Diana Davis Math 34 April 4, 2018 Class #28 ① §3.4

Mathematician spotlight: Dylan Thurston, Professor, University of Indiana (oops)

- studies topology, homology, geometry, ...
- is the (biological) child of my (mathematical) grandfather, William (Bill) Thurston, one of the greatest mathematicians of the $20^{th}$ century.

Today: "divergence" & "curl" of a vector field will help us describe the behavior of flows.

Vector fields! Example: Sketch the vector field $\vec{F}(x,y) = [y, x]$.

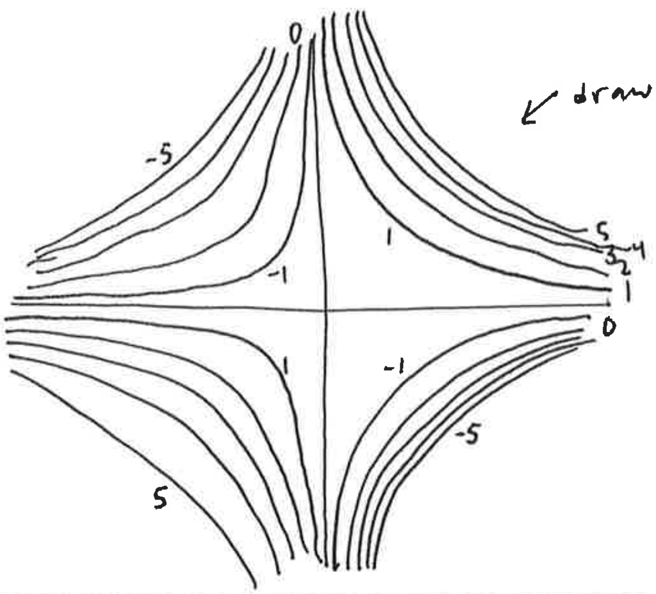
- Option 1: plot the vectors at some chosen points to get an idea of the picture.
- Option 2: view $\vec{F}$ as the gradient of some other function $f(x,y)$, ← not every $\vec{F}$ is the gradient of an f. this one is.
  plot level curves of $f(x,y)$, and draw in gradient vectors. ← direction of steepest ascent

→ New quest: find $f(x,y)$ so that $\nabla f = \vec{F}$, i.e. $[f_x, f_y] = [y, x]$.

Ok, so: $f_x = y \Rightarrow f(x,y) = x \cdot y + g_1(y)$ ← some function of y

$f_y = x \Rightarrow f(x,y) = y \cdot x + g_2(x)$ ← some function of x

$\Rightarrow \vec{F} = \nabla f(x,y)$, where $f(x,y) = x \cdot y$.




↙ draw in gradient vectors showing the direction of greatest ascent, at many example points of your choice.

Remember: the magnitude of the vector represents how steeply you are ascending.

Think! what would change about this picture if we chose $f(x,y) = xy + 10$, another function for which $\nabla f = \vec{F}$?

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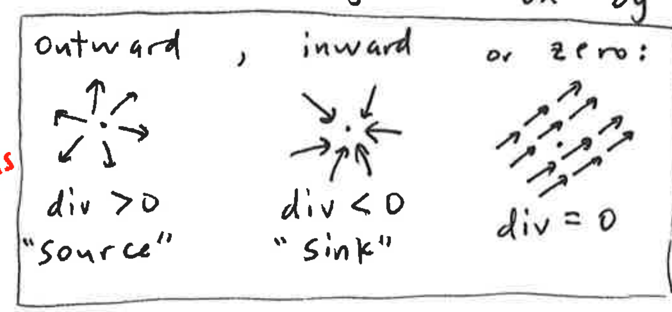
Divergence: The divergence of a vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

It measures whether the "net flow" is




new notation: $\nabla = [\partial/\partial x, \partial/\partial y, \partial/\partial z]$, ↖ treat this like a vector

so $\nabla \cdot \vec{F} = [\partial/\partial x, \partial/\partial y, \partial/\partial z] \cdot [P, Q, R]$

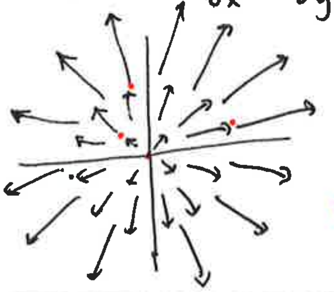
$= \frac{\partial}{\partial x} P + \frac{\partial}{\partial y} Q + \frac{\partial}{\partial z} R = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

Using this notation, $\mathrm{div}(\vec{F}) = \nabla \cdot \vec{F}$ ← a dot product of a "vector" with a vector.

Example: Compute the divergence of our two favorite vector fields from last time.

$\vec{F}(x,y) = [x, y]$ so $P(x,y) = x$, $Q(x,y) = y$

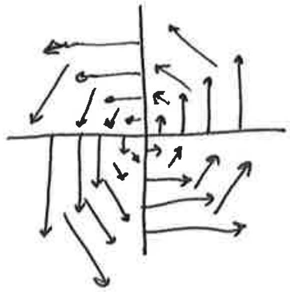
$\Rightarrow \operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 1 + 1 = 2.$



at every point, the vectors going "out" are longer than the vectors coming "in."

$\vec{G}(x,y) = [-y, x]$ so $P(x,y) = -y$, $Q(x,y) = x$

$\Rightarrow \operatorname{div} \vec{G} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 + 0 = 0.$



at every point, the vectors going "out" are the same size as the vectors going "in" — nothing is created or destroyed; the "water" just goes in a circle.

↳ we should quantify that:...

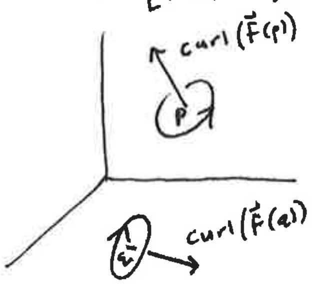
Curl. The curl of a vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is $\operatorname{curl}(\vec{F}) = (R_y - Q_z)\vec{i} - (R_x - P_z)\vec{j} + (Q_y - P_x)\vec{k}$.

how will I ever remember this?!

using our new notation $\nabla = [\partial/\partial x, \partial/\partial y, \partial/\partial z]$,

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

as the determinant of this matrix!




Geometrically: curl is a vector, measuring "circulation."

- its direction gives the axis of rotation (right hand rule)
- its length gives the speed/strength of rotation.

If $\vec{F}(x,y) = P\vec{i} + Q\vec{j}$ is a 2D vector field, curl $\vec{F}$ is in $z$-direction: $[0, 0, c]$.

$c > 0$ if the curl is CCW:

$c < 0$ " " " CW:

Let's find the curl of our favorite vector fields. For $\vec{G}$, we expect $c > 0$.

(1) $$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} + 0\vec{k} = \vec{0}.$$

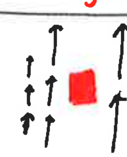
no rotation anywhere!

(2) $$\operatorname{curl}(\vec{G}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} + 2\vec{k} = [0, 0, 2].$$

CCW rotation everywhere!

How is it possible that it has CCW rotation everywhere? Imagine putting a small block of wood in a whirlpool.

zoom in



the current is stronger on the right than on the left, so it will spin CCW!

Do some algebra for a more complicated vector field:

$$\vec{F} = [x \cdot y \cdot \sin(z) + y,\; y - x \cdot e^z,\; x \cdot y \cdot z]$$

notice that div and curl are both functions of $x, y$ and $z$.
div($\vec{F}$) is a scalar function.
curl($\vec{F}$) is a vector function.

div($\vec{F}$) =

curl($\vec{F}$) =