<u>Mathematician spotlight:</u> Yitang (Tom) Zhang, Professor, Univ. of California Santa Barbara
- after PhD, didn't get math job; worked in restaurants & motels
- lecturer at UNH from 1999-2014, teaching calculus (etc)
- proved an extremely big theorem on gaps between prime numbers

---

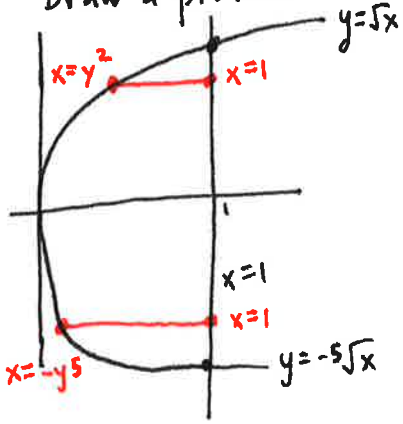
<u>Last time:</u> changing the order of integration for double integrals. How? Sketch the region!

<u>Today:</u> triple integrals! How? Visualize the surface in 3D, then set up the bounds.

<u>Example.</u> This is a toughie that will require all of the tools we have learned so far.

Compute $\displaystyle\int_0^1 \int_{-\sqrt[5]{x}}^{\sqrt{x}} \sin(y^3)\,dy\,dx.$ ← $\sin(y^3)$ does not have an antiderivative with respect to y, so we have to change the order of integration.

Draw a picture:



$$= \int_{y=-1}^{y=0} \int_{x=-y^5}^{x=1} \sin(y^3)\,dx\,dy \;+\; \int_{y=0}^{y=1} \int_{x=y^2}^{x=1} \sin(y^3)\,dx\,dy.$$

BOTTOM TOP

$$= \int_{y=-1}^{y=0} \left(\sin(y^3) + y^5 \sin(y^3)\right) dy + \int_{y=0}^{y=1} \left(\sin(y^3) - y^2 \sin(y^3)\right) dy$$

$$= \underbrace{\int_{y=-1}^{y=0} \sin(y^3)\,dy}_{①} + \underbrace{\int_{y=-1}^{y=0} y^5 \sin(y^3)\,dy}_{②} + \underbrace{\int_{y=0}^{y=1} \sin(y^3)\,dy}_{③} - \underbrace{\int_{y=0}^{y=1} y^2 \sin(y^3)\,dy}_{④}$$

$$= \underbrace{\int_{y=-1}^{y=1} \sin(y^3)\,dy}_{①+③} + \underbrace{\int_{y=-1}^{y=0} y^5 \sin(y^3)\,dy}_{②} - \underbrace{\int_{y=0}^{y=1} y^2 \sin(y^3)\,dy}_{④} = \underbrace{0}_{①+③} \overbrace{- \frac{1}{3} y^3 \cos(y^3)\Big|_{y=-1}^{y=0} + \frac{1}{3}\int_{y=-1}^{y=0} \cos(y^3)\cdot 3y^2\,dy}^{②} + ④$$

$$= -\frac{1}{3}\cdot 0 \cdot \cos 0 + \frac{1}{3}\cdot(-1)\cos(-1) + \frac{1}{3}\sin(y^3)\Big|_{y=-1}^{y=0} + \frac{1}{3}\cos(y^3)\Big|_{y=0}^{y=1}$$

$$= \cancel{\frac{1}{3}\cos(-1)} - \frac{1}{3}\sin(-1) + \cancel{\frac{1}{3}\cos(1)} - \frac{1}{3}\cos(0)$$

$$= \frac{1}{3}\sin(1) - \frac{1}{3} = \frac{1}{3}(\sin 1 - 1).$$

To compute ②, use Integration by Parts:

$u = y^3$ $\qquad v = -\frac{1}{3}\cos(y^3)$

$du = 3y^2\,dy$ $\qquad dv = y^2 \sin(y^3)\,dy$

$$\int u\,dv = uv - \int v\,du \Rightarrow \int y^5 \sin(y^3) = -\frac{1}{3} y^3 \cos(y^3) \cancel{-} + \int \cancel{-}\frac{1}{3}\cos(y^3)\cdot 3y^2\,dy$$

<u>Triple integrals!</u> We compute $\displaystyle\iiint_R f(x,y,z)\,dV$ over a 3D solid region R. There are 6 possible orders: $dx\,dy\,dz$, $dx\,dz\,dy$, $dy\,dx\,dz, \ldots$

<u>Example.</u> Compute $\displaystyle\iiint_{[0,1]\times[0,2]\times[0,3]} 8xyz\,dV$. This integrates $f(x,y,z) = 8xyz$ over the box $\begin{cases} 0 \le x \le 1 \\ 0 \le y \le 2 \\ 0 \le z \le 3. \end{cases}$ Let's do it in the order $dz\,dy\,dx$.

$$\int_{x=0}^{x=1}\int_{y=0}^{y=2}\int_{z=0}^{z=3} 8xyz\,dz\,dy\,dx = \int_0^1\int_0^2 \left(4z^2xy\Big|_{z=0}^{z=3}\right) dy\,dx = \int_{x=0}^{x=1}\int_{y=0}^{y=2} 36xy\,dy\,dx = \int_0^1 \left(18xy^2\Big|_{y=0}^{y=2}\right) dx = \int_{x=0}^{x=1} 72x\,dx$$

$$= 36x^2\Big|_{x=0}^{x=1} = 36.$$

What does a triple integral *mean*? It adds up function values over a solid region.

Special case: $f(x,y,z) = 1$ then

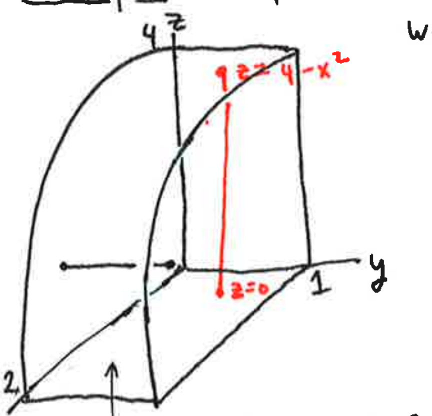
$$\iiint_R dV = \text{volume of region } R.$$

Example: $\displaystyle\iiint_{\substack{\text{solid ball}\\ \text{of radius } r}} dV = \frac{4}{3}\pi r^3$

Example: $\displaystyle\iiint_{[0,a]\times[0,b]\times[0,c]} dV = \underline{\qquad}$

In general: The function $f(x,y,z)$ gives the density of an object, or the electric charge, at each point. Then $\iiint_R f(x,y,z)\,dV$ gives the total mass of the object, or the total electric charge, etc.

Example. Compute the triple integral of $f(x,y,z) = x+y$ of the region R shown below.



front surface: $z = 4-x^2$

We want to integrate from $z=0$ to $z = 4-x^2$ over the rectangle $0 \le x \le 2$, $0 \le y \le 1$.

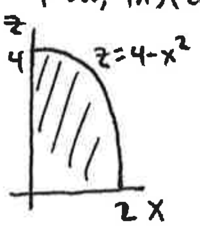
$$\iiint_R (x+y)\,dV = \int_{y=0}^{y=1}\int_{x=0}^{x=2}\int_{z=0}^{z=4-x^2} (x+y)\,dz\,dx\,dy = \int_{y=0}^{y=1}\int_{x=0}^{x=1} (x+y)(4-x^2)\,dx\,dy$$

$$= \int_{y=0}^{y=1}\int_{x=0}^{x=2} \left(4x + 4y - x^3 - x^2y\right)dx\,dy = \int_{y=0}^{y=1} \left(2x^2 + 4xy - \frac{1}{4}x^4 - \frac{1}{3}x^3y\right)\Big|_{x=0}^{x=2} dy$$

$$= \int_{y=0}^{y=1} \left(8 + 8y - 4 - \frac{8}{3}y\right) dy = 4y + 4y^2 - \frac{4}{3}y^2 \Big|_{y=0}^{y=1} = 4 + 4 - \frac{4}{3} = \frac{20}{3}.$$

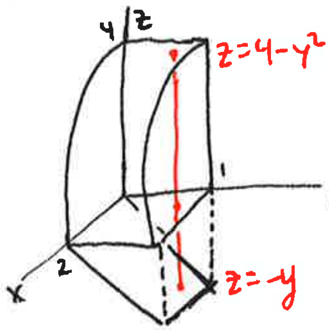
$8-4=4$

Now, instead of segments in the z-direction, let's use segments in the y-direction.



$$\iiint_R (x+y)\,dV = \underbrace{\int_{x=0}^{x=2}\int_{z=0}^{z=4-x^2}}_{\text{as in double integrals}}\int_{y=0}^{y=1} (x+y)\,dy\,dz\,dx = \dots = 20/3.$$

Example. Change the region so that the bottom surface is $z=-y$, not $z=0$, and set it up. (So, we just drop down the "floor" so it is slanted.)



So the rectangular region of integration in the xy-plane is the same, but now the z-segment starts lower down:

$$\iiint_R (x+y)\,dV = \int_{y=0}^{y=1}\int_{x=0}^{x=2}\int_{z=-y}^{z=4+y^2} (x+y)\,dz\,dx\,dy.$$

For a triple integral $\iiint_R f(x,y,z)\,dV$:

$$\underbrace{\int\int}_{}\int f(x,y,z)\,d(\text{inner variable})\,d(\text{middle variable})\,d(\text{outer variable})$$

inner bounds depend only on middle & outer variables

This is the "shadow" of R in the middle variable – outer var. plane.

- the outer bounds are always CONSTANT
- the middle bounds depend only on the OUTER variable