Diana Davis — Math 34 — March 26, 2018 — Class #24 ①
§5.5

Mathematician spotlight: Colin Adams, Thomas T. Read Professor of Mathematics, Williams
- studies knot theory, hyperbolic geometry
- giving Kitao lecture on Tuesday at 4:30 pm

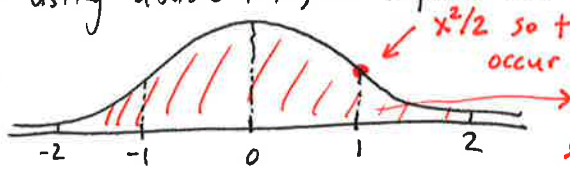
Last time: converting double integrals into polar coordinates

Recall: $dA = dx \cdot dy = r \cdot dr \cdot d\theta$

Today: changing coordinates to other convenient coordinates, other than polar.

Amazing & important application of using double integrals & polar coordinates: the bell curve.

given by $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$:



$x^2/2$ so that inflection points occur at $x = \pm 1$

need to find the area so we can normalize to area 1.

very important in statistics, economics, psychology...

We'll compute $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ ← essentially the bell curve, but with easier constants. Aww, shucks! It has no antiderivative.

Clever trick:

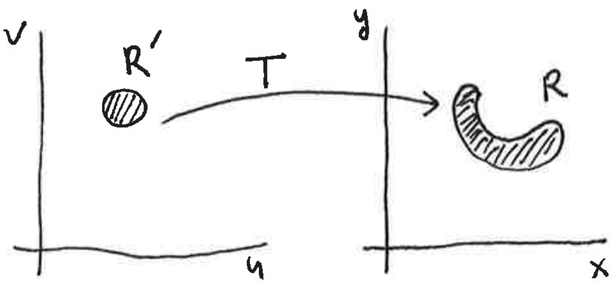
Let $A = \int_{-\infty}^{\infty} e^{x^2}\,dx$. Then $A^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right)$

$$\Rightarrow A^2 = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} e^{-x^2-y^2}\,dy\,dx = \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2}\cdot r\cdot dr\cdot d\theta = \int_{\theta=0}^{2\pi}\left(-\frac{1}{2}e^{-r^2}\Big|_{r=0}^{r=\infty}\right)d\theta = \int_{\theta=0}^{2\pi}\frac{1}{2}\,d\theta = \underline{\underline{\pi}}.$$

This is the easiest way to compute this integral.

Change of variables: When we change from (x,y) coordinates to (u,v) coordinates, we want to know what to do with dA: $dA = dx\cdot dy = \_\_\_\_ \cdot du\cdot dv$. (for example, $dA = \underline{r}\cdot dr\cdot d\theta$).

Think of a transformation from the uv-plane to the xy-plane:



$T(u,v) = (x(u,v),\, y(u,v))$

Define the notation:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det(DT) = \det\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$$

DT is the Jacobian matrix for the transformation T.

Then $\underline{\underline{dA = dx\cdot dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\cdot dv.}}$

→ For example, we computed that for the polar coordinates transformation $T(r,\theta) = (r\cdot\cos\theta,\, r\cdot\sin\theta)$, we have $\det(DT) = r$, so $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = |r| \Rightarrow dA = |r|\,dr\,d\theta$
$= r\,dr\,d\theta$ since $r \geq 0$.

Check this out: "change of variables" is just a two-dimensional u-substitution!

Let's apply the Jacobian method to the one-dimensional case (single-variable calculus):

Compute $\displaystyle\int_{x=0}^{x=4} x\cdot e^{x^2}\cdot dx$ Hmmm. Let's substitute $u=x^2$, so $x=\sqrt{u}=u^{1/2}$, so the Jacobian $\left(\frac{\partial x}{\partial u}\right)=\left(\frac{1}{2\sqrt{u}}\right)$.

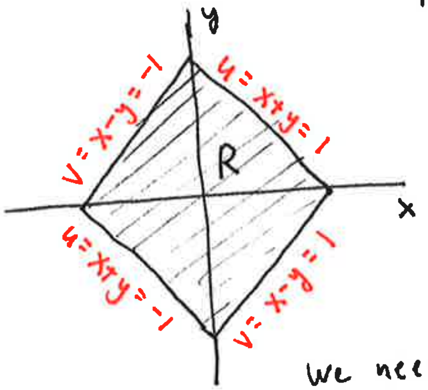
This is a 1×1 matrix with determinant $\frac{1}{2\sqrt{u}}$.

When x goes from 0 to 4, $u=x^2$ goes from 0 to 16, so we have:

$$\int_{x=0}^{x=4} x\cdot e^{x^2}\cdot dx=\int_{u=0}^{u=16}\sqrt{u}\cdot e^{u}\cdot \underbrace{\left(\frac{1}{2\sqrt{u}}\right)}_{\text{Jacobian determinant factor}} du = \int_0^{16}\tfrac{1}{2}e^{u}\,du,$$

just as you would do for a normal u-substitution! :)

Example. Compute $\displaystyle\iint_R (x^2-y^2)\,dA$, where R is the diamond with vertices $(\pm1,0),(0,\pm1)$.

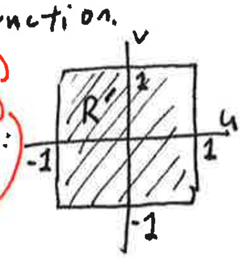


Option 1: Break up the region and compute the integral directly.

Option 2: Choose a change of variables that plays nicely with the region and with the integrand function.

Let $u=x+y$ and $v=x-y$. $\Rightarrow \begin{cases} x=\frac{1}{2}(u+v)\\ y=\frac{1}{2}(u-v)\end{cases}$

Then the boundary curves are $u=\pm1$, $v=\pm1$:



and $(x^2-y^2)=(x+y)(x-y)=u\cdot v$.

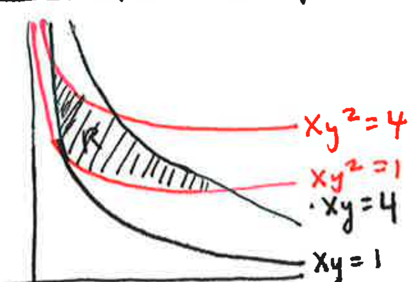
We need one more thing:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\left|\det\begin{pmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\end{pmatrix}\right| = \left|\det\begin{pmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{pmatrix}\right| = \left|-\tfrac{1}{2}\right| = \tfrac{1}{2}.$$

So now

$$\iint_R (x^2-y^2)\,dA = \iint_{R'} u\cdot v\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_{v=-1}^{v=1}\int_{u=-1}^{u=1} u\cdot v\cdot \underbrace{\tfrac{1}{2}}_{\substack{\text{Jacobian}\\ \text{det. factor}}}\,du\,dv = \int_{v=-1}^{v=1}\left(\tfrac{1}{4}u^2v\Big|_{u=-1}^{u=1}\right)dv = \int_{-1}^{1}0\,dv = \underline{\underline{0.}}$$

Example. Compute $\displaystyle\iint_R xy^2\,dA$, where D is in the first quadrant bounded by $\begin{cases} xy=1, & xy^2=1,\\ xy=4, & xy^2=4.\end{cases}$



Let's use the change of variables $u=xy$ and $v=xy^2$.

We'll need $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$, so we have to solve for x and y in terms of u and v.

Ugh! What a job! But wait – $T:(u,v)\to(x,y)$, so $T^{-1}:(x,y)\to(u,v)$.

We want $D(T^{-1})$, which is $(DT)^{-1}$. So let's compute:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \left|\det\begin{pmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial v/\partial y\end{pmatrix}\right|^{-1} = \left|\det\begin{pmatrix} y & x\\ y^2 & 2xy\end{pmatrix}\right|^{-1} = \frac{1}{xy^2} = \underline{\underline{\frac{1}{v}.}}$$

So

$$\iint_R xy^2\,dA = \int_{v=1}^{v=4}\int_{u=1}^{u=4} v\cdot\underbrace{\frac{1}{v}}_{\substack{\text{Jacobian}\\ \text{determinant factor}}}\,du\,dv = \int_{v=1}^{v=4}\int_{u=1}^{u=4}1\,du\,dv = \text{area of } 3\times3 \text{ rectangle} = \underline{\underline{9.}}\ :)$$

> So, a nice trick: if you've defined u and v in terms of x and y, use $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}$.