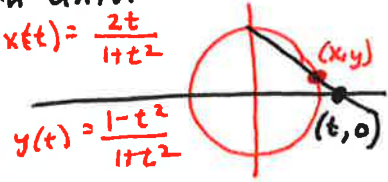
Diana Davis   Math 34   April 11, 2018   Class #30 ①
§6.1

Mathematician spotlight: Yajnaseni Dutta, graduate student, Northwestern Univ.
- studies birational geometry & Hodge theory
- birational geometry: studying which curves can be mapped to each other via "rational functions". $x(t) = \frac{2t}{1+t^2}$, $y(t) = \frac{1-t^2}{1+t^2}$



Recall: plan for rest of semester: integrate a scalar / vector function along a curve / surface — last time (scalar, curve); this time, and also the next two (Green's Thm) (vector, curve).

First, review: scalar line integrals

Example. Compute $\int_C (z^2 - 12y + z)\, ds$, where C is defined by $\vec{x}(t) = [t^3, 3t^2, 6t]$, $-1 \le t \le 2$.

① have parametric equations for C! ② we'll need $\|\vec{x}\,'(t)\|$: $\vec{x}\,'(t) = [3t^2, 6t, 6]$

$$\Rightarrow \|\vec{x}\,'(t)\| = \sqrt{9t^4 + 36t^2 + 36} = \sqrt{(3t^2+6)^2} = 3t^2 + 6.$$

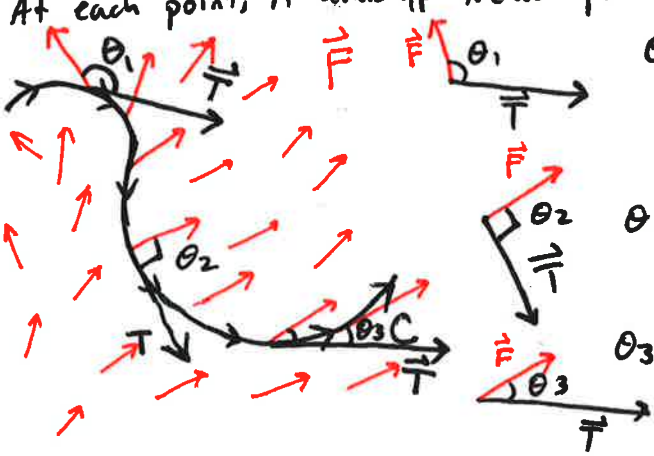
③ we'll need $f(\vec{x}(t))$: $f(x,y,z) = z^2 - 12y + z$ and $[x(t), y(t), z(t)] = [t^3, 3t^2, 6t]$, so

$$f(\vec{x}(t)) = (6t)^2 - 12(3t^2) + (6t) = 36t^2 - 36t^2 + 6t = 6t.$$

④ Now set up the integral:

$$\int_C f\, ds = \int_{t=-1}^{t=2} f(\vec{x}(t))\, \|\vec{x}\,'(t)\|\, dt = \int_{t=-1}^{t=2} \underbrace{6t}_{f(\vec{x}(t))}\, \overbrace{(3t^2+6)}^{\|\vec{x}\,'(t)\|}\, dt = \frac{1}{2}(3t^2+6)^2 \Big|_{t=-1}^{t=2} = \frac{1}{2}(18^2 - 9^2) = \frac{243}{2}.$$

Vector line integrals! Given a vector field $\vec{F}$ and a curve C with its direction specified, the vector line integral of $\vec{F}$ measures how much $\vec{F}$ "points in the same direction" as C. At each point, it adds up the dot product of $\vec{F}$ with the unit tangent vector to C.



$\theta_1 > \frac{\pi}{2}$ so $\vec{F} \cdot \vec{T} < 0$: $\vec{F}$ and C go different directions. "$\vec{F}$ goes against the flow"

$\theta_2 = \frac{\pi}{2}$ so $\vec{F} \cdot \vec{T} = 0$: $\vec{F}$ and C are perpendicular.

$\theta_3 < \frac{\pi}{2}$, so $\vec{F} \cdot \vec{T} > 0$: $\vec{F}$ and C go in the same direction. "F goes with the flow"

Symbolically, we compute $\int_C \vec{F} \cdot d\vec{s} = \int_C \vec{F} \cdot \vec{T}\, ds$, where $\vec{T}$ is the <u>unit</u> tangent vector at each point. We use the unit tangent vector because the "speed" along C should not affect how much C moves with/against $\vec{F}$.

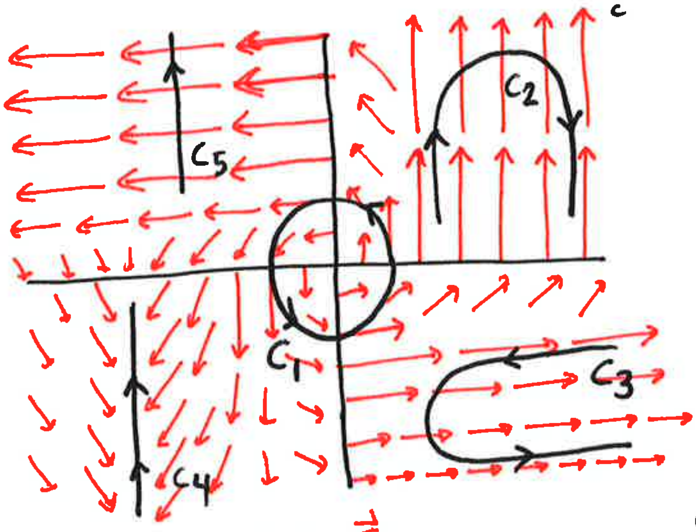
What if you reverse the direction of the curve?

Let $-C$ be the curve C, traversed in the opposite direction as C.

$$\int_{-C} \vec{F} \cdot d\vec{s} = -\int_C \vec{F} \cdot d\vec{s}$$

because it changes the sign of all the computations — "with the flow" becomes "against the flow," and vice-versa.

<u>Example.</u> Determine whether $\int_C \vec{F}\cdot d\vec{s}$ is positive, negative or zero for each curve.



vector field $\vec{F}$

C_1: The curve goes with the flow, so $\int_{C_1} \vec{F}\cdot d\vec{s}$ ____ 0.

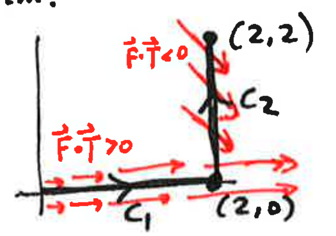
C_2: The curve goes with the flow at the beginning, and against the flow at the end, and they are equal and opposite, so $\int_{C_2} \vec{F}\cdot d\vec{s}$ ____ 0.

C_3: Curve goes against the flow at the beginning and with it at the end, but the flow against is stronger, so $\int_{C_3} \vec{F}\cdot d\vec{s}$ ____ 0.

C_4: Curve is always against the flow, so $\int_{C_4} \vec{F}\cdot d\vec{s}$ ____ 0.

C_5: Curve is always perpendicular to the flow, so $\int_{C_5} \vec{F}\cdot d\vec{s}$ ____ 0.

Now that we know what vector line integrals <u>mean</u>, let's <u>compute</u> them!



<u>Method 1:</u> compute by considering function and curve together.

<u>Example:</u> Compute the line integral of $\vec{F}=[x^2+y, -(x+1)y]$ over $C=C_1C_2$:

Along C_1, $y=0$, so $\vec{F}=[x^2,0]$, and $\vec{T}=[1,0]$, so $\vec{F}\cdot\vec{T}=[x^2,0]\cdot[1,0]=\underline{\underline{x^2}}$.

Along C_2, $x=2$, so $\vec{F}=[4+y,-3y]$, and $\vec{T}=[0,1]$, so $\vec{F}\cdot\vec{T}=[4+y,-3y]\cdot[0,1]=\underline{\underline{-3y}}$.

$$\text{So } \int_C \vec{F}\cdot d\vec{s} = \int_C \vec{F}\cdot\vec{T}\,d\vec{s} = \int_{C_1}\vec{F}\cdot\vec{T}\,ds + \int_{C_2}\vec{F}\cdot\vec{T}\,ds = \int_{x=0}^{x=2} x^2\,dx + \int_{y=0}^{y=2} -3y\,dy = \underbrace{+\frac{8}{3}}_{\vec{F}\text{ is with } C_1} + \overbrace{-6}^{\vec{F}\text{ is against } C_2} = \underbrace{-\frac{10}{3}}_{\text{net flow is against } C=C_1C_2.}$$

<u>Method 2:</u> Equation that always works. If C is defined by $\vec{x}(t)$, the tangent vector is $\vec{x}'(t)$, so the unit tangent vector is $\vec{T}=\dfrac{\vec{x}'(t)}{\|\vec{x}'(t)\|}$.

$$\text{So } \int_C \vec{F}\cdot\vec{T}\,ds = \int_{t=a}^{t=b} \underbrace{\vec{F}(\vec{x}(t))}_{\vec{F}} \underbrace{\bullet}_{\bullet} \underbrace{\frac{\vec{x}'(t)}{\cancel{\|\vec{x}'(t)\|}}}_{\vec{T}} \underbrace{\cancel{\|\vec{x}'(t)\|}\,dt}_{ds} = \int_{t=a}^{t=b} \vec{F}(\vec{x}(t))\cdot\vec{x}'(t)\,dt.$$

← if you do it this way, you don't have to worry about making a <u>unit</u> tangent vector!

<u>Example:</u> Compute the line integral of $\vec{F}=\left[\dfrac{-y\cdot\sin x}{x^2}, \dfrac{\cos x}{2x}\right]$ over the piece of $y=x^2$ from $\left(\frac{\pi}{2},\frac{\pi^2}{4}\right)$ to $\left(\frac{5\pi}{4},\frac{25\pi^2}{16}\right)$.

Let $\vec{x}(t)=[t,t^2]$, $\frac{\pi}{2}\le t\le\frac{5\pi}{4}$

$\Rightarrow \vec{x}'(t)=[1,2t]$.

Also, $\vec{F}(\vec{x}(t))=\vec{F}(t,t^2)=\left[\dfrac{-\cancel{t^2}\sin t}{\cancel{t^2}}, \dfrac{\cos t}{2t}\right]=\left[-\sin t, \dfrac{\cos t}{2t}\right]$.

$$\text{So } \int_C \vec{F}\cdot\vec{T}\,ds = \int_{t=\pi/2}^{t=5\pi/4} \vec{F}(\vec{x}(t))\cdot\vec{x}'(t)\,dt = \int_{t=\pi/2}^{t=5\pi/4}\left[-\sin t, \frac{\cos t}{2t}\right]\cdot[1,2t]\,dt = \int_{t=\pi/2}^{t=5\pi/4}(-\sin t+\cos t)\,dt = \cos t+\sin t\Big|_{t=\pi/2}^{t=5\pi/4}$$

$$= \underline{\underline{-\sqrt{2}-1}}.$$