Diana Davis   Math 34   2 March 2018   Class #17 ①
§5.2

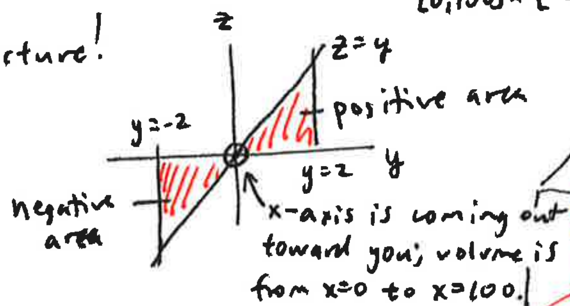
Mathematician spotlight: Autumn Kent, Associate Professor, Univ. of Wisconsin
- Studies geometry, topology, dynamical systems, knot theory
- Knot theory works on answering the (HARD!) question of when two given knots are the same.

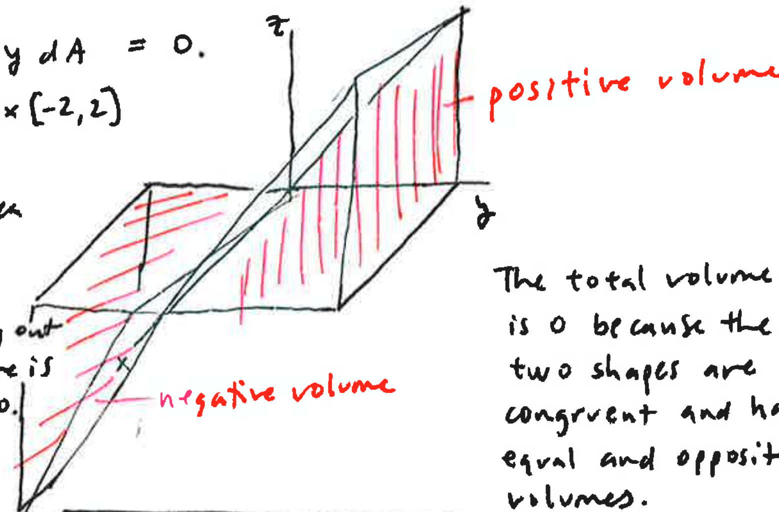
Last time: We introduced double integrals, computed them geometrically & algebraically, and stated Fubini's Theorem: ______________________

Today: More examples, plus the Riemann sum definition of double integrals.

Example: Justify geometrically why $\displaystyle\iint_{[0,100]\times[-2,2]} y \, dA = 0$.

Draw a picture!





The total volume is 0 because the two shapes are congruent and have equal and opposite volumes.

Example. Find the volume below the surface $z = xe^{xy}$ above $[1,2]\times[1,3]$ in the xy-plane.

This volume is given by $\displaystyle\iint_{[1,2]\times[1,3]} x e^{xy} \, dA$. Let's compute it in both orders:

$$\int_{x=1}^{x=2}\left(\int_{y=1}^{y=2} x e^{xy}\, dy\right) dx = \int_{x=1}^{x=2}\left(e^{xy}\Big|_{y=1}^{y=3}\right) dx = \int_{x=1}^{x=2}\left(e^{3x}-e^{x}\right)dx = \frac{1}{3}e^{3x}-e^{x}\Big|_{x=1}^{x=2} = \left(\frac{1}{3}e^{6}-e^{2}\right)-\left(\frac{1}{3}e^{3}-e\right)$$

$$= \underline{\underline{\frac{1}{3}e^{6}-\frac{1}{3}e^{3}-e^{2}+e.}}$$

x is a constant with respect to y

Now the other order:

$$\int_{y=1}^{y=3}\underbrace{\left(\int_{x=1}^{x=2} x e^{xy}\, dx\right)}_{\text{need integration by parts!}} dy = \int_{y=1}^{y=3}\left(\frac{x}{y}e^{xy}-\frac{1}{y^2}e^{xy}\Big|_{x=1}^{x=2}\right)dy = \int_{y=1}^{y=3}\left(\frac{2}{y}e^{2y}-\frac{1}{y^2}e^{2y}-\frac{1}{y}e^{y}+\frac{1}{y^2}e^{y}\right)dy$$

→ we cannot solve this, even with integration by parts. Conclusion: other order was easier!

Let $u = x$, $dv = e^{xy}\,dx$

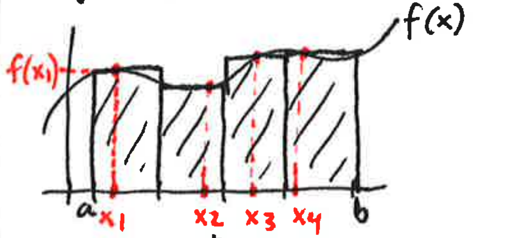
$\Rightarrow du = dx$, $v = \frac{1}{y}e^{xy}$

then $\boxed{\int u\,dv = uv - \int v\,du}$

$$\Rightarrow \int x e^{xy}\,dx = \frac{x}{y}e^{xy} - \int \frac{1}{y}e^{xy}\,dx = \underbrace{\frac{x}{y}e^{xy} - \frac{1}{y^2}e^{xy}}_{\text{use this above}}$$

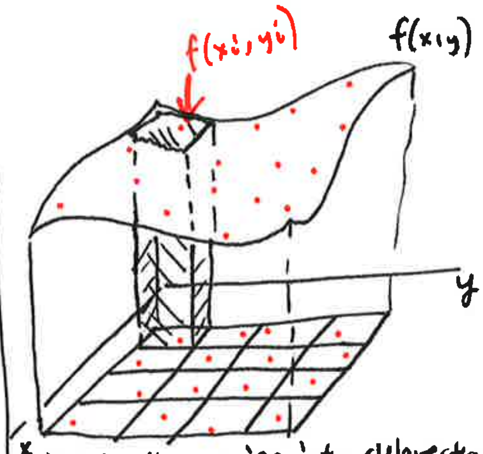
OK, now that we understand what double integrals do to find the volume under a surface, let's back up and define them more rigorously.

single-variable Riemann sum:



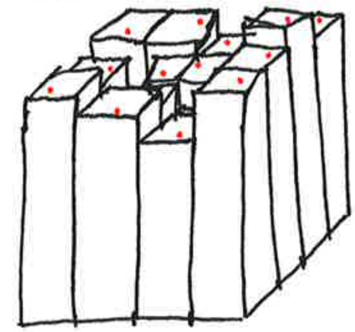
To compute $\int_a^b f(x)\,dx$, break the interval into subintervals, choose an x-value x_i in each, find each $f(x_i)$, and add up all the areas $f(x_i)\Delta x$ to get an area estimate.
As $\Delta x \to 0$, rectangle area $\to \int_a^b f(x)\,dx$.

multivariable Riemann sum:

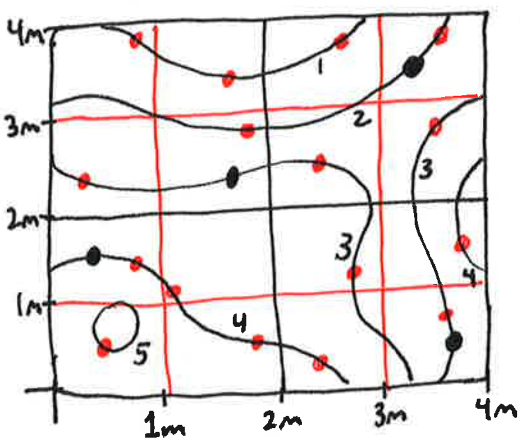


Break the region into subrectangles and choose an (x_i, y_i) point in each.
Add up the volumes $f(x_i,y_i)\Delta x \Delta y$ of boxes to get volume.

total volume of boxes approximates volume under surface. Becomes more accurate as rects become thinner.



Example. The map shows level curves for elevation above bedrock of granite at a new quarry site. Estimate the total volume of granite in the quarry.



① First, let's break the region into four 2x2 squares, and choose a sample point in each (black).

Riemann sum $= \sum f(\text{sample point}) \times (\text{area of rectangle})$
$= 3\times4 + 2\times4 + 4\times4 + 3\times4 = 48\,m^3$.

② Now let's break the region into sixteen 1x1 squares, and choose a sample point in each (red)

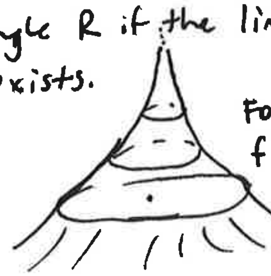
Riemann sum $= \sum f(\text{sample point}) \times (\text{area of rectangle})$
$= 1\cdot1 + 1\cdot1 + 1\cdot1 + 2\cdot1 + 3\cdot1 + 2\cdot1 + 3\cdot1 + 3\cdot1 + 4\cdot1 + 4\cdot1 + 3\cdot1 + 4\cdot1 + 5\cdot1 + 4\cdot1 + 4\cdot1 + 3\cdot1 = 47\,m^3$.

With finer rectangles and finer level curves, we would get even better approximations.

Integrability: Integrals are defined as limits of Riemann sums. What if the limit doesn't exist??

A function is integrable over a rectangle R if the limit of Riemann sums used to define the double integral exists.

Example: let $f(x,y) = \begin{cases} \frac{1}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

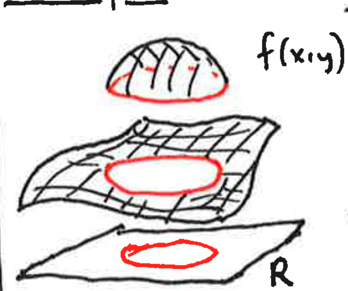


For any rectangle R containing (0,0), f is not integrable over R, because f(sample point) can be as large as you wish, so the limit does not exist.

Theorem: If f is bounded, and the set of points where f is not continuous has zero area, then f is integrable over such a region.

we usually consider continuous functions that are bounded over our region, so we won't worry much about integrability.

Example:



f is bounded over R, and f is only discontinuous on the circle, which is a curve, which has area 0.
$\Rightarrow$ f is integrable on R.