts and ends in the same place, $f(\text{end point}) = f(\text{start point})$,

So $\int_C \vec{F}\cdot\vec{T}\,ds = 0$.