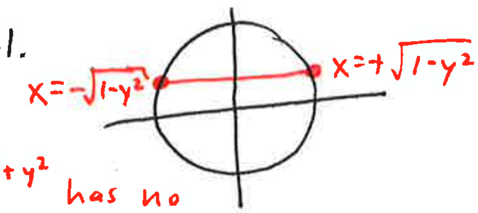
<u>Mathematician spotlight:</u> Evelyn Lamb, Freelance science writer for Scientific American and others
- Studies Teichmüller theory; postdoc at University of Utah
- explains research mathematics clearly & engagingly to a wide audience

<u>The past three classes:</u> Triple integrals, and solid regions of integration.
<u>Today & next time:</u> Tools for changing variables to make the integral easier.

<u>Example:</u> Converting from rectangular to polar coordinates (double integral).

Compute $\iint_D e^{x^2+y^2}\, dA$, where D is the unit disk $x^2+y^2 \le 1$.

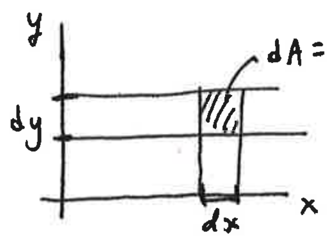


$$= \int_{y=-1}^{y=1} \int_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} e^{x^2+y^2}\, dx\, dy$$

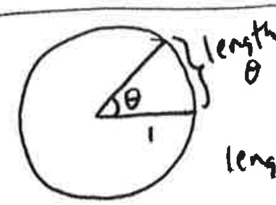
← impossible, because $e^{x^2+y^2}$ has no antiderivative with respect to x.
Maybe switch the order? No, that just exchanges x with y everywhere, so it's still impossible.

<u>Idea:</u> Maybe we can convert to polar coordinates. $e^{x^2+y^2} = e^{r^2}$,
$D = \begin{cases} 0 \le \theta \le 2\pi, \\ 0 \le r \le 1. \end{cases}$ that seems easier.

Ok, but what is dA?



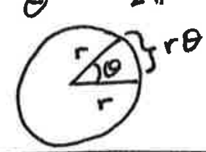
$dA = dx \cdot dy$, the area of an infinitesimal rectangle with side lengths dx and dy.



unit circle, angle measured in radians: length of subtended arc = θ

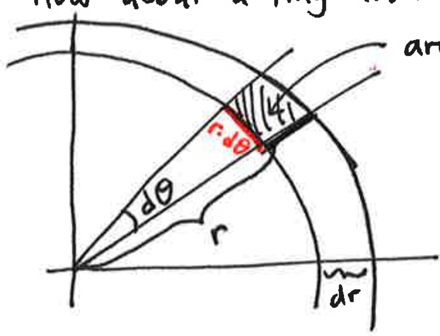
How do we see this? Call the length ℓ.

$\frac{\ell}{\theta} = \frac{2\pi r}{2\pi} = \frac{2\pi}{2\pi} = 1 \Rightarrow \ell = \theta.$



when the circle radius is r, the arc length scales up by a factor of r.

How about a tiny area in polar coords?



area of this ≈ width × length
$= dr \times r \cdot d\theta$

$\Rightarrow dA = r \cdot dr \cdot d\theta.$

$$\text{So } \iint_D e^{x^2+y^2}\, dA = \int_{\theta=0}^{\theta=2\pi} \left(\int_{r=0}^{r=1} e^{r^2} \cdot r \cdot dr \right) d\theta = \int_{\theta=0}^{\theta=2\pi} \left(\frac{1}{2} e^{r^2} \Big|_{r=0}^{r=1} \right) d\theta = \int_{\theta=0}^{\theta=2\pi} \left(\frac{1}{2} e - \frac{1}{2} \cdot 1 \right) d\theta$$

usually we use the order $dr\, d\theta$

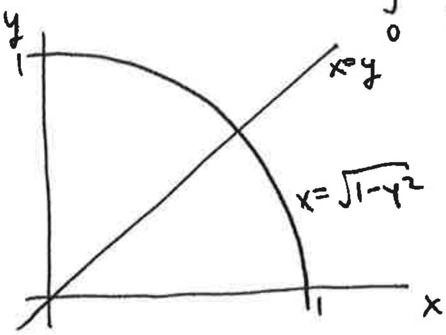
$$= \frac{1}{2}(e-1) \cdot \theta \Big|_{\theta=0}^{\theta=2\pi}$$

$$= \frac{1}{2}(e-1) \cdot 2\pi = \underline{\pi(e-1)}.$$

Converting to polar coordinates made this integral computable because the extra "r" from the dA term made the integrand $e^{r^2} \cdot r$, which has an antiderivative w.r.t. r.

Takeaway message: when converting to polar coordinates, $\boxed{dA = r \cdot dr \cdot d\theta} = r \cdot d\theta \cdot dr$.

Example. Compute $\displaystyle\int_0^{\sqrt{2}/2} \int_y^{\sqrt{1-y^2}} dx\, dy.$



sketch the curves: $x = y$

$x = \sqrt{1-y^2}$
$\Rightarrow x^2 = 1 - y^2$
$\Rightarrow x^2 + y^2 = 1$

⓪ Geometry: this is 1/8 of a unit disk, and we are finding its area, so the value is $\frac{1}{8} \cdot \pi \cdot 1^2 = \frac{\pi}{8}$.

① Compute as written: $\displaystyle\int_0^{\sqrt{2}/2} \left(\sqrt{1-y^2} - y\right) dy = \ldots$ requires a trig substitution :(

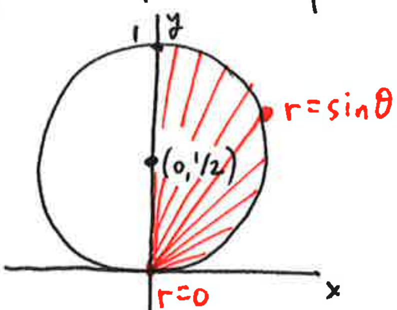
② Convert to polar coordinates:

$$\int_{\theta=0}^{\theta=\pi/4} \left(\int_{r=0}^{r=1} r\, dr \right) d\theta = \int_{\theta=0}^{\theta=\pi/4} \left(\frac{r^2}{2} \Big|_{r=0}^{r=1} \right) d\theta = \int_{\theta=0}^{\theta=\pi/4} \frac{1}{2}\, d\theta = \frac{1}{2}\theta \Big|_{\theta=0}^{\theta=\pi/4} = \pi/8.$$

☺

Example. Compute $\displaystyle\int_0^1 \int_0^{\sqrt{y-y^2}} \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy.$ ← what does this look like?!

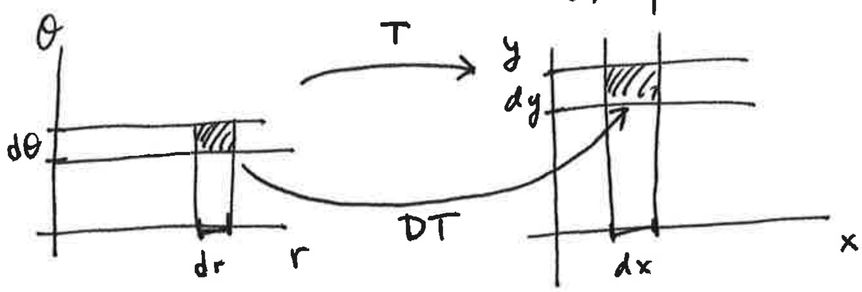
$x = \sqrt{y - y^2}$
$\Rightarrow x^2 = y - y^2 \Rightarrow x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$] circle
$\Rightarrow x^2 + y^2 = y \Rightarrow r^2 = r \sin\theta$
$\Rightarrow r = \sin\theta$] same circle, in polar



$$= \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=\sin\theta} \frac{1}{r} \cdot r\, dr\, d\theta = \int_{\theta=0}^{\theta=\pi/2} \sin\theta \cdot d\theta = -\cos\theta \Big|_{\theta=0}^{\theta=\pi/2} = -\cos\frac{\pi}{2} - -\cos 0 = -0 + 1 = 1.$$

Here is another way to derive $dA = r \cdot dr \cdot d\theta$, which will generalize to other changes of variables:

View the process of transforming (x,y) coords into (r,θ) coords as a transformation between the (r,θ) plane and the (x,y) plane (easier direction):



here $T(r,\theta) = (r \cdot \cos\theta,\ r \cdot \sin\theta)$
$= (x(r,\theta),\ y(r,\theta))$

The Jacobian DT of T describes the linear transformation sending the $dr \times d\theta$ rectangle to the $dx \times dy$ rectangle.
$\Rightarrow$ determinant of DT tells us the expansion factor!

$dA = |\det(DT)|$ (old area) $= |\det(DT)|\, dr \cdot d\theta$

$$DT = \begin{pmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \Rightarrow \det(DT) = r\cos^2\theta - -r\sin^2\theta$$
$$= r(\cos^2\theta + \sin^2\theta) = r.$$

so $\boxed{dA = |\det(DT)| \cdot dr \cdot d\theta = r \cdot dr \cdot d\theta.}$