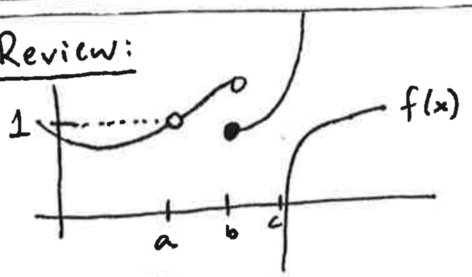
Diana Davis Math 34 January 31, 2018 Class #5 ①
§2.2

Mathematician spotlight: Emily Riehl, Assistant Professor, Johns Hopkins
· category theory, homotopy theory

Last time, we used cross-sections of a surface to figure out what it looks like.

This time, we'll think about what can "go wrong" when surfaces have vertical parts, etc.

Review:



$\lim_{x\to a} f(x) = 1$ (and it exists)

$\lim_{x\to b} f(x)$ does not exist (two different values) $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

$\lim_{x\to c} f(x)$ does not exist (vertical asymptote)

For limits of multivariable functions, you must approach the point from all directions.

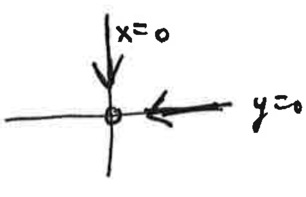
Example. $\lim_{(x,y)\to(1,2)} x^2+y^2 = 1^2+2^2 = 5$

If $f(x,y)$ is a product or sum of functions that are continuous at (a,b), just plug in (a,b) for the limit.

Example. $\lim_{(x,y)\to(0,0)} \frac{x+y}{2x+y}$

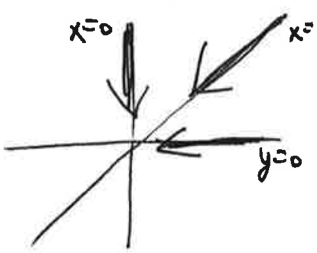
① Approach $(0,0)$ from the "east" $(y=0)$: $\lim_{(x,y)\to(0,0)} \frac{x+y}{2x+y} = \lim_{x\to 0} \frac{x+0}{2x+0} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$

② Approach $(0,0)$ from the "north" $(x=0)$: $\lim_{(x,y)\to(0,0)} \frac{x+y}{2x+y} = \lim_{y\to 0} \frac{0+y}{0+y} = \lim_{x\to 0} \frac{1}{1} = 1.$



The limit DOES NOT EXIST, because function value depends on the direction of approach.

Example. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$



① along $x=0$: $\lim_{y\to 0} \frac{0\cdot y}{0+y^2} = \lim_{y\to 0} \frac{0}{y^2} = \lim_{y\to 0} 0 = 0.$

② along $y=0$: $\lim_{x\to 0} \frac{x\cdot 0}{x^2+0} = \lim_{x\to 0} \frac{0}{x^2} = \lim_{x\to 0} 0 = 0.$

③ along $y=x$: $\lim_{x\to 0} \frac{x\cdot x}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$

again, limit DNE because value of f depends on direction of approach.

CLEVER TRICK: approach along all lines at once, using $y=mx$.
(except $x=0$)

④ along $y=mx$: $\lim_{x\to 0} \frac{x\cdot mx}{x^2+(mx)^2} = \lim_{x\to 0} \frac{mx^2}{x^2(1+m^2)} = \lim_{x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$

limit DNE, value depends on slope (direction)

TAKEAWAY MESSAGE: If you think the limit does not exist, prove it by approaching on lines $y=mx$ and showing that the value depends on m.

Okay, now we know a method for showing that a limit does not exist.
How do we show that it does exist? Is checking all lines enough?

1. No, lines are not enough, because you could also approach along other curves.
2. To show the limit does exist, convert to polar (or spherical in $\mathbb{R}^3$).

Example. Here, the limit when approaching along all lines is 0, but the limit when approaching along a parabola is not! So the limit DNE.

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^4+y^2)^3}$$

$$\begin{pmatrix}\text{along}\\ y=mx:\end{pmatrix} : \lim_{x\to 0} \frac{x^4(mx)^4}{(x^4+(mx)^2)^3} = \lim_{x\to 0} \frac{x^8\cdot m^4}{x^6(x^2+m^2)^3} = \lim_{x\to 0} x^2\left(\frac{m^4}{(x^2+m^2)^3}\right) = 0.$$

$$\begin{pmatrix}\text{along}\\ y=x^2:\end{pmatrix} : \lim_{x\to 0} \frac{x^4(x^2)^4}{(x^4+(x^2)^2)^3} = \lim_{x\to 0} \frac{x^{12}}{(2x^4)^3} = \lim_{x\to 0} \frac{x^{12}}{8x^{12}} = \lim_{x\to 0}\frac{1}{8} = \frac{1}{8}.$$

So the limit DNE, because it depends on the direction of approach.

CLEVER TRICK: (or, the only way to show that a limit exists): approach from all directions at once by converting to polar (r,θ) and do $\lim_{r\to 0}$.

Example.

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = \lim_{r\to 0}\frac{(r\cdot\cos\theta)^2}{\sqrt{r^2}} = \lim_{r\to 0}\frac{r^2\cdot\cos^2\theta}{r} = \lim_{r\to 0} \underbrace{r}_{\text{goes to } 0}\cdot\underbrace{\cos\theta}_{\text{finite}} = 0.$$

Also works in spherical coordinates: take $\lim_{\rho\to 0}$ to approach from all directions.

Example.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xz}{x^2+y^2+z^2} = \frac{(\rho\sin\varphi\cos\theta)(\rho\cos\varphi)}{\rho^2} = \lim_{\rho\to 0}\frac{\rho^2(\sin\varphi\cos\theta\cos\varphi)}{\rho^2}$$

$$= \lim_{\rho\to 0} \sin\varphi\cos\theta\cos\varphi \Rightarrow$$ limit DNE, because it depends on the direction of approach.

Good ways to make a computer draw graphs for you:

- Google (type in z = x^2 / (x^2 + y^2)^(1/2), for example)
- WolframAlpha.com
- Grapher (comes standard on every Apple computer)
- Many free apps