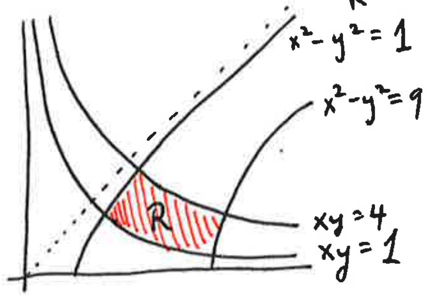
Diana Davis   Math 34   March 27, 2018   Class # 25 (1)
§5.5

Mathematician spotlight: Amie Wilkinson, Professor, University of Chicago
- dynamical systems, ergodic theory
- studies spaces of surfaces

Last time: change of variables, to polar coordinates (r,θ) and to general coordinates (u,v)

Today: integration in cylindrical & spherical coordinates.

Example: Compute $\iint_R (x^2+y^2)e^{x^2-y^2}\,dA$, where R is bounded by $\begin{matrix} x^2-y^2=1 & xy=1 \\ x^2-y^2=9 & xy=4 \end{matrix}$:



Based on the region, let's try the change of variables $\begin{cases} u=xy \\ v=x^2-y^2. \end{cases}$ Then $\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\det\begin{pmatrix} y & x \\ 2x & -2y\end{pmatrix}\right| = |-2y^2-2x^2| = 2(x^2+y^2)$, since $x,y>0$ in R.

So $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{1}{2(x^2+y^2)}$ ← we can't easily convert this to u & v, but luckily it cancels out in the integral.

our desired Jacobian expansion factor for area →

So now $\iint_R (x^2+y^2)e^{x^2-y^2}\,dA = \iint_{R'} \cancel{(x^2+y^2)}\,e^{x^2-y^2}\cdot\underbrace{\frac{1}{2\cancel{(x^2+y^2)}}}_{\text{Jacobian}}\,du\,dv = \int_{v=1}^{v=9}\int_{u=1}^{u=4}\frac{1}{2}e^v\,du\,dv = \int_{v=1}^{v=9}\frac{3}{2}e^v\,dv = \frac{3}{2}(e^9-e).$

For triple integrals, we use the 3×3 Jacobian expansion factor for volume:

Cylindrical coordinates: $\begin{cases} x=r\cdot\cos\theta \\ y=r\cdot\sin\theta \\ z=z \end{cases} \Rightarrow \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \left|\det\begin{pmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial z \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial z\end{pmatrix}\right| = \left|\det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1\end{pmatrix}\right| = |r| = r.$

So $\boxed{dV = r\cdot dz\cdot dr\cdot d\theta.}$ This makes sense because only x and y are affected when converting to cylindrical coordinates, so the expansion factor is the same as for polar coordinates.

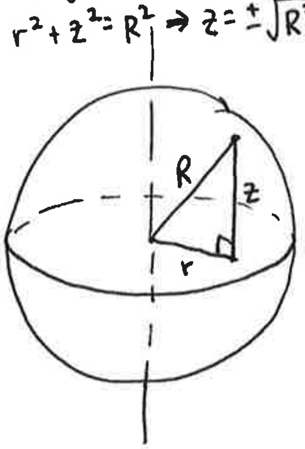
Spherical coordinates: $\begin{cases} x=\rho\cdot\sin\phi\cdot\cos\theta \\ y=\rho\cdot\sin\phi\cdot\sin\theta \\ z=\rho\cdot\cos\phi \end{cases} \Rightarrow \left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| = \left|\det\begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0\end{pmatrix}\right|$

$= |\rho^2\sin\phi| = \rho^2\sin\phi$ ← $0\le\phi\le\pi$, so $\sin\phi\ge 0$.

So $\boxed{dV = \rho^2\sin\phi\cdot d\rho\cdot d\phi\cdot d\theta.}$

Example. Let's test this out by finding the volume of the solid ball B of radius R (centered at $\vec{0}$).

In cylindrical coordinates:
$r^2+z^2=R^2 \Rightarrow z=\pm\sqrt{R^2-r^2}$



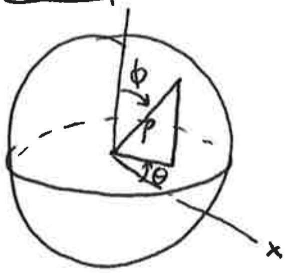
Shadow in xy-plane (or $r\theta$-plane): the disk of radius R, $r=0$ to $r=R$ and $\theta=0$ to $\theta=2\pi$.

So volume $= \iiint_B 1\,dV = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=R}\int_{z=-\sqrt{R^2-r^2}}^{z=+\sqrt{R^2-r^2}} 1\cdot\underbrace{r}_{\text{Jacobian}}\cdot dz\cdot dr\cdot d\theta$

$= \int_{\theta=0}^{2\pi}\int_{r=0}^{R} 2r\sqrt{R^2-r^2}\,dr\,d\theta = \underbrace{\int_{\theta=0}^{2\pi}d\theta}_{2\pi}\cdot\underbrace{\int_{r=0}^{R}2r\sqrt{R^2-r^2}\,dr}_{\text{compute this}} = 2\pi\left(-\frac{2}{3}(R^2-r^2)^{3/2}\Big|_{r=0}^{r=R}\right)$

$= 2\pi\left(\frac{2}{3}R^3\right) = \frac{4}{3}\pi R^3.$:)

Example. Volume of the solid ball B of radius R again, now in spherical coordinates.



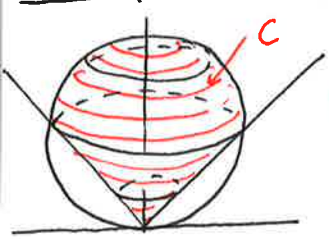
$$\iiint_B 1\,dV = \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi}\int_{\rho=0}^{\rho=R} 1\cdot \underbrace{\rho^2\sin\phi}_{\text{Jacobian}}\cdot d\rho\cdot d\phi\cdot d\theta$$

$$= \underbrace{\int_{\theta=0}^{2\pi} d\theta}_{2\pi} \cdot \int_{\phi=0}^{\phi=\pi}\sin\phi\,d\phi \cdot \int_{\rho=0}^{\rho=R}\rho^2\,d\rho = (2\pi)\left(-\cos\phi\Big|_{\phi=0}^{\phi=\pi}\right)\left(\frac{\rho^3}{3}\Big|_{\rho=0}^{\rho=R}\right)$$

$$= 2\pi \cdot 2 \cdot \frac{R^3}{3} = \frac{4}{3}\pi R^3.$$ ☺

Okay, so we have some faith that it works. Now let's integrate something more exotic.

Example. Integrate $f(x,y,z) = xz$ over the "ice cream cone" C bounded by the surfaces $z = \sqrt{x^2+y^2}$ and $x^2+y^2+(z-1)^2 = 1$, in all three coordinate systems.



$\Rightarrow z = \sqrt{r^2} = r$

$\Rightarrow r^2 + (z-1)^2 = 1$

$(z-1)^2 = 1 - r^2 \Rightarrow z = 1 + \sqrt{1-r^2}$

$\Rightarrow r^2 = 1-(z-1)^2 \Rightarrow r = \sqrt{1-(z-1)^2}$

1. Cylindrical, in order $dz\,dr\,d\theta$:

$$\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\int_{z=r}^{z=1+\sqrt{1-r^2}} \underbrace{(r\cdot\cos\theta\cdot z)}_{xz}\cdot \underbrace{r}_{\text{Jacobian}}\cdot dz\cdot dr\cdot d\theta = \underbrace{\int_{\theta=0}^{2\pi}\cos\theta\cdot d\theta}_{=0}\cdot\int_{r=0}^{r=1}\int_{z=r}^{z=1+\sqrt{1-r^2}} r^2 z\,dz\,dr = 0.$$

2. Cylindrical, in order $dr\,dz\,d\theta$:

$$\underbrace{\int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=1}\int_{r=0}^{r=z}(r\cdot\cos\theta\cdot z)\cdot r\cdot dz\cdot dr\cdot d\theta}_{\text{cone part}} + \underbrace{\int_{\theta=0}^{\theta=2\pi}\int_{z=1}^{z=2}\int_{r=0}^{r=\sqrt{1-(z-1)^2}}(r\cdot\cos\theta\cdot z)\cdot r\cdot dz\cdot dr\cdot d\theta}_{\text{ice cream (sphere) part}}$$

3. Spherical $d\rho\,d\phi\,d\theta$:

$$\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/4}\int_{\rho=0}^{\rho=2\cos\phi} \underbrace{\rho\sin\phi\cos\theta}_{x}\cdot\underbrace{\rho\cos\phi}_{z}\cdot\underbrace{\rho^2\sin\phi}_{\text{Jacobian}}\cdot d\rho\,d\phi\,d\theta.$$

$$= \underbrace{\int_{\theta=0}^{2\pi}\cos\theta\,d\theta}_{0}\cdot\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{\rho=2\cos\phi}\rho^4\cdot\sin^2\phi\cdot\cos\phi\cdot d\rho\cdot d\phi$$

$x^2+y^2+(z-1)^2=1$

$x^2+y^2+z^2-2z+1=1$

$x^2+y^2+z^2 = 2z$

$\rho^2 = 2\rho\cos\phi$

$\rho = 2\cos\phi$

solve for sphere equation in terms of ρ

Explain how you could figure out that $\iiint_C xz\,dV = 0$, without doing any calculations.