Diana Davis Math 34 16 February 2018 Class #12 ① §4.2

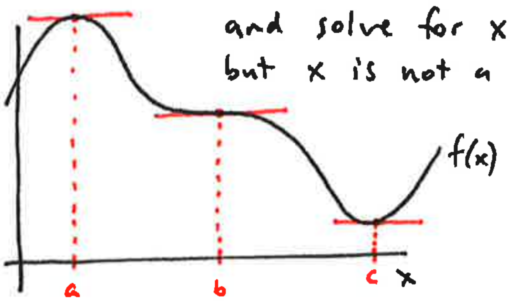
Midterm 1 a week from today in class – covering through class 11 / §2.6 / gradients.

Mathematician spotlight: Radia Perlman – math undergrad, CS Ph.D. (MIT)
- invented Spanning Tree Protocol – crucial to Internet ('70s)
- founding expert on network & security protocols

Last time: Using gradients to find · direction of greatest increase/decrease
· tangent lines & tangent planes to implicitly-defined {curves / surfaces.

This time: Optimization in multivariable situations! Finding extrema (max/min) of a function!

Review: In single-variable calculus, to find the maxes & mins of $f(x)$, you set $f'(x)=0$ and solve for x. Sometimes $f'(x)=0$ but x is not a max or min, as at b.



You also need to check the endpoints if you're optimizing on a closed, bounded set.

New: In multivariable calculus, to find the maxes & mins of $f(x,y)$, you set $f_x=0$ and $f_y=0$ and solve for x and y. As in single-variable calculus, this sometimes picks up points that are not maxes or mins, and we have to check the boundary if optimizing over a closed, bounded set.

Notice that at each critical point, the tangent plane is horizontal.

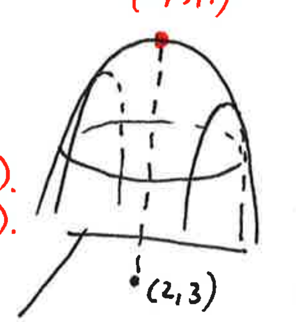
To find maxes & mins of a function, we find all the critical points, where $\begin{cases} f_x(x,y)=0 \\ f_y(x,y)=0 \end{cases} \Leftrightarrow \nabla f(x,y)=\vec{0}$, and then classify them as a max, min or saddle.

Example. Find all local extrema of $f(x,y)=4x+6y-x^2-y^2-12$.

$\nabla f=\begin{bmatrix} 4-2x \\ 6-2y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x=2 \\ y=3 \end{cases}$ is the only critical point of f.

Is $(2,3)$ a max, min or saddle?

Hmm. $f(x,y)=-(x^2-4x+4)-(y^2-6x+9)+1$
$=\underbrace{-(x-2)^2-(y-3)^2+1}_{\text{a downward-opening paraboloid whose maximum is at } (x,y)=(2,3)\text{. or } (x,y,z)=(2,3,1)\text{.}}$



So $(2,3)$ gives a maximum value for f.

→ Being able to complete the square was very lucky! What do we do in general??

Isn't there a more systematic way to do this? Maybe a multivariable second derivative test? YES!

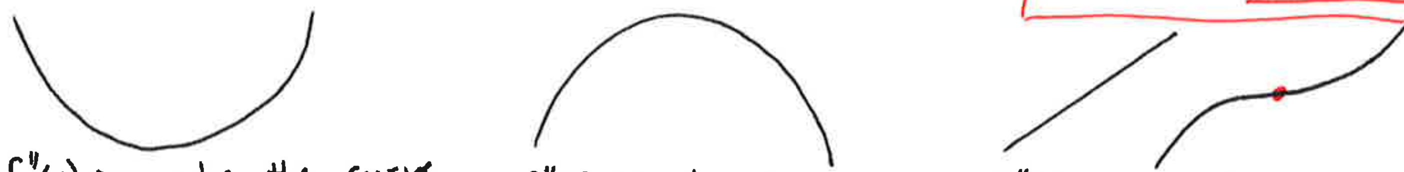
<u>Review:</u> second derivative test from single-variable calculus.

$f(x)$ - value of function - position

$f'(x)$ - rate of change of function - slope - velocity

$f''(x)$ - rate of change of slope - concavity - acceleration

If $f'(a)=0$, then
- $f''(a)>0 \Rightarrow a$ is a ______
- $f''(a)<0 \Rightarrow a$ is a ______
- $f''(a)=0 \Rightarrow$ ______



$f''(x)>0$ when the curve is concave-up

$f''(x)<0$ when the curve is concave-down

$f''(x)=0$ when the curve is linear, or instantaneously flat (infl. pt)

The multivariable second derivative test uses the "Hessian" matrix of partial derivatives:

$$Hf = \begin{bmatrix} f_{xx}(x,y) & f_{yx}(x,y) \\ f_{xy}(x,y) & f_{yy}(x,y) \end{bmatrix}$$

<u>Here's how it goes:</u> Set $\nabla f = \vec{0}$ and solve for critical points.

Let $\vec{a}$ be one of the critical points. Let's classify it.

$Hf(\vec{a})$ is a 2×2 matrix of numbers. It has 2 eigenvalues: λ_1, λ_2.

- if $\lambda_1, \lambda_2 > 0$, this corresponds to a local shape like a paraboloid opening up , so $\vec{a}$ is a local <u>min</u> for f.
- if $\lambda_1, \lambda_2 < 0$, this corresponds to a local shape like a paraboloid opening down , so $\vec{a}$ is a local <u>max</u> for f.
- if one is positive and one is negative, this corresponds to a shape like one parabola opening up and the other opening down, , like a hyperbolic paraboloid, so $\vec{a}$ is a saddle point.

<u>Example.</u> $f(x,y) = 4x + 6y - x^2 - y^2 - 12$ has one critical point, $\vec{a} = (2,3)$.

$$Hf(2,3) = \begin{bmatrix} \_ & \_ \\ \_ & \_ \end{bmatrix} \Rightarrow \lambda_1 = -2,\ \lambda_2 = -2,$$

so $(2,3)$ is a local <u>maximum</u> for f.

<u>Example.</u> The same works in higher dimensions, with criteria "all positive," "all negative" and "some pos, some neg."

$f(x,y,z) = xy + xz + 2yz - 1/x$

$$\nabla f(x,y,z) = \begin{bmatrix} y+z+1/x^2 \\ x+2z \\ x+2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \text{one critical point, } (1, -1/2, -1/2).$$

$$Hf = \begin{bmatrix} 2/x^3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \Rightarrow Hf(1,-1/2,-1/2) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

eigenvalues are $-2, -2, 2$

$\Rightarrow (1, -1/2, -1/2)$ is a <u>saddle point.</u>