Diana Davis — Math 34 — 26 February 2018 — Class #15 ① §4.3

Mathematician spotlight: Nsoki Mamie Mavinga - Associate* Professor, Swarthmore

- *Just promoted to Associate (so, tenured now) on Saturday!
- Studies nonlinear differential equations

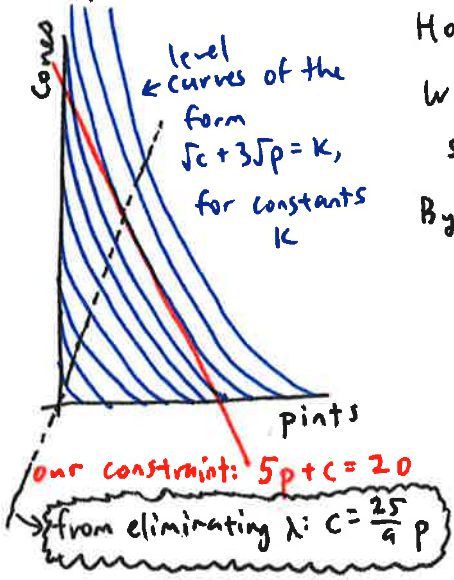
<u>Last time:</u> Introduction to Lagrange multipliers, for optimizing a function under a constraint.

<u>Today:</u> Fun applications, optimized using Lagrange multipliers!

<u>Example from economics:</u> Suppose that the amount of happiness (h) that you derive from consumption of ice cream pints (p) and cones (c) is described by the equation

$h(p,c) = \sqrt{c} + 3\sqrt{p}$ (an additional pint makes you happier than an additional cone).

Suppose that pints are \$5, cones are \$1, and you have \$20 to spend on them.



How many pints and cones should you buy, to maximize happiness???

We wish to maximize $h(p,c) =$ ________

subject to the constraint $g(p,c) =$ ________ $=$ ____.

By the Lagrange multipliers equation, $\nabla h = \lambda \nabla g$, so

$$\begin{bmatrix} \frac{1}{2}c^{-1/2} \\ \frac{3}{2}p^{-3/2} \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 5 \end{bmatrix} \Rightarrow \left.\begin{array}{l} \frac{1}{2\sqrt{c}} = \lambda \\ \frac{3}{2\sqrt{p}} = 5\lambda \end{array}\right\} \Rightarrow \frac{1}{2\sqrt{c}} = \frac{3}{10\sqrt{p}} \Rightarrow 10\sqrt{p} = 6\sqrt{c}$$

$$\Rightarrow 100p = 36c$$

$$\Rightarrow 25p = 9c$$

$c = \frac{25}{9}p \approx 3p$

$\underbrace{5p + c = 20}_{\text{3 eqns, 3 vars}}$

this is the happiness-maximizing ratio of pints to cones. Now plug into our personal budget constraint.

budget: $5p + c = 20$

$c = 20 - 5p$

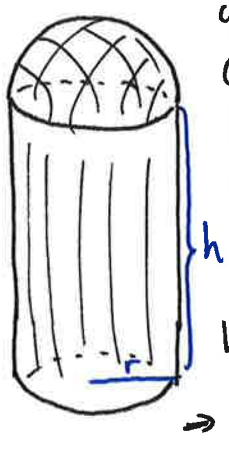
Lagrange multipliers: $25p = 9c = 9(20-5p) \Rightarrow 70p = 180 \Rightarrow p = \frac{18}{7} \approx \boxed{2.6 \text{ pints}}$

$\Rightarrow c = \frac{50}{7} \approx \boxed{7.1 \text{ cones.}}$

Is this actually a max, or perhaps a rogue min? $h(2.6, 7.1) = 7.5$ happiness

try a nearby point satisfying constraint: $h(2, 10) = 7.4$ happiness $\Rightarrow$ yay, a max indeed!

<u>Example from construction:</u> Suppose you are building a silo to store grain, which will be a cylinder with a hemispherical top, as shown. You have 600π square feet of corrugated aluminum to construct the sides and top (the floor is pre-existing). What dimensions should you make it to maximize the volume of storage, which is the volume of the cylindrical part (not the hemispherical part)?



circ of circle $= 2\pi r$
vol of cyl $= \pi r^2 h$
S.A. of sphere $= 4\pi r^2$

We wish to maximize $f(r,h) =$ ________

under the constraint $g(r,h) =$ ________ $=$ ____

Lagrange multipliers equation: $\nabla f = \lambda \nabla g$

$$\Rightarrow \begin{bmatrix} 2\pi r h \\ \pi r^2 \end{bmatrix} = \lambda \begin{bmatrix} 2\pi h + 4\pi r \\ 2\pi r \end{bmatrix} \Rightarrow \begin{array}{l} \cancel{2\pi} rh = \lambda(\cancel{2\pi} h + \overset{2}{\cancel{4\pi}} r) \\ \cancel{\pi} r^2 = \lambda \cdot 2\cancel{\pi} r \end{array} \Rightarrow \left.\begin{array}{l} rh = \lambda(h+2r) \\ r^2 = \lambda \cdot 2r \\ 2\pi r h + 2\pi r = 600\pi \end{array}\right\}$$

From the Lagrange multipliers equation:

① $rh = \lambda(h+2r)$

② $r^2 = \lambda \cdot 2r$ ← we know $r \neq 0$ at a maximum, since that would give zero volume, so solve: $\lambda = \frac{r^2}{2r} = \frac{r}{2}$. Plug into ①: $rh = \frac{r}{2}(h+2r)$

$2rh = r(h+2r)$

$2rh = rh + 2r^2$

$rh = 2r^2$ again, $r \neq 0$, so

$h = 2r$ ← volume-maximizing proportion!

Now plug this volume-maximizing proportion $h=2r$ into constraint $\cancel{2\pi} rh + \cancel{2\pi} r^2 = \cancel{600\pi}$ 300

simplify to $rh + r^2 = 300$

$r(2r) + r^2 = 300$

$3r^2 = 300 \Rightarrow \boxed{r = 10 \Rightarrow h = 20.}$

Check that this is indeed a max: volume $(r=10, h=20) = \pi \cdot 10^2 \cdot 20 = 2000\pi\ ft^3$ ← max! :)

compute at a nearby point on constraint: volume$(r=15, h=5) = \pi \cdot 15^2 \cdot 5 = 1125\pi\ ft^3$

Suppose that we allowed the hemispherical top to also be filled with grain. What would be the volume-maximizing radius and height then?

We wish to maximize $f(r,h) = \underbrace{\pi r^2 h}_{\text{cylinder}} + \underbrace{\frac{2}{3}\pi r^3}_{\text{hemisphere}}$

subject to constraint $g(r,h) = 2\pi rh + 2\pi r^2 = 600\pi$.

$\nabla f = \lambda \nabla g \Rightarrow \begin{bmatrix} 2\pi rh + 2\pi r^2 \\ \pi r^2 \end{bmatrix} = \lambda \begin{bmatrix} 2\pi h + 4\pi r \\ 2\pi r \end{bmatrix} \Rightarrow$ $\cancel{2\pi} rh + \cancel{2\pi} r^2 = \lambda(\cancel{2\pi} h + \cancel{4}^{2}\cancel{\pi} r)$

$\cancel{\pi} r^2 = \lambda(2\cancel{\pi} r)$

$\Rightarrow rh + r^2 = \lambda(h+2r)$ ①

$r^2 = \lambda(2r) \longrightarrow$ again, $r \neq 0$ so $\lambda = \frac{r^2}{2r} = \frac{r}{2}$ $\Rightarrow rh + r^2 = \frac{r}{2}(h+2r)$ ②

$2rh + 2r^2 = r(h+2r)$

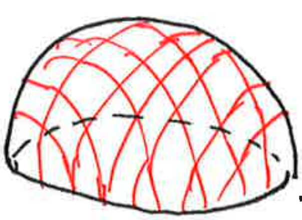
$2rh + \cancel{2r^2} = rh + \cancel{2r^2}$

$rh = 0 \Rightarrow r=0$ or $\boxed{h=0.}$?!

not possible because then volume $=0$.

So Lagrange multipliers says volume is maximized when $h=0$.

Is this a mistake? How could this be?

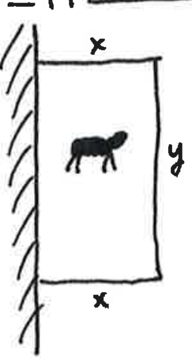


$h=0$

↙ Oh, right! A hemisphere is volume-maximizing for given surface area when the base is "free."

This is why soap bubbles, solving the reverse problem (minimizing surface area for fixed volume of air) make hemispheres on a soapy surface.

Application to animal husbandry: Suppose that you wish to make a rectangular pen along the side of a building, that encloses $32\ m^2$ of area with minimum fencing. How to do it?



minimize $f(x,y) = 2x + y$

subject to $g(x,y) = xy = 32$

$\nabla f = \lambda \nabla g \Rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} y \\ x \end{bmatrix} \Rightarrow$ $\left.\begin{matrix} 2 = \lambda y \\ 1 = \lambda x \\ xy = 32 \end{matrix}\right\}$ 3 eqns 3 vars

Mult. ① by x: $2x = \lambda y x$

mult. ② by y: $1y = \lambda xy$ $\Big\} \Rightarrow 2x = \lambda xy = 1y \Rightarrow \boxed{2x = y}$ fence-minimizing proportions

Plug into constraint: $xy = 32$

$x(2x) = 32 \Rightarrow 2x^2 = 32 \Rightarrow \boxed{x = 4 \Rightarrow y = 8.}$

Again, check nearby to ensure max, not min.