Diana Davis Math 34 March 30, 2018 Class#26 ①

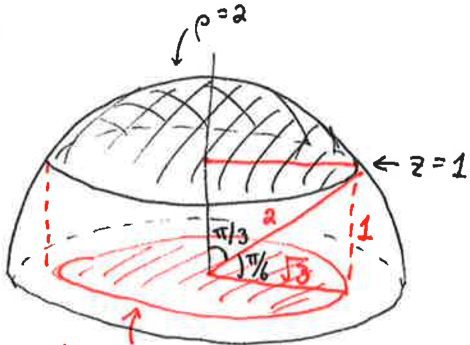
§3.1

Mathematician spotlight: Federico Ardila, Associate Professor, San Francisco State Univ.

- combinatorics of objects in algebra, geometry, topology, etc.
- uses polyhedra to understand power series e.g. $a_0 + a_1 x + a_2 x^2 + \dots$ (!)

Example. Set up an integral for the solid "spherical cap" inside $\rho = 2$ and above $z = 1$.

$x^2+y^2+z^2=2^2$
$z^2 = 4-x^2-y^2$
$z = \sqrt{4-x^2-y^2}$



shadow is disk of radius $\sqrt{3}$

Rectangular: $$\int_{x=-\sqrt{3}}^{x=\sqrt{3}} \int_{y=-\sqrt{3-x^2}}^{+\sqrt{3-x^2}} \int_{z=1}^{z=\sqrt{4-x^2-y^2}} f(x,y,z)\, dz\, dy\, dx$$

Cylindrical: $$\int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{3}} \int_{z=1}^{\sqrt{4-r^2}} f(x,y,z)\cdot r \cdot dz \cdot dr \cdot d\theta$$ $= \sqrt{4-r^2}$

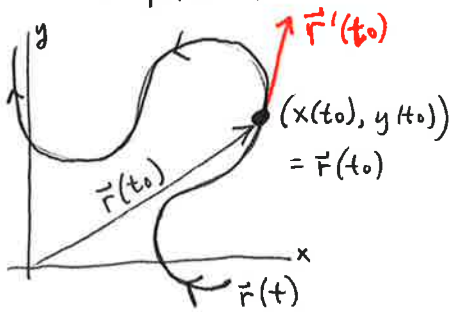
Spherical: $$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/3} \int_{\rho=1/\cos\phi}^{2} f(x,y,z)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

plane $z=1$:
$\Rightarrow \rho \cdot \cos\phi = 1$
$\Rightarrow \rho = \dfrac{1}{\cos\phi}$

Curves! Any curve can be described parametrically by $\vec{r}(t) = (x(t), y(t))$ (ant on paper)

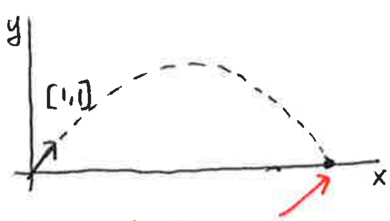
$\vec{r}(t) = (x(t), y(t), z(t))$ (fly in air)

For example, in 2D:



At any time t, the location of the fly is $\vec{r}(t) = (x(t), y(t))$;
the direction of travel is given by $\vec{r}\,'(t) = (x'(t), y'(t))$, the tangent vector to the curve at the point $\vec{r}(t)$.
The magnitude of the tangent vector, $|\vec{r}\,'(t)|$, is the speed of the fly.
So $\vec{r}\,'(t)$ is the velocity (direction and speed) vector, and $\vec{r}\,''(t)$ is the acceleration vector.

Example. Suppose you fire a pebble from a slingshot at the origin, at an angle of $45°$ with a speed of $\sqrt{2}$ meters/sec. Assume that the only force acting on the pebble is gravity, at g meters/sec². Find parametric equations for its position.



when and where does the pebble hit the ground?
when $y(t)=0$.

- Start with acceleration: $\vec{r}\,''(t) = [0, -g]$
- integrate to get velocity: $\vec{r}\,'(t) = [0 + c_1, -gt + c_2]$ ← c_1, c_2 are the "+c" integration constants
  we know $\vec{r}\,'(0) = [1,1]$: $\vec{r}\,'(0) = [c_1, c_2] = [1,1] \Rightarrow c_1 = 1, c_2 = 1$
  $\Rightarrow \vec{r}\,'(t) = [1, -gt+1]$
- integrate to get position: $\vec{r}(t) = [t + c_3, -\frac{1}{2}gt^2 + t + c_4]$ ← c_3, c_4 are the integration constants
  we know $\vec{r}(0) = [0,0]$: $\vec{r}(0) = [c_3, c_4] = [0,0] \Rightarrow c_3 = 0, c_4 = 0$
  $\Rightarrow \vec{r}(t) = [t, -\frac{1}{2}gt^2 + t]$.

$$\Rightarrow \begin{cases} x(t) = t \\ y(t) = t - \frac{1}{2}gt^2 \end{cases} = 0$$ to see when it hits the ground $\Rightarrow t(1 - \frac{1}{2}gt) = 0$

$\Rightarrow t = 0$ or $t = 2/g$

start   hits at ≈ 0.2 sec.

How far does the pebble / bug / fly travel? We need to find the arclength.

Recall: distance = rate × time, so distance traveled in time "dt" is speed × dt = $\|\vec{r}\,'(t)\|\cdot dt$.

magnitude of velocity vector

So the total distance (arclength) traveled from time $t=a$ to $t=b$ is

$$\int_{t=a}^{t=b} \underbrace{\|\vec{r}\,'(t)\|}_{\text{rate}}\, \underbrace{dt}_{\text{time}}.$$

Let's test it out on a circle to see if it works.

Example: Find the length of a circle of radius R. (we expect the answer to be $2\pi R$.)

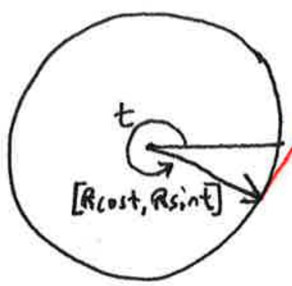
$\vec{r}(t) = [R\cos(t), R\sin(t)]$

$\Rightarrow \vec{r}\,'(t) = [-R\cdot\sin(t), R\cdot\cos(t)]$

$\Rightarrow \|\vec{r}\,'(t)\| = \sqrt{(-R\cdot\sin(t))^2 + (R\cdot\cos(t))^2} = R.$

So its length is $\displaystyle\int_{t=0}^{t=2\pi} R\cdot dt = Rt\Big|_{t=0}^{t=2\pi} = \underline{\underline{2\pi R}}.$ ☺ (it works!)

Picture:

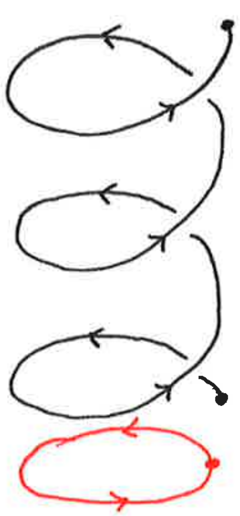


$\vec{r}\,'(t) = [-R\sin t, R\cos t]$ is the direction vector of the particle. It has a constant speed of $\|\vec{r}\,'(t)\| = R$.

Check this out: $\vec{r}\,''(t) = [-r\cos t, -r\sin t] = -\vec{r}(t)$. So the acceleration vector points toward the center of the circle.

↖acceleration

Example: A lazy housefly is languidly circling toward the ceiling light in a helix (corkscrew), with her path described by $\vec{r}(t) = [\cos 3t, \sin 3t, t]$, with distance in feet and time in minutes. Find how far she travels from time $t=0$ to time $t=2\pi$.



OK, we have $\vec{r}(t) = [\cos 3t, \sin 3t, t]$

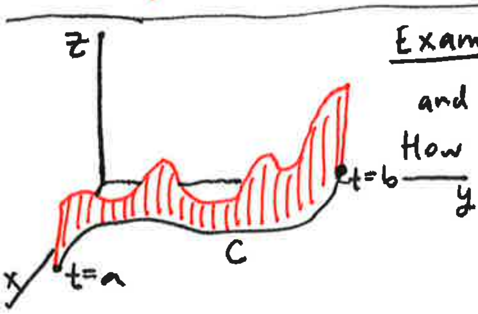
so $\vec{r}\,'(t) = [-3\sin 3t, 3\cos 3t, 1]$

$\Rightarrow \|\vec{r}\,'(t)\| = \sqrt{(-3\sin 3t)^2 + (3\cos 3t)^2 + 1^2} = \sqrt{9+1} = \sqrt{10}.$ ← constant speed of $\sqrt{10}$ ft/min

So the length is $\displaystyle\int_{t=0}^{t=2\pi} \underset{\text{ft/min}}{\|\vec{r}\,'(t)\|}\, \underset{\text{min}}{dt} = \int_{t=0}^{t=2\pi} \sqrt{10}\, dt = \sqrt{10}\,t\Big|_{t=0}^{t=2\pi} = 2\pi\sqrt{10}$. feet.

Check this out: $\vec{r}\,''(t) = [-9\cos 3t, -9\sin 3t, 0]$ ← not accelerating in the z-direction.

← shadow of path in the xy-plane is $(\cos 3t, \sin 3t)$ – a triple-speed unit circle.



Example: Suppose $f(x,y)$ = depth of gold dust at point (x,y) and you vacuum along the curve $C = \{\vec{r}(t)$ from $t=a$ to $t=b\}$. How much gold dust will you suck up?

$$\int_C \underset{\text{depth}}{f(x,y)}\, dS$$ ← distance parameter along the curve C