Diana Davis Math 34 April 2, 2018 Class #27

§3.3 ①

Mathematician spotlight: Rachel Epstein, Assistant Professor, Georgia College

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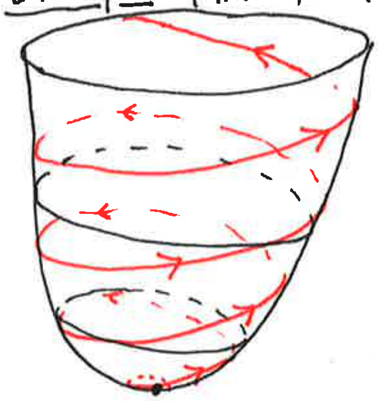
- Studies computably enumerable sets: those whose elements can be listed by some algorithm.
- amazingly, there are (way) more numbers that <u>cannot</u> be listed by an algorithm, than numbers that can be.

<u>Today</u>: A bit more on curves, and then vector fields! (like wind)

<u>Example</u>: Find the tangent line to the curve $\vec{r}(t) = [t\cdot\cos t,\ t\cdot\sin t,\ t^2]$ at $t = \pi$.



First, what does it look like?
- x and y are spiraling outward
- z is increasing
- $x^2+y^2=z$, so it lies on a paraboloid!

To find the tangent line, we need a <u>point</u> and a <u>direction vector</u>.

point: $\vec{r}(\pi) = [\pi\cdot\cos\pi,\ \pi\cdot\sin\pi,\ \pi] = [-\pi, 0, \pi^2]$.

direction vector: $\vec{r}'(t) = [\cos t + t\cdot\sin t,\ \sin t - t\cos t,\ 2t]$

so $\vec{r}'(\pi) = [\cos\pi + \pi\cdot\sin\pi,\ \sin\pi - \pi\cdot\cos\pi,\ 2\pi] = [-1, \pi, 2\pi]$

So the tangent line equation is $[x(t), y(t), z(t)] = [-\pi, 0, \pi^2] + t[-1, \pi, 2\pi] = \begin{bmatrix} -\pi - t \\ \pi t \\ \pi^2 + 2\pi t \end{bmatrix}$.

<u>Check this out</u>: For a circle, or for any curve on a sphere, $\vec{r}(t)$ is perpendicular to the tangent vector $\vec{r}'(t)$. Are circles and spheres the only case where this happens? Yes!

<u>Proposition</u>: If $\vec{r}(t)$ is a curve with the property that $\vec{r}(t)\bullet\vec{r}'(t) = 0$, then $\|\vec{r}(t)\|$ is a constant, i.e. $\vec{r}(t)$ lies on a circle, sphere, etc.

<u>Proof</u>: $\|\vec{r}(t)\|^2 = \vec{r}(t)\bullet\vec{r}(t)$ ← a vector dotted w/ itself gives its length squared

$\Rightarrow \frac{d}{dt}\|\vec{r}(t)\|^2 = \frac{d}{dt}\left(\vec{r}(t)\bullet\vec{r}(t)\right) = \vec{r}'(t)\bullet\vec{r}(t) + \vec{r}(t)\bullet\vec{r}'(t)$ ← product rule

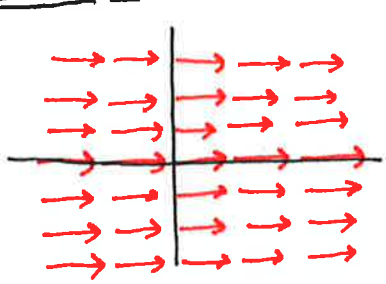
$= 2\left(\vec{r}(t)\bullet\vec{r}'(t)\right)$ ← combine like terms

$= 0$ ← we assumed $\vec{r}(t)$ is perp. to $\vec{r}'(t)$

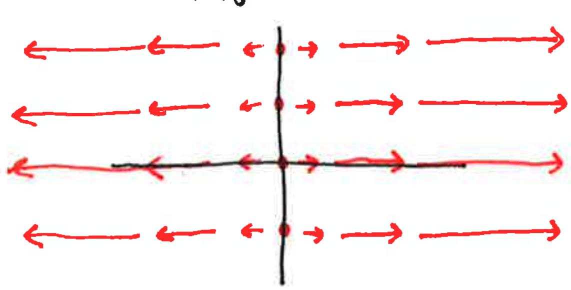
$\Rightarrow \frac{d}{dt}\|\vec{r}(t)\| = 0 \Rightarrow \|\vec{r}(t)\|$ is a constant, $\Rightarrow \vec{r}(t)$ lies on a circle/sphere!

<u>Vector fields</u>: Associates a <u>vector</u> to each <u>point</u> in a space.

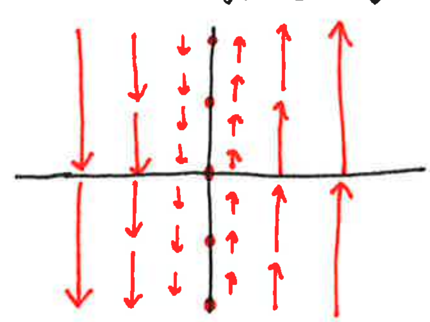
<u>Example</u>: $\vec{F}(x,y) = [1, 0]$



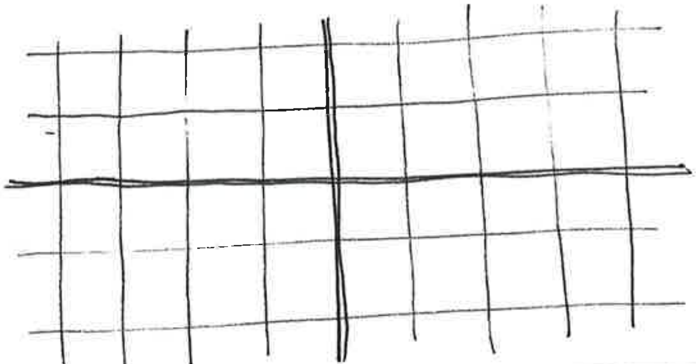
$\vec{F}(x,y) = [x, 0]$



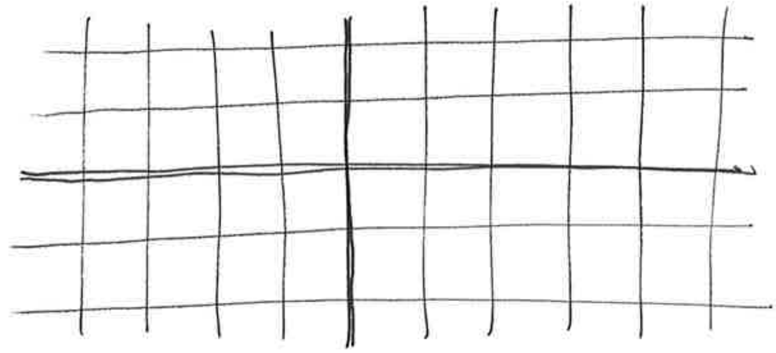
$\vec{F}(x,y) = [0, x]$



Now you try: Sketch $\vec{F}(x,y) = [x, y]$



Sketch $\vec{F}(x,y) = [-y, x]$

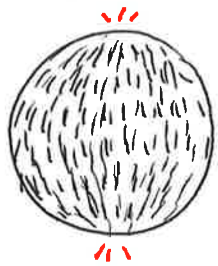


The above examples will be our favorites, which we will use repeatedly to understand new ideas.

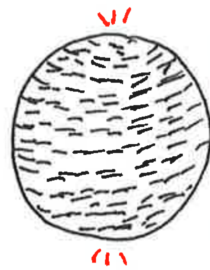
Application of vector fields: The Hairy Ball Theorem: Given any continuous tangent vector field on a sphere, there must be at least one point on the sphere where the tangent vector is $\vec{0}$.

Corollary: If you have a hairy ball (koosh ball, coconut, etc.) and you wish to comb down all the hair so that it lies flat, there will be at least one point where it goes wrong — this goal is impossible to achieve.

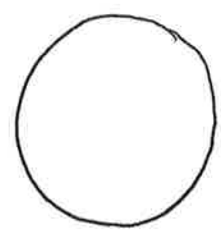
Examples:



Comb all the hair "down": problem at north & south poles ⍨



Comb all the hair "east": problem at the poles again ⍨



You try!

Corollary: There is always at least one point on Earth where the wind isn't blowing.

Application to economics: Suppose you have three products x_1, x_2, x_3.

Put their prices into a vector: $[p_1, p_2, p_3] = \vec{p}$.

Corresponding to each product there is a demand, and in particular an excess demand, with vector $[d_1, d_2, d_3] = \vec{d}$.

we want the excess demand to be $\vec{0}$.

Walras's Law says $\vec{p} \cdot \vec{d} = 0$.

So, on the sphere of possible price vectors, the excess demand vector forms a (continuous) tangent vector field!

By the Hairy Ball Theorem, there is some point on the sphere where $\vec{d} = \vec{0}$! At this point, there is no excess demand!

So that point gives the optimal prices for the products. :)