Diana Davis   Math 34   19 February 2018   Class #13 ①
§4.2

Exam in class on Friday – through gradients / class 11 / §2.6

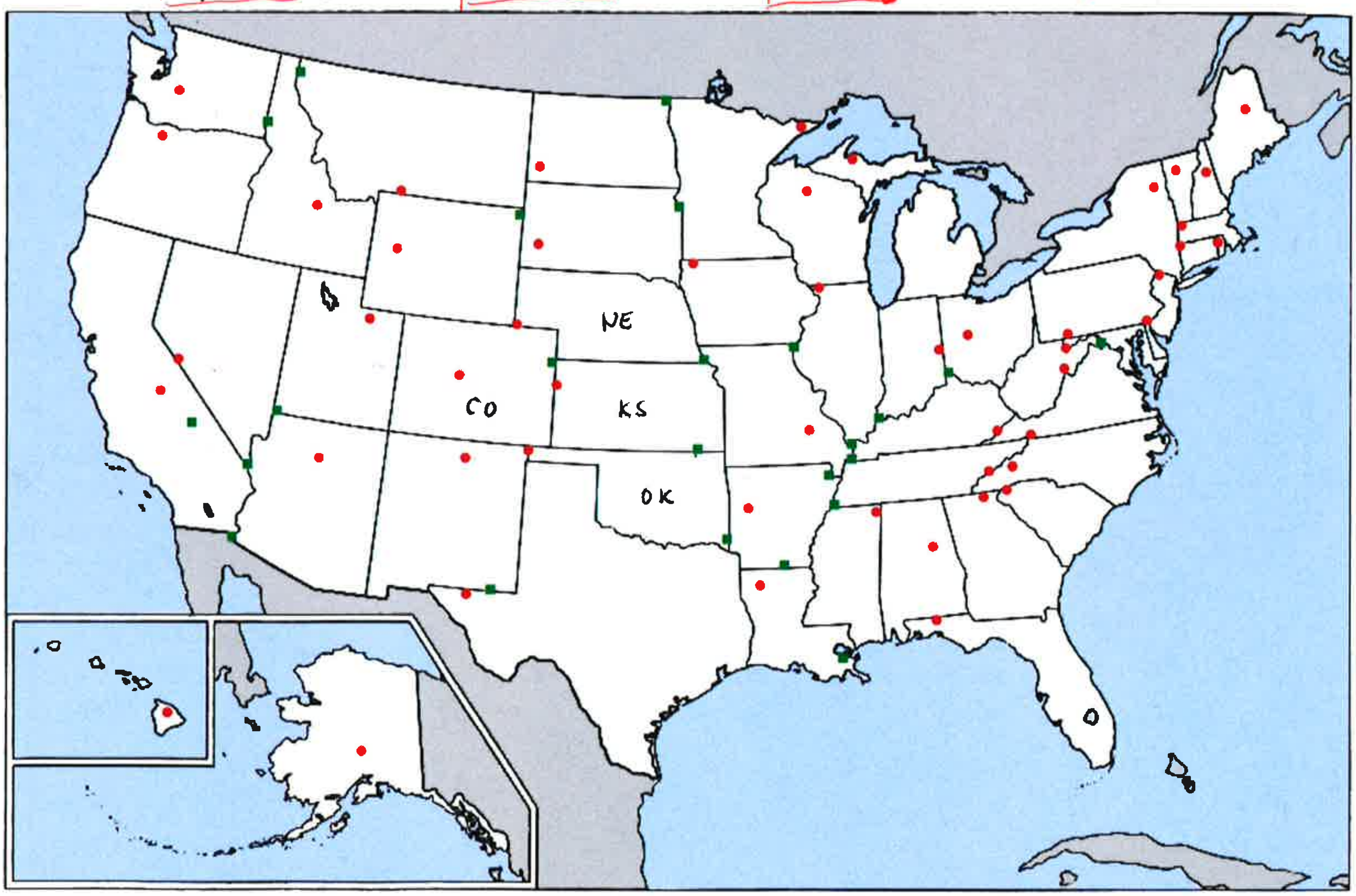
Mathematician spotlight: Kathryn Lindsey, Boston College (Williams math undergrad)
- dynamical systems, complex dynamics
- showed that you can get a Julia set in any shape, incl. cat.

Last time: Find & classify critical points (where $\nabla f = 0$) of multiv. fcns using eigenvalues of Hf.

Today: Find absolute max & min of a function on a constrained (closed, bounded) region.

The map below marks the highest (red) & lowest (green) point of each U.S. state.
Choose 10 states, and for each one, say whether the highest point is:

at an interior point   along an edge   at a corner   somewhere else ?



Record your results & observations:

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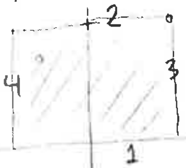
Starting from the high point of Colorado (CO), sketch in plausible level curves for elevation (topo lines) that result in the high point locations for NE, KS and OK.

Example. Find the absolute extrema of $f(x,y) = x^2 + xy + y^2 - 6y$ over the rectangle $-3 \le x \le 3$, $0 \le y \le 5$:



① Find critical points of f that are inside the region.

$\nabla f = \begin{bmatrix} 2x+y \\ x+2y-6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} y=-2x \\ x+2y-6=0 \end{matrix} \Rightarrow x+2(-2x)-6=0 \Rightarrow -3x=6 \Rightarrow x=-2 \Rightarrow y=4.$

so $(-2,4)$ is the only C.P.

notice that it is inside the rectangle.

② Check the four boundary components for critical points along the region.

1: $y=0$ so $f(x,0) = x^2$
$f'(x,0) = 2x = 0 \Rightarrow x=0 \Rightarrow (0,0)$

notice that these are on the boundary of the rectangle.

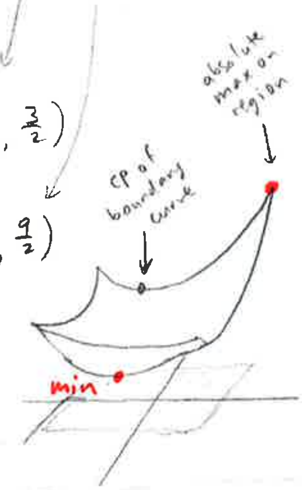
2: $y=5$ so $f(x,5) = x^2+5x+25-30$
$f'(x,5) = 2x+5 = 0 \Rightarrow x = -\frac{5}{2} \Rightarrow (-\frac{5}{2}, 5)$

3: $x=3$ so $f(3,y) = 9+3y+y^2-6y$
$f'(3,y) = 3+2y-6 = 2y-3 = 0 \Rightarrow y = \frac{3}{2} \Rightarrow (3, \frac{3}{2})$

4: $x=-3$ so $f(-3,y) = 9-3y+y^2-6y$
$f'(-3,y) = -3+2y-6 = 2y-9=0 \Rightarrow y = \frac{9}{2} \Rightarrow (-3, \frac{9}{2})$

③ check the four corners, in case the extreme value is not a critical point of the boundary curve:

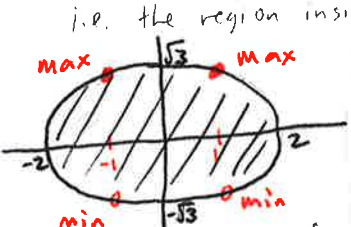
List of candidates	value of f there	
$(-2,4)$	-12	← min; interior
$(0,0)$	0	edges
$(-\frac{5}{2},5)$	-11.25	
$(3,\frac{3}{2})$	6.75	
$(-3,\frac{9}{2})$	-11.25	
$(3,0)$	9	corners
$(3,5)$	19	← max
$(-3,5)$	-11	
$(-3,0)$	9	



So the max value of f is 19, attained at $(3,5)$, and the min value of f is -12, at $(-2,4)$.

Example. Find the absolute extrema of $f(x,y) = x^2 y$ over the region $3x^2+4y^2 \le 12$,

i.e. the region inside (and including the boundary) of the ellipse $3x^2+4y^2=12$.



① Find critical points of f that are inside the region.

$\nabla f = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x=0 \Rightarrow$ the entire y-axis is critical points of f.

② Find critical points of the function restricted to the boundary:

Method 1: $3x^2+4y^2=12 \Rightarrow x^2 = \frac{12-4y^2}{3} = 4 - \frac{4}{3}y^2$

So on the boundary, $f(x,y) = x^2 y$ is just

$f(y) = (4-\frac{4}{3}y^2)y = 4y - \frac{4}{3}y^3$

$f'(y) = 4-4y^2 = 0 \Rightarrow y = \pm 1 \Rightarrow 3x^2+4=12 \Rightarrow x^2 = \frac{8}{3} \Rightarrow x = \pm\sqrt{8/3}$

List of candidates	value of f there	
y-axis $(0,y)$	0	
$(\sqrt{8/3}, 1)$	$8/3$	← max
$(-\sqrt{8/3}, 1)$	$8/3$	← max
$(\sqrt{8/3}, -1)$	$-8/3$	← min
$(-\sqrt{8/3}, -1)$	$-8/3$	← min

Method 2: Parameterize the boundary:

$x = 2\cos t$, $y = \sqrt{3}\sin t$ $\Rightarrow f(x,y) = f(t) = (2\cos t)^2 \cdot \sqrt{3}\sin t = 4\sqrt{3}\cos^2 t \sin t$

$\Rightarrow f'(t) = 4\sqrt{3}(-2\cos t \sin^2 t + \cos^3 t) = 0$

$\Rightarrow -2\cos t \sin^2 t + \cos^3 t = 0$

$\Rightarrow -2\cos t(1-\cos^2 t) + \cos^3 t = 0$

$\Rightarrow -2\cos t + 2\cos^3 t + \cos^3 t = 0$

$\Rightarrow \cos t(-2+3\cos^2 t) = 0$ $\qquad 3\cos^2 t = 2$

$\Rightarrow \cos t = 0$ or $\cos t = \sqrt{2/3} \Rightarrow \sin t = \pm\sqrt{1-(\sqrt{2/3})^2} = \pm\sqrt{\frac{1}{3}}$

$\Rightarrow x=0, y=\pm\sqrt{3}$ (ALREADY HAVE)

$\Rightarrow x = 2\sqrt{2/3} = \sqrt{8/3}$, $y = \sqrt{3}\cdot\pm\sqrt{\frac{1}{3}} = \pm 1$ } as in Method 1

③ check corners: no corners!

So the absolute max of f is $\sqrt{8/3}$, attained at $(1, \sqrt{8/3})$ and $(-1, \sqrt{8/3})$, and the absolute min of f is $-\sqrt{8/3}$, attained at $(1, -\sqrt{8/3})$ and $(-1, -\sqrt{8/3})$.