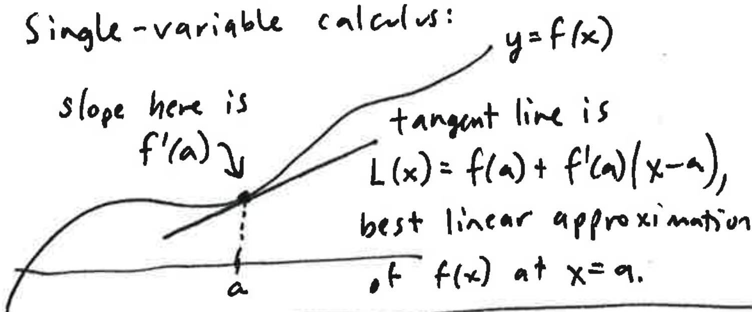
Diana Davis Math 34 February 2, 2018 Class #5 (1)
§2.3

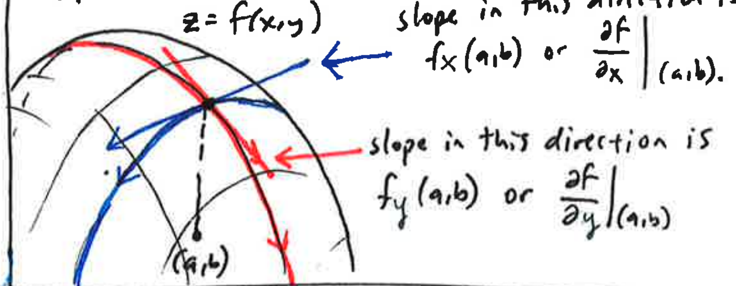
Mathematician spotlight: Rich Schwartz, Brown University
- geometry, dynamical systems
- children's books explaining serious math (e.g. infinity)

Last time: multivariable limits.
This time: partial derivatives, like "slopes" in a given direction, and tangent planes.

Single-variable calculus:



Multivariable calculus:



To find the "slope" or "tilt" of the surface at the desired point, we give the slope in the two axis-parallel directions:
- slope in positive x-direction is f_x
- " " y-direction is f_y.

We can use these to find the tangent plane to $z=f(x,y)$ at the point $(a,b,f(a,b))$, which is the best linear approximation of f at (a,b).

Example. Find the partial derivatives of $f(x,y)=5-x^2-2y^2$ at $(1,1)$.

(1) To find it in the x-direction, we can set $y=1$ and take the derivative with respect to x:

$$f(x,y)=5-x^2-2y^2$$
$$f(x,1)=5-x^2-2=3-x^2$$
$$\frac{d}{dx}f(x,1)=-2x$$
at $x=1$, $\frac{d}{dx}f(x,1)=-2=f_x(1,1)$.

(2) To find it in the y-direction, do the same for y, setting $x=1$: $f(1,y)=5-1-2y^2=4-2y^2$

$$\frac{d}{dy}f(1,y)=-4y$$
at $y=1$, $\frac{d}{dy}f(1,y)=-4=f_y(1,1)$.

In practice, we don't plug in a number for the other variable; we just treat it "as a constant": $f(x,y)=5-x^2-2y^2$

$$f_x(x,y)=-2x \qquad f_y(x,y)=-4y$$
$$f_x(1,1)=-2 \qquad f_y(1,1)=-4$$

Notation: $\left.\frac{\partial f}{\partial x}\right|_{(a,b)}=f_x(a,b)$ mean the same thing.
$\frac{\partial f}{\partial x}$ is partial derivative, different from total derivative $\frac{df}{dx}$.


"del"

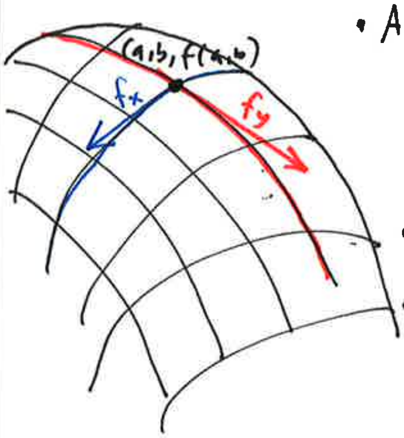
Practice taking a partial derivative. Note that chain rule, product rule, etc. still apply.

$g(x,y) = x^2 y + e^x \cdot \sin(xy).$

$\frac{\partial g}{\partial x} =$ ______________________

$\frac{\partial g}{\partial y} = x^2 + e^x \cdot \cos(xy) \cdot x.$

Now we want to find the tangent plane to $z = f(x,y)$ at the point $(a, b, f(a,b))$.



- A vector tangent to the surface in the x-direction is given by $\begin{bmatrix} 1 \\ 0 \\ f_x(a,b) \end{bmatrix}$ ← y is not changing; ← amount of rise, for run of 1 in the x-direction
- A vector tangent to the surface in the y-direction is given by $\begin{bmatrix} 0 \\ 1 \\ f_y(a,b) \end{bmatrix}$.
- To find the tangent plane to the surface at (a,b), we need:
  ① a point on the plane: ______________
  ② a normal vector to the plane: ______________

Find the normal vector:

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x(a,b) \\ 0 & 1 & f_y(a,b) \end{vmatrix} = \vec{i}\left(-f_x(a,b)\right) - \vec{j}\left(f_y(a,b)\right) + \vec{k}(1) = \begin{bmatrix} -f_x(a,b) \\ -f_y(a,b) \\ 1. \end{bmatrix}$$

The equation for the plane through (x_0, y_0, z_0) with $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$,

so " " " " $(a, b, f(a,b))$ " $\vec{n} = \begin{bmatrix} -f_x(a,b) \\ -f_y(a,b) \\ 1 \end{bmatrix}$ is given by

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + 1(z - f(a,b)) = 0.$$

rearranging the terms: $z = \underbrace{f(a,b)}_{\text{number}} + \underbrace{f_x(a,b)}_{\text{number}}(x - \underset{\uparrow \text{number}}{a}) + \underbrace{f_y(a,b)}_{\text{number}}(x - \underset{\nwarrow \text{number}}{b}).$

Example: Find the tangent plane to $z = 5 - x^2 - 2y^2$ at $(1,1,2)$.

We know: $f_x(1,1) = -2$, $f_y(1,1) = -4$ $\Rightarrow$ $z = 2 + (-2)(x-1) + (-4)(y-1)$
$= 8 - 2x - 4y.$

This is the best linear approximation of $f(x,y) = 5 - x^2 - 2y^2$ at $(1,1)$:
- the value is the same
- the partial derivatives are the same.

$L(x,y) = 8 - 2x - 4y$
$L(1,1) = 2$ ✓
$L_x(1,1) = -2$ ✓
$L_y(1,1) = -4$ ✓
} same as for $f(x,y)$.