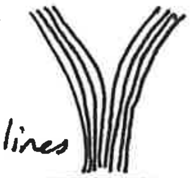
Mathematician spotlight: Eriko Hironaka, Professor Emerita, Florida State University
- Studies dynamical systems
- also "train tracks" – like a doodle of many non-crossing lines

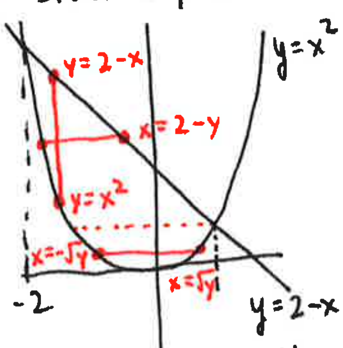


Last time, we explored how to integrate over a general region in the xy-plane.

Today, we'll discover how to change the order of integration: first, draw a picture to figure out what your region is, and then switch the order.

Example. Find the area of the region between $y=x^2$ and $y=2-x$.

Draw a picture:



intersection points:

$x^2 = 2-x$

$x^2+x-2=0$

$(x+2)(x-1)=0$

$\Rightarrow x=-2$ or $x=1$

$\Rightarrow (-2,4)$ and $(1,1)$.

① Using vertical slices:

$$\iint_D dA = \int_{x=-2}^{x=1}\int_{y=x^2}^{y=2-x} dy\,dx = \int_{x=-2}^{x=1}(2-x-x^2)\,dx = 2x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3\Big|_{x=-2}^{x=1}$$

$$=\left(2-\tfrac{1}{2}-\tfrac{1}{3}\right)-\left(-4-2+\tfrac{8}{3}\right)$$

$$=2-\tfrac{1}{2}-\tfrac{1}{3}+4+2-\tfrac{8}{3} = 4\tfrac{1}{2}.$$

② using horizontal slices: need 2 pieces.

$$\iint_D dA = \underbrace{\int_{y=0}^{y=1}\int_{x=-\sqrt{y}}^{x=+\sqrt{y}} dx\,dy}_{\text{BOTTOM}} + \underbrace{\int_{y=1}^{y=4}\int_{x=-\sqrt{y}}^{x=2-y} dx\,dy}_{\text{TOP}} = \ldots = \frac{4}{3}+\frac{19}{6} = 4\tfrac{1}{2}.$$

When would I use double integrals in real life? To compute probabilities!

Example. Your checkout line at the store has an average wait of 10 minutes, and your friend's of 5 minutes. What is the prob. that you check out first?

① Reality check: we expect an answer between ____ and ____.

② let x = amount of time until you check out
y = " " " friend checks out.

we want: prob. $x<y$.

For one person, $P_\tau(t) = \begin{cases} \frac{1}{\tau}e^{-t/\tau} & \text{if } t\geq 0 \\ 0 & \text{if } t<0\end{cases}$ ← the probability that you check out at t minutes, in a line whose average wait is τ minutes.

For two people, $p(x,y) = P_{10}(x)\cdot P_5(y)$ is the probability of the person in the 10-minute line checking out in x minutes, and the person in the 5-minute line checking out in y minutes.

We want: $\displaystyle\iint_{x<y} p(x,y)\,dA.$

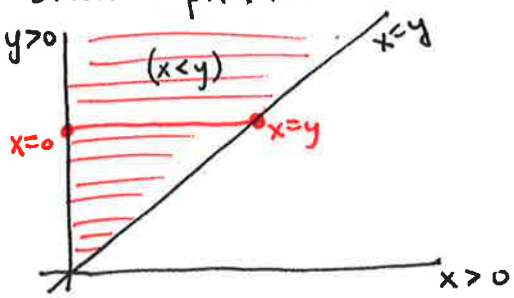
$$= \int_{y=0}^{y=\infty}\int_{x=0}^{x=y}\tfrac{1}{10}e^{-x/10}\cdot\tfrac{1}{5}e^{-y/5}\,dx\,dy = \int_{y=0}^{y=\infty} -e^{-x/10}\cdot\tfrac{1}{5}e^{-y/5}\Big|_{x=0}^{x=y}dy$$

$$= \int_{y=0}^{y=\infty} -\tfrac{1}{5}e^{-y/5}\left(e^{-y/10}-1\right)dy = \int_{y=0}^{y=\infty}\left(-\tfrac{1}{5}e^{-3y/10}-\tfrac{1}{5}e^{-y/5}\right)dy = \tfrac{2}{3}e^{-3y/10}+e^{-y/5}\Big|_{y=0}^{y\to\infty}$$

$$=(0-0)-\left(\tfrac{2}{3}-1\right) = 1/3.$$

So there is a 1/3 probability that you'll check out before your friend.

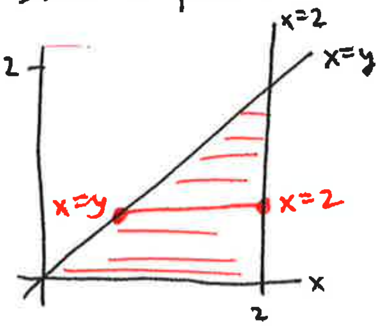
Draw a picture:



Now let's change the order of integration.

<u>Example.</u> Compute $\int_{y=0}^{y=2}\int_{x=y}^{x=2} e^{x^2}\,dx\,dy$. ← it is <u>impossible</u> to find an antiderivative for e^{x^2} with respect to x. Our only hope is to change the order.

Draw a picture:



Rewrite in the other order, using vertical slices:

$$=\int_{x=0}^{x=2}\int_{y=0}^{y=x} e^{x^2}\,dy\,dx = \int_{x=0}^{x=2}\left(y\cdot e^{x^2}\Big|_{y=0}^{y=x}\right)dx = \int_{x=0}^{x=2} x\cdot e^{x^2}\,dx = \frac{1}{2}e^{x^2}\Big|_{x=0}^{x=2}$$

$$=\frac{1}{2}e^4-\frac{1}{2}=\frac{1}{2}(e^4-1).$$

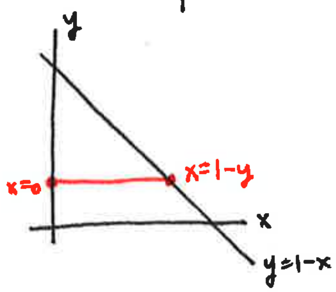
<u>Note:</u> we were lucky here, that changing the order yielded a computable integral. It happened that our region and our integrand played nicely together.

When changing the order of integration, use the limits of integration to sketch the region, and use the region to determine the new limits of integration.

A "good order" of integration generally depends on the function <u>and</u> on the region.

<u>Example.</u> Compute $\int_{x=0}^{x=1}\int_{y=0}^{y=1-x} \cos(1-y)^2\,dy\,dx$. ← it is impossible to find an antiderivative for $\cos(1-y)^2$ w.r.t. y, so we must change the order.

Draw a picture:



$$=\int_{y=0}^{y=1}\int_{x=0}^{x=1-y}\cos(1-y)^2\,dx\,dy = \int_{y=0}^{y=1}\left(\cos(1-y)^2\cdot(1-y)\right)dy = -\frac{1}{2}\sin(1-y)^2\Big|_{y=0}^{y=1} = -\frac{1}{2}(\sin 0-\sin 1)$$

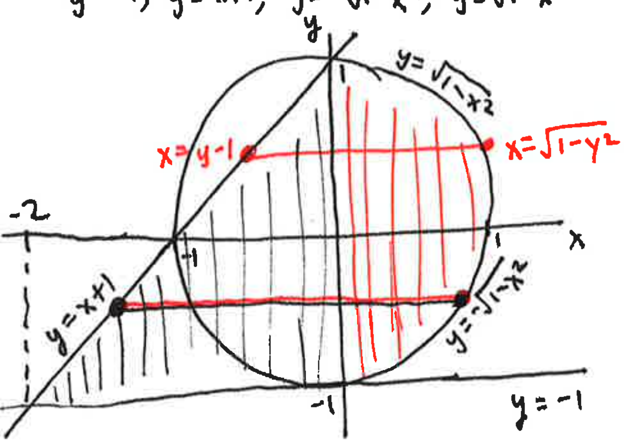
$$=\frac{1}{2}\sin 1.$$

<u>Example.</u> Consolidate into a single integral $\int_{x=-2}^{x=0}\int_{y=-1}^{y=x+1} f(x,y)\,dy\,dx + \int_{x=0}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx$.

FIRST SECOND

First, draw all the bounding curves:

$y=-1$, $y=x+1$, $y=-\sqrt{1-x^2}$, $y=\sqrt{1-x^2}$



$$=\int_{y=-1}^{y=1}\int_{x=y-1}^{x=\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$

Now, shade the region corresponding to each integral: FIRST, SECOND.

Finally, draw a horizontal slice and solve for the bounds.