Mathematician spotlight: Steve Robinson, Professor, Wake Forest University

- Studies partial differential equations (PDEs) - similar to Profs. Mavinga & Morris
- applies the fact that the game Hex always has a winner, to prove mathematical theorems! → Colloquium Tuesday 4:30pm SCI 199

For the last few weeks of the course, we will study how to integrate a {function (today) / vector field} over a {curve / surface} ← 2×2 = 4 things.

From last time: divergence = div($\vec{F}$) = "amount of stuff created" a scalar function.

curl = curl($\vec{F}$) = "direction of the axis of rotation" a vector function.

Suppose that $\vec{F}$ and $\vec{G}$ are vector fields, and f is a function. Which of the following make sense?

1. div(curl($\vec{F}$))
2. curl(div($\vec{F}$))
3. curl(div(f))
4. div($\vec{F} \bullet \vec{G}$)
5. curl(∇f)
6. div(∇f)

Let's compute: For $f(x,y,z)$, and for $F = [P, Q, R]$, (assuming f is reasonably nice)

$$\text{curl}(\nabla f) = \text{curl}([f_x, f_y, f_z]) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix} = [f_{zy} - f_{yz}, -(f_{zx} - f_{xz}), f_{yx} - f_{xy}] = [0,0,0]$$

by Clairaut's Theorem!

(assuming P, Q, R are reasonably nice)

$$\text{div}(\text{curl}(\vec{F})) = \text{div}(R_y - Q_z, P_z - R_x, Q_x - P_y) = R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0$$

by Clairaut's Theorem!

Theorem. For any function f, curl(∇f) = $\vec{0}$. ← vector

For any vector field $\vec{F}$, div(curl($\vec{F}$)) = 0. ← number

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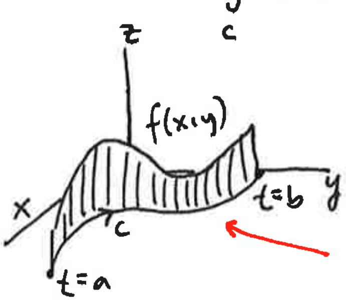
Scalar line integrals! Integrating a function ("charge", "snow") over a curve ("wire", "path").

Given a function $f(x,y)$ and a curve C in the xy-plane with parametric equations $\vec{r}(t)$ for $a \le t \le b$, the scalar line integral of f over C is

$$\int_C f(x,y)\, ds = \int_{t=a}^{t=b} f(\vec{r}(t))\, \|\vec{r}\,'(t)\|\, dt$$

- $f(\vec{r}(t))$: the height of the function at the point where you are at time t.
- $\|\vec{r}\,'(t)\|\, dt$: ↳ ds, an infinitesimal piece of arclength = the distance you travel in "dt" seconds

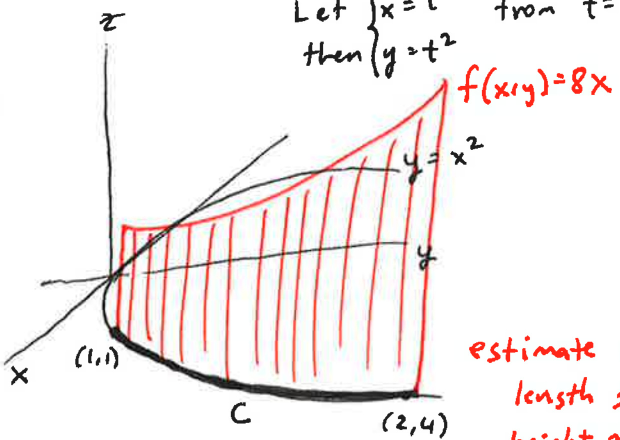


← the integral gives the "total area of the fence." (charge / snow; wire / path)

Example. Compute the line integral of $f(x)=8x$, along the curve C consisting of the part of the parabola $y=x^2$ from $x=1$ to $x=2$.

First step: find parametric equations for C. *tip: make one variable "t", then solve for the other variables in terms of t.*

Let $\begin{cases} x=t \\ y=t^2 \end{cases}$ from $t=1$ to $t=2$.
then



estimate answer:
length ≈ 3
height ≈ 12
⇒ expect answer ≈ 36

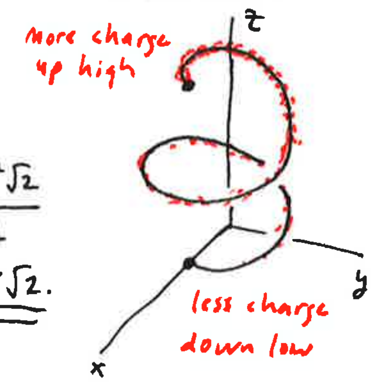
So $\int_C f(x,y)\,ds = \int_{t=1}^{t=2} f(\vec{r}(t))\|\vec{r}\,'(t)\|\,dt$

$= \int_{t=1}^{t=2} f(t,t^2)\|[1,2t]\|\,dt = \int_{t=1}^{t=2} 8t\sqrt{1+4t^2}\,dt$

$= \frac{2}{3}(1+4t^2)^{3/2}\Big|_{t=1}^{t=2} = \underline{\underline{\frac{2}{3}\left(17^{3/2}-5^{3/2}\right)}}$. ≈ 39 ☺

Example. A coil of wire (spring) is shaped like two turns of the helix $\vec{x}(t)=(\cos t, \sin t, t)$, so from $t=0$ to $t=4\pi$. The amount of charge at any point is given by $f(x,y,z)=z$, due to a nearby charged object. Compute the total charge on the wire.

Let's compute $\vec{x}\,'(t)=(-\sin t, \cos t, 1)$
$\Rightarrow \|\vec{x}\,'(t)\| = \sqrt{(-\sin t)^2+(\cos t)^2+1^2} = \sqrt{1+1} = \sqrt{2}$.



So $\int_C f(x,y,z)\,ds = \int_{t=0}^{t=4\pi} f(\cos t, \sin t, t)\|\vec{x}\,'(t)\|\,dt = \int_{t=0}^{t=4\pi} t\sqrt{2}\,dt = \frac{t^2\sqrt{2}}{2}\Big|_{t=0}^{t=4\pi} = \frac{(4\pi)^2\sqrt{2}}{2}$

$= \underline{\underline{8\pi^2\sqrt{2}}}$.

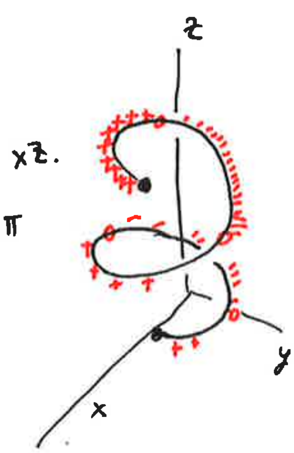
Example. If we integrate $\int_C 1\,ds$, what does it mean? The <u>length</u> of C.

For the curved wire C from above, its length is

$\int_{t=0}^{t=4\pi} 1\,\|\vec{x}\,'(t)\|\,dt = \int_{t=0}^{4\pi} \sqrt{2}\,dt = 4\pi\sqrt{2}$.

(same as our arclength formula) (4π: circumference of 2 unit circles; $\sqrt{2}$: from being stretched "up" in the z-direction)

Example. Find the total charge on the wire if charge is given by $f(x,y,z)=xz$.



$\int_C f(x,y,z)\,ds = \int_{t=0}^{t=4\pi} t\cdot\cos t\cdot\sqrt{2}\,dt = \sqrt{2}\int_{t=0}^{t=4\pi} t\cdot\cos t\cdot dt = \sqrt{2}\left(t\sin t+\cos t\right)\Big|_{t=0}^{t=4\pi}$

integration by parts!

$= \sqrt{2}\left(4\pi\sin 4\pi+\cos 4\pi-(0\sin 0+\cos 0)\right) = \sqrt{2}(0) = \underline{\underline{0}}$.