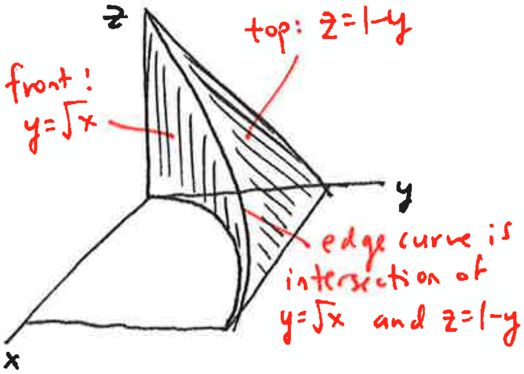
Diana Davis | Math 34 | 21 March 2018 | Class # 22 ①
§5.4

Mathematician spotlight: Jayadev Athreya, Associate Professor, Univ. of Washington
- studies flows on surfaces, billiards
- recent short paper proved existence of vertex-to vertex path on dodecahedron by showing it!

Last time: Setting up and interpreting triple integrals over solid regions

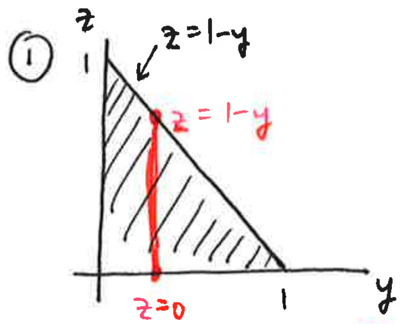
Today: More setting up triple integrals, including breaking a 3D region into two pieces

Recall: Last time, we sketched the region of integration for $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy\, dx$.



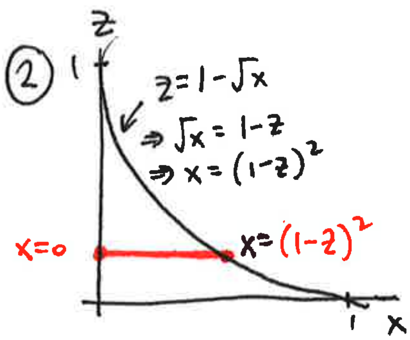
Now, let's rewrite it in the orders ① $dx\,\underbrace{dz\,dy}$ and ② $dy\,\underbrace{dx\,dz}$.

shadow plane is yz | shadow plane is xz.



$$\underbrace{\int_{y=0}^{y=1} \int_{z=0}^{z=1-y}} \underbrace{\int_{x=0}^{x=y^2}} f(x,y,z)\, dx\, dz\, dy$$

x goes from back (yz-plane, x=0) to front: surface $y=\sqrt{x}$ $\Rightarrow y^2 = x$



what is the shadow of the curved edge in the xz-plane? Eliminate y from the equations.

$y=\sqrt{x}$ and $z=1-y \Rightarrow \underline{z = 1-\sqrt{x}}.$

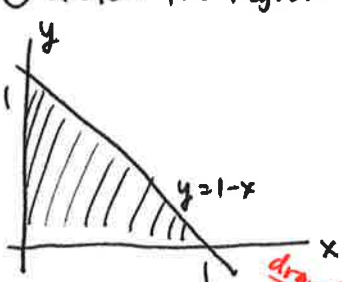
$$\underbrace{\int_{z=0}^{z=1} \int_{x=0}^{x=(1-z)^2}} \underbrace{\int_{y=\sqrt{x}}^{y=1-z}} f(x,y,z)\, dy\, dx\, dz$$

choose a point in the xz-plane. Draw a line in the y-direction. It pops INTO the solid at the surface $y=\sqrt{x}$, and it pops OUT at the surface $z=1-y \Rightarrow y=1-z$.

Example. Rewrite $\int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x,y,z)\, dz\, \underbrace{dy\, dx}$ in orders ① $dy\, \underbrace{dx\, dz}$ and ② $dx\, dz\, dy$.

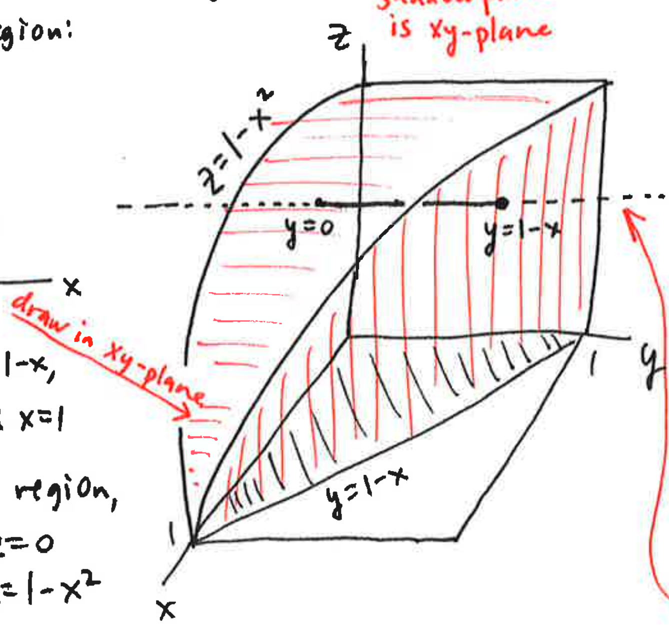
shadow plane is xy-plane | shadow plane is xz

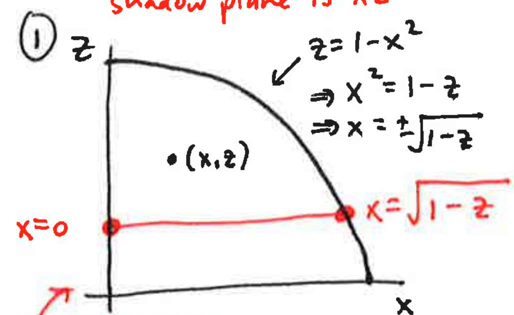
⓪ Sketch the region:



from $y=0$ to $y=1-x$, between $x=0$ and $x=1$

Then over this region, z goes from $z=0$ up to $z=1-x^2$

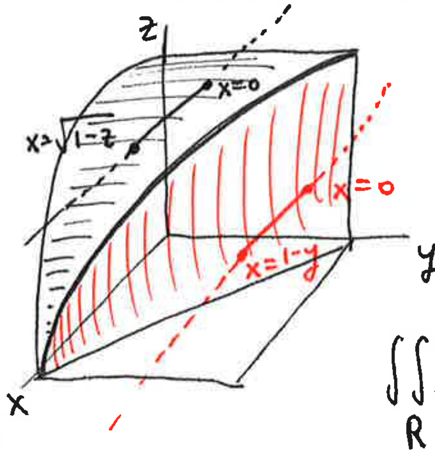


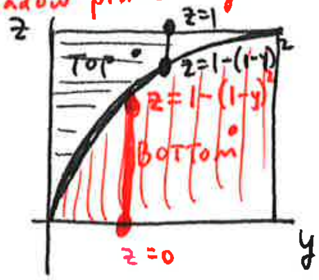


$$\underbrace{\int_{z=0}^{z=1} \int_{x=0}^{x=\sqrt{1-z}}} \underbrace{\int_{y=0}^{y=1-x}} f(x,y,z)\, dy\, dx\, dz$$

for a point (x,z) in the shadow of R, a line in the y-direction enters R at y=0 and exits at y=1-x.

(Continued) (2) order $dx\,dz\,dy$: $\underbrace{dz\,dy}_{\text{shadow plane is } yz}$





For a point (y,z) in the shadow of R, a line parallel to the x-axis enters R at $x=0$, and exits at

- $x=1-y$ if in the red region (bottom)
- $x=\sqrt{1-z}$ if in the black region (top).

So we need to set up a sum of two integrals.

$$\iiint_R f(x,y,z)\,dV = \int_{y=0}^{y=1}\int_{z=0}^{z=1-(1-y)^2}\int_{x=0}^{x=1-y} f(x,y,z)\,dx\,dz\,dy + \int_{y=0}^{y=1}\int_{z=1-(1-y)^2}^{z=1}\int_{x=0}^{x=\sqrt{1-z}} f(x,y,z)\,dx\,dz\,dy.$$

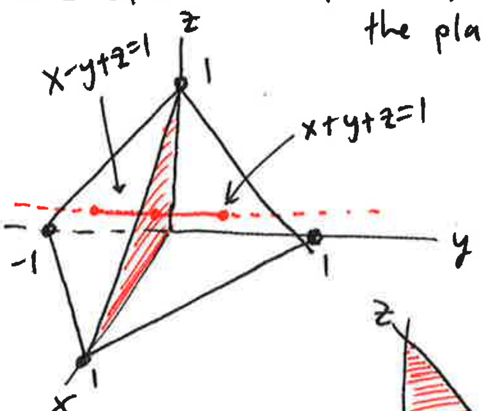
The intersection of the surfaces $z=1-x^2$ and $y=1-x$ in the yz-plane: eliminate x.

$y=1-x \Rightarrow x=1-y$

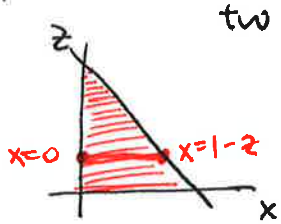
$z=1-x^2=1-(1-y)^2$

Example. Set up a triple integral for the region bounded by the xy-plane, the yz-plane, the plane $\underbrace{x+y+z=1}$, and the plane $\underbrace{x-y+z=1}$.

3 points on plane: (1,0,0), (0,1,0), (0,0,1) $\longrightarrow$ now we can plot these planes $\longleftarrow$ 3 points on plane: (1,0,0), (0,-1,0), (0,0,1)



Notice that if your shadow plane is xy or yz, you'll need two integrals. So let's use xz as the shadow plane.

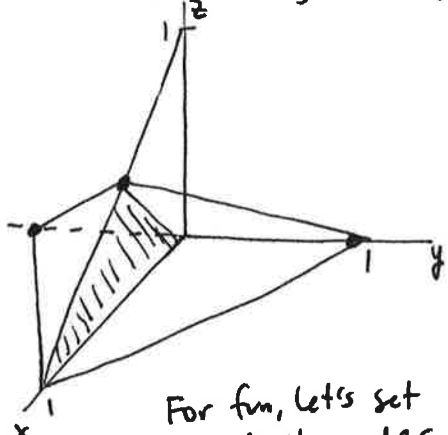


$$\int_{z=0}^{z=1}\int_{x=0}^{x=1-z}\int_{y=x+z-1}^{y=1-x-z} f(x,y,z)\,dy\,dx\,dz$$

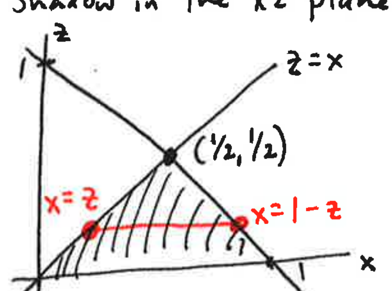
$y=1-x-z$ ← solve right plane $x+y+z=1$ for y

$y=x+z-1$ ← solve left plane $x-y+z=1$ for y

Now let's integrate over the part (half) of that region below the plane $z=x$:



Shadow in the xz plane:

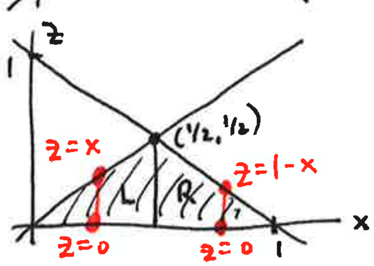


$$\int_{z=0}^{z=1/2}\int_{x=z}^{x=1-z}\int_{y=x+z-1}^{y=1-x-z} f(x,y,z)\,dy\,dx\,dz$$

$y=1-x-z$ ← as above

$y=x+z-1$ ← as above

For fun, let's set it up in the order $dy\,dz\,dx$: So, vertical segments.



$$\int_{x=0}^{x=1/2}\int_{z=0}^{z=x}\int_{y=x+z-1}^{y=1-x-z} f\ldots\,dy\,dz\,dx + \int_{x=1/2}^{x=1}\int_{z=0}^{z=1-x}\int_{y=x+z-1}^{y=1-x-z} f\cdot dy\,dz\,dx$$

LEFT RIGHT