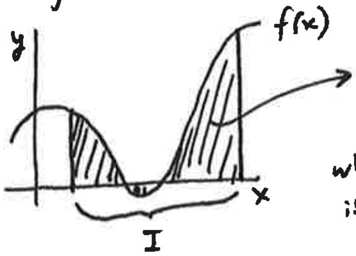
Mathematician spotlight: Erik Demaine – Professor, MIT (electrical eng. & C.S.)
– PhD thesis proved the "Fold and One Cut Theorem"
– child prodigy; works with father Martin; creates art

Theme of course: Take ideas from single-variable calculus, generalize to multivariable calculus.

So far, we've done this with: functions, limits, derivatives

Today, we begin: integrals!

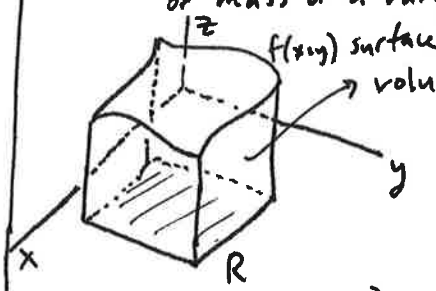
Single-variable integration: area under a curve



Area $= \int_I f(x)\,dx$

where area below the x-axis is taken to be negative

multivariable integration: volume under a surface or mass of a varying-density material



(rectangular region)

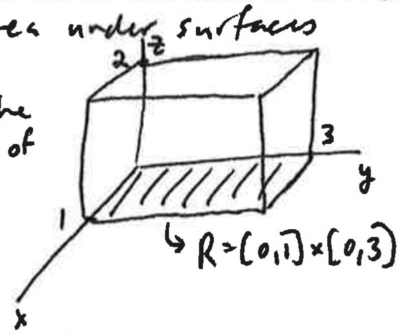
volume $= \iint_R f(x,y)\,dA$

where volume below the xy-plane is taken to be negative

Examples: finding area under surfaces

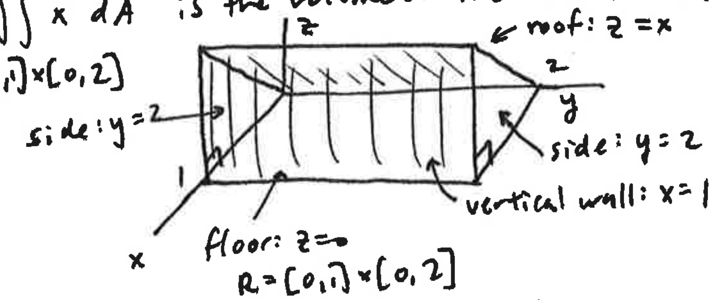
$\iint_{[0,1]\times[0,3]} 2\,dA$ ← this is the volume of

$[0,1]$ ↑ x between 0 and 1; ↖ y between 0 and 3



So the volume of this box $= \iint_{[0,1]\times[0,3]} 2\,dA =$ ______.

$\iint_{[0,1]\times[0,2]} x\,dA$ is the volume of the "lean-to shed"



The volume is (area of Δ side) × (length) $= \frac{1}{2} \times 2 = 1$
or half of (rectangular box) $= \frac{1}{2} \times 1 \times 1 \times 2 = 1$.
so $\iint_{[0,1]\times[0,2]} x\,dA = 1$.

It is elegant to use geometry to find volumes, but it is often not possible; we have to use calculus to compute it algebraically. We use a double integral: first do the inner, then the outer.

$$\iint_{[0,1]\times[0,2]} x\,dA = \int_{y=0}^{2}\int_{x=0}^{1} x\,dx\,dy = \int_{y=0}^{y=2}\underbrace{\left(\int_{x=0}^{x=1} x\,dx\right)}_{\text{do this first}}dy = \int_{y=0}^{y=2}\left(\frac{x^2}{2}\Big|_{x=0}^{x=1}\right)dy = \int_{y=0}^{y=2}\left(\frac{1}{2}\right)dy = \frac{1}{2}y\Big|_{y=0}^{y=2} = 1-0 = 1.$$

or we could integrate first with respect to y:

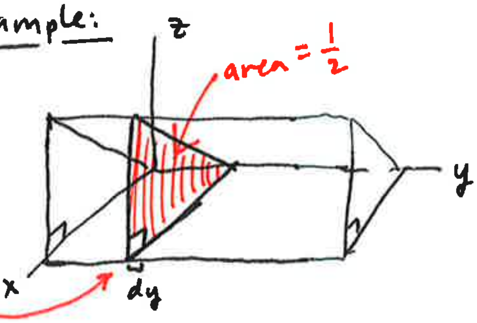
$$\iint_{[0,1]\times[0,2]} x\,dA = \int_{x=0}^{x=1}\int_{y=0}^{y=2} x\,dy\,dx = \int_{x=0}^{x=1}\underbrace{\left(\int_{y=0}^{y=2} x\,dy\right)}_{\text{do this first}}dx = \int_{x=0}^{x=1}\underbrace{\left(xy\Big|_{y=0}^{y=2}\right)}_{x\text{ is a constant with respect to }y}dx = \int_{x=0}^{x=1}(2x)\,dx = x^2\Big|_{x=0}^{x=1} = 1-0 = 1.$$

What do the inner and outer integrals mean?

- the inner integral gives the area of a slice of the region.
- the outer integral gives each slice a tiny thickness and sums up the volumes.

For our shed example:

$$\int_{y=0}^{y=2} \underbrace{\left(\int_{x=0}^{x=1} x \, dx \right)}_{\text{area of slice of volume for a fixed } y} dy :$$

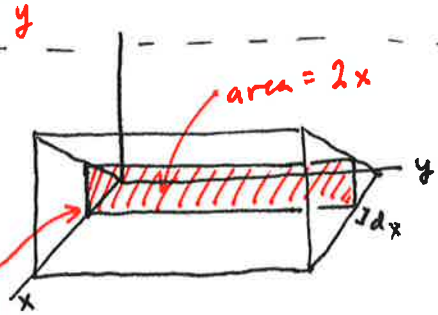


volume of solid
= sum of (area of slice) × (tiny thickness dy)

$$= \int_{y=0}^{y=2} \tfrac{1}{2} \, dy = \tfrac{1}{2} y \Big|_{y=0}^{y=2} = \tfrac{1}{2} \cdot 2 = 1.$$

In this case, the area of each slice is the same, $\frac{1}{2}$. However, the area may also depend on the outer variable of integration:

$$\int_{x=0}^{x=1} \underbrace{\left(\int_{y=0}^{y=1} x \, dy \right)}_{\text{area of slice of volume for a fixed } x} dx$$



volume of solid
= sum of (area of slice) × (tiny thickness dx)

$$= \int_{x=0}^{x=1} 2x \, dx = x^2 \Big|_{x=0}^{x=1} = 1.$$

If our region of integration is of the form $[a,b] \times [c,d]$, we can integrate in either order, as in the example above. It always works out the same in both ways:

Fubini's Theorem: If $f(x,y)$ is a sufficiently well-behaved function (as ours will always be), then

$$\iint_{[a,b]\times[c,d]} f(x,y) \, dA = \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y) \, dx \, dy = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y) \, dy \, dx.$$

> For a double integral over a rectangular region, you can change the order of integration.

Why would we want to change it? To turn an impossible integral into another order, which may be possible.

$$\int_{y=-1}^{y=1} \int_{x=0}^{x=1} y \cdot \underbrace{\sin(x^2) \, dx}_{} \, dy = \int_{x=0}^{x=1} \int_{y=-1}^{y=1} y \cdot \sin(x^2) \, dy \, dx = \int_{x=0}^{x=1} \left(\frac{y^2}{2} \sin(x^2) \Big|_{y=-1}^{y=1} \right) dx = \int_{x=0}^{x=1} \left(\tfrac{1}{2}\sin(x^2) - \tfrac{1}{2}\sin(x^2) \right) dx$$

$\sin(x^2)$ has no antiderivative with respect to x, so we are stuck. change the order!

$$= \int_{x=0}^{x=1} 0 \, dx = 0.$$

So the other order was not only possible, it was in the end quite easy.