Diana Davis Math 34 21 February 2018 Class #14 ① §4.3

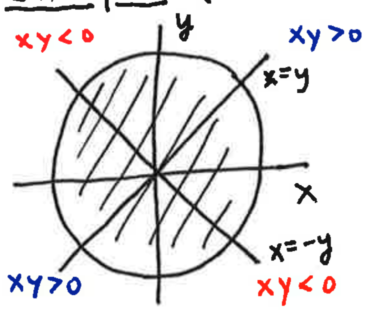
Today: office hours 1:00-2:30, 3:00-4:15; review 8pm SCI 101
Friday: Midterm 1 in class

Mathematician spotlight: Kwadwo Antwi-Fordjour, Earlham College
- modeling head re-growth of hydra using differential equations
- speaking TODAY 4:30 pm in SCI 181 (VAP job candidate)

Last time: To find absolute extrema of a function on a bounded region, you have to check for critical points on interior, critical pts of boundary curve(s), and corners.

Today & Monday: Absolute extrema on a constraint curve using Lagrange multipliers.

Example. (review using strategy from last time) Find absolute extrema of $f(x,y)=xy$ over the closed unit disk $x^2+y^2\le 1$.



① Find critical points on interior by setting $\nabla f=\vec{0}$:

$$\nabla f=\begin{bmatrix} y \\ x\end{bmatrix}=\begin{bmatrix} 0 \\ 0\end{bmatrix} \Rightarrow (0,0) \text{ is the only critical point.}$$

② Find critical points of f on the boundary curve:

boundary curve is unit circle: $\begin{cases} x=\cos\theta \\ y=\sin\theta\end{cases}$ so on the boundary, $f(x,y)=xy$

$\Rightarrow f(\theta)=\cos\theta\cdot\sin\theta$

$f'(\theta)=-\sin^2\theta+\cos^2\theta=0 \Rightarrow \cos^2\theta=\sin^2\theta$

$\Rightarrow x^2=y^2$

$\Rightarrow x=\pm y$

$\Rightarrow \left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$

Now check value of f at these points:

$f(0,0)=0$ ← neither a max nor a min

$f\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)=f\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)=\frac{1}{2}$ ← two equal maxima

$f\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)=f\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)=-\frac{1}{2}$ ← two equal minima

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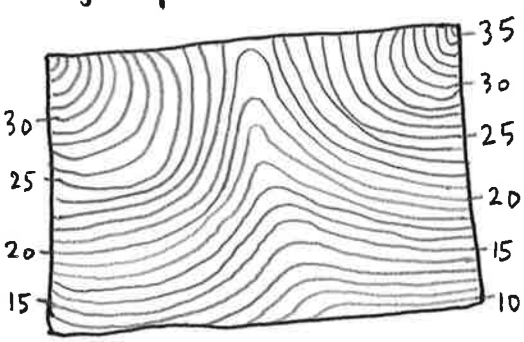
Lagrange's great idea: a new strategy for finding critical points of f on the boundary.



← This is a topographical (elevation) map of a piece of land.

Draw a lumpy blob outline. This is the fence that encloses your herd of sheep.

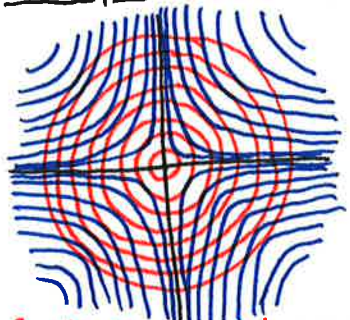
Find the points of highest & lowest elevation of the fence. How did you find them? ________________

Lagrange's idea: At a critical point of the function $f(x,y)$ on the constraint $g(x,y)=k$,
- the boundary curve g is tangent to a level curve of f
- ∇f and ∇g point in the same direction: they are multiples of each other.

So, to optimize $f(x,y)$ or $f(x,y,z)$ subject to constraint $g(x,y)=k$ or $g(x,y,z)=k$, solve the Lagrange multipliers equation $\nabla f(x,y)=\lambda\,\nabla g(x,y)$.

← λ lambda: some constant

Example. Find absolute extrema of $f(x,y)=xy$ on the unit circle. (same as before)



family of constraint curves
family of level curves

We wish to maximize/minimize the function $f(x,y)=$ __________

subject to the constraint $g(x,y)=$ __________ $=$ _____.

By the Lagrange multipliers equation,

$$\nabla f=\lambda\cdot\nabla g \Rightarrow \begin{bmatrix} y \\ x \end{bmatrix}=\lambda\begin{bmatrix} 2x \\ 2y \end{bmatrix} \Rightarrow \underbrace{\begin{matrix} y=\lambda\cdot 2x \\ x=\lambda\cdot 2y \\ x^2+y^2=1. \end{matrix}}_{\text{3 eqns, 3 variables}}$$

Usual strategy: find a way to eliminate λ, because we don't care what it is.

Idea: multiply 1st equation by y, 2nd equation by x, so RHS will be equal.

$$\begin{matrix} y^2=\lambda\cdot 2xy \\ x^2=\lambda\cdot 2xy \end{matrix} \Rightarrow y^2=x^2. \Rightarrow y=\pm x.$$

now, this tells us that the places where the constraint curves of the form $x^2+y^2=r^2$ are tangent to the level curves of the function $g(x,y)=k$ occur along the lines $y=\pm x$.

Can you see this in the picture above?

Finally, plug this into the constraint equation to solve for points:

$y=x$ and $x^2+y^2=1 \Rightarrow (\sqrt{2}/2, \sqrt{2}/2)$ and $(-\sqrt{2}/2, -\sqrt{2}/2)$ maxes

$y=-x$ and $x^2+y^2=1 \Rightarrow (\sqrt{2}/2, -\sqrt{2}/2)$ and $(-\sqrt{2}/2, \sqrt{2}/2)$ mins

as before.

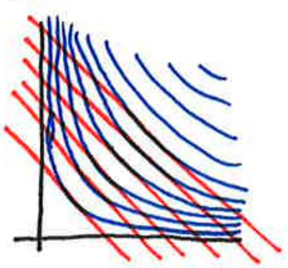
Example. Find the largest possible product of three numbers whose sum is 100.

→ not possible because ____________________

How about: Find the largest possible product of three positive numbers whose sum is 100.

Let the three numbers be x,y,z. We seek to maximize $f(x,y,z)=$ __________

under the constraint $g(x,y,z)=$ ________ $=$ ____.



family of constraint curves
family of level curves

By the Lagrange multipliers equation, $\nabla f=\lambda\nabla g$, so

$$\begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}=\lambda\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \underbrace{\begin{matrix} yz=\lambda \\ xz=\lambda \\ xy=\lambda \\ x+y+z=100 \end{matrix}}_{\text{4 eqns, 4 variables}}$$

$yz=\lambda,\ xz=\lambda \Rightarrow \frac{y\cancel{z}}{\cancel{z}}=\frac{x\cancel{z}}{\cancel{z}} \Rightarrow x=y.$ We can divide by z because __________

$xz=\lambda,\ xy=\lambda \Rightarrow \frac{\cancel{x}z}{\cancel{x}}=\frac{\cancel{x}y}{\cancel{x}} \Rightarrow z=y.$

together, $x=y=z$.
This is where the constraint & level curves are tangent.

Now plug into our constraint: $x+y+z=100 \Rightarrow x+x+x=100 \Rightarrow x=\frac{100}{3}. \Rightarrow$ numbers are all $100/3$.

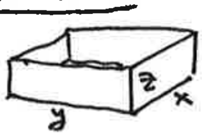
How do we know it's a max, not a min? $f(\frac{100}{3},\frac{100}{3},\frac{100}{3})\approx 37037$

$f(33,33,34)=37026$ ← function value at nearby point is lower, so it's a max!

Example. Find the largest volume of a rectangular box whose length, width, height sum to 100.

→ Did that!

How about: Find the largest volume of an open-top rectangular box made of $12\,ft^3$ of cardboard.



We wish to maximize $f(x,y,z)=$ __________

subject to constraint $g(x,y,z)=$ __________ $=$ ____.

so $\nabla f=\lambda\nabla g \Rightarrow \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}=\lambda\begin{bmatrix} y+2z \\ x+2z \\ 2x+2y \end{bmatrix}$

... etc.