Diana Davis   Math 34   February 14, 2018   Class #11 ①
§2.6

Mathematician spotlight: Ron Buckmire ♥ Dean Elzinga

Happy Valentine's Day!

Ron: NSF program officer for financially supporting student math.

Dean: Opera singer (20 years), now senior machine learning engineer.

Last time: The gradient of $f(x,y)$ is $\nabla f(x,y) = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix}$ or $\nabla f(x,y,z) = \begin{bmatrix} f_x(x,y,z) \\ f_y(x,y,z) \\ f_z(x,y,z) \end{bmatrix}$ etc.

The directional derivative of f at (x_0,y_0) in the direction of unit vector $\vec{u}$ is

$$D_{\vec{u}} f(x_0,y_0) = \nabla f(x_0,y_0) \bullet \vec{u}.$$

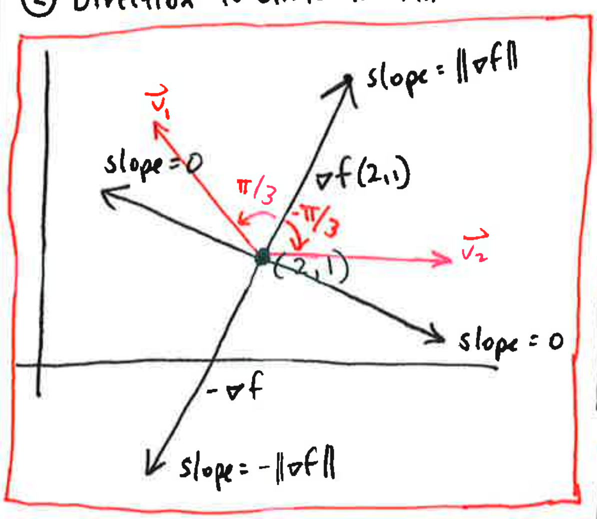
Today: The gradient is perpendicular to the level curves/surfaces, which gives us a new way to find the tangent plane to a surface, even one that isn't the graph of a surface (e.g. sphere).

Example. Let $f(x,y) = xy$. Find the direction of steepest ascent at $(2,1)$.
Then find the direction(s) you could go to climb at half this steepness.

① Direction of steepest ascent is ______________ so let's compute:

$\nabla f(x,y) = \begin{bmatrix} \\ \end{bmatrix} \Rightarrow \nabla f(2,1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The rate of increase, or slope, in this direction, is ________.

② Direction to climb at half the maximum steepness: find direction $\vec{u}$ so that



$$D_{\vec{u}} f(2,1) \overset{\text{same}}{=} \nabla f(2,1) \bullet \vec{u} \overset{\text{same}}{=} \|\nabla f(2,1)\| \|\vec{u}\| \cos\theta \overset{\text{same}}{=} \|\nabla f(2,1)\| \cos\theta$$

want $\frac{1}{2}$ maximum steepness: $= \frac{1}{2}\|\nabla f(2,1)\|$.

$\Rightarrow \cos\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3}$. So, rotate:

rotate by $\pi/3$:

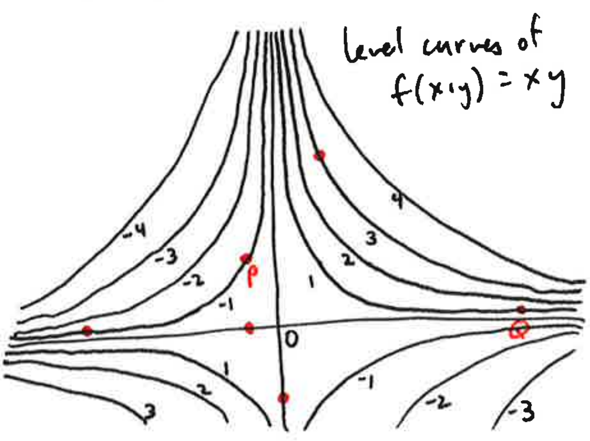
$$\begin{bmatrix} \cos\pi/3 & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} \frac{1-2\sqrt{3}}{2\sqrt{5}} \\ \frac{2+\sqrt{3}}{2\sqrt{5}} \end{bmatrix} = \vec{v}_1$$

divide by $\sqrt{5}$ to make it a unit vector

rotate by $-\pi/3$:

$$\begin{bmatrix} \cos-\pi/3 & -\sin-\pi/3 \\ \sin-\pi/3 & \cos-\pi/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} \frac{1+2\sqrt{3}}{2\sqrt{5}} \\ \frac{2-\sqrt{3}}{2\sqrt{5}} \end{bmatrix} = \vec{v}_2$$

Use level curves to sketch the gradient vector at some points. (these: •)



- $\nabla f(x,y)$ should · be perpendicular to level curve
- point in direction of ascent
- be appropriately scaled.

which vector has greater magnitude, $\nabla f(P)$ or $\nabla f(Q)$? ________

why? ______________________

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Why is the gradient perpendicular to the level sets (level curves, level surfaces)?

① ∇f points in the direction of steepest increase, and $-\nabla f$ points in the direction of steepest decrease, so the direction "between," perpendicular to both, has no change.

② The level curve of a function $f(x,y)$ at level k has equation $f(x,y)=k$.

Suppose the level curve has equation $\vec{r}(t)=\begin{bmatrix}x(t)\\y(t)\end{bmatrix}$, so $f(x(t),y(t))=k$.

Differentiate both sides with respect to t: $\frac{d}{dt}\left(f(x(t),y(t))\right)=\frac{d}{dt}(k)$

$$\Rightarrow \frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}=0$$

$$\Rightarrow \left[\partial f/\partial x, \partial f/\partial y\right]\cdot\left[dx/dt, dy/dt\right]=0 \Rightarrow \nabla f\cdot\begin{bmatrix}x'(t)\\y'(t)\end{bmatrix}=0 \Rightarrow \nabla f\cdot\vec{r}\,'(t)=0$$

tangent vector to the level curve!

Example. Consider the implicitly-defined curve $e^{xy}-xy=1$. Find the tangent line at $(0,1)$.

We need: ① a point on the line: ____

② the tangent vector to the curve. Set $F(x,y)=e^{xy}-xy$.

Our curve is the level set $F(x,y)=$ ____. So $\nabla F(0,1)$ is perpendicular to our desired tangent vector:

$$\nabla F(x,y)=\begin{bmatrix}\quad\end{bmatrix}\Rightarrow\nabla F(0,1)=\begin{bmatrix}\quad\end{bmatrix}\Rightarrow \text{tangent vector is }\begin{bmatrix}\quad\end{bmatrix}\text{ so }\begin{bmatrix}x(t)\\y(t)\end{bmatrix}=\begin{bmatrix}\quad\end{bmatrix}+\begin{bmatrix}\quad\end{bmatrix}t=\begin{bmatrix}\quad\end{bmatrix}.$$

Example. Find an equation for the tangent plane to $x^2+y^2+z^2=3$ at $(1,1,1)$.

The sphere is a level surface of $g(x,y,z)=x^2+y^2+z^2$ at level ____.

For a tangent plane, we need ① a point $(1,1,1)$ and ② a normal vector, $\nabla g(1,1,1)$:

$$\nabla g(x,y,z)=\begin{bmatrix}2x\\2y\\2z\end{bmatrix}\Rightarrow\nabla g(1,1,1)=\begin{bmatrix}2\\2\\2\end{bmatrix}\Rightarrow \text{equation is}$$

$$2(x-1)+2(y-1)+2(z-1)=0.$$

Example. On the surface $xyz=8$, which points have a tangent plane parallel to $x+2y+4z=100$? → find points whose tangent vector to surface is multiple of $\begin{bmatrix}1\\2\\4\end{bmatrix}$.

View this surface as a level surface of $g(x,y,z)=$ ______ at level ____.

$$\nabla h(x,y,z)=\begin{bmatrix}yz\\xz\\xy\end{bmatrix}=\lambda\begin{bmatrix}1\\2\\4\end{bmatrix}\Rightarrow\begin{cases}yz=\lambda\\xz=2\lambda\\xy=4\lambda\end{cases}$$

gradient can be any (nonzero) multiple of $\begin{bmatrix}1\\2\\4\end{bmatrix}$

notice: ① x,y,z cannot be zero since $xyz=8$

② Solve to get $x^3=64\Rightarrow x=4$
$\Rightarrow y=2,\ z=1$

So $(4,2,1)$ is the unique (only) point whose tangent plane is parallel to $x+2y+4z=100$. ☺