Diana Davis   Math 34   5 March 2018   Class #18 ① §5.2

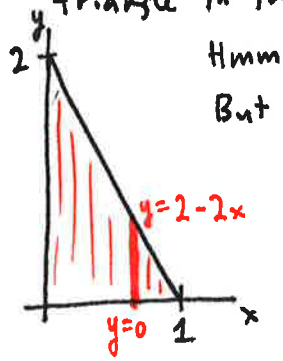
Mathematician spotlight: Rodrigo Treviño, Assistant Professor, Univ. of Maryland

- ergodic theory, dynamical systems, geometry, math-physics
- ergodic theory studies how "well mixed" a system becomes

<u>Past two classes:</u> Double integrals over rectangular regions; Riemann sums

<u>Today:</u> Double integrals over general regions - circles, triangles, anything you want!

<u>Example:</u> Find the volume of the solid under the surface $f(x,y)=2xy$, above the triangle in the xy-plane bounded by the x and y axes and the line $y=2-2x$.



Hmm! If we were integrating over the entire rectangle, it would be $\int_{x=0}^{x=1}\int_{y=0}^{y=1}(2-2x)\,dy\,dx$

But y doesn't go all the way to 2 – only up to $2-2x$.

So we compute:

$$\int_{x=0}^{x=1}\int_{y=0}^{y=2-2x} 2xy\;dy\,dx = \int_{x=0}^{x=1}\left(xy^2\Big|_{y=0}^{y=2-2x}\right)dx = \int_{x=0}^{x=1} x(2-2x)^2\,dx$$

$$= \int_{x=0}^{x=1}(4x-8x^2+4x^3)\,dx = 2x^2-\frac{8}{3}x^3+x^4\Big|_{x=0}^{x=1} = 2-\frac{8}{3}+1 = \frac{1}{3}.$$

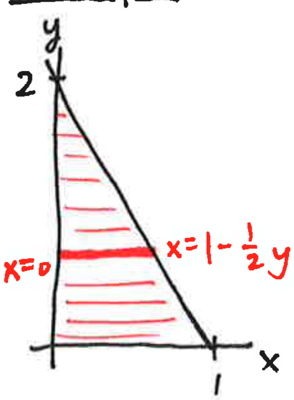
For a region D in the xy-plane and a function $f(x,y)$,

$$\iint_D f(x,y)\,dA = \int_{\text{min value of outer variable}}^{\text{max value of outer variable}} \int_{\text{lower bound of inner variable in terms of outer variable}}^{\text{upper bound of inner variable in terms of outer variable}} f(x,y)\;d(\text{inner})\;d(\text{outer}).$$

outer bounds are ALWAYS CONSTANTS

inner bounds can ONLY depend on the OUTER variable

<u>Example.</u> Re-do the example above, in the other order of integration (dx dy).



To find the upper bound for x in terms of y, we write down the equation of the line ($y=2-2x$) and solve for x: $x=1-\frac{1}{2}y$.

$y=2-2x$
$2x=2-y$
$x=1-\frac{1}{2}y$

$$\iint_D 2xy\,dA = \int_{y=0}^{y=2}\int_{x=0}^{x=1-\frac{1}{2}y} 2xy\;dx\,dy = \int_{y=0}^{y=2}\left(x^2y\Big|_{x=0}^{x=1-\frac{1}{2}y}\right)dy$$

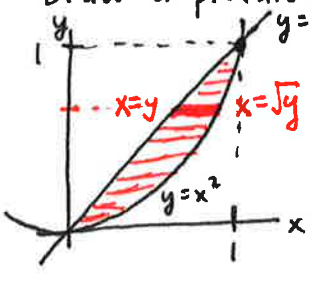
$$= \int_{y=0}^{y=2}\left(1-\tfrac{1}{2}y\right)^2\cdot y\;dy = \int_{y=0}^{y=2}\left(y-y^2+\tfrac{1}{4}y^3\right)dy = \frac{1}{2}y^2-\frac{1}{3}y^3+\frac{1}{16}y^4\Big|_{y=0}^{y=2}$$

$$= 2-\frac{8}{3}+1 = 1/3.$$

Same as above! ☺

Example. Compute $\iint_D (2y-x)\,dA$, where D is the domain in the first quadrant between $y=x$ and $y=x^2$.

Draw a picture:



① with x as the inner variable and y as the outer:

$$\int_{y=0}^{y=1}\int_{x=y}^{x=\sqrt{y}}(2y-x)\,dx\,dy=\int_{y=0}^{y=1}\left(2xy-\tfrac12x^2\right)\Big|_{x=y}^{x=\sqrt{y}}dy=\int_{y=0}^{y=1}\left(2y\sqrt{y}-\tfrac12(\sqrt{y})^2\right)-\left(2y\cdot y-\tfrac12y^2\right)dy$$

$$=\int_{y=0}^{y=1}\left(2y^{3/2}-\tfrac12y-\tfrac32y^2\right)dy=\tfrac45y^{5/2}-\tfrac14y^3-\tfrac12y^2\Big|_{y=0}^{y=1}=\tfrac45-\tfrac14-\tfrac12=\tfrac{1}{20}.$$

$$\tfrac{16}{20}-\tfrac{5}{20}-\tfrac{10}{20}$$

② with y as the inner variable and x as the outer:

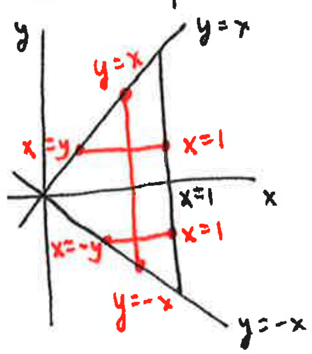
$$\int_{x=0}^{x=1}\int_{y=x^2}^{y=x}(2y-x)\,dy\,dx=\int_{x=0}^{x=1}\left(y^2-xy\Big|_{y=x^2}^{y=x}\right)dx=\int_{x=0}^{x=1}(x^2-x^2)-(x^4-x^3)\,dx=\int_{x=0}^{x=1}(x^3-x^4)\,dx=\frac{x^4}{4}-\frac{x^5}{5}\Big|_0^1=\frac14-\frac15=\frac{1}{20}.$$

Same again! yay.

Example. Compute $\iint_D dA$, where D is the triangle formed by the lines $y=x$, $y=-x$, and $x=1$.

↖ this is integrating $\iint_D 1\,dA$, which gives the area of the region D.

Draw a picture:



⓪ Geometry: area $=\frac12$ base $\cdot$ height $=\frac12\cdot2\cdot1=1.$

① using vertical segments: $\displaystyle\int_{x=0}^{x=1}\int_{y=-x}^{y=x}dy\,dx=\int_{x=0}^{x=1}(x--x)\,dx=\int_{x=0}^{x=1}2x\,dx=x^2\Big|_{x=0}^{x=1}=1.$

② using horizontal segments: we have to break the region into two pieces!

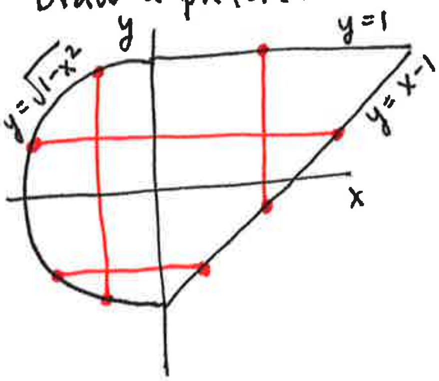
$$\underbrace{\int_{y=-1}^{y=0}\int_{x=-y}^{x=1}dx\,dy}_{\text{BOTTOM}}+\underbrace{\int_{y=0}^{y=1}\int_{x=y}^{x=1}dx\,dy}_{\text{TOP}}=\int_{y=-1}^{y=0}(1+y)\,dy+\int_{y=0}^{y=1}(1-y)\,dy$$

$$=\left(y+\tfrac12y^2\Big|_{y=-1}^{y=0}\right)+\left(y-\tfrac12y^2\Big|_{y=0}^{y=1}\right)$$

$$=(0+0)-\left(-1+\tfrac12\right)+\left(1-\tfrac12\right)-(0-0)=1.$$

Example. Set up integrals in both orders for $\iint_D f(x,y)\,dA$, where D is the domain consisting of the left half of the unit disk, plus the triangle between $y=x-1$ and $y=1$.

Draw a picture:



① using horizontal segments: solve for x in terms of y:

$x^2+y^2=1 \Rightarrow x^2=1-y^2 \Rightarrow x=\pm\sqrt{1-y^2}$

$y=x-1 \Rightarrow x=y+1$

So $\displaystyle\int_{y=-1}^{y=1}\int_{x=-\sqrt{1-y^2}}^{x=y+1}f(x,y)\,dA.$

② using vertical segments, we need two pieces:

$$\underbrace{\int_{x=-1}^{x=0}\int_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}}f(x,y)\,dA}_{\text{LEFT (semi circle)}}+\underbrace{\int_{x=0}^{x=2}\int_{y=x-1}^{y=1}f(x,y)\,dA}_{\text{RIGHT (triangle)}}.$$