Diana Davis　　Math 34　　12 February 2018　　Class #10 ①
§2.6

Mathematician spotlight: David Rockoff, University of Arizona
- differential item functioning - fairness of tests
- uses randomization, simulations & data

Last time: chain rule for multivariable functions

This time: vector-valued chain rule example, plus directional derivatives!

Review: Suppose $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$. Then $\frac{d}{dx}f(g(x)) = \underbrace{\frac{d}{dx}(f\circ g)(x)}_{\text{function composition notation}} = f'(g(x))\cdot g'(x) = \frac{df}{dg}\cdot\frac{dg}{dx}$.

New: Suppose $f:\mathbb{R}^m\to\mathbb{R}^p$, $g:\mathbb{R}^n\to\mathbb{R}^m$.

Then the Jacobian of $f\circ g:\mathbb{R}^n\to\mathbb{R}^p$ is

$$D(f\circ g) = \begin{bmatrix} Df \\ (p\times m)\end{bmatrix}\begin{bmatrix} Dg \\ (m\times n)\end{bmatrix} = \begin{bmatrix} Df\cdot Dg \\ (p\times n)\end{bmatrix}.$$

Let's do an example!

Example. Suppose $f(x,y,z) = \begin{bmatrix} xyz \\ x+y\end{bmatrix} = \begin{bmatrix} f_1(x,y,z) \\ f_2(x,y,z)\end{bmatrix}$ and $x(s,t)=st$, $y(s,t)=s+t$, $z(s,t)=s^2-t^2$, so $g(s,t)=\begin{bmatrix} st \\ s+t \\ s^2-t^2\end{bmatrix}$.

Consider $f\circ g = f(g):\mathbb{R}^2\to\mathbb{R}^2$, because $\mathbb{R}^2 \xrightarrow{g} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}^2$.

Then $D(f\circ g) = Df\cdot Dg = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial z\end{bmatrix}\begin{bmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y/\partial t \\ \partial z/\partial s & \partial z/\partial t\end{bmatrix} = \underbrace{\begin{bmatrix} yz & xz & xy \\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix} t & s \\ 1 & 1 \\ 2s & -2t\end{bmatrix}}_{\text{now multiply out and put everything in terms of } s \text{ and } t.}$.

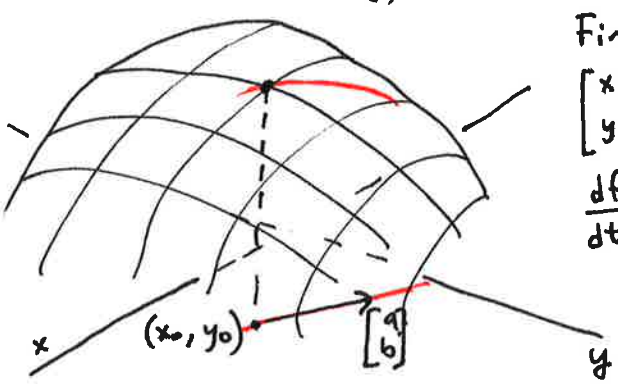
If you want to find this Jacobian matrix at a point $(s,t)=(1,1)$, compute $(x,y,z)=(1,2,0)$ and plug in:

$$D(f\circ g)(\vec{a}) = Df(g(\vec{a}))\cdot Dg(\vec{a})$$

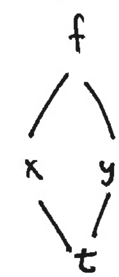
$$\left.\begin{aligned} D(f\circ g)(1,1) &= Df(g(1,1))\cdot Dg(1,1) \\ &= Df(1,2,0)\cdot Dg(1,1)\end{aligned}\right\} = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 2 & -2\end{pmatrix} = \begin{pmatrix} 4 & -4 \\ 2 & 2\end{pmatrix} = \begin{pmatrix} \partial f_1/\partial s & \partial f_1/\partial t \\ \partial f_2/\partial s & \partial f_2/\partial t\end{pmatrix}.$$

Today: Directional derivatives! We know the "slope" of $z=f(x,y)$ in the positive x-direction is f_x, and in pos y-direction is f_y, but what about the slope in other directions?

$z=f(x,y)$



Find "slope" at (x_0,y_0) in direction $\begin{bmatrix} a \\ b\end{bmatrix}$: This line is $\begin{bmatrix} x(t) \\ y(t)\end{bmatrix} = \begin{bmatrix} x_0+at \\ y_0+bt\end{bmatrix}$ and we want $\frac{d}{dt}f(x(t),y(t))$.



$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = \frac{\partial f}{\partial x}\cdot a + \frac{\partial f}{\partial y}\cdot b$$

$$= \left[\partial f/\partial x,\ \partial f/\partial y\right]\cdot\begin{bmatrix} a \\ b\end{bmatrix}. \Rightarrow \boxed{D_{\vec{u}}f = \nabla f\bullet\vec{u}}$$

Directional derivative of f in direction $\vec{u}=\begin{bmatrix} a \\ b\end{bmatrix}$.

Note that the "slope" should not depend on the length of the direction vector $\vec{u}$ that we choose, so we always take $\vec{u}$ to be a unit vector:

$\boxed{\text{The directional derivative of } f(x,y) \text{ at } (x_0,y_0) \text{ in the direction of the unit vector } \vec{u} \text{ is } D_{\vec{u}} f(x_0,y_0) = \nabla f(x_0,y_0) \bullet \vec{u}.}$

<u>Example.</u> Let $f(x,y) = x^2 - y^2$. Find the directional derivative of f at $(3,1)$ in the direction $\begin{bmatrix}1\\2\end{bmatrix}$.

① Find $\nabla f = \begin{bmatrix}2x\\-2y\end{bmatrix}$ ② Find $\nabla f(3,1) = \begin{bmatrix}6\\-2\end{bmatrix}$ ③ Find a unit direction vector: $\vec{u} = \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}$

④ Put them together! $D_{\vec{u}} f(3,1) = \nabla f(3,1) \bullet \vec{u} = \begin{bmatrix}6\\-2\end{bmatrix} \bullet \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix} = 6/\sqrt{5} - 4/\sqrt{5} = \underline{\underline{2/\sqrt{5}}}$.

⑤ What does it mean? ______________________

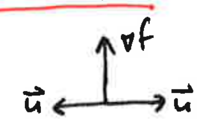
<u>Pondering the geometry:</u> $D_u f(x_0,y_0) = \nabla f(x_0,y_0) \bullet \vec{u} = \|\nabla f(x_0,y_0)\| \cdot \|\vec{u}\| \cdot \cos\theta$ ← where θ is the angle between $\nabla f(x_0,y_0)$ and $\vec{u}$

$= \|\nabla f(x_0,y_0)\| \cdot \cos\theta$ ← since $\|\vec{u}\| = 1$

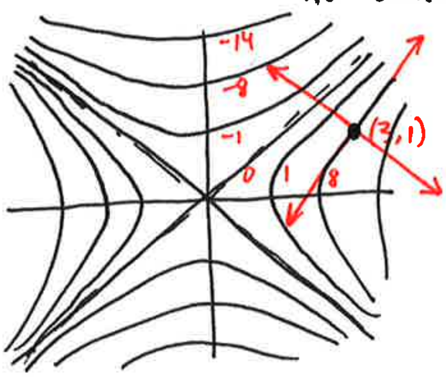
- When is $D_{\vec{u}} f$ the largest positive number? → when $\underbrace{\cos\theta = 1}$, $D_{\vec{u}} f$ is $\|\nabla f\|$, the largest
  ↳ happens when $\theta = 0$, i.e. ∇f and $\vec{u}$ in same direction
  ⇒ <u>gradient points in direction of steepest ascent.</u>

- When is $D_{\vec{u}} f$ the largest negative number? → when $\underbrace{\cos\theta = -1}$, $D_{\vec{u}} f$ is $-\|\nabla f(x_0,y_0)\|$
  ↳ happens when $\theta = \pi$, i.e. ∇f and $\vec{u}$ in opposite directions
  ⇒ <u>direction of steepest descent is $-\nabla f(x_0,y_0)$.</u>

- When is $D_u f = 0$, i.e. which direction(s) can you go to stay on the same level? → ① when $\cos\theta = 0$, so when $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, so when ∇f and $\vec{u}$ are perpendicular
  → ② when you go along a level curve.



<u>Example.</u> For $f(x,y) = x^2 - y^2$, find directions of steepest ascent, steepest descent, and no change, and the directional derivatives (slopes) in those directions.



- Steepest ascent: $\nabla f(3,1) = \begin{bmatrix}6\\-2\end{bmatrix}$ and slope is $\|\nabla f(3,1)\| = \sqrt{6^2+(-2)^2} = \sqrt{40}$.
- steepest descent: ______________________
- no change: ______________________