Diana Davis — Math 34 — February 9, 2018 — Class #9 ① §2.5

Mathematician spotlight: Sarah Koch, Associate Professor, University of Michigan
- complex dynamics, complex analysis
- comes with beautiful pictures – search for "Julia set" online

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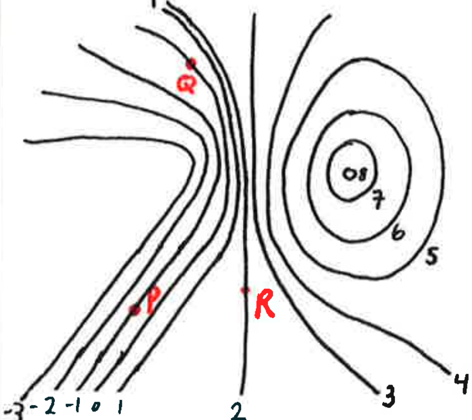
<u>Last time:</u> ① linear approximation of a function $f:\mathbb{R}^m \to \mathbb{R}^n$
② second partial derivatives f_{xx}, f_{xy}, f_{yy} (etc. for more variables).

<u>This time:</u> Explore those more, plus the <u>Chain Rule.</u>

What partial derivatives mean, geometrically: Fill in with ">", "<" or "=".



Level curves of $f(x,y)$

$f_x(R)$ ___ 0 slope if you walk in $+x$ direction

$f_y(P)$ ___ 0 slope if you walk in $+y$ direction

$f_{xx}(P)$ ___ 0 rate of change of slope in $+x$ direction

$f_{yy}(Q)$ ___ 0 rate of change of slope in $+y$ direction

$f_{xy}(R)$ ___ 0 how the slope of an eastward $(+x)$ path changes, if you move your path slightly to the north $(+y)$.

$f_y(R)$ ___ 0 slope if you walk in $+y$ direction

<u>Example of linear approximation:</u> Find the (x,y) coordinates of the following points:

① $(r,\theta) = (2, \frac{\pi}{3}) \Rightarrow (x,y) =$ ________ $(r,\theta) = (2.1, \frac{\pi}{3} - 0.1) \Rightarrow (x,y) =$ ________

Hmmm, need a calculator... or linear approximation!

Let $f:\mathbb{R}^2 \to \mathbb{R}^2$ be defined by
$$f(r,\theta) = \begin{bmatrix} r\cdot\cos\theta \\ r\cdot\sin\theta \end{bmatrix} = \begin{bmatrix} x(r,\theta) \\ y(r,\theta) \end{bmatrix}.$$
Then $Df = \begin{bmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{bmatrix} = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}.$

We want the linear approximation at $\vec{a} = (2, \frac{\pi}{3})$. So let's plug in:

$$L(r,\theta) = f(\vec{a}) + Df(\vec{a})(\vec{r} - \vec{a})$$
$$\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + \begin{bmatrix} 1/2 & -\sqrt{3} \\ \sqrt{3}/2 & 1 \end{bmatrix}\begin{bmatrix} r-2 \\ \theta - \frac{\pi}{3} \end{bmatrix}$$
$$\Rightarrow L(2.1, \tfrac{\pi}{3} - 0.1) = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + \begin{bmatrix} 1/2 & -\sqrt{3} \\ \sqrt{3}/2 & 1 \end{bmatrix}\begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix} \approx \begin{bmatrix} 1.223 \\ 1.719 \end{bmatrix}$$

actual value is $\begin{bmatrix} 1.226 \\ 1.705 \end{bmatrix}$

wow, so close!

It seems like we can differentiate anything! How about functions of other functions?

<u>Example.</u> Let $f(x,y) = x^2 y$ and suppose x and y are also functions:
$x(s,t) = st$
$y(s,t) = e^{st}$.

To find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$, we can just plug in:

$f(s,t) = (st)^2 \cdot e^{st}$
$\Rightarrow \partial f/\partial s = \dots$

This looks like a tedious pain.
Let's use the Chain Rule!

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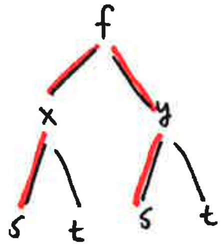
Recall: $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\cdot g'(x)$. In Leibniz notation: $\frac{df}{dx} = \frac{df}{dg}\cdot\frac{dg}{dx}$.

f
| depends on
g
| depends on
x

For more variables: $\frac{\partial f}{\partial s} = \underbrace{\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}} + \underbrace{\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}}$



$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$

Find the dependence of f on s from both contributions.

$f(x,y) = x^2y \qquad x(s,t) = st \qquad y(s,t) = e^{st}$

$\partial f/\partial x = \qquad \partial x/\partial s = \qquad \partial y/\partial s =$

$\partial f/\partial y = \qquad \partial x/\partial t = \qquad \partial y/\partial t =$

So now we can compute:

$\partial f/\partial s = (2xy)(t) + (x^2)(t e^{st})$ ← not done; need in terms of s and t

$= 2\cdot st\cdot e^{st}\cdot t + (st)^2\cdot t\cdot e^{st}$

$\partial f/\partial t = (2xy)(s) + (x^2)(s e^{st})$

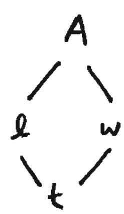
$= 2\cdot st\cdot e^{st}\cdot s + (st)^2\cdot s\cdot e^{st}$.

Example. A patch of moss at the Scott Arboretum is rectangular. In February 2018, its length was 5 meters and increasing by 1 meter per month, and its width was 4 meters and decreasing by ½ meter per month. At what rate was its area changing?

We know: $\ell = 5 \qquad w = 4 \qquad A = \ell w$

$d\ell/dt = 1 \qquad dw/dt = -1/2 \qquad \partial A/\partial \ell = w \qquad \partial A/\partial w = \ell$

← ∂ for partial derivative: function depends on ℓ and w.



We want: $\frac{dA}{dt}$ ← d for total derivative: function only depends on t

Chain Rule: $\frac{dA}{dt} = \frac{\partial A}{\partial \ell}\cdot\frac{d\ell}{dt} + \frac{\partial A}{\partial w}\cdot\frac{dw}{dt} = (w)(1) + (\ell)(-1/2) = 4\times 1 + 5\times\frac{-1}{2} = 4 - 2.5 = \underline{+1.5}$ square m per month.

What if we have more functions and more variables? Make a tree; follow the branches.

Example. $g(x,y,z) = xy^2z^3$

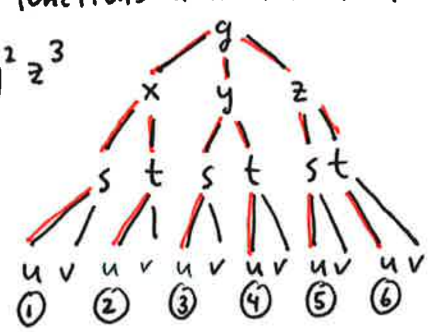
$x(s,t) = st$

$y(s,t) = e^{st}$

$z(s,t) = s+t$

$s(u,v) = u+v$

$t(u,v) = 2u-v$



$\frac{\partial g}{\partial u} = \underset{①}{\frac{\partial g}{\partial x}\frac{\partial x}{\partial s}\frac{\partial s}{\partial u}} + \underset{②}{\frac{\partial g}{\partial x}\frac{\partial x}{\partial t}\frac{\partial t}{\partial u}} + \dots$ (4 more terms)

These functions ↗ are all scalar-valued: $g:\mathbb{R}^3\to\underline{\mathbb{R}}$ and the rest are from $\mathbb{R}^2\to\underline{\mathbb{R}}$.

What about vector-valued functions from $\mathbb{R}^m\to\mathbb{R}^n$? Use matrices, not scalars:

If $f:\mathbb{R}^m\to\mathbb{R}^p$ and $g:\mathbb{R}^n\to\mathbb{R}^m$, then the Jacobian $D(f\circ g)$ for $f\circ g:\mathbb{R}^n\to\mathbb{R}^p$ is:

$$D(f\circ g) = \underset{p\times m}{\begin{bmatrix} Df \end{bmatrix}}\underset{m\times n}{\begin{bmatrix} Dg \end{bmatrix}} = \underset{p\times n}{\begin{bmatrix} Df\cdot Dg \end{bmatrix}}.$$

Matrix multiplication instead of scalar multiplication. Same idea, higher dimensions.