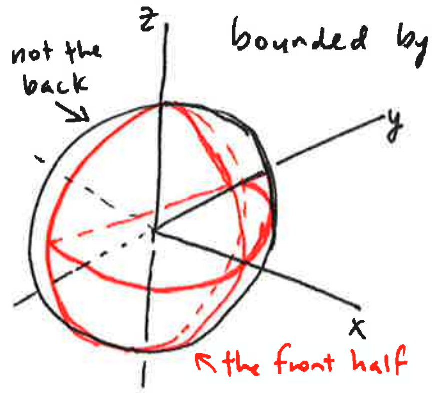
Diana Davis | Math 34 | 19 March 2018 | Class # 21 §5.4 | ①

Mathematician spotlight: Dusa McDuff, Professor, Barnard College
- symplectic geometry & topology, very eminent mathematician
- applies to even dimensions, e.g. 2, 4, 6, ... how do we visualize these?

Last time: Introduction to triple integrals, for finding "total mass" (or just volume)

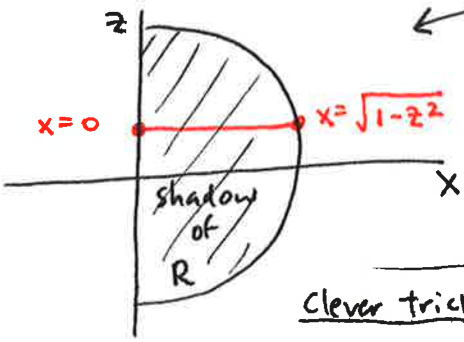
Today: More triple integrals; in particular, how to use the solid region of integration to set up the limits of integration.

Example. Set up the integral $\iiint_R \left(e^{xz^2} \cdot y \cdot \cos(xy) + 3\right) dV$, where R is the solid half ball bounded by the yz-plane and the unit sphere, on the positive-x side.



Let's do the order $dy\ \underbrace{dx\ dz}$ so, we integrate over the "shadow" of R in the xz-plane.

$$\underbrace{\int_{z=-1}^{z=1} \int_{x=0}^{x=\sqrt{1-z^2}}}_{\text{determine this just like a double integral}} \underbrace{\int_{y=-\sqrt{1-x^2-z^2}}^{y=\sqrt{1-x^2-z^2}}} \left(e^{xz^2} \cdot y \cdot \cos(xy) + 3\right) dy\ dx\ dz$$



→ for a fixed point (x, z) in the shadow of R, the y-value goes from
- the back surface: $y = -\sqrt{1-x^2-z^2}$

to
- the front surface: $x^2 + y^2 + z^2 = 1$

$y = +\sqrt{1-x^2-z^2}$

$y^2 = 1 - x^2 - z^2$

$y = \pm\sqrt{1-x^2-z^2}$

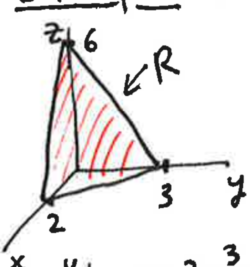
Clever trick: What if you actually wanted to compute this?

Notice that
- R is symmetric with respect to y ($y=0$ is a mirror for it)
- $e^{xz^2} \cdot y \cdot \cos(xy)$ is odd with respect to y (plug in $-y$, get the opposite of when you plug in y)

So $\iiint_R e^{xz^2} \cdot y \cdot \cos(xy)\, dV = 0$

So the value of the integral is $\iiint_R 3\, dV = 3(\text{volume of } R)$
$= 3 \cdot \frac{1}{2} \cdot \frac{4}{3}\pi \cdot 1^3 = \underline{\underline{2\pi}}.$

Example. Find the volume of the region bounded by the coordinate planes and $3x + 2y + z = 6$.



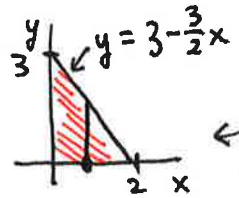
(0) Geometry: volume of pyramid $= \frac{1}{3}(\text{base area})(\text{height}) = \frac{1}{3} \cdot 3 \cdot 6 = \underline{\underline{6}}.$ $\Rightarrow z = 6 - 3x - 2y$

(1) Let's do $dz\ \underbrace{dy\ dx}$ → use shadow of R in the xy-plane.

$$\underbrace{\int_{x=0}^{x=2} \int_{y=0}^{y=3-\frac{3}{2}x}}_{\text{set up double integral}} \int_{z=0}^{z=6-3x-2y} 1\ dz\ dy\ dx = \int_{x=0}^{x=2} \int_{y=0}^{y=3-\frac{3}{2}x} (6 - 3x - 2y)\, dy\ dx$$

← the volume over the triangular region in the xy-plane, under the plane $z = 6 - 3x - 2y$.

$$= \ldots = \underline{\underline{6}}.$$



Now, let's go the other way. Decipher the bounds!

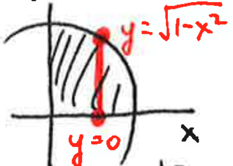
Example. Describe the region of integration for the integral $\displaystyle\int_{x=0}^{x=1}\int_{y=0}^{y=\sqrt{1-x^2}}\int_{z=0}^{z=\sqrt{x^2+y^2}} f(x,y,z)\,dz\,\underbrace{dy\,dx}$

shadow plane is xy-plane

① the shadow is in the xy-plane, with — I suggest to always write these in:

y from 0 to $\sqrt{1-x^2}$ } unit circle ← helpful to sketch bounding curves $y=0$, $y=\sqrt{1-x^2}$.

x from 0 to 1

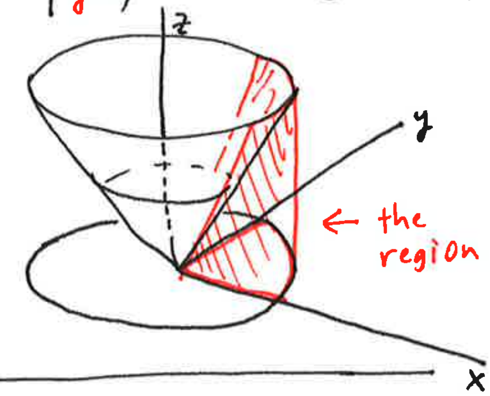


② the z-values go from $z=0$ (in xy-plane) up to $z=\sqrt{x^2+y^2}$. ← this surface is a CONE!

③ Let's put it together: we have the solid region in the first quadrant of the xy-plane,

under the cone $z=\sqrt{x^2+y^2}$,

inside the cylinder $x^2+y^2=1$.

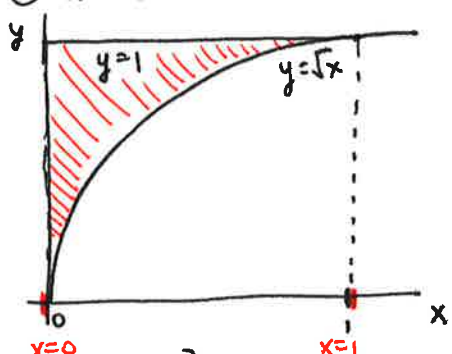


← the region

Example. Sketch the region of integration for the integral $\displaystyle\int_{x=0}^{x=1}\int_{y=\sqrt{x}}^{y=1}\int_{z=0}^{z=1-y} f(x,y,z)\,dz\,\underbrace{dy\,dx}$

shadow plane is xy-plane

① the shadow of R is in the xy-plane, from $y=\sqrt{x}$ to $y=1$, with x from 0 to 1.



② the z-values go from $z=0$ (so, the xy-plane) up to $z=1-y$, which is a PLANE.

③ To put it together and sketch the solid region, we need to determine where the (vertical) surface $y=\sqrt{x}$ intersects the plane $z=1-y$:

- in the yz-plane: $x=0 \Rightarrow y=\sqrt{x}=0 \Rightarrow z=1-y=1-0=0 \Rightarrow (0,0,1)$
- in the xy-plane: $z=0 \Rightarrow z=1-y=0 \Rightarrow y=1 \Rightarrow 1=\sqrt{x} \Rightarrow x=1 \Rightarrow (1,1,0)$

The intersection of the surfaces $y=\sqrt{x}$ and $z=1-y$ is a curve from $(0,0,1)$ to $(1,1,0)$. (dashed)

Our solid region is the "curved tetrahedron" shape.