Diana Davis Math 34 April 13, 2018 Class #31 ① §6.2

**Mathematician spotlight:** Pamela Harris, Assistant Professor, Williams College

- studies algebraic combinatorics, representation theory
- example: visibility problems
  - ☺ you
  - • trees you can see
  - • trees you can't see (blocked)

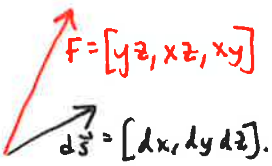
In the whole orchard, what proportion of trees can you see? $\frac{6}{\pi^2}$ ☺

**Today:** integrating vector fields over (closed) curves $\xleftrightarrow[\text{Green's Theorem, as}]{\text{is the same, by}}$ integrating scalar functions over the area region inside

First, two warm-up vector line integrals.

**Example.** Compute $\int_C \underbrace{yz\,dx + xz\,dy + xy\,dz}$ where $C$ is $\vec{x}(t) = [t, t^2, t^3]$ for $0 \le t \le 2$.

What is this?! Rewrite: $yz\,dx + xz\,dy + xy\,dz = [yz, xz, xy] \cdot [dx, dy, dz]$
$= \vec{F} \cdot d\vec{s}$. Oh. Familiar!



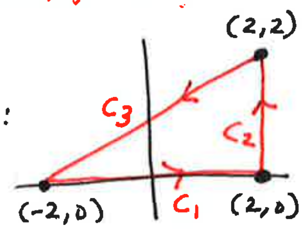

Here we have $x = t \quad y = t^2 \quad z = t^3$
$\Rightarrow dx = dt \quad dy = 2t\,dt \quad dz = 3t^2\,dt$.

**So compute:** $$\int_C yz\,dx + xz\,dy + xy\,dz = \int_{t=0}^{t=2} \underbrace{(t^2)(t^3)}_{t^5}dt + \underbrace{(t)(t^3)2t}_{2t^5}\,dt + \underbrace{(t)(t^2)3t^2}_{3t^5}\,dt = \int_{t=0}^{t=2} 6t^5\,dt = t^6\Big|_0^2 = \underline{\underline{64}}.$$

This notation is actually very nice, because it tells us exactly what we need to multiply and add up, to get the line integral over our vector field for our curve.

---

**Example.** Compute the line integral of $\vec{F} = [2x^2 - 3y^2, 2x + 3y^2]$ over the curve $C$:




Let's do each of these separately, $C_1$ and $C_2$ cleverly and $C_3$ with the equation:

- On $C_1$, $y = 0$, so $\vec{F} = [2x^2, 2x]$. Also, $C$ is in the positive $x$-direction, so $\vec{T} = [1, 0]$ and $ds = dx$. So

$$\int_{C_1} \vec{F} \cdot \vec{T}\,ds = \int_{x=-2}^{x=2} [2x^2, 2x] \cdot [1, 0]\,dx = \int_{-2}^{2} 2x^2\,dx = \frac{2}{3}x^3\Big|_{x=-2}^{x=2} = \underline{\underline{\frac{32}{3}}}.$$

- On $C_2$, $x = 2$, so $\vec{F} = [8 - 3y^2, 4 + 3y^2]$. Also, $C$ is in the positive $y$-direction, so $\vec{T} = [0, 1]$ and $ds = dy$.

$$\int_{C_2} \vec{F} \cdot \vec{T}\,ds = \int_{y=0}^{y=2} [8 - 3y^2, 4 + 3y^2] \cdot [0, 1]\,dy = \int_{y=0}^{y=2} (4 + 3y^2)\,dy = 4y + y^3\Big|_{y=0}^{y=2} = \underline{\underline{16}}.$$

- We need to parameterize $C_3$. We start at $(2,2)$ and go in direction $[-4, -2]$, so we have

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + t\begin{bmatrix} -4 \\ -2 \end{bmatrix} = \begin{pmatrix} 2 - 4t \\ 2 - 2t \end{pmatrix} \text{ for } 0 \le t \le 1.$$

Check: $t = 0 \Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $t = 1 \Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$ ü

$\Rightarrow \vec{x}'(t) = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$ and $\vec{F}(x,y) = [2x^2 - 3y^2, 2x + 3y^2] \Rightarrow \vec{F}(2-4t, 2-2t) = [2(2-4t)^2 - 3(2-2t)^2, 2(2-4t) + 3(2-2t)^2]$.

$$\int_{C_3} \vec{F} \cdot \vec{T}\,ds = \int_{t=a}^{t=b} \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\,dt = \int_{t=0}^{t=1} [32t^2 - 32t + 8 - 12t^2 + 24t - 12, 4 - 8t + 12t^2 - 24t + 12] \cdot [-4, -2]\,dt$$

$$= \int_{t=0}^{t=1} (-104t^2 + 96t - 16)\,dt = \underline{\underline{\frac{-8}{3}}}.$$

$$\Rightarrow \int_C \vec{F} \cdot \vec{T}\,ds = \frac{32}{3} + 16 - \frac{8}{3} = \underline{\underline{24}}.$$

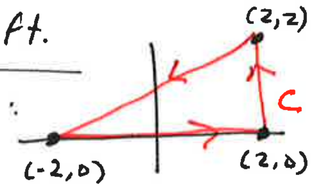
Wow, that was so tedious. There must be a better way. Yes!

**Green's Theorem.** For a vector field $\vec{F} = [P, Q]$, where $P$ and $Q$ have continuous partial derivatives throughout a region $D$ in the $xy$-plane whose boundary $\partial D$ consists of simple, closed curves, we have

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA,$$

no self intersections. has to enclose a region!

when $\partial D$ is oriented so that, moving along it, you always have $D$ on the left.




First, let's try it! For our previous example: $\vec{F} = [\underbrace{2x^2-3y^2}_{P}, \underbrace{2x+3y^2}_{Q}]$ on $C$:

$Q_x = 2$ and $P_y = -6y$, so

$$\iint_D (Q_x - P_y)\,dA = \iint_D (2+6y)\,dA = \int_{y=0}^{y=2}\int_{x=2y-2}^{x=2} (2+6y)\,dx\,dy = \int_{y=0}^{y=2} (2+6y)(2-(2y-2))\,dy = \int_0^2 (-12y^2+20y+8)\,dy = \underline{\underline{24}}.$$

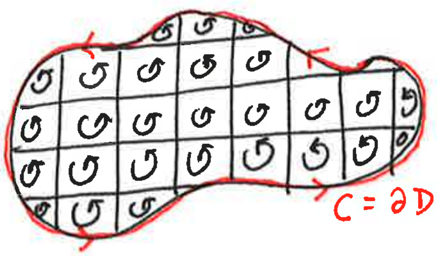
$= -3$

So much easier!

OK, now why does Green's Theorem work?

- If $\vec{F} = [P, Q]$, we can write $\vec{F} = [P, Q, 0]$ and compute $\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix} = \underbrace{[0, 0, Q_x - P_y]}_{\text{a vector in the } z\text{-direction}}$.
- So Green's Theorem days $\int_C \vec{F}\cdot\vec{T}\,ds \xrightarrow{\text{rewrite}} \int_{\partial D} P\,dx + Q\,dy \xrightarrow{\text{Green's Theorem}} \iint_D (Q_x - P_y)\,dA \xrightarrow{\text{rewrite}} \iint_D \text{“}\operatorname{curl}(\vec{F})\text{”}\,dA$

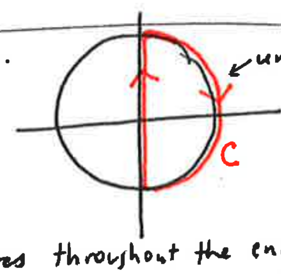
just the z-component




- $\iint_D \operatorname{curl}(\vec{F})\,dA$ adds up the circulation at each point in $D$. It cancels out along each interior boundary, so all you get is the circulation around the outside curve $C = \partial D$!

Let's do one more example, and use the theorem.

**Example.** Compute $\int_C -y\,dx + x\,dy$ where $C$ is:




**Estimate:** $[-y, x]$ is one of our favorite vector fields, CCW circulation, so we expect our answer to be ____ 0.

Check the conditions of Green's Theorem:

- $P = -y$ and $Q = x$ have continuous partial derivatives throughout the enclosed region: ✓
- The boundary consists of simple, closed curves: ✓
- The boundary curve is oriented so that, moving along it, the enclosed region is on the left: X
  → so we need to change the sign of the double integral. No problem!

$$\int_C -y\,dx + x\,dy = -\int_{-C} -y\,dx + x\,dy = -\iint_D (Q_x - P_y)\,dA = -\iint_D (1-(-1))\,dA = -\iint_D 2\,dA = -2\,(\text{area of } D) = -2\left(\tfrac{\pi}{2}\right) = \underline{\underline{-\pi}}.$$

__Note:__ this one would not have been too hard to do directly as a vector line integral, setting it up over the curved part and over the line part, but Green's Theorem allows us to do it all in one step.