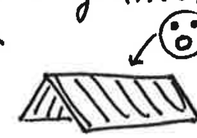
Mathematician spotlight: Piper Harron, postdoc, University of Hawaii
- algebraic number theory (PhD at Princeton)
- intersectional radical feminism, anti-racism

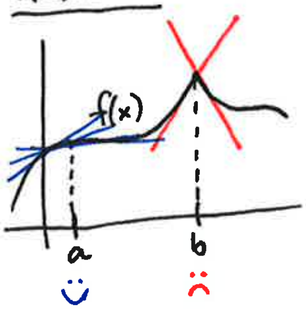
Previously: Limits - exploring one way in which functions can fail to be "nice".

for function values:
- function graph has a "hole" - e.g. $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{(x,y)\to(0,0)} x-y = \underline{\underline{0}}$. Can fill in, no problem.
- function graph has a "vertical part" - e.g. $\lim_{(x,y)\to(0,0)} \frac{x+y}{2x+y}$ DNE - no way to fix ☹

(Partial) Derivatives - rate of change of function (slope) in x- or y-direction
- if function is "smooth", we know what to do. 
- if function has "creases," the derivative is not defined. 

Today: Differentiability & non-differentiability of multivariable functions.

Review: single-variable calculus



$f(x)$ is differentiable at p if:
- f has a well-defined tangent line at p, or (equivalently)
- $\lim_{x\to p^-} f'(x) = \lim_{x\to p^+} f'(x)$, or (equivalently)
- $\lim_{x\to p} \frac{f(x) - (\text{tangent line at } p)}{x-p} = 0.$

New: multivariable calculus

$f(x,y)$ is differentiable at (a,b) if f has a tangent plane at $(a,b,f(a,b))$ that is a good approximation of f, i.e.

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - (\text{tangent plane at }(a,b))}{\underbrace{\|(x,y)-(a,b)\|}_{\text{distance between points.}}} = \underline{\underline{0}}.$$

Example: The nice, everywhere-differentiable function from last time.
$f(x,y) = 5 - x^2 - y^2$ we found the tangent plane at $(1,1,3)$ to be
$L(x,y) = 7-2x-2y = z$.

$\boxed{\text{note: I am using } 5-x^2-y^2 \text{ instead of } 5-x^2-2y^2 \text{ so things work out cleaner.}}$

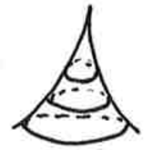
To show that $f(x,y)$ is differentiable at $(1,1)$, we see if the tangent plane is a good approximation by taking the limit:

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\|(x,y)-(a,b)\|} = \lim_{(x,y)\to(1,1)} \frac{(5-x^2-y^2)-(7-2x-2y)}{\sqrt{(x-1)^2+(y-2)^2}} = \lim_{(x,y)\to(1,1)} \frac{-(x^2-2x+1+y^2-2y+1)}{\sqrt{(x-1)^2+(y-2)^2}}$$

$$= \lim_{(x,y)\to(1,1)} \frac{-((x-1)^2+(y-1)^2)}{\sqrt{(x-1)^2+(y-1)^2}} = \lim_{(x,y)\to(1,1)} \frac{-\left(\sqrt{(x-1)^2+(y-1)^2}\right)^{\cancel{2}}}{\cancel{\sqrt{(x-1)^2+(y-1)^2}}} = \lim_{(x,y)\to(1,1)} -\sqrt{(x-1)^2+(y-1)^2} = \underline{\underline{0}}.$$

So the plane is a good approximation, so f is diff'ble!

So, what does it look like when a function is not differentiable at a point?

Geometrically: a sharp cusp  or crease 

Analytically: the limit $\displaystyle\lim_{(x,y)\to(a,b)} \frac{f(x,y) - (\text{tangent plane at } (a,b))}{\|(x,y)-(a,b)\|} \neq 0$ because no tangent plane approximates the function well there.

Example. $f(x,y) = \big||x|-|y|\big| - |x| - |y|$ will turn out to be non-differentiable at the origin (see cardboard model in class).

First, let's find a candidate tangent plane at $(0,0)$.
- taking the partial derivative with respect to x while treating y as constant seems complicated here, so let's use the definition: set $y=0$ and take the limit as $x \to 0$.

at $(0,0)$: $f_x(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \big||x|-|0|\big| - |x| - |0| = \lim_{x\to 0} \big||x|\big| - |x| = \lim_{x\to 0} |x| - |x| = \lim_{x\to 0} 0 = \underline{\underline{0}}.$

now let's do the same for y: set $x=0$ and take the limit as $y \to 0$.

at $(0,0)$: $f_y(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \big||0|-|y|\big| - |0| - |y| = \lim_{y\to 0} \big||y|\big| - |y| = \lim_{y\to 0} |y| - |y| = \lim_{y\to 0} 0 = \underline{\underline{0}}.$

and the function value:

$f(0,0) = \big||0|-|0|\big| - |0| - |0| = 0.$

So our candidate tangent plane equation is

$$\begin{aligned} z = L(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) \\ &= 0 + 0\,(x-0) + 0\,(y-0) \\ &= \underline{\underline{0}}. \end{aligned}$$

OK, now let's write down the limit that we expect to come out $\neq 0$:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - (\text{tangent plane at } (0,0))}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{\big||x|-|y|\big| - |x| - |y|}{\|(x,y)\|}$$

For the limit to exist and be 0, it has to exist and be 0 from every direction of approach. Let's approach along the line $y = x$.

$$\left(\begin{array}{l}\text{along} \\ y=x:\end{array}\right) \quad \lim_{x\to 0} \frac{\big||x|-|x|\big| - |x| - |x|}{\|(x,x)\|} = \lim_{x\to 0} \frac{-2|x|}{\sqrt{2}\,|x|} = \lim_{x\to 0} -\sqrt{2} = \underline{\underline{-\sqrt{2}.}} \quad \neq 0$$

So the candidate tangent plane does not approximate the function well at $(0,0)$, so the function is not differentiable at $(0,0)$.